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CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 9: ZEROS OF THE PRIME ZETA FUNCTION IN ITS HALF-PLANE OF ABSOLUTE CONVERGENCE

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ABSTRACT. The prime zeta function $P(s)=\sum_p p^{-s}$ has zeros in its half-plane of absolute convergence. Belovas, Čepaitytė and Sabaliauskas proved that $P\neq 0$ for $\operatorname{Re} s>\sigma_P=1.7795446535\ldots$, the solution of $P(\sigma)=2^{1-\sigma}$. We show that this is sharp: the closure of the real parts of the zeros in $\operatorname{Re} s>1$ is exactly $[1,\sigma_P]$, there are no zeros on $\operatorname{Re} s=\sigma_P$, and zeros occur arbitrarily close to every line $\operatorname{Re} s=\sigma$ with $1<\sigma<\sigma_P$ (Bohr's method). By Jessen's theorem the density of real parts is $J''(\sigma)/2\pi$, where $J(\sigma)=-\sigma\log 2+\mathbb{E}\log^+(2^\sigma|Y_\sigma|)$ and $Y_\sigma=\sum_{p\geq 3}p^{-\sigma}e^{i\theta_p}$ has independent uniform phases. J is smooth, $\sigma_P=\inf\{\sigma: J(\sigma)+\sigma\log 2=0\}$, and the density vanishes at σ_P faster than any power. The argument principle in ball arithmetic certifies exactly 68 zeros with $1.1\leq\operatorname{Re} s\leq 1.85$ below height 2000 (70.0 predicted) and 12 with $\operatorname{Re} s\geq 1.5$ below height 5000 (12.0 predicted). A heuristic large-deviation extrapolation predicts zeros with real part above 1.7 about once every $6\cdot 10^6$ units of height. In $\frac12<\operatorname{Re} s<1$, P is single-valued on the half-plane minus $(\frac12,1]$ if and only if RH holds (a reformulation of a classical equivalence), its zeros are canonical below height $3\cdot 10^{12}$, and they are the solutions of $\zeta(s)=e^{A(s)}$, $A(s)=\sum_{m\geq 2}P(ms)/m$, with $|P(s)|<2\pi$. Below height 600 their counts for $\operatorname{Re} s\geq 0.7$ agree numerically with the random model. The zeros are cancellations between 2^{-s} and the remaining primes, the prime-sum analogue of the Euler-product interference behind the off-line zeros of the Davenport–Heilbronn function. Version 2 adds the critical line: the partial prime zeta function weights each zero by $1/((\rho-s)\log x)$, and a Gaussian slice of the primes weights it by $e^{-\varepsilon^2(\gamma-T)^2}$. In a numerical experiment at height 1000, the partial prime zeta function aimed at an off-line pair detects it sooner than a wide slice ($\varepsilon=0.02$) and later than a narrow one ($\varepsilon=1$); aimed 30 away, the wide slice is the best of the three.

1. INTRODUCTION

The prime zeta function

$$P(s)=\sum_p p^{-s}=\sum_{k\geq 1}\frac{\mu(k)}{k}\log\zeta(ks)$$

converges absolutely for $\operatorname{Re} s>1$. The second expression continues it to $\operatorname{Re} s>0$, where it has logarithmic branch points at $s=\rho/k$ and $s=1/k$ (k squarefree, ρ a zero of ζ) and a natural boundary on $\operatorname{Re} s=0$ [LW]. Fröberg remarked in 1968 that very little was known about its roots [Fr]. Recently Belovas, Čepaitytė and Sabaliauskas [BCS] proved that $P(s)\neq 0$ for $\operatorname{Re} s>\sigma_P$, with σ_P as below, computed more than 10^4 zeros in $(0.1,1.65)\times(0,10^4)$, found the largest real part $1.6826\ldots$ for $|\operatorname{Im} s|<2\cdot 10^5$, and tested conjectures on the distribution of the zeros statistically.

