

REFLEXIVE DESCENT TO MINIMAL SEEDS IN THE KREUZER–SKARKE LIST

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ABSTRACT. Kreuzer and Skarke built the list of reflexive polytopes by growing it from small seed polytopes. We ask how far a polytope can be from its seeds when it is shrunk through reflexive polytopes only, one at a time and without skipping any. Such a descent always exists and always ends at a seed. In four dimensions no descent takes more than 348 steps, which proves the conjecture of a uniform transition depth, and in two and three dimensions the exact maxima are five and eleven. In four dimensions two polytopes with thirty-seven lattice points need exactly twenty-two steps, so the maximum lies between twenty-two and 348.

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1. INTRODUCTION

A lattice polytope is reflexive when it contains the origin in its interior and every facet lies on a hyperplane $\langle x, y \rangle = -1$ with y a lattice vector; equivalently, its polar dual is again a lattice polytope. Batyrev showed that a reflexive polytope Δ of dimension four gives a family of Calabi–Yau threefolds, the anticanonical hypersurfaces in the toric variety of the fan over the faces of Δ^* , and that the exchange of Δ and Δ^* is mirror symmetry [1]. Kreuzer and Skarke classified these polytopes: there are 16 in dimension two, 4,319 in dimension three [3] and 473,800,776 in dimension four [4], up to unimodular equivalence. We write KS_d for the list in dimension d .

The classification rests on a finiteness principle. A lattice polytope with the origin in its interior is called *minimal* when removing any one of its vertices and taking the convex hull of the others loses that property, and every reflexive polytope contains a minimal one spanned by some of its vertices [2, 4]. In dimension four Kreuzer and Skarke worked with a sharper notion: a reflexive polytope is *r-minimal* when it contains no reflexive polytope other than itself, and every reflexive polytope contains an *r-minimal* one. Both notions

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describe the seeds from which the whole list is grown, and the question studied here is how far a polytope is from its seeds.

The seed universality preprint [15] proposed that the list is generated from at most 64 seed orbits with Hodge numbers constant on each orbit. A later note [16] refuted that proposal, kept what is true near it, and isolated the following statement as the part that remained open.

Conjecture 1 (Uniform transition depth [16]). There is an absolute constant D such that every $\Delta \in \text{KS}_4$ is connected to some minimal seed by a chain of at most D subpolytope steps $\Delta_i \supset \Delta_{i+1}$ in which every intermediate hull is reflexive after unimodular equivalence. The stratification data suggest a value of D of order a few tens.

The note also asserted that descent through reflexive polytopes alone is not always possible, and offered a vertex-deletion chain with non-reflexive intermediate hulls as the substitute. We show that the assertion is false, prove the conjecture with an explicit constant, and determine the constant exactly in dimensions two and three.

To make the statement precise we use the order of the reflexive polytopes by inclusion. A reflexive polytope $\Gamma \subsetneq \Delta$ is a *cover* of Δ when no reflexive polytope lies strictly between them.¹ A *reflexive descent* of Δ is a chain

$$\Delta = \Delta_0 \supsetneq \Delta_1 \supsetneq \cdots \supsetneq \Delta_k$$

in which each Δ_{i+1} is a cover of Δ_i and Δ_k is r -minimal. Its length is k , and the *transition depth* $\delta(\Delta)$ is the least length of a reflexive descent of Δ . A descent never skips a reflexive polytope, so it is the strictest form of the chains in Conjecture 1; without the cover condition every polytope reaches a seed in one step, as we explain in Section 3. We write D_d for the largest transition depth in KS_d , and $|P|$ for the number of lattice points of a polytope P .

Theorem 2. *Every reflexive polytope has a reflexive descent, every reflexive descent ends at an r -minimal polytope, and every r -minimal polytope contains a minimal polytope. Hence every $\Delta \in \text{KS}_4$ is joined to a minimal seed by a chain in which every intermediate hull is reflexive and no reflexive polytope fits between two consecutive members.*

Theorem 3. *No reflexive descent in dimension four has more than 348 steps. More precisely, every reflexive descent of $\Delta \in \text{KS}_4$ has length at most $\min(|\Delta| - 6, 680 - |\Delta^*|)$ and at most the pair bound $L(|\Delta|, |\Delta^*|)$ of Section 4. The first number never exceeds 654 and the second never exceeds 348, and each maximum is attained by a single polytope. Hence $D_4 \leq 348$, and Conjecture 1 holds with $D = 349$.*

Theorem 4. $D_2 = 5$ and $D_3 = 11$. *In dimension two the maximum is attained only by the triangle with ten lattice points; in dimension three it is attained by exactly two polytopes, with 35 and 31 lattice points, both of which are contained in no larger reflexive polytope.*

Theorem 5. *The polytopes P_1 and P_2 of Section 7.2, with 37 lattice points each, have transition depth exactly 22, and P_1 is unimodularly equivalent to the polytope reached after ten steps of a reflexive descent of the simplex Δ_{680} with 680 lattice points, the largest reflexive polytope in dimension four. The transition depth of Δ_{680} is at most 31. Hence $22 \leq D_4 \leq 348$.*

Theorem 3 bounds every descent, not only the shortest one, so it also bounds the longest chain of covers below any polytope. The existence of some constant D is the easy part of the conjecture: the list is finite and Theorem 2 gives each polytope a descent. The content of Theorem 3 is that the constant can be read off from two numbers attached to each polytope, its lattice points and the lattice points of its dual, without examining any polytope. The same count, with Blichfeldt's bound on lattice points and the volume theorem of Balletti, Kasprzyk and Nill in place of the list, bounds the depth in every dimension: $D_d \leq 2(s_d - 1)^2 - 2$

¹Here and below all polytopes live in the same lattice M and have the origin as their only interior lattice point; a subpolytope of Δ is a lattice polytope contained in Δ . Reflexivity of a subpolytope is always meant with respect to M and the origin, which is the reading of “after unimodular equivalence” in Conjecture 1: a unimodular map that carries Γ into Δ fixes the origin and the lattice.

for $d \geq 4$, where s_d is Sylvester’s sequence (Proposition 11). The guess of “a few tens” in the conjecture is neither confirmed nor refuted. The exact values 5 and 11 in lower dimension, the depth 22 of two polytopes with thirty-seven points and the descent of length 31 from the largest polytope are consistent with it, but we do not know where D_4 lies between 22 and 348.

Theorem 2 is a short argument in the inclusion order (Section 3). Theorem 3 counts lattice points on both sides of the duality: along a descent the polytope loses points and its dual gains them, and both counts must stay among the pairs that occur in the list (Section 4). For the lower bounds we need all covers of a given polytope. Section 5 describes them as cuts along dual lattice points; the volume bound of Balletti, Kasprzyk and Nill [12] leaves finitely many cuts to try, and a climbing lemma certifies one cover without listing the others. In dimension two these tools give $D_2 = 5$ by hand, and in dimensions three and four they give the exact values and constructions of Sections 6 and 7. A lower bound for a depth is a statement about every descent, and in dimension four we prove it with a certificate: a finite set of polytopes with their complete lists of covers and a bound attached to each, subject to a condition that can be checked one polytope at a time (Lemma 17). The polytopes P_1 and P_2 are skew bipyramids over a three-dimensional reflexive tetrahedron, and their certificate contains 5,592 polytopes. Section 8 lists what remains open.

History. Reflexive polytopes entered algebraic geometry through Batyrev’s construction of mirror pairs of Calabi–Yau hypersurfaces [1]; the lattice-point formulas for $h^{1,1}$ and $h^{2,1}$ in that paper made the polytopes themselves the objects of study. Kreuzer and Skarke set out to list them. Their first paper [2] introduced the minimal polytopes, showed that a minimal polytope is either a simplex or a union of lower-dimensional simplices sharing the origin as an interior point, and attached a weight system to each simplex, so that the minimal polytopes in a given dimension come from finitely many combined weight systems. The three-dimensional list of 4,319 polytopes followed in 1998 [3]. In dimension four the combined weight systems number 201,346, too many to use directly, and Kreuzer and Skarke replaced minimality by r-minimality: only 308 combined weight systems give r-maximal polytopes, that is, duals of r-minimal ones, and together with 25 sublattice versions and 7 exceptional polytopes they give the 340 r-maximal reflexive polytopes of dimension four [4]. Every reflexive four-polytope is a subpolytope of one of these, and the complete list of 473,800,776 was obtained by enumerating subpolytopes with the program PALP [5]. The lattice-point counts range from 6 to 680 [4].

The inclusion order was used from the start to argue that the Calabi–Yau hypersurfaces form a connected web. Candelas, Green and Hübsch had shown that the complete intersections in products of projective spaces are connected by conifold transitions [6]; Avram, Kreuzer, Mandelberg and Skarke showed that the toric hypersurfaces from weighted projective spaces form a web connected by inclusions of polytopes [7]; and Kreuzer and Skarke observed that connectedness of all reflexive four-polytopes reduces to connectedness of the 340 r-maximal ones, which they checked during the classification [4]. Reid’s question whether all Calabi–Yau threefolds lie in one family connected by such transitions [8] remains open in general. Databases built on the list [9] made the joint statistics of polytopes and Hodge numbers easy to read, and the list is now distributed in a form that can be streamed in a few minutes [10].

The extremes of the list have a separate history. Lagarias and Ziegler bounded the volume of lattice polytopes with a fixed number of interior points [11]; for reflexive polytopes the sharp bound was proved by Balletti, Kasprzyk and Nill [12], who showed that in every dimension $d \geq 4$ the largest reflexive polytope is a simplex built from Sylvester’s sequence. In dimension four it is the simplex Δ_{680} with 680 lattice points, the Newton polytope of the hypersurface of degree 84 in the weighted projective space $\mathbb{P}(1, 1, 12, 28, 42)$, whose mirror is the Calabi–Yau threefold with the largest value of $h^{1,1}$ in the list.

