

Projective-plane determinant bounds for sparse binary matrices

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Abstract

For an integer $r \geq 3$ put $t = r - 1$, $v = r^2 - r + 1$ and $D_r = r t^{(v-1)/2}$, the determinant magnitude of a projective-plane incidence matrix of order t whenever such a plane exists. We prove that for every $3 \leq r \leq 14$ and every $n \times n$ matrix A with entries in $\{0, 1\}$ and e ones,

$$|\det A| \leq \exp(n \ell_r + \alpha_r (e - rn)), \quad \ell_r = \frac{1}{v} \log D_r,$$

with an explicit constant $\alpha_r > 0$. In particular $e \leq rn$ implies $|\det A| \leq D_r^{n/v}$, with no restriction on individual row or column degrees. Equality holds exactly for direct sums of incidence matrices of projective planes of order $r - 1$ (up to row and column permutations). The bound is attained at positive orders divisible by v for $r \in \{3, 4, 5, 6, 8, 9, 10, 12, 14\}$, is never attained at positive order for $r \in \{7, 11\}$, and is attained at some positive order for $r = 13$ if and only if a projective plane of order 12 exists. The case $r = 3$, $|\det A| \leq 24^{n/7}$, is the sparse determinant conjecture of Bruhn and Rautenbach, recorded as Conjecture 1 by Araujo, Balogh and Wang. The proof combines a degree-normalised Gram matrix, a quadratic Hermite majorant of log, an edgewise convex-tangent decomposition and cofactor peeling of low-degree lines. The only computer-assisted step is a finite exact-integer certificate of an explicit two-variable inequality for each r , with rigorous coverage of every unbounded degree tail.

1 Introduction

A bound on the total number of ones permits arbitrarily dense individual rows or columns. Thus total sparsity is distinct from a fixed row sum or simultaneous local degree caps. The strengthened all-count inequality below is essential: cofactor minors need not retain an original average-degree bound.

Ryser's fixed-order, fixed-total determinant bound has symmetric designs as its equality matrices [6, Theorem 3]. At projective-plane parameters $n = r^2 - r + 1$ and $e = rn$, it gives the value D_r below. Projective-plane blocks and their determinant rates are classical; see also Scheinerman [7, Section 2]. The question here is a uniform inequality for every order and degree distribution.

Bruhn and Rautenbach [2] obtained $6^{n/6}$ for at most $2n$ ones and proposed the Fano-plane rate for at most $3n$ ones. Shitov [9] proved $3^{(e-n)/4}$ for $e \geq n$. Araujo, Balogh and Wang [1, Theorem 3] improved this to $2^{(e-n)/3}$, giving $2^{2n/3}$ when $e \leq 3n$, and recorded the Fano statement as Conjecture 1 in their Section 6. Scheinerman [7, Theorem 7.1] also treats unrestricted total- kn sparsity, with bound c_k^n , where

$$c_k = (k^2 - 1)^{(k-1)/(4k)} k^{1/(2k)} < \sqrt{k}.$$

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His results are not confined to fixed row sums. Semonsen [8] gives further fixed-row-sum estimates through linear programming. Hu and Zuo [4, Theorem 4] also give a more recent unrestricted edge-count determinant estimate, alongside their separate work on C_4 -free perfect matchings. Their displayed determinant estimate does not give the projective-plane constants proved here. Our result supplies the projective-plane rate for the finite density range below and identifies all equality matrices.

All logarithms are natural. Binary entries are regarded as real numbers when taking norms and logarithms. We use the empty determinant convention $\det(\emptyset) = 1$. Blocks in a direct sum need not be isomorphic.