In 2018 the author computed zeros of P numerically and plotted them on a logarithmic scale [C0]. That computation found a cluster of zeros with $\operatorname{Re} s<1.6$ and $\operatorname{Im} s<800$. It also reported

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three further clusters with $\operatorname{Re} s > 10^9$ or $\operatorname{Im} s > 10^{13}$. This note explains the first cluster and shows that the other three are numerical artifacts.

We restrict mainly to the half-plane $\operatorname{Re} s > 1$, where P is single-valued and its Dirichlet series converges absolutely. There the zeros can be described completely.

1. **The right edge (Theorem 2.1).** Let σ_P be the unique real solution of $P(\sigma) = 2^{1-\sigma}$, that is, $2^{-\sigma} = \sum_{p\geq 3} p^{-\sigma}$. Then $\sigma_P = 1.7795446535\ldots$ and P has no zeros with $\operatorname{Re} s \geq \sigma_P$. Conversely, for every $1 < \sigma < \sigma_P$ there are zeros arbitrarily close to the line $\operatorname{Re} s = \sigma$; we do not claim that every such line contains a zero. So the closure of the real parts of the zeros in $\operatorname{Re} s > 1$ is exactly $[1, \sigma_P]$.
2. **The density (Section 3).** By Jessen's theorem on the zeros of almost periodic functions [Je, JT], the zeros with $\operatorname{Re} s$ in an interval have a frequency given by the second derivative of the mean $J(\sigma)$ of $\log|P|$ on vertical lines. We show that $J(\sigma) = -\sigma \log 2 + \mathbb{E} \log^+ \left(2^\sigma |Y_\sigma|\right)$, where $Y_\sigma = \sum_{p\geq 3} p^{-\sigma} e^{i\theta_p}$ has independent uniform phases, that J is C^∞, and that $\sigma_P = \inf\{\sigma > 1 : J(\sigma) + \sigma \log 2 = 0\}$; we compute J through a Bessel-product formula.
3. **Certified counts (Section 4).** By the argument principle in ball arithmetic there are exactly 68 zeros (with multiplicity) with $1.1 \leq \operatorname{Re} s \leq 1.85$ and $0 < \operatorname{Im} s < 2000$, against 70.0 predicted, and exactly 12 with $1.5 \leq \operatorname{Re} s \leq 1.85$, $0 < \operatorname{Im} s < 5000$, against 12.0 predicted. The distribution of real parts matches the prediction bin by bin (numerically). The edge is sharp but practically invisible: by a heuristic large-deviation extrapolation (not certified), zeros with $\operatorname{Re} s > 1.7$ are expected about once every $6 \cdot 10^6$ units of height.

What is new. The zero-free half-plane $\operatorname{Re} s > \sigma_P$ is due to [BCS]. Theorem 2.1 adds the boundary line and shows that the half-plane is sharp; it is an instance of the classical theory of Bohr, Jessen and Tornehave for Dirichlet series with linearly independent frequencies [Bo, JT], in which a line $\operatorname{Re} s = \sigma$ carries zeros nearby exactly when no single term dominates the sum of the others, and we include a self-contained proof. New here are also the Jessen density of the real parts (Section 3), including its edge characterization and its flatness at σ_P; the certified counts (Section 4); the heuristic edge asymptotics; and the analysis inside the critical strip (Section 5). The value 1.6826 found in [BCS] below height $2 \cdot 10^5$ is consistent with Section 3, which predicts about 4 zeros with real part above 1.65 and about 0.4 above 1.68 in that range.

The mechanism is the one studied in [CE46] for the Davenport–Heilbronn function: P is a sum, not a product, and its zeros are cancellations between the term 2^{-s} and the other primes. An Euler product cannot vanish where it converges absolutely, but a sum over primes can.