The question of distance to the seeds came from the seed universality preprint [15]. Its orbit picture combined polar duality, stellar subdivision and flips, and claimed constant Hodge numbers on at most 64 orbits; the correcting note [16] disposed of that claim with two observations (duality exchanges the Hodge pairs $(1, 101)$ and $(101, 1)$ of the quintic and its mirror, and the list realises 30,108 Hodge pairs), proved the statements that survive at a fixed polytope, and stated Conjecture 1. The present paper answers it. Related work of the first author on Calabi–Yau geometry and the landscape includes [17].

2. REFLEXIVE POLYTOPES AND THEIR ORDER

Let $M \cong \mathbb{Z}^d$ be a lattice, $N = \text{Hom}(M, \mathbb{Z})$ its dual and $\langle \cdot, \cdot \rangle$ the pairing. For a polytope $P \subset M_{\mathbb{R}}$ with the origin in its interior the polar dual is $P^* = \{y \in N_{\mathbb{R}} : \langle x, y \rangle \geq -1 \text{ for all } x \in P\}$. A lattice polytope Δ is reflexive when Δ^* is a lattice polytope; then $\Delta^{**} = \Delta$, the facets of Δ are the sets $\{\langle x, v \rangle = -1\} \cap \Delta$ for the vertices v of Δ^* , and the origin is the only interior lattice point. For a reflexive Δ let $R(\Delta)$ be the set of reflexive subpolytopes of Δ .

Lemma 6. *Let $\Gamma \subsetneq \Delta$ be lattice polytopes with the origin in their interiors. Then*

- (i) $|\Gamma| < |\Delta|$;
- (ii) $\Delta^* \subsetneq \Gamma^*$, and if both are reflexive then $|\Delta^*| < |\Gamma^*|$;
- (iii) $R(\Delta)$ is finite, and every element of $R(\Delta)$ other than Δ lies in a cover of Δ .

Proof. (i) If every vertex of Δ lay in Γ , the convex hull Δ would lie in Γ ; so some vertex of Δ , a lattice point, is missing from Γ . (ii) Polarity reverses inclusion and is injective on polytopes with the origin inside, and (i) applies to the lattice polytopes $\Delta^* \subsetneq \Gamma^*$. (iii) A reflexive subpolytope is the convex hull of a set of lattice points of Δ , so there are at most $2^{|\Delta|}$ of them. Given $\Gamma \in R(\Delta)$ with $\Gamma \neq \Delta$, a maximal element of $\{\Gamma' \in R(\Delta) : \Gamma \subseteq \Gamma' \neq \Delta\}$ exists by finiteness and is a cover of Δ . $\square$

A polytope is r -minimal exactly when it has no cover. By Lemma 6(iii) a reflexive polytope that is not r -minimal has at least one cover, and we will need all of them.

3. THE LITERAL CONJECTURE AND REFLEXIVE DESCENT

Proof of Theorem 2. Let Δ be reflexive. If Δ is r -minimal the descent of length zero will do. Otherwise Lemma 6(iii) gives a cover Δ_1 , and we repeat with Δ_1 . By Lemma 6(i) the number of lattice points drops at each step, so the process stops, and it stops exactly at an r -minimal polytope. Any reflexive descent ends at an r -minimal polytope by definition.

Let Δ_k be r -minimal and let V be its vertex set. Remove vertices from V one at a time, keeping the origin in the interior of the convex hull of what is left, until no vertex can be removed. The hull S of the remaining set is a minimal polytope, its vertices are vertices of Δ_k , and $S \subseteq \Delta_k$. If $S \neq \Delta_k$ the step $\Delta_k \supsetneq S$ leaves no room for a reflexive polytope between the two, because Δ_k has no reflexive subpolytope other than itself. Appending this step to a reflexive descent gives the chain of the theorem. $\square$

Three readings of Conjecture 1 are now settled at once. If a “subpolytope step” may be any inclusion, the chain $\Delta \supsetneq S$ of the second paragraph applied to Δ itself has no intermediate hull at all, and $D = 1$; this is the minimal-polytope theorem of Kreuzer and Skarke [2] restated. If the seeds are the r -minimal polytopes, which are the seeds of the four-dimensional classification [4], a reflexive descent ends at a seed. If the seeds are minimal polytopes, at most one further step is needed. In every reading the chain length is controlled by the transition depth, which is the quantity bounded in Theorem 3.²

The correcting note [16] stated that descent through reflexive polytopes only is not always possible. Theorem 2 shows that it always is. What fails is descent by deleting vertices, which is the operation the note had in mind, and it fails in two ways.

Proposition 7. (i) *If Δ is a reflexive simplex, the convex hull of any proper subset of its vertices is a face of Δ and contains no interior point; so a reflexive simplex that is not r -minimal admits no descent by convex hulls of vertex subsets, although it has a reflexive descent.*
(ii) *Among the 4,303 reflexive three-polytopes that are not r -minimal, exactly 74 have no reflexive subpolytope of the form $\text{conv}((\Delta \cap M) \setminus \{v\})$ with v a vertex, and exactly 354 have none of the form $\text{conv}(V \setminus \{v\})$ with V the vertex set.*

²In dimensions two and three we checked which r -minimal polytopes are themselves minimal. The three r -minimal polygons are minimal; in dimension three, twelve of the sixteen r -minimal polytopes are minimal and four contain a smaller minimal polytope that is not reflexive. In dimension four Kreuzer and Skarke found seven r -minimal polytopes that do not come from weight systems, and each of them contains a single minimal subpolytope, a simplex that is not reflexive [4].

Proof. Part (i) is immediate: a proper subset of the vertices of a simplex spans a face of positive codimension. Part (ii) is a finite check on the list of Kreuzer and Skarke, recorded in the code archive [20]. $\square$

Example 8. The smallest polytope in part (ii) has 15 lattice points and vertices $(1, 0, 0)$, $(0, 1, 0)$, $(-1, -1, 0)$, $(0, 0, 1)$, $(-3, 2, -2)$, $(-2, 3, -2)$, $(-4, 4, -3)$. Removing any one vertex and taking the hull of the remaining fourteen lattice points gives a polytope that is not reflexive, yet the polytope has a reflexive descent of length 5. In dimension four the largest polytope Δ_{680} is a simplex, so by part (i) it cannot be reduced by vertex deletion at all, while Theorem 5 gives it a descent of length 31.

The vertex-deletion chain of the note does reach a minimal polytope, and its length is less than the number of vertices [16]; in dimension four the vertex counts range from 5 to 33 [4], so such chains have at most 28 steps. Its intermediate hulls are not reflexive in general, which is why the conjecture was stated with reflexive intermediates.

4. COUNTING LATTICE POINTS

The lattice points of a polytope and of its dual move in opposite directions along a descent, and in dimension four both counts are known for every polytope of the list. This is enough to bound the length of every descent. We write $\mathcal{P}_4 \subset \mathbb{Z}^2$ for the set of pairs $(|\Delta|, |\Delta^*|)$ with $\Delta \in \text{KS}_4$; both numbers are invariant under unimodular maps, and $\mathcal{P}_4$ is symmetric under $(p, q) \mapsto (q, p)$ because the list is closed under duality.

Definition 9. For $(p, q) \in \mathcal{P}_4$ let $L(p, q)$ be the largest k for which there are pairs $(p_0, q_0), \dots, (p_k, q_k)$ in $\mathcal{P}_4$ with $(p_0, q_0) = (p, q)$, $p_0 > p_1 > \dots > p_k$ and $q_0 < q_1 < \dots < q_k$. We call $L(p, q)$ the *pair bound* and write $L(\Delta) = L(|\Delta|, |\Delta^*|)$.

The pair bound satisfies $L(p, q) = 1 + \max L(p', q')$ over the pairs with $p' < p$ and $q' > q$, and $L(p, q) = 0$ when there is no such pair. Evaluating this recursion in increasing order of p computes L on all of $\mathcal{P}_4$ at once.

Lemma 10. Let $\Delta_0 \supsetneq \Delta_1 \supsetneq \dots \supsetneq \Delta_k$ be reflexive polytopes of dimension four, not necessarily a descent. Then

$$k \leq L(\Delta_0) \leq \min(|\Delta_0| - 6, 680 - |\Delta_0^*|).$$

Proof. By Lemma 6 the counts $|\Delta_i|$ strictly decrease and the counts $|\Delta_i^*|$ strictly increase. Each Δ_i is unimodularly equivalent to a member of KS_4 , so each pair $(|\Delta_i|, |\Delta_i^*|)$ lies in $\mathcal{P}_4$, and the pairs form a chain as in Definition 9. Hence $k \leq L(\Delta_0)$. In any such chain of pairs the first coordinates are distinct integers between p_k and p_0 and the second are distinct integers between q_0 and q_k . Every reflexive polytope of dimension four has at least six lattice points and at most 680 [4], which gives the second inequality.³ $\square$

The set $\mathcal{P}_4$ was computed by reading the two point counts of all 473,800,776 polytopes of the list [10]. It has 71,870 elements, its extreme values 6 and 680 agree with those recorded by Kreuzer and Skarke [4], and the counts add up to the size of the list. Table 1 and Fig. 1 summarise both bounds over the list.

Proof of Theorem 3. Lemma 10 applies to every reflexive descent. Over $\mathcal{P}_4$ the largest value of $\min(p - 6, 680 - q)$ is 654, attained only at $(680, 26)$, and this pair belongs to one polytope, the simplex Δ_{680} ; the next values are 634 at $(640, 26)$ and 633 at $(639, 27)$. The largest value of L is 348, attained only at $(521, 22)$, again by a single polytope, with vertices

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (1, 1, 1, 2), \\ (-3, -3, 1, -2), (-45, -38, -6, -30), (15, 22, -6, 30)$$

and Hodge numbers $(h^{1,1}, h^{2,1}) = (10, 376)$ for the hypersurface of which it is the Newton polytope. So every reflexive descent in dimension four has at most 348 steps, $D_4 \leq 348$, and with one more step to a minimal polytope (Theorem 2) Conjecture 1 holds with $D = 349$. $\square$

³The lower bound six is elementary: a full-dimensional lattice polytope in dimension four with an interior point has at least five vertices, and the origin is a further lattice point. The upper bound 680 is attained only by the simplex Δ_{680} of Section 7, as the list shows; Balletti, Kasprzyk and Nill proved that this simplex also has the largest volume among reflexive four-polytopes [12].

