

# The book Ramsey number $R(B_{57}, B_{58}) = 231$

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## Abstract

We construct a graph on 230 vertices with no book  $B_{57}$  whose complement contains no  $B_{58}$ . With the upper bound of Rousseau and Sheehan this gives  $R(B_{57}, B_{58}) = 231$ , the smallest case of Wesley's conjecture  $R(B_{n-1}, B_n) = 4n - 1$  that was open. The graph is a two-block circulant graph over  $\mathbb{Z}_{115}$ , and the proof reduces to 57 inequalities.

*Keywords:* Ramsey number, book graph, block-circulant graph, local search, formal verification

*2020 MSC:* 05C55, 05C25, 05B10

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## 1. Introduction

The book  $B_k$  is the graph  $K_2 + \overline{K_k}$ :  $k$  triangles sharing one edge, its spine. For graphs  $G$  and  $H$ , the Ramsey number  $R(G, H)$  is the least  $N$  such that every graph on  $N$  vertices contains  $G$  or has a complement containing  $H$ . Rousseau and Sheehan [9] proved, from Goodman's count of monochromatic triangles [6], that

$$R(B_{n-1}, B_n) \leq 4n - 1 \quad (n \geq 2). \quad (1.1)$$

Wesley [12] proved equality for  $n \leq 20$  and whenever  $2n - 1$  is a prime power congruent to 1 modulo 4, and conjectured equality for every  $n$ ; the conjecture is one of the open problems of Epoch AI's FrontierMath collection [5].

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Turturean [11] constructed extremal graphs for every  $n \leq 57$  and for fourteen further values below 100, partly through a cyclotomic construction of Szekeres [10]; Chung, Du and Wesley [3] found three further infinite families and settled  $n = 67$  by a finite certificate [4], for which [2] gives a proof checkable by hand; and Price obtained  $n = 70$  [5]. Before the present note the smallest open case was  $n = 58$ .<sup>1</sup>

**Theorem.**  $R(B_{57}, B_{58}) = 231$ .

The upper bound is (1.1), and the lower bound is a graph on 230 vertices with no  $B_{57}$  whose complement has no  $B_{58}$ . For odd  $n$  the graph of [2] and the Paley graphs of [12] are regular two-block circulant graphs over  $\mathbb{Z}_{2n-1}$  whose two blocks are complementary circulants of degree  $n - 1$ . When  $n$  is even such a graph does not exist, because a symmetric subset of  $\mathbb{Z}_{2n-1} \setminus \{0\}$  has even size. Turturean's examples for even  $n$  [11] enlarge the first block by one element and shrink the second by one, and our graph has that shape. Section 2 reduces the two block conditions for it to one inequality (2.1) on the connection sets, a reduction that underlies the search of [11] and that we prove here in full. Section 3 gives the connection sets for  $n = 58$ , and Table 1 verifies (2.1) at the 57 values of  $d$  that matter. The graph has also been checked by two independent programs and in Lean 4 (see the end of the paper).

## 2. Two-block circulant graphs of even order

Throughout,  $n \geq 4$  is even,  $v = 2n - 1$  and  $G = \mathbb{Z}_v$ . For  $X, Y \subseteq G$  and  $d \in G$  put

$$N_{X,Y}(d) = |X \cap (Y + d)| = \#\{y \in Y : y + d \in X\}.$$

Translating by  $-d$  shows  $N_{X,X}(-d) = N_{X,X}(d)$ , and  $\sum_{d \in G} N_{X,X}(d) = |X|^2$ .

Let  $A \subseteq G \setminus \{0\}$  with  $A = -A$  and  $|A| = n$ , let  $C = G \setminus (A \cup \{0\})$ , so that  $C = -C$  and  $|C| = n - 2$ , and let  $B \subseteq G$  with  $|B| = n - 1$ . The graph  $\Gamma = \Gamma(A, B)$  has vertex set  $G \times \{0, 1\}$ , and

$$(x, 0) \sim (y, 0) \iff y - x \in A, \quad (x, 1) \sim (y, 1) \iff y - x \in C,$$

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<sup>1</sup>The values  $n \leq 100$  that none of the constructions cited here covers were 58, 59, 60, 64, 65, 68, 72, 73, 76, 78, 80, 81, 84, 88, 89, 92, 93, 94, 96 and 100. This list is our reading of [3, 4, 5, 11, 12] as of 15 October 2026; it agrees with the ledger in [4, Section 6.1].