Place in the book. This is Part 9 of the series, in block II (Primes and Euler products). It builds on Parts 1 and 8 (the cross-energy and the Euler product in the strip). The prime zeta function is used by Parts 8 and 17, by Part 10 (whose Caceres prime function has an exact prime zeta structure), and by Parts 13 and 31. It also uses Part 46 [CE46]. Section 6 (version 2) reads P on the critical line against the smooth slices of Parts 3 and 37 [CE3, CE37].

2. The right edge

Theorem 2.1. Let $\sigma_P > 1$ be defined by $2^{-\sigma_P} = \sum_{p\geq 3} p^{-\sigma_P}$.

(a) $P(s) \neq 0$ for $\operatorname{Re} s \geq \sigma_P$.
(b) For every $1 < \sigma < \sigma_P$ and every $\varepsilon > 0$, the set of t for which P has a zero in the disc $|s - (\sigma + it)| < \varepsilon$ has positive lower density.
(c) Consequently $\overline{\{\operatorname{Re}\rho : P(\rho) = 0,\ \operatorname{Re}\rho > 1\}} = [1, \sigma_P]$.

The number σ_P is well defined. The function $\sigma \mapsto \sum_{p\geq 3} (2/p)^\sigma$ is strictly decreasing, tends to $+\infty$ as $\sigma \to 1^+$, and tends to 0 as $\sigma \to \infty$. We computed σ_P from $P(\sigma) = \sum_k \mu(k) k^{-1} \log\zeta(k\sigma)$ (it agrees with [BCS]); a direct sum over the primes below $2 \cdot 10^6$, without tail correction, confirms it to about six digits.

Proof of (a). For $\operatorname{Re} s = \sigma \geq \sigma_P$,

$$|P(s)| \geq 2^{-\sigma} - \sum_{p \geq 3} p^{-\sigma} \geq 0.$$

The second inequality is strict for $\sigma > \sigma_P$. Let $\sigma = \sigma_P$ and $P(s) = 0$, $s = \sigma_P + it$. Equality in the triangle inequality forces every p^{-s}, $p \geq 3$, to point opposite to 2^{-s}, that is, $t \log(p/2) \equiv \pi \pmod{2\pi}$ for all $p \geq 3$. If $t = 0$ this is impossible, since $P(\sigma_P) > 0$. If $t \neq 0$, write $t \log \frac{3}{2} = b\pi$ and $t \log \frac{5}{2} = a\pi$ with a, b odd integers. Then $a, b \neq 0$ have the same sign and $b \log \frac{5}{2} = a \log \frac{3}{2}$, so $(5/2)^{|b|} = (3/2)^{|a|}$, that is, $5^{|b|}2^{|a|} = 3^{|a|}2^{|b|}$. The left side is divisible by 5 and the right side is not, which contradicts unique factorization. □

Lemma 2.2 (polygon lemma). Let $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_n > 0$, $n \geq 2$, with $\ell_1 \leq \ell_2 + \cdots + \ell_n$. Then $\{\sum_j \ell_j e^{i\theta_j} : \theta_j \in \mathbb{R}\}$ is the closed disc of radius $\sum_j \ell_j$ centred at 0.

The hypothesis is necessary: for lengths 2 and 1 the set of sums is the annulus $1 \leq |z| \leq 3$.

Proof. The set K of sums is the image of a torus, so it is compact and connected, and it is invariant under rotations; so $\{|z| : z \in K\}$ is an interval containing $\sum_j \ell_j$, and it suffices to show $0 \in K$. Choose signs $\varepsilon_2, \ldots, \varepsilon_n$ greedily ($\varepsilon_j = -1$ if the partial sum $\sum_{2 \leq i < j} \varepsilon_i \ell_i$ is positive, $+1$ otherwise); then $|\sum_{j \geq 2} \varepsilon_j \ell_j| \leq \ell_2 \leq \ell_1$. The moduli of the sums of $\ell_2, \ldots, \ell_n$ therefore form an interval containing a value $\leq \ell_1$ and the value $\ell_2 + \cdots + \ell_n \geq \ell_1$, hence containing ℓ_1. Rotating, some sum of $\ell_2, \ldots, \ell_n$ equals $-\ell_1$, and $0 \in K$. □