TABLE 1. Number of polytopes in KS_4 whose bound on the length of a reflexive descent exceeds a given value, for the simple bound $\min(|\Delta| - 6,680 - |\Delta^*|)$ and for the pair bound $L(\Delta)$. The note below the table gives the largest value of each bound and the point pair where it is attained.

bound exceeds	$\min(\Delta - 6,680 - \Delta^*)$	$L(\Delta)$
100	7,159,636	6,838,181
150	893,324	825,559
200	136,519	117,767
250	30,306	23,814
300	7,755	3,556
320	4,763	1,028
340	2,705	106
348	2,197	0
500	39	0
654	0	0

simple bound 654 at $(680, 26)$, the simplex Δ_{680} ; pair bound 348 at $(521, 22)$, a single polytope with seven vertices.

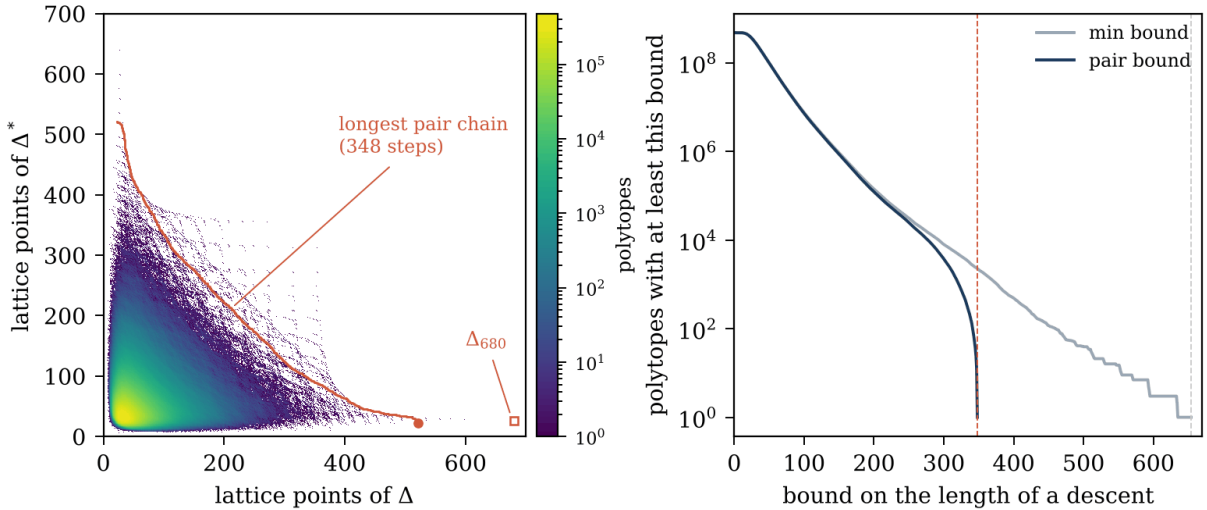


FIGURE 1. Left: the number of reflexive four-polytopes with a given number of lattice points (horizontal) and dual lattice points (vertical), on a logarithmic colour scale. A reflexive descent moves left and up through occupied cells. The red staircase is a longest chain of occupied cells, with 348 steps from $(521, 22)$ to its mirror image $(22, 521)$; the dot marks its start and the square marks the simplex Δ_{680} at $(680, 26)$. Right: for each value X , the number of polytopes whose bound on the length of a descent is at least X , for the simple bound and the pair bound.

The pair bound of Δ_{680} is 347. A longest chain of pairs from $(521, 22)$ runs close to the upper edge of the occupied region and ends at the mirror pair $(22, 521)$ (Fig. 1). The chain is a property of the point counts only; nothing in the argument says that a descent follows it, and the descents we can compute are much shorter. For the polytope at $(521, 22)$, two covers found by the method of the next section lead to a polytope with 257 lattice points and 24 dual points, whose pair bound is 247, so that polytope has a descent of length at most 249.

In other dimensions the joint distribution of point counts is not known, but the extremes are, and they already bound the depth. Let $s_1 = 2$ and $s_{k+1} = s_k^2 - s_k + 1$ be Sylvester's sequence, so that $s_2, s_3, s_4, s_5 = 3, 7, 43, 1807$.

Proposition 11. *Let V_d be the largest normalised volume of a reflexive polytope of dimension d . Every chain $\Delta_0 \supsetneq \Delta_1 \supsetneq \dots \supsetneq \Delta_k$ of reflexive polytopes of dimension d has*

$$k \leq V_d - 2.$$

In particular $D_d \leq V_d - 2$ for every $d \geq 2$, and $D_d \leq 2(s_d - 1)^2 - 2$ for every $d \geq 4$.

Proof. A lattice polytope P of dimension d has at most $\text{nvol } P + d$ lattice points; this is Blichfeldt's inequality [13].⁴ So $|\Delta_0| \leq V_d + d$. A reflexive polytope of dimension d has at least $d + 1$ vertices and the origin as a further lattice point, so $|\Delta_k| \geq d + 2$. By Lemma 6(i) the counts $|\Delta_i|$ strictly decrease along the chain, hence

$$k \leq |\Delta_0| - |\Delta_k| \leq (V_d + d) - (d + 2) = V_d - 2.$$

A reflexive descent is such a chain, which gives $D_d \leq V_d - 2$. Every reflexive polytope Δ is the dual of the reflexive polytope Δ^* , and reflexive polytopes are canonical Fano polytopes, so the theorem of Ballelli, Kasprzyk and Nill [12], that the dual of a canonical Fano polytope of dimension $d \geq 4$ has normalised volume at most $2(s_d - 1)^2$, gives $V_d \leq 2(s_d - 1)^2$ for $d \geq 4$. $\square$

The bound is doubly exponential in d , and in low dimensions it is weak: $V_2 = 9$ and $V_3 = 72$ give $D_2 \leq 7$ and $D_3 \leq 70$ against the exact values 5 and 11 of Theorem 4, and in dimension four it gives $D_4 \leq 3526$ against 348. Its point is that it needs no classification. The dual side gives nothing better: the counts $|\Delta_i^*|$ rise from at least $d + 2$ to at most $V_d + d$, which is the same bound.

5. COVERS AS CUTS

To compute transition depths we need all covers of a reflexive polytope. In dimension two they are easy to list (Section 6), but in dimension four a polytope can have hundreds of lattice points and enumerating subsets is out of the question. The covers turn out to be cuts by single dual lattice points.

For a reflexive Δ and $q \in N$ put

$$H_q = \text{conv}\{x \in \Delta \cap M : \langle x, q \rangle \geq -1\},$$

the part of Δ that survives the cut by the hyperplane $\langle x, q \rangle = -1$.

Lemma 12 (Cut lemma). *Let $\Gamma \subseteq \Delta$ be reflexive polytopes and q a lattice point of Γ^* . Then $\Gamma \subseteq H_q$.*

Proof. Every lattice point x of Γ lies in Δ and satisfies $\langle x, q \rangle \geq -1$ because $q \in \Gamma^*$, so it is among the points whose convex hull is H_q ; and Γ is the convex hull of its lattice points. $\square$

For a vertex v of Δ let $F_v = \{y \in \Delta^* : \langle v, y \rangle = -1\}$ be the dual facet and $w_v = \text{nvol conv}(\{0\} \cup F_v)$ its normalised volume seen from the origin, so that $\text{nvol } \Delta^* = \sum_v w_v$. Let V_d be the largest normalised volume of a reflexive polytope of dimension d . The classification gives $V_2 = 9$ and $V_3 = 72$, and Ballelli, Kasprzyk and Nill proved $V_4 = 2 \cdot 42^2 = 3528$, attained by Δ_{680} [12].

Proposition 13. *Let Γ be a cover of the reflexive polytope $\Delta \subset M_{\mathbb{R}}$ of dimension d , and let q be a vertex of Γ^* that does not lie in Δ^* . Then*

- (a) $q \in N$, and $\langle v, q \rangle \leq -2$ for some vertex v of Δ ;
- (b) $\langle x, q \rangle = -1$ for d linearly independent lattice points x of Δ ;
- (c) $\sum_v w_v \max(1, -\langle v, q \rangle) \leq V_d$, the sum running over the vertices of Δ ;
- (d) $\Gamma \subseteq H_q \subsetneq \Delta$, and either $H_q = \Gamma$, or H_q is not reflexive and Γ is a maximal reflexive polytope inside it.

⁴Blichfeldt states it as $|P \cap \mathbb{Z}^d| \leq d! \text{vol } P + d$ for a convex body containing $d + 1$ affinely independent lattice points, and $d! \text{vol}$ is the normalised volume nvol . The bound is attained by the simplices $\text{conv}(0, e_1, \dots, e_d)$ and is far from attained by reflexive polytopes: in dimension four it gives 3532 against the true maximum 680.

Proof. Since Γ is reflexive, Γ^* is a lattice polytope and its vertex q lies in N . As $q \notin \Delta^* = \{y : \langle v, y \rangle \geq -1 \text{ for all vertices } v\}$, some vertex has $\langle v, q \rangle < -1$, hence ≤ -2 . This is (a). The vertex q of Γ^* corresponds to a facet of Γ lying on the hyperplane $\langle x, q \rangle = -1$. That facet is a lattice polytope of dimension $d-1$, so it has d affinely independent lattice points; they are linearly independent because the hyperplane misses the origin, and they are lattice points of Δ . This is (b).