and  $(x, 0) \sim (y, 1)$  if and only if  $y - x \in B$ . Since  $A$  and  $C$  are symmetric and avoid 0,  $\Gamma$  is a simple graph on  $4n - 2$  vertices. A vertex  $(x, 0)$  has  $|A| + |B| = 2n - 1$  neighbours and a vertex  $(x, 1)$  has  $|C| + |B| = 2n - 3$ ; the translations  $(x, i) \mapsto (x + t, i)$  are automorphisms. Write  $\lambda(p, q)$  for the number of common neighbours of two vertices,  $\mu(p, q)$  for the number of vertices other than  $p$  and  $q$  adjacent to neither,  $[P]$  for the indicator of a statement  $P$ , and

$$f(d) = N_{A,A}(d) + N_{B,B}(d) \quad (d \in G).$$

**Lemma 2.1.** *For  $d \neq 0$ ,*

$$\lambda((0, 0), (d, 0)) = f(d), \quad \lambda((0, 1), (d, 1)) = f(d) - 3 + 2[d \in A],$$

*and for every  $d \in G$ ,  $\lambda((0, 0), (d, 1)) = n - 1 - [d \in B]$ .*

*Proof.* A vertex  $(z, 0)$  is a common neighbour of  $(0, 0)$  and  $(d, 0)$  when  $z \in A$  and  $z - d \in A$ , and  $(z, 1)$  is one when  $z \in B$  and  $z - d \in B$ ; this gives  $N_{A,A}(d) + N_{B,B}(d)$ .

For  $(0, 1)$  and  $(d, 1)$ , the common neighbours  $(z, 1)$  are counted by  $N_{C,C}(d)$ . A vertex  $(z, 0)$  is adjacent to  $(0, 1)$  when  $-z \in B$  and to  $(d, 1)$  when  $d - z \in B$ ; with  $w = -z$  these say  $w \in B$  and  $w + d \in B$ , so there are  $N_{B,B}(-d) = N_{B,B}(d)$  of them. To compute  $N_{C,C}(d)$ , let  $S = A \cup \{0\}$ , so that  $|S| = n + 1$  and  $C = G \setminus S$ . Then  $|C \cap (C + d)| = v - |S \cup (S + d)| = v - 2(n + 1) + |S \cap (S + d)|$ . The set  $S \cap (S + d)$  is the disjoint union of  $A \cap (A + d)$ , of  $\{0\}$  when  $-d \in A$ , and of  $\{d\}$  when  $d \in A$ ; neither 0 nor  $d$  lies in  $A \cap (A + d)$ , because  $0 \notin A$ . As  $A = -A$  and  $v - 2(n + 1) = -3$ , we get  $N_{C,C}(d) = N_{A,A}(d) - 3 + 2[d \in A]$ .

For  $(0, 0)$  and  $(d, 1)$ , a vertex  $(z, 0)$  is a common neighbour when  $z \in A$  and  $d - z \in B$ , that is,  $z = d - b$  with  $b \in B$  and  $d - b \in A$ ; since  $A = -A$  the last condition reads  $b - d \in A$ . A vertex  $(z, 1)$  is a common neighbour when  $z \in B$  and  $z - d \in C$ . Together,  $\lambda = \#\{b \in B : b - d \in A \cup C\} = \#\{b \in B : b \neq d\} = n - 1 - [d \in B]$ .  $\square$

**Lemma 2.2.** *Let  $p, q$  be distinct non-adjacent vertices of  $\Gamma$ . Then  $\mu(p, q) = \lambda(p, q) - 2$  if both lie in  $G \times \{0\}$ ,  $\mu(p, q) = \lambda(p, q) + 2$  if both lie in  $G \times \{1\}$ , and  $\mu(p, q) = \lambda(p, q)$  otherwise.*

*Proof.* The neighbourhoods of  $p$  and  $q$  lie among the other  $4n - 4$  vertices, and their union has  $\deg p + \deg q - \lambda(p, q)$  elements, so  $\mu(p, q) = 4n - 4 - \deg p - \deg q + \lambda(p, q)$ . Insert the degrees  $2n - 1$  and  $2n - 3$ .  $\square$

**Proposition 2.3.**  $\Gamma(A, B)$  contains no  $B_{n-1}$  and its complement contains no  $B_n$  if and only if

$$f(d) \leq n + 1 - 3[d \in A] \quad \text{for every } d \in G \setminus \{0\}. \quad (2.1)$$

In that case  $R(B_{n-1}, B_n) = 4n - 1$ .