Proof of (b). Write $a_p = p^{-\sigma}$. Since $a_2 < \sum_{p \geq 3} a_p$, we can choose N with $a_2 < \sum_{3 \leq p \leq N} a_p$ and, since $\sum_p a_p < \infty$, also with $\sum_{p > N} a_p < a_2$. The largest of the lengths a_p ($p \leq N$) is a_2, so the polygon inequality $2 \max_{p \leq N} a_p < \sum_{p \leq N} a_p$ holds strictly, and by Lemma 2.2 the set $\{\sum_{p \leq N} a_p e^{i\theta_p}\}$ is the closed disc of radius $\sum_{p \leq N} a_p \geq a_2$.

Fix any phases θ_p for $p > N$. The tail $R = \sum_{p > N} a_p e^{i\theta_p}$ has modulus at most $\sum_{p > N} a_p < a_2$, so $-R$ lies in the disc, and we can choose the phases for $p \leq N$ with $\sum_{p \leq N} a_p e^{i\theta_p} = -R$. This gives phases with

$$F(s) := \sum_p e^{i\theta_p} p^{-s}, \qquad F(\sigma) = 0.$$

F is analytic in $\operatorname{Re} s > 1$ and not identically zero ($2^s F(s) \to e^{i\theta_2}$ as $\operatorname{Re} s \to \infty$), so there is $\delta < \min(\varepsilon, \sigma - 1)$ with $m := \min_{|s - \sigma| = \delta} |F(s)| > 0$.

The numbers $\log p$ are linearly independent over $\mathbb{Q}$. By the Kronecker–Weyl theorem, for every Y and $\eta > 0$ the set of t with $|p^{-it} - e^{i\theta_p}| < \eta$ for all $p \leq Y$ has positive density (the measure of the corresponding box in the torus $\mathbb{T}^{\pi(Y)}$). For such t and $|s - \sigma| = \delta$,

$$|P(s + it) - F(s)| \leq \eta \sum_{p \leq Y} p^{-\sigma + \delta} + 2 \sum_{p > Y} p^{-\sigma + \delta}.$$

Take Y large and η small to make the right side smaller than m. By Rouché's theorem $P(\cdot + it)$ has a zero in $|s - \sigma| < \delta$. □

Proof of (c). By (a) and (b), the closure contains $(1, \sigma_P)$, hence $[1, \sigma_P]$, and is contained in $[1, \sigma_P]$. □

This is Bohr's classical argument [Bo], [T, Ch. XI]. Mantovanelli recently used the same scheme to find the exact right edge of the zeros of the Fibonacci zeta function [Ma]. For P no Diophantine obstruction arises, because all the frequencies $\log p$ are independent. The arithmetic enters only through unique factorization, in (a) and in the Kronecker–Weyl theorem.

3. The Jessen function and the density of zeros

For $\sigma > 1$ put

$$J(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \log|P(\sigma + it)|\, dt.$$