For (c), Γ^* is a convex set containing Δ^* and q , so it contains $C = \text{conv}(\Delta^* \cup \{q\})$. The polytope C is the union of Δ^* and the pyramids with apex q over the facets F_v visible from q , that is, those with $\langle v, q \rangle < -1$, and these pieces meet only along their boundaries. The lattice distance from q to the hyperplane of F_v is $-\langle v, q \rangle - 1$, while the distance from the origin is 1, so the pyramid over F_v has normalised volume $w_v(-\langle v, q \rangle - 1)$. Adding $\text{nvol} \Delta^* = \sum_v w_v$ gives $\text{nvol} C = \sum_v w_v \max(1, -\langle v, q \rangle)$. Since Γ^* is reflexive, $\text{nvol} C \leq \text{nvol} \Gamma^* \leq V_d$.

For (d), the cut lemma gives $\Gamma \subseteq H_q$. The vertex v of (a) is a lattice point of Δ that is not in H_q , so $H_q \subsetneq \Delta$. If H_q is reflexive, it lies strictly between Γ and Δ or equals Γ , and the first is excluded because Γ is a cover. Otherwise any reflexive Γ' with $\Gamma \subseteq \Gamma' \subseteq H_q$ satisfies $\Gamma \subseteq \Gamma' \subsetneq \Delta$, so $\Gamma' = \Gamma$. $\square$

Conditions (a) and (c) confine q to a bounded set: by (c) each $-\langle v, q \rangle$ is at most V_d/w_v , and the vertices of Δ span $M_{\mathbb{R}}$ positively. So only finitely many cuts have to be tried.

Corollary 14. *Let $\mathcal{C}(\Delta)$ be the set of reflexive polytopes obtained as follows: for every q satisfying (a), (b) and (c), take H_q if it is reflexive, and otherwise its maximal reflexive subpolytopes. The covers of Δ are exactly the maximal elements of $\mathcal{C}(\Delta)$ under inclusion.*

Proof. Every element of $\mathcal{C}(\Delta)$ is a reflexive polytope strictly inside Δ , and by Proposition 13 every cover belongs to $\mathcal{C}(\Delta)$. A cover is maximal among all reflexive proper subpolytopes, hence maximal in $\mathcal{C}(\Delta)$. Conversely, a maximal element X of $\mathcal{C}(\Delta)$ lies in some cover C by Lemma 6(iii), and $C \in \mathcal{C}(\Delta)$, so $X = C$. $\square$

Fig. 2 shows a cut in dimension two. In dimension two the second case of Proposition 13(d) never occurs: H_q contains Γ , hence the origin in its interior, and lies in Δ , so the origin is its only interior lattice point, and a lattice polygon with exactly one interior lattice point is reflexive [14]. From dimension three on a lattice polytope with one interior lattice point need not be reflexive, and the argument fails. Yet the second case does not occur in dimension three either: over all 4,319 reflexive polyhedra, every cut H_q formed by the procedure of Corollary 14 is reflexive, and for each of the 32,129 pairs of a cover Γ and a vertex q of Γ^* outside Δ^* the cut H_q is Γ itself. In dimension four the same holds at each of the 5,250 polytopes of the certificate of Proposition 18 that have covers: every cut H_q formed by the procedure is reflexive, and for each of the 50,280 pairs of a cover Γ and a vertex q of Γ^* outside Δ^* the cut H_q is Γ itself. Whether the second case occurs anywhere in dimension four is open. The programs allow for the second case and then find the maximal reflexive subpolytopes by removing vertices one at a time.

Listing all covers is still expensive for the largest polytopes. To walk down from a polytope it is enough to certify one cover at a time, and for that the following lemma needs only the dual lattice points of the candidate.

Lemma 15 (Climbing lemma). *Let $\Gamma \subsetneq \Delta$ be reflexive. If $H_{q'} = \Gamma$ for every lattice point q' of Γ^* outside Δ^* , then Γ is a cover of Δ .*

Proof. Suppose $\Gamma \subsetneq \Gamma' \subsetneq \Delta$ with Γ' reflexive. Then $\Delta^* \subsetneq \Gamma'^* \subsetneq \Gamma^*$, so the lattice polytope Γ'^* has a vertex $q' \notin \Delta^*$, and q' is a lattice point of Γ'^* and of Γ^* . The cut lemma, applied to $\Gamma' \subseteq \Delta$ and q' , gives $\Gamma' \subseteq H_{q'} = \Gamma$, a contradiction. $\square$

The lemma turns into a procedure. Start from a cut q with H_q reflexive. For each lattice point q' of Γ^* outside Δ^* , the polytope $H_{q'}$ contains Γ by the cut lemma; if it is reflexive and larger, replace Γ by it and start again. Since the polytopes grow inside Δ , this stops. When it stops with $H_{q'} = \Gamma$ for every such q' , the lemma certifies Γ as a cover; every step of the descent of Section 7 that is attributed to the climbing lemma was certified in this way. Choosing the first cut so that Γ is small gives descents that lose many points per step, which is how the descent of Section 7 was found.

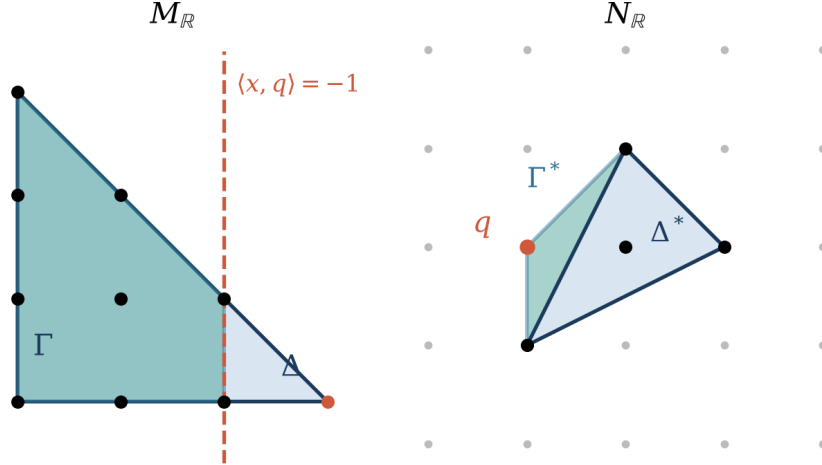


FIGURE 2. A cover as a cut. Left: a reflexive triangle Δ with ten lattice points and the cover Γ (shaded) obtained by removing the lattice points with $\langle x, q \rangle < -1$. Right: the duals; Γ^* is Δ^* with the new vertex q , and the added piece is the pyramid of Proposition 13(c) over the facet of Δ^* visible from q .

6. DIMENSIONS TWO AND THREE

In dimension two covers are simple.

Proposition 16. *If Γ is a cover of a reflexive polygon Δ , then $|\Gamma| = |\Delta| - 1$. Every r -minimal polygon $E \subseteq \Delta$ is the last member of some reflexive descent of Δ , and the transition depth of a reflexive polygon is $\delta(\Delta) = |\Delta| - m(\Delta)$, where $m(\Delta)$ is the largest number of lattice points of an r -minimal polygon inside Δ .*

Proof. A lattice polygon with exactly one interior lattice point is reflexive [14]. Let $\Gamma \subsetneq \Delta$ be reflexive. Some vertex v of Δ is not in Γ , and $\Delta' = \text{conv}((\Delta \cap M) \setminus \{v\})$ contains Γ , hence the origin in its interior, and has no other interior lattice point; so Δ' is reflexive, contains Γ , and has exactly $|\Delta| - 1$ lattice points, since a vertex is not in the hull of the other lattice points. If Γ is a cover, then $\Gamma = \Delta'$.

The same argument, applied to any reflexive P with $E \subsetneq P \subseteq \Delta$ and a vertex of P outside E , gives a cover of P that still contains E , so repeating it gives a descent of Δ that ends at E , of length $|\Delta| - |E|$. Every descent loses one point per step and ends at an r -minimal polygon, which gives the formula. $\square$

Proof of Theorem 4 in dimension two. There are sixteen reflexive polygons, listed in Appendix A with their covers; three are r -minimal, the square $\text{conv}(\pm e_1, \pm e_2)$ and the triangle with vertices $(1, 0), (0, 1), (-2, -1)$ with five lattice points each, and the triangle with vertices $(1, 0), (0, 1), (-1, -1)$ with four. Every polygon with at least six lattice points contains one of the two r -minimal polygons with five points (the table names one), so by Proposition 16 its transition depth is $|\Delta| - 5$. The largest polygon is the triangle $\text{conv}\{(-1, -1), (2, -1), (-1, 2)\}$ with ten lattice points; it contains the square, so its depth is 5, and every other polygon has at most nine points and depth at most 4. Hence $D_2 = 5$, attained only by this triangle. It also contains the triangle with four points, so its longest descent has 6 steps. $\square$

Fig. 3 shows the whole order. The argument uses only that in the plane one interior point forces reflexivity; that fails from dimension three on, and there covers can drop many points at once.

In dimension three we computed the covers of all 4,319 reflexive polytopes in two independent ways. The first enumerates, for each polytope, all lattice polytopes obtained by repeatedly removing a vertex and taking the convex hull of the remaining lattice points, as long as the origin stays inside; up to unimodular equivalence this gives 665,599 polytopes, among them every reflexive subpolytope, and the covers are read off as maximal elements. The second uses Corollary 14 with $V_3 = 72$. The two agree on all 4,303 polytopes that are not r -minimal, and give 21,841 covers in all.