Theorem 1. *Let r be an integer with $3 \leq r \leq 14$, let $n \geq 0$ be an integer, put $t = r - 1$, $R = r^2$, $v = R - t$, and define*

$$\gamma_r = \frac{v/t - \log(R/t)}{v^2}, \quad \alpha_r = \frac{1}{r} - \frac{1}{2t} + \left(r - \frac{1}{2}\right)\gamma_r, \quad \ell_r = \frac{\log r + \frac{v-1}{2} \log t}{v}.$$

Then $\alpha_r > 0$ (since $1/r - 1/(2t) = (r-2)/(2rt) > 0$ and $\gamma_r > 0$, as $x-1 > \log x$ at $x = R/t > 1$), and every $n \times n$ binary matrix A with e ones satisfies

$$|\det A| \leq \exp(n\ell_r + \alpha_r(e - rn)). \quad (1)$$

Equality holds if and only if, after independent permutations of rows and columns, A is a direct sum of incidence matrices of projective planes of order $r-1$ (the empty matrix included).

Corollary 2. *For integers $3 \leq r \leq 14$ and $e \leq rn$, $|\det A| \leq D_r^{n/v}$. The bound is attained, for every n divisible by v , when $r-1$ is a prime power, i.e. for $r \in \{3, 4, 5, 6, 8, 9, 10, 12, 14\}$. For $r = 7$ and $r = 11$ it is never attained at positive order, since there are no projective planes of orders 6 (Bruck–Ryser [3]) and 10 (Lam, Thiel and Swiercz [5]). For $r = 13$ it is attained at some positive order if and only if a projective plane of order 12 exists. Moreover, $e < rn$ or $v \nmid n$ forces strict inequality in the total-budget bound.*

The existence classifications in this corollary are external design-theoretic facts. In particular, nonattainment does not by itself prove a uniform improvement in the asymptotic exponential rate. No such improvement is claimed here for $r = 7, 11$, or 13 .

Proof outline. Degree normalization gives a spectral cap without bounding any original degree. A logarithmic majorant converts the determinant into weighted Gram moments. A convex tangent decomposition couples the row and column deficits along actual incidences. A finite exact scalar certificate, including the entire degree tail, then gives the bound whenever all degrees exceed a small threshold. Cofactor expansion handles the remaining degrees. Vanishing of every lost term at equality forces actual isolated projective-plane blocks.

2 The normalised Gram matrix and the vertex deficit

Fix r and drop the subscript. Let A be nonsingular with all row degrees $d_i \geq 1$ and column degrees $c_j \geq 1$. Put $M_{ij} = A_{ij}/\sqrt{d_i c_j}$ and $G = R M M^\top$.

Lemma 3 (contraction). *The Euclidean operator norm satisfies $\|M\| \leq 1$; hence the eigenvalues of G lie in $(0, R]$.*

r	v	$D_r^{1/v}$	K	Anchor	$10^7 \min T_r$ exceeds
3	7	1.574610	1	linear	17482
4	13	1.847235	1	linear	37649
5	21	2.089198	1	linear	18842
6	31	2.308410	1	linear	10371
7	43	2.510041	1	linear	6167
8	57	2.697612	1	linear	3894
9	73	2.873633	1	linear	2579
10	91	3.039955	1	quadratic	1999
11	111	3.197992	2	linear	1262
12	133	3.348848	2	cubic	1035
13	157	3.493405	2	cubic	772
14	183	3.632379	2	cubic	588

Table 1: Finite-window certificate, excluding (r, r) . The final column gives strict rational bounds with denominator 10^7 . Rate decimals are illustrative only. All tail-box and peeling-margin lower bounds exceed $3/100$.

Proof. By Cauchy–Schwarz, $(\sum_j M_{ij}x_j)^2 \leq (\sum_j A_{ij}/d_i)(\sum_j A_{ij}x_j^2/c_j) = \sum_j A_{ij}x_j^2/c_j$. Summing over i gives $\sum_j x_j^2$. Positivity follows from $\det A \neq 0$. $\square$

Since $\det G = R^n(\det A)^2/(\prod d_i \prod c_j)$,

$$\log |\det A|^2 = \log \det G + \sum_i \log \frac{d_i}{r} + \sum_j \log \frac{c_j}{r}. \quad (2)$$