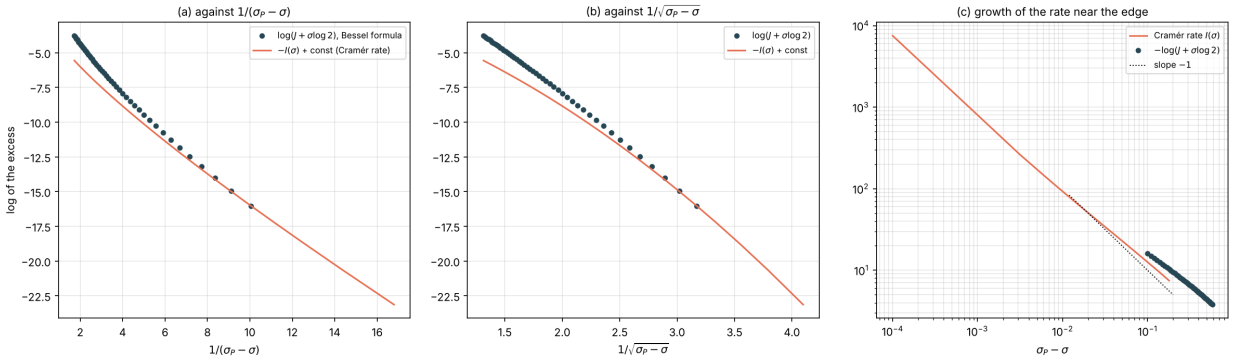


FIGURE 1. The Jessen excess near the edge (numerical, not certified). (a), (b): $\log(J(\sigma) + \sigma\log 2)$ from the Bessel formula, for $1.2 \le \sigma \le 1.68$ (above the error floor), against $1/(\sigma_P - \sigma)$ and $1/\sqrt{\sigma_P - \sigma}$, with $-I(\sigma)$ shifted by a constant, I the Cramér rate of Section 3. (c): $I(\sigma)$ and $-\log(J + \sigma\log 2)$ against $\sigma_P - \sigma$ on logarithmic axes.

The theorem used. On every closed strip $\sigma_1 \le \operatorname{Re} s \le \sigma_2$ with $\sigma_1 > 1$, P is the sum of a uniformly and absolutely convergent Dirichlet series, hence an analytic almost periodic function in the sense of Bohr, and it is not identically zero. For such functions Jessen [Je] proved: the limit $J(\sigma)$ exists for every σ, uniformly on compact sets; J is convex; and the zeros, *counted with multiplicity*, have a frequency in the following sense. If ν is the measure $J''/2\pi$ (the distributional second derivative, a positive measure because J is convex), then for all $1 < \sigma_1 < \sigma_2$ at which J is differentiable,

$$\begin{aligned} N(\sigma_1, \sigma_2; T) &:= \#\{\rho :\ \sigma_1 < \operatorname{Re}\rho < \sigma_2,\ 0 < \operatorname{Im}\rho < T\} \\ &= \frac{T}{2\pi}\big(J'(\sigma_2) - J'(\sigma_1)\big) + o(T) = T\,\nu\big((\sigma_1, \sigma_2)\big) + o(T). \end{aligned}$$

The count for $0 < t < T$ follows from the version for $|t| < T$ because $P(\bar{s}) = \overline{P(s)}$. For a Dirichlet series with linearly independent frequencies the mean equals an expectation over independent uniform phases, by the equidistribution of $(p^{-it})_p$ on the infinite torus (Jessen and Tornehave [JT]; Bohr and Jessen [BJ] for $\log\zeta$):

$$J(\sigma) = \mathbb{E}\log\Big|\sum_p p^{-\sigma} e^{i\theta_p}\Big|.$$

Lemma 3.3 below shows that J is C^∞, so the formula holds for all $\sigma_1 < \sigma_2$, and ν has the continuous density $d_P(\sigma) = J''(\sigma)/2\pi$.

Proposition 3.1. Let $Y_\sigma = \sum_{p\ge 3} p^{-\sigma} e^{i\theta_p}$. Then

$$J(\sigma) = -\sigma\log 2 + \mathbb{E}\log^+\big(2^\sigma |Y_\sigma|\big).$$

In particular $J(\sigma) = -\sigma\log 2$ exactly for $\sigma \ge \sigma_P$, and $J(\sigma) > -\sigma\log 2$ for $1 < \sigma < \sigma_P$.

Proof. Condition on Y_σ and average over θ_2 alone. Jensen's formula for the linear function $z \mapsto 2^{-\sigma} z + Y$ on the unit circle gives $\frac{1}{2\pi}\int_0^{2\pi} \log|2^{-\sigma} e^{i\theta} + Y|\, d\theta = \max(\log 2^{-\sigma}, \log|Y|)$.