*Proof.* An edge lying in  $k$  triangles is the spine of a  $B_k$ , and a non-edge  $pq$  is the spine of a  $B_k$  in the complement exactly when  $\mu(p, q) \geq k$ . So the two conditions say that  $\lambda(p, q) \leq n - 2$  for every edge and  $\mu(p, q) \leq n - 1$  for every pair of distinct non-adjacent vertices. By translation it suffices to take  $p = (0, i)$ , and we use Lemmas 2.1 and 2.2.

For  $p = (0, 0)$  and  $q = (d, 0)$  with  $d \neq 0$ , the pair is an edge if and only if  $d \in A$ . For  $d \in A$  the condition is  $f(d) \leq n - 2$ , and for  $d \notin A$  it is  $f(d) - 2 \leq n - 1$ ; together they are (2.1). For  $p = (0, 1)$  and  $q = (d, 1)$ , the pair is an edge if and only if  $d \in C$ , that is,  $d \notin A$ . For  $d \notin A$  the condition is  $f(d) - 3 \leq n - 2$ , and for  $d \in A$  it is  $(f(d) - 1) + 2 \leq n - 1$ ; together they are again (2.1). For  $p = (0, 0)$  and  $q = (d, 1)$  the pair is an edge if and only if  $d \in B$ ; then  $\lambda = n - 2$ , and otherwise  $\mu = \lambda = n - 1$ , so the condition holds with equality in both cases. The pairs  $(0, 1), (d, 0)$  are translates of these. The last sentence follows from (1.1), since  $\Gamma$  has  $4n - 2$  vertices.  $\square$

*Remark 2.4.* Summing over  $d \neq 0$  gives  $\sum f(d) = n(n - 1) + (n - 1)(n - 2) = 2(n - 1)^2$  and  $\sum (n + 1 - 3[d \in A]) = (n + 1)(2n - 2) - 3n = (n - 2)(2n + 1)$ . So when (2.1) holds, the slacks  $s(d) = n + 1 - 3[d \in A] - f(d)$  are non-negative integers with sum  $n - 4$ , and (2.1) is an equality for at least  $n + 2$  of the  $2n - 2$  values of  $d$ . For the regular graphs used when  $n$  is odd [2, 12] the corresponding sum is  $n - 1$ , so the even case is tighter. The cross pairs never matter:  $B$  enters only through its autocorrelation  $N_{B,B}$ . Since  $f(-d) = f(d)$  and  $[-d \in A] = [d \in A]$ , it suffices to check (2.1) for  $1 \leq d \leq n - 1$ .

### 3. The case $n = 58$

Let  $n = 58$  and  $v = 115 = 5 \cdot 23$ . Let  $A = \pm A_+$  and let  $B$  be the sets

$$\begin{aligned} A_+ &= \{3, 10, 11, 14, 16, 18, 21, 23, 24, 25, 26, 27, 28, 29, \\ &\quad 30, 32, 36, 37, 38, 43, 44, 46, 47, 48, 49, 52, 54, 55, \\ &\quad 56\}, \\ B &= \{0, 2, 5, 6, 7, 9, 10, 11, 15, 17, 20, 21, 22, 23, \\ &\quad 24, 26, 30, 34, 38, 43, 44, 46, 47, 50, 52, 53, 54, 56, \\ &\quad 59, 65, 66, 67, 69, 71, 72, 73, 74, 75, 80, 81, 83, 84, \\ &\quad 85, 88, 90, 92, 93, 97, 100, 102, 105, 106, 107, 109, 110, 111, \\ &\quad 114\}. \end{aligned}$$

The 29 elements of  $A_+$  lie in  $\{1, \dots, 57\}$ , so  $A = A_+ \cup (-A_+)$  has 58 elements, avoids 0 and satisfies  $A = -A$ ; and  $|B| = 57$ . Written out,

$$\begin{aligned} A &= \{3, 10, 11, 14, 16, 18, 21, 23, 24, 25, 26, 27, 28, 29, \\ &\quad 30, 32, 36, 37, 38, 43, 44, 46, 47, 48, 49, 52, 54, 55, \\ &\quad 56, 59, 60, 61, 63, 66, 67, 68, 69, 71, 72, 77, 78, 79, \\ &\quad 83, 85, 86, 87, 88, 89, 90, 91, 92, 94, 97, 99, 101, 104, \\ &\quad 105, 112\}. \end{aligned}$$

Figure 1 draws both sets on the torus  $\mathbb{Z}_5 \times \mathbb{Z}_{23}$ .