Hermite majorant. Let $q(z) = a + bz - \gamma z^2$ with $b = 1/t + 2\gamma t$, $a = \log t - bt + \gamma t^2$. Then $q(t) = \log t$, $q'(t) = 1/t$, $q(R) = \log R$. More explicitly, for $z > 0$,

$$q(z) - \log z = (z - t)^2(R - z) \int_0^\infty \frac{ds}{(s + t)^2(s + R)(s + z)}.$$

This identity follows by subtracting the Hermite interpolant of $1/(s + z)$ inside the integral representation of $\log z$. The integral converges and is strictly positive. Hence $\log z \leq q(z)$ on $(0, R]$, with equality exactly at $z \in \{t, R\}$. Applying this to every eigenvalue gives $\log \det G \leq na + b \operatorname{tr} G - \gamma \operatorname{tr} G^2$.

Moments. Let $u_i = \sum_j A_{ij}/(d_i c_j)$ and $w_j = \sum_i A_{ij}/(d_i c_j)$. Then $\operatorname{tr} G = R \sum_i u_i = R \sum_j w_j$, $\sum_i d_i u_i = \sum_j c_j w_j = n$, and expanding $\operatorname{tr}(MM^\top)^2$ over index quadruples gives

$$\operatorname{tr} G^2 = R^2 \left(\sum_i u_i^2 + \sum_j w_j^2 - \sum_{A_{ij}=1} \frac{1}{d_i^2 c_j^2} + 4W \right), \quad W \geq 0,$$

where $Q = \sum_{A_{ij}=1} 1/(d_i^2 c_j^2)$ and $W = \sum 1/(d_i d_{i'} c_j c_{j'})$ over $i < i'$, $j < j'$ with $A_{ij} = A_{ij'} = A_{i'j} = A_{i'j'} = 1$. The equal-row terms contribute $\sum_i u_i^2$, the equal-column terms contribute $\sum_j w_j^2$, and their intersection is subtracted once. Each rectangle has four orientations. This accounts for every term and the factor 4. Since $\frac{1}{d^2 c^2} \leq \frac{1}{2} \left(\frac{1}{d^3 c} + \frac{1}{d c^3} \right)$ (difference $\frac{(d-c)^2}{2d^3 c^3}$), we obtain the following inequality. The upper bound for Q has the correct sign: it gives a lower bound for $\operatorname{tr} G^2$, whose coefficient is $-\gamma$. The nonnegative rectangle loss is discarded.

$$\log |\det A|^2 \leq na + \sum_i F(d_i, u_i) + \sum_j F(c_j, w_j), \quad F(d, u) = \left(\frac{bR}{2} + \frac{\gamma R^2}{2d^2} \right) u - \gamma R^2 u^2 + \log \frac{d}{r}. \quad (3)$$

The vertex deficit. With $F_0 = F(r, 1/r) = br/2 - \gamma(R - r/2)$, define

$$q_r(d, u) = F_0 + \alpha(d - r) + \beta(du - 1) - F(d, u), \quad \beta = \frac{r}{2t} - \gamma\left(R + \frac{r}{2}\right).$$

The constants α, β are exactly those making q_r vanish to first order at the plane point $(d, u) = (r, 1/r)$, and $a + 2F_0 = 2\ell$. Indeed

$$a + 2F_0 = \log t + \frac{1}{t} - v\gamma = \log t + \frac{\log(R/t)}{v} = 2\ell.$$

These are algebraic identities; a symbolic replayer is supplied as a control. Summing (3) with $\sum_i d_i = \sum_j c_j = e$ and $\sum d_i u_i = \sum c_j w_j = n$ yields

$$\log |\det A| \leq n\ell + \alpha(e - rn) - \frac{1}{2} \left(\sum_i q_r(d_i, u_i) + \sum_j q_r(c_j, w_j) \right). \quad (4)$$