The event $|Y_\sigma| > 2^{-\sigma}$ has positive probability exactly when $\sup|Y_\sigma| = \sum_{p\ge3} p^{-\sigma}$ exceeds $2^{-\sigma}$, that is, when $\sigma < \sigma_P$: Y_σ is a continuous function on the torus $\mathbb{T}^\infty$, so $\{|Y_\sigma| > 2^{-\sigma}\}$ is open, and Haar measure charges every nonempty open set. □

Corollary 3.2 (edge characterization). $J(\sigma)+\sigma\log 2 \ge 0$ for all $\sigma > 1$, with equality exactly for $\sigma \ge \sigma_P$; hence

$$\sigma_P = \inf\{\sigma > 1 :\ J(\sigma) + \sigma\log 2 = 0\}.$$

Moreover J is strictly convex on $(1, \sigma_P]$ and affine on $[\sigma_P, \infty)$.

Proof. The first statements are Proposition 3.1. By Theorem 2.1(b) every strip $\sigma - \varepsilon < \operatorname{Re} s < \sigma + \varepsilon$ with $1 < \sigma < \sigma_P$ contains zeros with positive lower frequency (a set of t of lower density $c > 0$ gives at least $cT/(2\varepsilon) + o(T)$ zeros, since each zero serves a t-interval of length at most 2ε), so ν charges every open subinterval of $(1, \sigma_P)$ and J' is strictly increasing there; on $[\sigma_P, \infty)$, $J = -\sigma\log 2$. □

Probability and density are different quantities. The excess $J(\sigma)+\sigma\log 2 = \mathbb{E}\log^+(2^\sigma|Y_\sigma|)$ is positive exactly when the event $|Y_\sigma| > 2^{-\sigma}$ has positive probability, so this event decides *where* zeros can occur. The density of zeros, however, is $d_P(\sigma) = J''(\sigma)/2\pi$, the second derivative of the excess; it is not the probability $\Pr(|Y_\sigma| > 2^{-\sigma})$. Both are determined by the distribution of Y_σ, and both collapse as $\sigma \to \sigma_P$, because the event requires almost all the phases to align.

An exact formula. For $r > 0$, $\int_0^\infty (e^{-x} - J_0(rx))\,\frac{dx}{x} = \log\frac{r}{2}$. Since $Z = \sum_p p^{-\sigma} e^{i\theta_p}$ is isotropic with characteristic function $\mathbb{E}J_0(x|Z|) = \prod_p J_0(p^{-\sigma}x)$, taking expectations (Fubini on $[0, X]$, then $X \to \infty$) gives

$$J(\sigma) = \log 2 + \int_0^\infty \Big(e^{-x} - \prod_p J_0\big(p^{-\sigma}x\big)\Big)\frac{dx}{x}.$$

Lemma 3.3. $J \in C^\infty(1, \infty)$.

Proof. Let $\Phi(x, \sigma) = \prod_p J_0(p^{-\sigma}x)$ and $1 < \sigma_0 \le \sigma \le \sigma_1$. Since $|J_0| \le 1$, $|J_0(y)| \le (2/(\pi y))^{1/2}$ for $y > 0$ [W, §13.74], so $|J_0(y)| < 0.4$ for $y \ge 4$, and $|J_0^{(j)}| \le 1$, every derivative $\partial_\sigma^k\Phi$ is a sum, absolutely convergent because $\sum_p \log^k p\, p^{-\sigma_0} < \infty$, of terms bounded by $C_k x^k$ times a product of all but at most k of the factors. At least $\pi((x/4)^{1/\sigma_1}) - k$ of the remaining factors are below 0.4 in modulus, so $\partial_\sigma^k\Phi(x, \sigma)$ decays faster than any power of x, uniformly in σ. Near $x = 0$, $\partial_\sigma^k\Phi = O(x^2)$ uniformly, since $\Phi = 1 - \frac{x^2}{4}\sum_p p^{-2\sigma} + O(x^4)$. So all derivatives may be taken under the integral sign. □