**Theorem 3.1.** *The sets  $A$  and  $B$  above satisfy (2.1) for  $n = 58$ . Consequently  $\Gamma(A, B)$  is a graph on 230 vertices, with degrees 115 and 113, that contains no  $B_{57}$  and whose complement contains no  $B_{58}$ , and  $R(B_{57}, B_{58}) = 231$ .*

*Proof.* By Remark 2.4 it suffices to check (2.1) for  $1 \leq d \leq 57$ , and Table 1 lists  $N_{A,A}(d)$  and  $N_{B,B}(d)$  there; each entry is the number of elements of  $A$ , or of  $B$ , whose translate by  $d$  stays in the set.<sup>2</sup> In every row  $f(d) \leq 59 - 3[d \in A]$ . Proposition 2.3 gives the rest.  $\square$

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<sup>2</sup>For instance  $N_{A,A}(1) = 32$  is the number of pairs  $a, a+1$  of consecutive residues in  $A$ : the runs 10, 11; 23, ..., 30; 36, 37, 38; 43, 44; 46, ..., 49; 54, 55, 56; 59, 60, 61; 66, ..., 69; 71, 72; 77, 78, 79; 85, ..., 92; 104, 105 contribute  $1+7+2+1+3+2+2+3+1+2+7+1 = 32$ , and neither 0 nor 114 lies in  $A$ .

Table 1: The inequality (2.1) for  $n = 58$  at  $1 \leq d \leq 57$ , with  $b(d) = 59 - 3[d \in A]$  and the slack  $s(d) = b(d) - f(d)$  of Remark 2.4. The slacks add up to 27, half of  $n - 4 = 54$ .

| $d$ | $[d \in A]$ | $N_{A,A}$ | $N_{B,B}$ | $f$ | $b$ | $s$ | $d$ | $[d \in A]$ | $N_{A,A}$ | $N_{B,B}$ | $f$ | $b$ | $s$ |
|-----|-------------|-----------|-----------|-----|-----|-----|-----|-------------|-----------|-----------|-----|-----|-----|
| 1   | 0           | 32        | 27        | 59  | 59  | 0   | 30  | 1           | 26        | 28        | 54  | 56  | 2   |
| 2   | 0           | 32        | 27        | 59  | 59  | 0   | 31  | 0           | 30        | 29        | 59  | 59  | 0   |
| 3   | 1           | 29        | 27        | 56  | 56  | 0   | 32  | 1           | 27        | 28        | 55  | 56  | 1   |
| 4   | 0           | 28        | 30        | 58  | 59  | 1   | 33  | 0           | 30        | 28        | 58  | 59  | 1   |
| 5   | 0           | 31        | 27        | 58  | 59  | 1   | 34  | 0           | 30        | 27        | 57  | 59  | 2   |
| 6   | 0           | 31        | 27        | 58  | 59  | 1   | 35  | 0           | 30        | 27        | 57  | 59  | 2   |
| 7   | 0           | 33        | 26        | 59  | 59  | 0   | 36  | 1           | 27        | 29        | 56  | 56  | 0   |
| 8   | 0           | 32        | 27        | 59  | 59  | 0   | 37  | 1           | 26        | 29        | 55  | 56  | 1   |
| 9   | 0           | 28        | 31        | 59  | 59  | 0   | 38  | 1           | 28        | 27        | 55  | 56  | 1   |
| 10  | 1           | 26        | 29        | 55  | 56  | 1   | 39  | 0           | 31        | 28        | 59  | 59  | 0   |
| 11  | 1           | 35        | 21        | 56  | 56  | 0   | 40  | 0           | 32        | 26        | 58  | 59  | 1   |
| 12  | 0           | 30        | 29        | 59  | 59  | 0   | 41  | 0           | 31        | 28        | 59  | 59  | 0   |
| 13  | 0           | 30        | 29        | 59  | 59  | 0   | 42  | 0           | 31        | 27        | 58  | 59  | 1   |
| 14  | 1           | 30        | 26        | 56  | 56  | 0   | 43  | 1           | 29        | 27        | 56  | 56  | 0   |
| 15  | 0           | 26        | 32        | 58  | 59  | 1   | 44  | 1           | 26        | 28        | 54  | 56  | 2   |
| 16  | 1           | 28        | 27        | 55  | 56  | 1   | 45  | 0           | 30        | 28        | 58  | 59  | 1   |
| 17  | 0           | 29        | 30        | 59  | 59  | 0   | 46  | 1           | 29        | 27        | 56  | 56  | 0   |
| 18  | 1           | 30        | 26        | 56  | 56  | 0   | 47  | 1           | 26        | 29        | 55  | 56  | 1   |
| 19  | 0           | 29        | 30        | 59  | 59  | 0   | 48  | 1           | 27        | 28        | 55  | 56  | 1   |
| 20  | 0           | 31        | 26        | 57  | 59  | 2   | 49  | 1           | 28        | 28        | 56  | 56  | 0   |
| 21  | 1           | 25        | 31        | 56  | 56  | 0   | 50  | 0           | 29        | 30        | 59  | 59  | 0   |
| 22  | 0           | 31        | 28        | 59  | 59  | 0   | 51  | 0           | 29        | 30        | 59  | 59  | 0   |
| 23  | 1           | 29        | 27        | 56  | 56  | 0   | 52  | 1           | 27        | 29        | 56  | 56  | 0   |
| 24  | 1           | 28        | 28        | 56  | 56  | 0   | 53  | 0           | 32        | 27        | 59  | 59  | 0   |
| 25  | 1           | 28        | 28        | 56  | 56  | 0   | 54  | 1           | 27        | 29        | 56  | 56  | 0   |
| 26  | 1           | 28        | 27        | 55  | 56  | 1   | 55  | 1           | 27        | 28        | 55  | 56  | 1   |
| 27  | 1           | 27        | 29        | 56  | 56  | 0   | 56  | 1           | 27        | 29        | 56  | 56  | 0   |
| 28  | 1           | 27        | 29        | 56  | 56  | 0   | 57  | 0           | 31        | 28        | 59  | 59  | 0   |
| 29  | 1           | 27        | 29        | 56  | 56  | 0   |     |             |           |           |     |     |     |