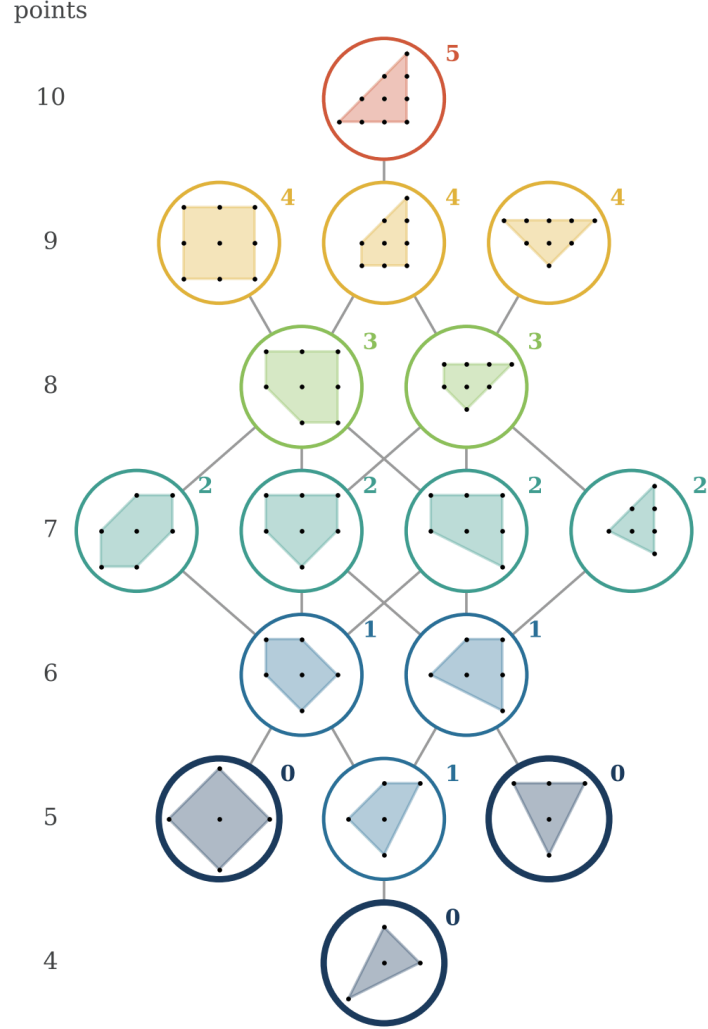


FIGURE 3. The sixteen reflexive polygons ordered by inclusion, by number of lattice points; edges are covers and the number beside each polygon is its transition depth. The three r -minimal polygons are drawn with thick circles.

Proof of Theorem 4 in dimension three. With the cover relation known, transition depths follow by induction on the number of lattice points: $\delta(\Delta) = 0$ if Δ has no cover and $\delta(\Delta) = 1 + \min \delta(\Gamma)$ over its covers otherwise. The maximum is 11, attained by exactly two polytopes. The first has 35 lattice points and vertices

$$(1, 0, 0), (1, 2, 0), (-3, -2, 0), \\ (1, 2, 4), (-3, -2, -4),$$

and the second has 31 lattice points and vertices

$$(1, 0, 0), (0, 1, 0), (1, 0, 2), \\ (1, -4, -2), (-2, 3, 2), (-2, -1, -2).$$

Neither is contained in a larger reflexive polytope, and both have longest descents of 23 steps. $\square$

The distribution of depths in dimension three is shown in Fig. 4. The longest descent of any three-polytope has 28 steps, against a simple counting bound of 33, and sixteen polytopes are r -minimal. Shortest and

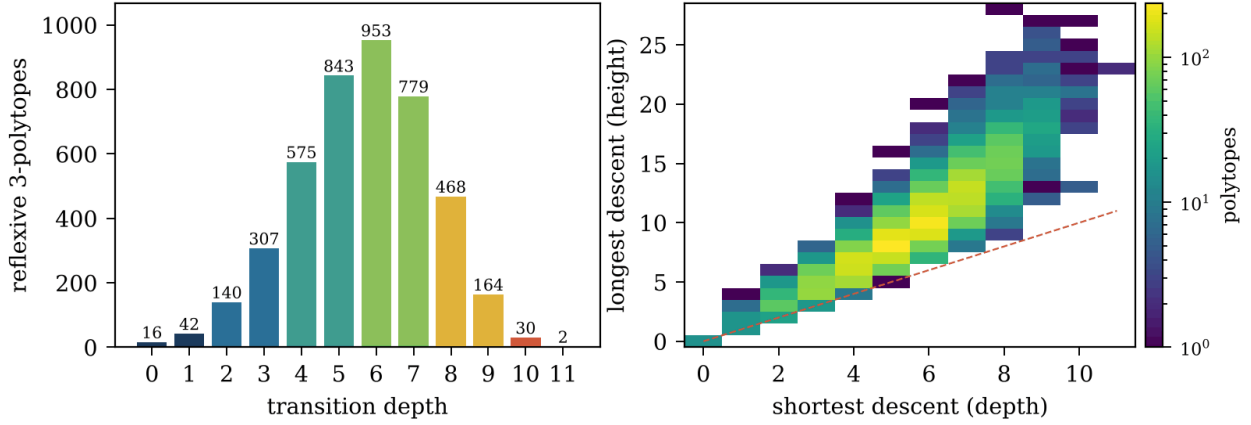


FIGURE 4. Reflexive three-polytopes. Left: the number of polytopes with each transition depth. Right: transition depth against the length of the longest descent, with the diagonal dashed.

TABLE 2. The sixteen reflexive polygons. Here $|\Delta|$ is the number of lattice points, c the number of covers, δ the transition depth and h the length of a longest descent; the last column gives an r -minimal subpolygon with five lattice points. The three r -minimal polygons are the last two rows with five points and the row with four.

vertices of Δ	$ \Delta $	c	δ	h	five-point r -minimal polygon inside Δ
$(-2, -3), (1, 0), (1, 3)$	10	1	5	6	$(-1, -2), (0, 1), (1, 0)$
$(-2, 1), (0, 1), (1, -2), (1, 0)$	9	2	4	5	$(-2, 1), (0, 1), (1, -1)$
$(-1, -2), (-1, 0), (1, 0), (1, 2)$	9	1	4	5	$(-1, -2), (0, 1), (1, 0)$
$(-3, -2), (1, 0), (1, 2)$	9	1	4	5	$(-2, -1), (0, 1), (1, 0)$
$(-1, -2), (-1, 1), (0, 1), (1, 0)$	8	3	3	4	$(-1, -2), (0, 1), (1, 0)$
$(-1, -1), (-1, 1), (0, 1), (1, -1), (1, 0)$	8	3	3	4	$(-1, -1), (0, 1), (1, -1)$
$(-3, -2), (0, 1), (1, 0)$	7	1	2	3	$(-2, -1), (0, 1), (1, 0)$
$(-1, -1), (-1, 1), (0, -1), (0, 1), (1, 0)$	7	2	2	3	$(-1, -1), (-1, 1), (1, 0)$
$(-2, 1), (0, -1), (0, 1), (1, 0)$	7	2	2	3	$(-2, 1), (0, -1), (1, 0)$
$(-1, 0), (-1, 1), (0, -1), (0, 1), (1, -1), (1, 0)$	7	1	2	3	$(-1, 1), (0, -1), (0, 1), (1, -1)$
$(-1, -1), (-1, 1), (0, 1), (1, 0)$	6	2	1	2	$(-1, -1), (-1, 1), (1, 0)$
$(-1, 0), (0, -1), (0, 1), (1, 0), (1, 1)$	6	2	1	2	$(-1, 0), (0, -1), (0, 1), (1, 0)$
$(-1, 0), (0, 1), (1, -1), (1, 0)$	5	1	1	1	—
$(-2, -1), (0, 1), (1, 0)$	5	0	0	0	—
$(-1, 0), (0, -1), (0, 1), (1, 0)$	5	0	0	0	—
$(-1, -1), (0, 1), (1, 0)$	4	0	0	0	—

longest descents differ a great deal. The mean transition depth is 5.67, while the mean length of a longest descent is 10.23, so a well-chosen descent is on average a little more than half as long as a badly chosen one.

7. DIMENSION FOUR

The largest reflexive four-polytope is the simplex

$$\Delta_{680} = \text{conv}\{(-1, -1, -1, -1), (83, -1, -1, -1), (-1, 6, -1, -1), (-1, -1, 2, -1), (-1, -1, -1, 1)\}.$$

TABLE 3. The reflexive descent $\Gamma_0 = \Delta_{680} \supsetneq \Gamma_1 \supsetneq \cdots \supsetneq \Gamma_{38}$: lattice points of Γ_i and of Γ_i^* , and vertices of Γ_i .

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$ \Gamma_i $	680	422	236	207	137	122	103	78	57	44	37	36	35
$ \Gamma_i^* $	26	27	28	29	30	32	33	34	35	36	37	38	39
vert.	5	7	7	9	11	9	7	9	7	7	5	6	6
i	13	14	15	16	17	18	19	20	21	22	23	24	25
$ \Gamma_i $	34	33	32	31	30	29	28	27	26	25	24	23	22
$ \Gamma_i^* $	40	42	44	46	48	51	54	58	62	66	70	80	86
vert.	6	7	7	7	7	7	7	7	7	7	7	8	8
i	26	27	28	29	30	31	32	33	34	35	36	37	38
$ \Gamma_i $	21	20	19	18	17	16	15	14	13	12	11	10	9
$ \Gamma_i^* $	88	92	104	116	128	140	152	164	179	188	196	200	201
vert.	7	7	8	8	8	8	8	7	7	7	7	6	5

Its lattice points are the exponents of the monomials of degree 84 in the weighted projective space $\mathbb{P}(1, 1, 12, 28, 42)$, shifted by the interior point $(1, 1, 1, 1)$, so it has 680 lattice points. The dual simplex has vertices e_1, e_2, e_3, e_4 and $(-1, -12, -28, -42)$, 26 lattice points and normalised volume $1 + 1 + 12 + 28 + 42 = 84$, and Δ_{680} has normalised volume $3528 = 2 \cdot 42^2$, the largest possible [12]. The weights divided by their sum are $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{1}{84}, \frac{1}{84}$, which add up to 1 because $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = 1$; here 2, 3, 7, 43 are the first terms of Sylvester's sequence and $42 = 43 - 1$. The hypersurfaces have Hodge numbers $(h^{1,1}, h^{2,1}) = (11, 491)$, and their mirror has the largest $h^{1,1}$ in the list.