3 Edgewise convex tangents

For fixed d , $q_r(d, \cdot)$ is a convex quadratic ($\partial_u^2 q_r = 2\gamma R^2 > 0$). For an anchor function $x \mapsto A(x)$ with $A(1/r) = 1/r$, the supporting tangent at $A(1/d)$ gives $q_r(d, u) \geq q_r(d, A) + \partial_u q_r(d, A)(u - A)$ for all real u . A row vertex of degree d has $u = \frac{1}{d} \sum_{j \sim i} \frac{1}{c_j}$, so the right side splits over its d incident ones. Doing the same for columns, the total deficit in (4) is at least $\sum_{A_{ij}=1} T_r(1/d_i, 1/c_j)$ with

$$T_r(x, y) = g(x, y) + g(y, x), \quad g(x, y) = x \left[q_r\left(\frac{1}{x}, A(x)\right) + \partial_u q_r\left(\frac{1}{x}, A(x)\right) (y - A(x)) \right].$$

Multiplying through by x removes all singularities: g is a polynomial in x, y plus $x \log(rx)$. The anchors are

$$A_{\text{lin}}(x) = \frac{2}{5r} + \frac{3}{5}x, \quad A_{\text{quad}}(x) = -\frac{1}{100} + \frac{6}{5}x - x^2 \quad (r = 10), \quad A_{\text{cub}}(x) = x - \frac{2}{3}\left(x - \frac{1}{r}\right)^2 - \frac{3}{2}\left(x - \frac{1}{r}\right)^3.$$

At the plane point, $q_r = \partial_u q_r = 0$, hence $T_r(1/r, 1/r) = 0$ exactly. For the finite certificate we use the expanded expression

$$\begin{aligned} T_r(x, y) = & 2\alpha + (F_0 - r\alpha)(x + y) + x \log(rx) + y \log(ry) - bRxy \\ & + \gamma R^2 \left[2xy\{A(x) + A(y)\} - xA(x)^2 - yA(y)^2 - \frac{xy(x^2 + y^2)}{2} \right]. \end{aligned} \quad (5)$$

Set $x \log(rx) = 0$ at $x = 0$; the formula then extends continuously to the closed nonnegative quadrant. This explicit nonsingular form also specifies exactly what the certificate evaluates.

Lemma 4 (edge certificate). *For each $3 \leq r \leq 14$, with the anchor and integer K of Table 1, $T_r(1/d, 1/c) > 0$ for all integers $d, c \geq K + 1$ with $(d, c) \neq (r, r)$.*

Exact finite certificate. Put $N = 12r$ and $H = 1/(N + 1)$. By symmetry, all integer reciprocal degree pairs are covered by: (i) the points $(1/d, 1/c)$ with $K + 1 \leq d \leq c \leq N$; (ii) the boxes $\{1/d\} \times [0, H]$ for $K + 1 \leq d \leq N$; and (iii) the box $[0, H]^2$. Exclude the known zero (r, r) from (i).

The standard-library program `verify_family_exact.py` evaluates (5) by integer interval arithmetic with scale 10^{50} . Every logarithm is bounded using a rational odd-power series and an explicit tail. Appendix A gives the arithmetic and tail enclosure. Every listed point and box has a strictly positive lower endpoint; no box requires subdivision. Table 1 gives conservative rational finite-window bounds. Every mixed or double-tail lower bound exceeds $3/100$. The whole infinite degree range is therefore covered, rather than sampled. The plane point is zero by exact tangency, not by numerical enclosure. $\square$

4 Cofactor peeling

Lemma 5. *Suppose (1) holds for all binary matrices of order $n - 1$, and A (order n) has a row or column of degree k with $\log k + \alpha(r - k) < \ell$. Then (1) holds strictly for A .*

Proof. Transpose if needed. Laplace expansion along the line gives $|\det A| \leq \sum_j |\det A^{(j)}|$ over its k nonzero positions; $A^{(j)}$ has order $n - 1$ and at most $e - k$ ones. By induction and $\alpha > 0$, $|\det A| \leq k \exp((n - 1)\ell + \alpha(e - k - r(n - 1)))$, which is strictly below the right side of (1) by the hypothesis. $\square$

The certified margins $\ell - \alpha(r - k) - \log k$ are positive for $k = 1$ (all r) and for $k = 2$ when $11 \leq r \leq 14$; the smallest is 0.0352 ($r = 11, k = 2$).