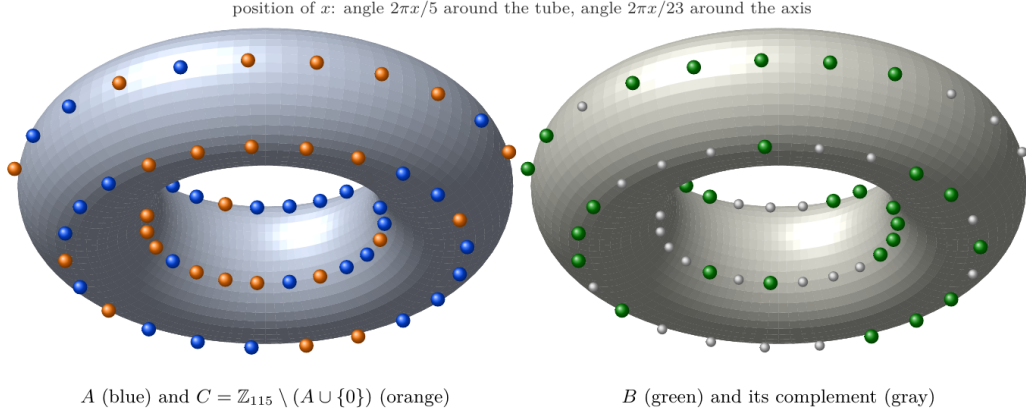


Figure 1: The sets  $A$  and  $B$  for  $n = 58$ . The residue  $x \in \mathbb{Z}_{115}$  sits at angle  $2\pi x/5$  around the tube and  $2\pi x/23$  around the axis.

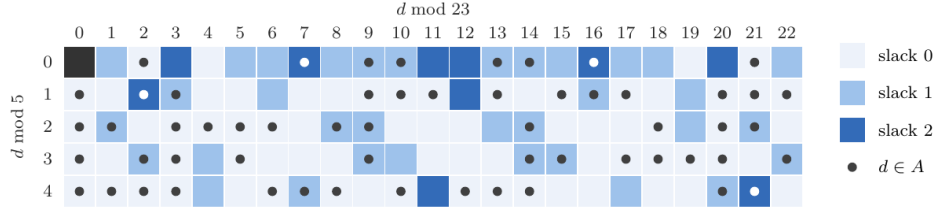


Figure 2: The slack  $s(d) = 59 - 3[d \in A] - f(d)$  for  $n = 58$  at every  $d \neq 0$ , with  $d$  placed by its residues modulo 5 and 23; the black cell is  $d = 0$ , and a dot marks  $d \in A$ . Equality holds for 70 values of  $d$ , the slack is 1 for 34 and 2 for 10.