We evaluate the product exactly for $p \le 10^4$. For $10^4 < p \le 2\cdot10^6$ we use $\log J_0(y) = -y^2/4 - y^4/64 - y^6/576 + O(y^8)$, and we add an integral estimate for $p > 2\cdot10^6$. For $\sigma \ge 1.8$, where the excess is exactly 0, the computed $J(\sigma) + \sigma\log 2$ is below 10^{-8} in modulus; this is the error floor of the computation (double precision, not certified). The values also agree with the time average of $\log|P(\sigma + it)|$ over $0 < t < 2000$ to within 10^{-3} at $\sigma = 1.3$ and 1.5.

σ	$J(\sigma) + \sigma\log 2$	$J'(\sigma)$	$J''(\sigma)/2\pi$
1.10	$4.13\cdot10^{-2}$	-0.9131	0.1200
1.20	$2.28\cdot10^{-2}$	-0.8445	0.0997
1.30	$1.06\cdot10^{-2}$	-0.7870	0.0834
1.40	$3.68\cdot10^{-3}$	-0.7404	0.0641
1.50	$7.13\cdot10^{-4}$	-0.7083	0.0366
1.60	$3.41\cdot10^{-5}$	-0.6947	0.0085
1.70	$\approx 6\cdot10^{-9}$ (extrapolated)	-0.6931	$\approx 2.7\cdot10^{-5}$ (extrapolated)

Since $J'(\sigma_P) = -\log 2$, the number of zeros per unit height with real part greater than σ is $(-\log 2 - J'(\sigma))/2\pi$:

$\operatorname{Re}\rho >$	1.1	1.3	1.5	1.6	1.65	1.7	1.72
zeros per unit height	0.0350	0.0149	0.0024	$2.4\cdot10^{-4}$	$2.3\cdot10^{-5}$	$\approx 1.6\cdot10^{-7}$	$\approx 3\cdot10^{-9}$

The entries for $\sigma \geq 1.7$ are below the error floor of the Bessel computation and are extrapolations by the large-deviation rate described next (heuristic). With them, a zero with real part above 1.7 is expected about once every $6\cdot10^6$ units of height, and one above 1.72 about once every $3\cdot10^8$.

Corollary 3.4 (flatness at the edge). For every N, $J(\sigma)+\sigma\log 2$, $J'(\sigma)+\log 2$ and $J''(\sigma)$ are $O((\sigma_P-\sigma)^N)$ as $\sigma\to\sigma_P^-$. In particular the density of the real parts of the zeros and the number of zeros per unit height with real part above σ tend to 0 faster than any power of $\sigma_P-\sigma$, and J is not real-analytic at σ_P.

Proof. By Lemma 3.3 and Corollary 3.2, $g(\sigma)=J(\sigma)+\sigma\log 2$ is C^∞ on $(1,\infty)$ and vanishes identically on $[\sigma_P,\infty)$; so all its derivatives vanish at σ_P, and Taylor's theorem gives the bounds. If J were real-analytic at σ_P, g would vanish near σ_P, contradicting Corollary 3.2. □

Behaviour at the edge (heuristic). The event $|Y_\sigma|>2^{-\sigma}$ is a large deviation. Its Cramér rate in a fixed direction,

$$I(\sigma)=\sup_{\lambda>0}\Big(\lambda 2^{-\sigma}-\sum_{p\geq 3}\log I_0(\lambda p^{-\sigma})\Big),$$

can be computed accurately for every $\sigma<\sigma_P$, and $I(\sigma)\to\infty$ as $\sigma\to\sigma_P$. On the range where the Bessel values are reliable ($1.2\leq\sigma\leq 1.68$), $\log(J(\sigma)+\sigma\log 2)+I(\sigma)$ varies only from -1.6 to -3.4 while I goes from 2.2 to 12.6 (Figure 1(a),(b)), consistent with $\log(J(\sigma)+\sigma\log 2)=-I(\sigma)(1+o(1))$. A saddle-point analysis of I (the primes $p\lesssim\lambda^{1/\sigma}$ are nearly aligned and the remaining tail is of the size of the deficit $\sum_{p\geq3}p^{-\sigma}-2^{-\sigma}\asymp\sigma_P-\sigma$) suggests