5 Proof of Theorem 1

Induct on n ; the empty matrix gives equality and a singular matrix is trivial. A nonsingular matrix has every line degree ≥ 1 . If some line has degree k with $1 \leq k \leq K$, Lemma 5 gives strict inequality. Otherwise all degrees are $\geq K + 1$, and (4) with Lemma 4 gives (1).

6 Equality and actual incidence blocks

Suppose equality holds at positive order. The matrix is nonsingular and peeling is strict, so every degree exceeds K . Every incidence must have $T_r = 0$; hence all row and column degrees are r , and $G = AA^\top$. Equality in the Hermite step puts the spectrum of G in $\{t, R\}$. Equality in the discarded rectangle term forces $W = 0$, since $\gamma > 0$ and all rectangle weights are positive. Distinct rows therefore meet in at most one column.

The symmetric matrix $C = AA^\top - tI$ is binary with diagonal entries one and

$$C^2 = vC, \quad C\mathbf{1} = v\mathbf{1}.$$

The first identity follows from its spectrum $\{0, v\}$, and the second from r -regularity on both sides. Each row of C has v ones. If $C_{ij} = 1$, the dot product of rows i and j is v , so those binary rows are identical. If $C_{ij} = 0$, their dot product is zero. Since $C_{ii} = 1$, this partitions the rows into classes of exactly v elements, with C a direct sum of J_v .

Rows from different classes have Gram entry zero; a column cannot meet two classes. A class of v rows has rv incidences, while each associated column has degree r . It therefore has exactly v associated columns. Independent row and column permutations put A into square nonsingular blocks B with

$$BB^\top = tI + J_v, \quad B\mathbf{1} = B^\top\mathbf{1} = r\mathbf{1}.$$

Since $BJ_v = J_vB = rJ_v$, we also have $B^\top B = B^{-1}(tI + J_v)B = tI + J_v$. Thus each block is the incidence matrix of a symmetric $2-(v, r, 1)$ design, equivalently a projective plane of order t . This derives actual isolated blocks, rather than only their spectral parameters.

Conversely such a block has $|\det B|^2 = Rt^{v-1}$, so $|\det B| = D_r$, and it has rv ones. Determinants multiply and the affine exponent adds under block sums. The empty sum gives the order-zero equality. This completes the theorem.

7 Scope and limitations

Remark 6. With these anchors the edge inequality fails at (3,3) for $r = 15, 16$, while degree-3 cofactor peeling is not available there; the method as stated stops at $r = 14$. For $r \leq 6$ a simpler separable argument (bounding each $q_r(d, u)$ directly) already suffices; it fails for $r \geq 7$ near $d \approx 2r$.

The certificate package contains a fresh exact-integer scalar replayer and a separate replay of the earlier outward-rounded interval certificate. Finite-field constructions for $q \in \{2, 3, 4, 5, 7, 8, 9, 11, 13\}$ verify $AA^\top = qI + J$ and the determinant by exact Bareiss elimination. Symbolic identities are checked for general r . Exhaustive small-matrix numerical checks and hill-climbing searches are controls, not proof premises. No bounded matrix census establishes an unbounded matrix-order quantifier.

Density 16 has a separate proof using additional positive spectral terms and local resource allocation. A separate technical companion covers the unrestricted inequality, actual-plane equality classification, and literal-real-entry endpoints. That separate result neither enlarges the density range of Theorem 1 nor establishes the density-15 member or an all-density theorem. The classical projective-plane examples and the total-sparsity antecedents above are not claimed as new.

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A Exact certificate arithmetic

The only computational obligation in the proof is the explicit scalar lemma and the finitely many peeling margins. The supplied verifier uses only the Python standard library; all acceptance decisions compare integers.