Because Δ_{680} is a simplex, no proper subset of its vertices spans a polytope with the origin inside, so vertex deletion cannot even start (Proposition 7). Reflexive descent works (Fig. 5). Ten steps of the climbing procedure of Section 5 take the number of lattice points through

$$680, 422, 236, 207, 137, 122, 103, 78, 57, 44, 37,$$

each step certified as a cover by Lemma 15. From 37 points on, all covers were computed by Corollary 14; each step removes one point, and after 38 steps the descent reaches a simplex with nine lattice points and vertices

$$(-1, -1, -1, 1), (-1, -1, 2, -1), (-1, 0, 1, -1), \\ (-1, 6, -1, -1), (1, 0, 0, 0),$$

which has no reflexive subpolytope and is itself minimal. The dual counts rise from 26 to 201 along the way. Appendix B lists the whole descent.

7.1. A certificate for lower bounds. An upper bound for a transition depth needs one descent, a lower bound a statement about all of them. The following lemma reduces a lower bound to a finite object that can be checked one polytope at a time.

Lemma 17 (Certificate lemma). *Let $\mathcal{N}$ be a set of reflexive polytopes and $b: \mathcal{N} \rightarrow \mathbb{Z}_{\geq 0}$ a function such that for every $\Gamma \in \mathcal{N}$ with $b(\Gamma) > 0$ the polytope Γ has a cover, and every cover of Γ is unimodularly equivalent to some $\Gamma' \in \mathcal{N}$ with $b(\Gamma') \geq b(\Gamma) - 1$. Then $\delta(\Gamma) \geq b(\Gamma)$ for every $\Gamma \in \mathcal{N}$.*

Proof. A unimodular map fixes the lattice and the origin and preserves inclusion, so it carries covers to covers and r -minimal polytopes to r -minimal ones; hence δ is constant on classes of unimodularly equivalent polytopes. We use induction on $|\Gamma|$. If $b(\Gamma) = 0$ there is nothing to prove. Otherwise Γ has a cover, so it contains a reflexive polytope other than itself and is not r -minimal, and every descent $\Gamma = \Gamma_0 \supsetneq \Gamma_1 \supsetneq \cdots \supsetneq \Gamma_k$

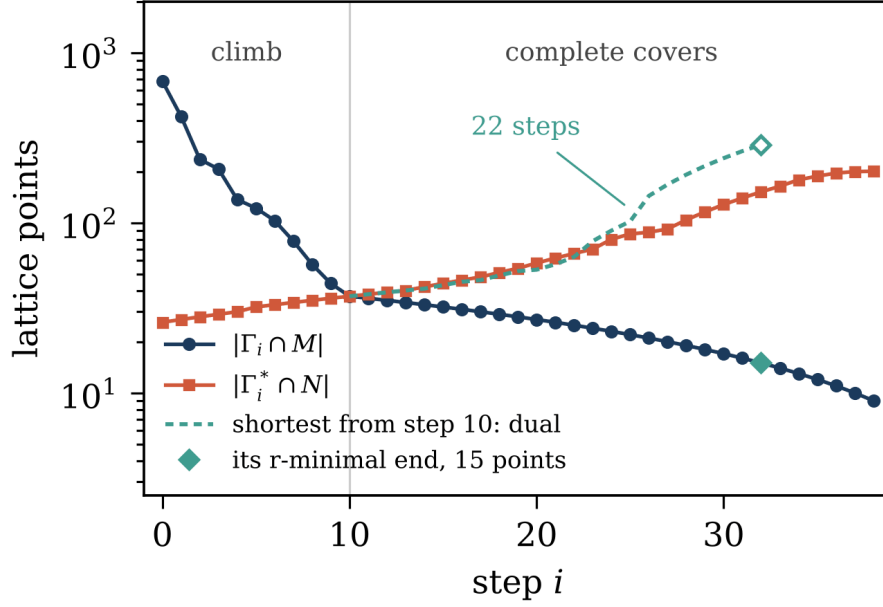


FIGURE 5. The descent from Δ_{680} : lattice points of Γ_i and of its dual at each step. The first ten covers were certified by the climbing lemma, the remaining ones by listing all covers. The polytope Γ_{10} is the polytope P_1 of Section 7.2; a shortest descent of it has 22 steps, removes one lattice point at each step like the descent drawn, and ends at an r-minimal polytope with 15 lattice points (filled diamond) whose dual has 288 (open diamond). The dashed line shows its dual counts.

has $k \geq 1$. Its first step Γ_1 is a cover of Γ , so it is equivalent to some $\Gamma' \in \mathcal{N}$ with $b(\Gamma') \geq b(\Gamma) - 1$, and $|\Gamma'| = |\Gamma_1| < |\Gamma|$ by Lemma 6(i). The chain $\Gamma_1 \supsetneq \dots \supsetneq \Gamma_k$ is a descent of Γ_1 , so the induction hypothesis gives $k - 1 \geq \delta(\Gamma_1) = \delta(\Gamma') \geq b(\Gamma') \geq b(\Gamma) - 1$, and $k \geq b(\Gamma)$. $\square$

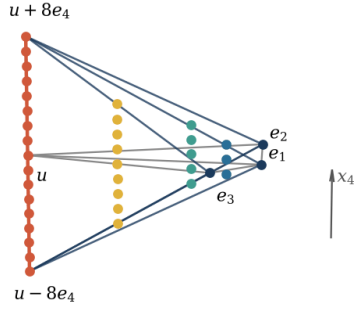
The lemma asks for a complete list of covers only at the polytopes with positive bound, and Corollary 14 provides such lists. The polytopes need not be the same as those of any particular descent, and the bounds need not be exact depths. A certificate is produced by a bounded search: to show $\delta(\Gamma) > \ell$ it lists the covers of Γ and shows $\delta(\Gamma') > \ell - 1$ for each of them, identifying unimodularly equivalent polytopes by their normal forms and keeping every bound it has proved, so that a polytope reached along many paths is examined once.⁵ The certificate is then checked by a program that does not repeat the search: it verifies the hypothesis of the lemma polytope by polytope and recomputes the lists of covers.

7.2. Two polytopes of depth twenty-two. Let $e_1, \dots, e_4$ be the standard basis of $M = \mathbb{Z}^4$, put $u = (-4, -2, -1, 0)$, and let

$$\begin{aligned} P_1 &= \text{conv}\{e_1, e_2, e_3, u + 8e_4, u - 8e_4\}, \\ P_2 &= \text{conv}\{e_1, e_2, e_3, u, u + 8e_4, e_3 - 8e_4\}. \end{aligned}$$

Both polytopes contain $Q \times \{0\}$, where the tetrahedron $Q = \text{conv}\{e_1, e_2, e_3, (-4, -2, -1)\}$ lies in $\mathbb{R}^3$ with the lattice $\mathbb{Z}^3$, and both are bipyramids over it with apexes at height ± 8 : P_1 is the convex hull of $Q \times \{0\}$ and the two points $((-4, -2, -1), \pm 8)$, and P_2 that of $Q \times \{0\}$ and the points $((-4, -2, -1), 8)$ and $(e_3, -8)$. The

⁵The search is a depth-first search with a memory of exact depths and of lower bounds. For a bound ℓ it tries the covers of Γ in the order of their known depths, smallest first, and stops as soon as one of them has depth at most $\ell - 1$; if none has, the bound $\delta(\Gamma) \geq \ell + 1$ is recorded. Repeating with $\ell = 0, 1, 2, \dots$ gives the exact depth, and the recorded bounds of all polytopes that were reached form a function b as in Lemma 17. Exact depths found on the way are recorded as well; they give the descents that prove the matching upper bounds.

(a) P_1 : columns of heights 8, 4, 2, 1, 0

(b) shortest descents

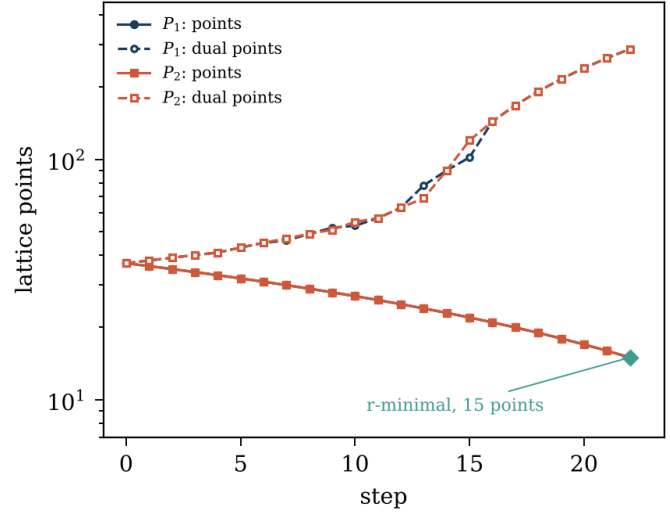


FIGURE 6. (a) The polytope P_1 as a bipyramid over the tetrahedron $Q \times \{0\}$ (grey edges), with the fourth coordinate vertical. Its 37 lattice points lie on vertical lines over the seven lattice points of Q , with 17, 9, 5 and 3 points over $(-4, -2, -1)$, $(-2, -1, 0)$, $(-1, 0, 0)$ and 0 and one point at each of e_1, e_2, e_3 ; the half-lengths 8, 4, 2, 1, 0 of these lines are 8 times the barycentric coordinate at the vertex $(-4, -2, -1)$ of Q . (b) Lattice points (solid) and dual lattice points (dashed) along shortest descents of P_1 and P_2 . Each step removes one lattice point, and both descents end at the same r-minimal polytope, with 15 lattice points and 288 dual lattice points (diamond).

tetrahedron Q is reflexive, it is the simplex of the weights $(4, 2, 1, 1)$ because $4e_1 + 2e_2 + e_3 + (-4, -2, -1) = 0$, and it has the seven lattice points

$$e_1, e_2, e_3, (-4, -2, -1), (-2, -1, 0), (-1, 0, 0), 0.$$

It contains no reflexive polytope other than itself, so it is r-minimal and its transition depth in dimension three is 0.