#### 4. Remarks

*Remark 4.1.* The function  $f$  is not constant on  $A$  or on  $C$  (Figure 2): on the 58 elements of  $A$  it takes the values 54, 55 and 56, for 4, 18 and 36 of them, and on the 56 elements of  $C$  the values 57, 58 and 59, for 6, 16 and 34 of them. So the example is not a difference-set construction of the kind behind the infinite families in [10, 11, 12].

*Remark 4.2.* Unlike the graph for  $n = 67$  in [2], this one has no multiplier symmetry beyond  $d \mapsto -d$ : the only units  $u$  of  $\mathbb{Z}_{115}$  with  $uA = A$  are  $\pm 1$ , the only one with  $uB = B$  is 1, and  $B$  meets  $-B$  in 29 elements. So the 57 rows of Table 1 cannot be grouped into orbits as in [2].

*Remark 4.3.* We found the example by a computer search following the two-phase scheme of Turturean [11]. The first phase chooses  $A$  so that the bounds

$N_{B,B}(d) \leq n + 1 - 3[d \in A] - N_{A,A}(d)$  leave a large margin in Fourier space, where they become linear conditions on the power spectrum of  $B$ . The second phase fixes  $A$  and searches for  $B$  by local search over exchanges of one element of  $B$  with one outside it, guided by weights on the inequalities (2.1); when no exchange lowers the weighted excess, the weights of the violated inequalities are raised, as in the breakout method of Morris [8]. No step of the proof above depends on the search.

**Question 4.4.** The smallest open case of  $R(B_{n-1}, B_n) = 4n - 1$  is now  $n = 59$ , with  $v = 117$ ; there  $n$  is odd, and the regular two-block circulant graphs of [2, 12] are available. Is there a two-block circulant graph over  $\mathbb{Z}_{117}$ , or over  $\mathbb{Z}_3^2 \times \mathbb{Z}_{13}$ , that settles it?

### Formal verification

Theorem 3.1 and the equality  $R(B_{57}, B_{58}) = 231$  are also proved in Lean 4 [7], without any library beyond the Lean core. For the lower bound the development defines the graph  $\Gamma(A, B)$  on the vertices  $0, \dots, 229$  by the two bit masks of  $A$  and  $B$ , and the kernel evaluates the common neighbours and common non-neighbours of all 26,335 pairs of vertices and checks that the adjacency is symmetric: every edge has at most 56 common neighbours and every pair of distinct non-adjacent vertices has at most 57 common non-neighbours. For the upper bound (1.1) the development proves Goodman's count in the form used by Rousseau and Sheehan: if every edge of a graph on  $N$  vertices lies in at most  $a$  triangles and every non-edge has at most  $b$  common non-neighbours, then counting the ordered triples of distinct vertices by the colours of their three pairs gives

$$N(N-1)(N-2) \leq \sum_w (3d(w)\bar{d}(w) + a d(w) + b \bar{d}(w)),$$

where  $d(w)$  is the degree of  $w$  and  $\bar{d}(w) = N - 1 - d(w)$ . For  $N = 231$ ,  $a = 56$  and  $b = 57$  each term on the right is at most  $230 \cdot 229$ , with equality only when  $d(w) = 115$ , and then the degrees would add up to the odd number  $231 \cdot 115$ . The final theorem states that 231 is the least  $N$  such that every graph on  $N$  vertices contains  $B_{57}$  or has a complement containing  $B_{58}$ ; it depends only on the standard axioms `propext`, `Classical.choice` and `Quot.sound`. The graph is also checked by two independent programs, in Python and in C, which read the sets  $A$  and  $B$  and test all pairs directly. The development, the programs and the search are archived in [1].

## Declaration of competing interest

The author has no competing interests to declare.

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## Data availability

The data, the Lean 4 development, the C and Python checks and the search are release v2.1.0 of the repository <https://github.com/creelie/Breakthrough-problems-2>, <https://github.com/creelie/Breakthrough-problems-2/releases/tag/v2.1.0>.

## Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used Claude (Anthropic) to search for the example, to run the computer checks and the Lean formalization, and to draft the text. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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