For rational $q \in [1, 2]$, set $z = (q - 1)/(q + 1)$. With $M = 100$,

$$2 \sum_{j=0}^{M-1} \frac{z^{2j+1}}{2j+1} \leq \log q \leq 2 \sum_{j=0}^{M-1} \frac{z^{2j+1}}{2j+1} + \frac{2z^{2M+1}}{(2M+1)(1-z^2)}.$$

This is the positive-term $2 \operatorname{arctanh} z$ series; bounding all tail denominators below by $2M + 1$ gives the geometric remainder. Inversion and powers of 2 reduce every positive rational argument to $[1, 2]$; the same formula encloses $\log 2$.

Intervals are stored as integer endpoints divided by $S = 10^{50}$. Rational lower endpoints are rounded down and upper endpoints up. Addition is exact on the grid. For multiplication, take the minimum and maximum of the four endpoint products and round outward after division by S . For division by an interval of one sign, enclose its reciprocal then multiply. An interval containing zero is never inverted. Induction on the expression therefore proves enclosure for every polynomial operation in (5), including the coefficients γ, α, b, F_0 .

At a point $x = 1/d$, enclose $x \log(rx)$ directly by the series. For a tail, let $\hat{H} \geq 1/(N + 1)$ be the grid-rounded upper endpoint. The verifier checks $3r\hat{H} < 1$ and $\log 3 > 1$ exactly. Thus the derivative $\log(rx) + 1$ is negative on $(0, \hat{H}]$, so

$$x \log(rx) \in [\hat{H} \log(r\hat{H}), 0] \quad (0 \leq x \leq \hat{H}).$$

The lower endpoint is again enclosed by the rational series. The slightly enlarged closed interval preserves every reciprocal tail point; neither Euler's number nor a transcendental minimizer enters the program.

The program evaluates every unordered finite pair, $N - K$ mixed boxes, and the single double-tail box. Every retained lower endpoint must be positive. A timeout, an uncertain sign, or an unresolved box cannot count as a pass. Three deliberately altered configurations must be rejected: linear anchors at $r = 10$ and $r = 12$, and omitted degree-two peeling at $r = 13$. Rejection means not certified; it is not in itself a determinant counterexample. The exact zero at the plane point remains an algebraic identity.

The complete receipt records scale, series length, source SHA-256, pair counts, all lower bounds, and rejected controls. Standard and optimized Python runs agree byte for byte. A separate arithmetic review checks interval containment against exact fractions and derives (5) symbolically from the original tangent definition. The README supplies the build commands and the frozen artifact manifest.

B Formal statement correspondence

The companion Lean formalization provides the following statement correspondence. In namespace `SparseComponent.TotalFamily`:

- `determinant_total_family` proves the all-count integer bound for $3 \leq r \leq 14$ with only binary entries as a matrix hypothesis. `determinant_total_family_budget` specializes the budget, and `literal_total_family` states the explicit design base.
- `determinant_total_family_equality_iff` and its budget counterpart identify equality through actual row-overlap components and their projective-plane incidence property.
- `determinant_real_total_family` is a separately proved literal-real-entry interface. Equality is expressed on a canonical integer binary representative proved entrywise identical to the real matrix, with matching determinant and total count.
- `determinant_family_order_multiple` proves $n = (r^2 - r + 1) \#\{\text{actual components}\}$ at equality. Its nondivisibility counterpart gives strictness.

The formal scalar route uses exact rational kernel-reduced gates through $3r$ and an analytic reciprocal tail. It is distinct from the $12r$ cover in this paper. Concrete endpoints discharge every scalar premise; a generic arbitrary-density interface retaining such premises is not an all-density result.

The formalization uses only the standard axioms `propext`, `Classical.choice`, and `Quot.sound`. External projective-plane existence and nonexistence results are not formalized. Source versions, dependency pins, and reproducible build instructions are supplied with the companion code.

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