The polytope P_1 is a simplex, and $8e_1 + 4e_2 + 2e_3 + (u + 8e_4) + (u - 8e_4) = 0$, so it is the simplex of the weights $(8, 4, 2, 1, 1)$. Its lattice points lie on vertical lines over the lattice points of Q . A point (x, t) with $x \in Q$ lies in P_1 exactly when $|t| \leq 8\lambda(x)$, where $\lambda(x)$ is the barycentric coordinate of x in Q at the vertex $(-4, -2, -1)$; indeed, writing $(x, t) = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \alpha(u + 8e_4) + \beta(u - 8e_4)$ with nonnegative coefficients of sum 1 gives $\alpha + \beta = \lambda(x)$ and $t = 8(\alpha - \beta)$. The barycentric coordinates of the four lattice points $(-4, -2, -1)$, $(-2, -1, 0) = \frac{1}{2}(-4, -2, -1) + \frac{1}{2}e_3$, $(-1, 0, 0) = \frac{1}{4}(-4, -2, -1) + \frac{1}{2}e_2 + \frac{1}{4}e_3$ and $0 = \frac{1}{8}(-4, -2, -1) + \frac{1}{2}e_1 + \frac{1}{4}e_2 + \frac{1}{8}e_3$ at that vertex are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, so the lines over them carry 17, 9, 5 and 3 lattice points, and the three points e_1, e_2, e_3 carry one each (Fig. 6(a)). Hence

$$|P_1| = 17 + 9 + 5 + 3 + 3 = 37.$$

The dual simplex P_1^* has the vertices $(-1, -1, -1, \pm 1)$, $(-1, -1, 7, 0)$, $(-1, 3, -1, 0)$ and $(1, -1, -1, 0)$, read off from the five facets of P_1 , and it also has 37 lattice points. The polytope P_2 has six vertices, and $|P_2| = |P_2^*| = 37$. For both polytopes the Calabi–Yau hypersurfaces have Hodge numbers $(h^{1,1}, h^{2,1}) = (34, 34)$.

The two polytopes were found by a search over skew bipyramids $\text{conv}(Q' \times \{0\} \cup \{(a, k), (b, -k)\})$ over three-dimensional reflexive polytopes Q' , with a, b lattice points of Q' and $k \geq 1$, ranked by the length of a

descent found greedily with the cuts of Section 5; the exact depths of the highest ranked were then computed with complete lists of covers.⁶

Proposition 18. $\delta(P_1) = \delta(P_2) = 22$.

Proof. For the upper bound, each of P_1 and P_2 has a descent of 22 steps, each step a cover by Corollary 14. Both remove one lattice point at each step, so the numbers of lattice points run through $37, 36, \dots, 15$, while the dual counts rise from 37 to 288 (Fig. 6(b)). Both end at the same polytope, with vertices

$$(-4, -2, -1, 7), (-2, -1, 0, -1), (-2, -1, 0, 4), e_1, e_2, e_3,$$

15 lattice points and no cover.

For the lower bound we apply Lemma 17 to a set $\mathcal{N}$ of 5,592 polytopes containing P_1 and P_2 , with $b(P_1) = b(P_2) = 22$. Of these, 5,250 have positive bound; for each of them the set $\mathcal{N}$ contains a representative of every cover, given by the complete list of Corollary 14, 23,646 covers in all, and each cover has bound at least one less. The lemma gives $\delta(P_1), \delta(P_2) \geq 22$. $\square$

The polytopes of the certificate have between 14 and 37 lattice points; the most frequent sizes are 17 to 20 points, with between 472 and 494 polytopes each, and only 61 polytopes have more than 31 points.

Proposition 19. *The matrix*

$$G = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & 6 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$$

is unimodular, and $x \mapsto Gx$ carries P_1 onto the polytope Γ_{10} of the descent of Δ_{680} in Table 3. Hence $\delta(\Gamma_{10}) = 22$ and $\delta(\Delta_{680}) \leq 32$.

Proof. The matrix

$$H = \begin{pmatrix} 0 & 3 & 7 & 11 \\ 0 & 2 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ -1 & -6 & -14 & -21 \end{pmatrix}$$

is integral and $GH = 1$, so G is unimodular. The map sends $e_1, e_2, e_3, u + 8e_4$ and $u - 8e_4$ to

$$\begin{aligned} &(-1, -1, -1, 1), (-1, -1, 2, -1), (-1, 6, -1, -1), \\ &(-1, 0, 1, -1), (15, 0, 1, -1), \end{aligned}$$

which are the five vertices of the simplex Γ_{10} . Since Γ_{10} is reached from Δ_{680} by ten covers, $\delta(\Delta_{680}) \leq 10 + \delta(\Gamma_{10}) = 10 + 22$. $\square$

Three of the vertices of Γ_{10} are vertices of Δ_{680} , and the other two lie on its facet $x_4 = -1$. One step higher the same search gives a descent of 22 steps from Γ_9 , which has 44 lattice points: it removes one or two points at a time and ends at an r -minimal polytope with 19 lattice points. Hence $\delta(\Delta_{680}) \leq 9 + 22 = 31$.

⁶The search was organised on the Anaxia engine of the BLLM programme [18, 19], release 3.0.5, its whole record was rerun on release 3.1.1, and it is kept in release 3.2.1. Anaxia runs a failure-driven loop: each strategy for a target is a list of checks, a computation or a proof with a named source, a failed check is recorded together with the change of strategy it forces, all checks are rerun from the archived data, and the label of a target (open, candidate, proved) is computed from the record, never assigned by hand. For the upper bound of Theorem 3 its record shows the first two strategies failing: descent by deleting one vertex at a time fails on 7 of 400 sampled three-dimensional polytopes, which have no reflexive vertex deletion, and cuts at dual lattice points of gauge two miss covers on 73 of the 400; the change forced by the second failure was the complete set of cuts of Corollary 14, which agrees with the vertex-deletion order on all of them. For the sharp form of Conjecture 1 the first strategy, a sweep of the whole list, failed on cost, since 162,328,050 polytopes have $\min(|\Delta| - 6, 680 - |\Delta^*|) > 40$; the second and third gave the descent of Δ_{680} with exact depths of small polytopes along it; the fourth was the search over skew bipyramids that produced P_1 and P_2 , with the certificate of Proposition 18 and the identification of Proposition 19 as its checks. Its check that the sharp constant is determined fails, and that target stays open. The engine organises and reruns the computations; every statement of this section is proved in the text, and the certificate is checked by the programs of Appendix C, which do not use the engine.

Proof of Theorem 5. Proposition 18 gives $\delta(P_1) = \delta(P_2) = 22$, so $D_4 \geq 22$, and Proposition 19 identifies P_1 with Γ_{10} . The nine covers from Δ_{680} to Γ_9 in Table 3, each certified by the climbing lemma, followed by the descent of 22 steps from Γ_9 , form a descent of Δ_{680} of length 31. Finally $D_4 \leq 348$ by Theorem 3. $\square$

Smaller polytopes along the descent of Δ_{680} have smaller depth: the same search gives $\delta(\Gamma_{27}) = 9$ for the polytope with twenty lattice points and $\delta(\Gamma_{32}) = 4$ for the one with fifteen, and among the four-polytopes with at most eleven lattice points the largest transition depth is 5.

8. OPEN QUESTIONS

The main open problem is the exact value of D_4 . We know $22 \leq D_4 \leq 348$. The upper bound comes from point counts alone and is attained by no descent that we know; the lower bound comes from two polytopes with thirty-seven points, one of which lies ten covers below Δ_{680} . Two improvements look within reach. The upper bound can be lowered polytope by polytope: a few certified covers bring a polytope to one with a small pair bound, as for the polytope at (521, 22) in Section 4, and only 106 polytopes have pair bound above 340. The lower bound rises as soon as one polytope is found all of whose covers have depth at least 22, and Lemma 17 turns such a search into a checkable certificate. The reflexive polytopes that contain P_1 or P_2 , and the polytopes $\Gamma_9, \dots, \Gamma_1$ of the descent of Δ_{680} , are the first candidates; the cost of the search lies in listing covers of polytopes with many lattice points.

Three further questions seem natural. First, how does D_d grow with the dimension? Proposition 11 gives $D_d \leq 2(s_d - 1)^2 - 2$ for $d \geq 4$, a doubly exponential bound; the pair bound of Section 4 would do far better as soon as the joint distribution of point counts is known, and that distribution is not known in dimension five. We know no lower bound for D_d that grows with d .

Second, how far apart can the shortest and the longest descent of one polytope be? In dimension two they differ by at most one. In dimension three the largest difference is 20, for the simplex with vertices $(1, 0, 0)$, $(1, 2, 0)$, $(1, 2, 4)$, $(-7, -6, -4)$ and 35 lattice points, whose depth is 8 and whose longest descent has 28 steps (Fig. 4); it is not the polytope with 35 points of Theorem 4.

Third, do the polytopes of maximal depth share a structure? In dimension two the extreme polygon is the triangle of the weights $(1, 1, 1)$, and in dimension three both extremes are r -maximal. In dimension four the polytope P_1 is the simplex of the weights $(8, 4, 2, 1, 1)$ and lies inside Δ_{680} , so it is not r -maximal; whether the polytope of largest depth in dimension four is r -maximal is open.

DATA AVAILABILITY

The code and data behind every computation in this paper are archived in release 1.2.0 of the repository, <https://github.com/creelie/seed/releases/tag/v1.2.0> [20]. The archive contains the joint distribution of point counts over the whole list, the pair bounds, the cover relation in dimensions two and three, the descent of Δ_{680} with its certificates, the certificate of Lemma 17 for P_1 and P_2 , the checks of the cuts formed by Corollary 14 in dimension three and at the certificate polytopes, and independent checks in C, C++, Python, Julia, Macaulay2, shell and Lean 4. The Kreuzer–Skarke list was read from its public copy [10]. The search for long descents was organised on the Anaxia engine, release 3.0.5, and rechecked on release 3.1.1; the engine is archived in release 3.2.1 [18] and described in [19], and its record of strategies, failed checks and the changes they forced is part of the archive. The computations and the preparation of the text were carried out with AI assistance from Claude (Anthropic).

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APPENDIX A. THE REFLEXIVE POLYGONS

Table 2 lists the sixteen reflexive polygons by their vertices, with the number of lattice points, the number of covers, the transition depth δ and the length h of a longest descent. For each polygon with at least six points the last column gives the vertices of an r -minimal polygon with five lattice points inside it, which is what the proof of Theorem 4 needs. Each entry can be checked by hand.

APPENDIX B. THE DESCENT FROM THE LARGEST POLYTOPE

Table 3 lists the descent of Section 7: for each step i the number of lattice points of Γ_i and of its dual and the number of vertices of Γ_i . Steps 1 to 10 were certified by the climbing lemma and steps 11 to 38 by listing all covers. The lattice points of every Γ_i are in the archive [20].

APPENDIX C. HOW THE COMPUTATIONS WERE CHECKED

Every computational statement in the paper was obtained by one method and confirmed by a second, written independently, where that was practical.

Point counts. The joint distribution of $|\Delta|$ and $|\Delta^*|$ was read from all 473,800,776 entries of the list [10]. The total, the extreme values 6 and 680, and the single polytope at (680, 26) agree with Kreuzer and Skarke [4]. The pair bound was computed by the recursion of Section 4 and again by a direct quadratic search over all pairs, with the same result; a separate program checks the bounds 654 and 348 and the counts of Table 1.

Covers. In dimension two the sixteen polygons and their covers were computed from scratch and agree with Table 2. In dimension three the cover relation obtained by removing lattice points and the one obtained from Corollary 14 agree on every polytope. Unimodular equivalence was decided with the normal forms of PALP [5].

The simplex Δ_{680} . Its 680 lattice points, the 26 points of its dual and the volumes 3528 and 84 were checked by four separate programs, and the arithmetic of the weights by a proof assistant. Each step of the descent in Table 3 was certified as a cover, and the last polytope was checked to have no reflexive proper subpolytope. Exact depths. The values $\delta(\Gamma_{27}) = 9$ and $\delta(\Gamma_{32}) = 4$ come from breadth-first searches through complete lists of covers, in which unimodularly equivalent polytopes are identified. The search stops at the first level that contains a polytope without covers, so it gives the least length of a descent.

The certificate for P_1 and P_2 . The archive stores the 5,592 polytopes of the certificate by their lattice points and PALP normal forms, the list of covers of each of the 5,250 polytopes with positive bound (23,646 covers in all) and of one polytope without covers, the bound of every polytope, and the two descents of length 22. A separate program checks, without PALP, that every stored point set is the set of lattice points of a reflexive polytope, that the roots are P_1 and P_2 , that every cover has fewer lattice points than the polytope it covers, that the bounds satisfy the hypothesis of Lemma 17 and equal 22 at the roots, that both descents end at a polytope without covers, and that the matrix G of Proposition 19 is unimodular and carries the lattice points of P_1 onto those of Γ_{10} . With PALP it recomputes every normal form and every list of covers by Corollary 14, and all 5,251 recomputed lists agree with the stored ones. The descent of length 22 from Γ_9 is stored as a chain of subpolytopes of Γ_9 , and each of its steps is checked to be a cover in the same way. Lemma 17, the point counts along the descents and the arithmetic of G are also theorems of the Lean file.

Every dimension. The step from Blichfeldt’s inequality to the bound of Proposition 11 is a theorem of the Lean file for every d and every V : a chain of point counts that strictly decrease, stay at least $d + 2$ and start at most at $V + d$ has at most $V - 2$ steps. A short program compares the bound with the posets of dimensions two and three, whose longest chains of covers have 6 and 28 steps, and with the point counts of the list in dimension four.

REFERENCES

- [1] V. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties, *J. Algebraic Geom.* **3** (1994), 493–535.

- [2] M. Kreuzer and H. Skarke, On the classification of reflexive polyhedra, *Commun. Math. Phys.* **185** (1997), 495–508, doi:[10.1007/s002200050100](https://doi.org/10.1007/s002200050100).
- [3] M. Kreuzer and H. Skarke, Classification of reflexive polyhedra in three dimensions, *Adv. Theor. Math. Phys.* **2** (1998), 853–871, doi:[10.4310/ATMP.1998.v2.n4.a5](https://doi.org/10.4310/ATMP.1998.v2.n4.a5).
- [4] M. Kreuzer and H. Skarke, Complete classification of reflexive polyhedra in four dimensions, *Adv. Theor. Math. Phys.* **4** (2000), 1209–1230, doi:[10.4310/ATMP.2000.v4.n6.a2](https://doi.org/10.4310/ATMP.2000.v4.n6.a2).
- [5] M. Kreuzer and H. Skarke, PALP: A package for analysing lattice polytopes with applications to toric geometry, *Comput. Phys. Commun.* **157** (2004), 87–106, doi:[10.1016/S0010-4655\(03\)00491-0](https://doi.org/10.1016/S0010-4655(03)00491-0).
- [6] P. Candelas, P. S. Green and T. Hübsch, Rolling among Calabi–Yau vacua, *Nucl. Phys. B* **330** (1990), 49–102, doi:[10.1016/0550-3213\(90\)90302-T](https://doi.org/10.1016/0550-3213(90)90302-T).
- [7] A. C. Avram, M. Kreuzer, M. Mandelberg and H. Skarke, The web of Calabi–Yau hypersurfaces in toric varieties, *Nucl. Phys. B* **505** (1997), 625–640, doi:[10.1016/S0550-3213\(97\)00582-8](https://doi.org/10.1016/S0550-3213(97)00582-8).
- [8] M. Reid, The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible, *Math. Ann.* **278** (1987), 329–334, doi:[10.1007/BF01458074](https://doi.org/10.1007/BF01458074).
- [9] R. Altman, J. Gray, Y.-H. He, V. Jejjala and B. D. Nelson, A Calabi–Yau database: threefolds constructed from the Kreuzer–Skarke list, *J. High Energy Phys.* **2015**(2) (2015), 158, doi:[10.1007/JHEP02\(2015\)158](https://doi.org/10.1007/JHEP02(2015)158).
- [10] Calabi–Yau data, Reflexive polytopes in four dimensions (the Kreuzer–Skarke list), Hugging Face dataset [calabi-yau-data/polytopes-4d](https://huggingface.co/datasets/calabi-yau-data/polytopes-4d), licence CC BY-SA 4.0, <https://huggingface.co/datasets/calabi-yau-data/polytopes-4d>.
- [11] J. C. Lagarias and G. M. Ziegler, Bounds for lattice polytopes containing a fixed number of interior points in a sublattice, *Canad. J. Math.* **43** (1991), 1022–1035, doi:[10.4153/CJM-1991-058-4](https://doi.org/10.4153/CJM-1991-058-4).
- [12] G. Balletti, A. M. Kasprzyk and B. Nill, On the maximum dual volume of a canonical Fano polytope, *Forum Math. Sigma* **10** (2022), e109, doi:[10.1017/fms.2022.93](https://doi.org/10.1017/fms.2022.93).
- [13] H. F. Blichfeldt, A new principle in the geometry of numbers, with some applications, *Trans. Amer. Math. Soc.* **15**, 227–235 (1914), doi:[10.1090/S0002-9947-1914-1500976-6](https://doi.org/10.1090/S0002-9947-1914-1500976-6).
- [14] B. Poonen and F. Rodriguez-Villegas, Lattice polygons and the number 12, *Amer. Math. Monthly* **107** (2000), 238–250, doi:[10.1080/00029890.2000.12005186](https://doi.org/10.1080/00029890.2000.12005186).
- [15] D. Bhattacharjee, Seed universality for reflexive polytopes in dimension four: generation of the Kreuzer–Skarke Calabi–Yau landscape via toric seed orbits, *Preprints.org* (2026), doi:[10.20944/preprints202603.1056.v1](https://doi.org/10.20944/preprints202603.1056.v1).
- [16] D. Bhattacharjee, Seeds and orbits in the Kreuzer–Skarke landscape, manuscript, *Preprints.org* submission 229062 (2026).
- [17] D. Bhattacharjee, Calabi–Yau saturation universality: toric mirror laws, Hodge statistics, and the higher-dimensional landscape, *Int. J. Prof. Stud.* **21**(1) (2026), 211–256, doi:[10.37648/ijps.v21i01.017](https://doi.org/10.37648/ijps.v21i01.017).
- [18] D. Bhattacharjee, BLLM: Beyond the LLM with Deep Human-Mind-AI, the Anaxia engine, release 3.2.1, *Zenodo* (2026), doi:[10.5281/zenodo.23524945](https://doi.org/10.5281/zenodo.23524945); all versions doi:[10.5281/zenodo.23286641](https://doi.org/10.5281/zenodo.23286641).
- [19] D. Bhattacharjee and U. Bhattacharya, BLLM: Beyond the LLM with Deep Human-Mind-AI, paper distributed with the engine [18] (2026).
- [20] D. Bhattacharjee, Reflexive descent to minimal seeds in the Kreuzer–Skarke list: code, data and checks, release 1.2.0, <https://github.com/creelie/seed/releases/tag/v1.2.0> (2026).

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