$$I(\sigma)\asymp x^{-1/(\sigma_P-1)}\big(\log\tfrac{1}{x}\big)^{-\sigma_P/(\sigma_P-1)},\qquad x=\sigma_P-\sigma,$$

with $1/(\sigma_P-1)=1.283$. The local exponent $d\log I/d\log\frac{1}{x}$ computed from I is 0.88, 0.95, 1.02 at $x=10^{-2},10^{-3},10^{-4}$, against 0.79, 0.95, 1.03 for $1.283-2.283/\log\frac{1}{x}$. Decay faster than any power is proved (Corollary 3.4); if the heuristic holds, the excess decays eventually faster than $e^{-c/(\sigma_P-\sigma)}$ for every c, which is not proved. Figure 1 shows $\log(J+\sigma\log 2)$ against $1/(\sigma_P-\sigma)$ and $1/\sqrt{\sigma_P-\sigma}$.

4. Numerical verification

Evaluation of P. For $\operatorname{Re}s\geq 1.1$ we use

$$P(s)=\operatorname{Log}\zeta(s)+\sum_{p\leq M}\big(\log(1-p^{-s})+p^{-s}\big)+E,\qquad |E|\leq\frac{M^{1-2\sigma}}{2(2\sigma-1)(1-M^{-\sigma})},$$

which follows from $\log\zeta(s)=-\sum_p\log(1-p^{-s})$ by bounding $\sum_{p>M}\sum_{m\geq2}p^{-m\sigma}/m$. Here Log is the principal branch: for $\operatorname{Re}s\geq1.1$, $|\operatorname{Im}\log\zeta(s)|\leq\log\zeta(1.1)<\pi$. We take $M=10^4$ for $\operatorname{Re}s<1.3$ and $M=2000$ otherwise, and evaluate ζ, the logarithms and the exponentials in ball arithmetic (Arb). A second, independent floating-point implementation ($P(s)=\log\zeta(s)-\sum_{k=2}^{39}P(ks)/k$, Euler–Maclaurin for ζ) reproduces the zeros to six digits.

Certified counting. On a side of a rectangle with $\operatorname{Re}s\geq s_0$ we use $|P'(s)|\leq\sum_p\log p\,p^{-s_0}\leq-\zeta'/\zeta(s_0)=:B$. From a point s_j where the ball for $P(s_j)$ has modulus at least l_j, we step by $h_j\leq 0.45\,l_j/B$. On $[s_j,s_{j+1}]$ the values of P stay in a disc of radius $<l_j/2$ around $P(s_j)$, which does not contain 0; so P does not vanish on the contour, and the change of $\arg P$ over the step is the principal $\operatorname{Arg}(P(s_{j+1})/P(s_j))$, of modulus $<\pi/3$. Summing (in chunks, so that the error does not accumulate; Appendix A) gives the winding numbers as balls of radius at most 0.03, each containing exactly one integer. At the nodes s_j, $|P|\geq0.0013$ on $\operatorname{Re}s=1.1$, $|P|\geq2.6\cdot10^{-4}$ on $\operatorname{Re}s=1.5$, $|P|\geq0.0035$ on $\operatorname{Re}s=1.6$ and $|P|\geq0.05$ on $\operatorname{Re}s=1.85$; hence $|P|$ is at least 0.55 times these values on the whole contour. The corners of the rectangles are the binary floating-point

numbers nearest to the decimal values; since no zero lies within 10^{-5} of the sides, the counts are the same for the decimal rectangles.

region	zeros (certified)	Jessen prediction
$1.1 \le \operatorname{Re} s \le 1.85$, $0 < t < 1000$	31	35.0
$1.1 \le \operatorname{Re} s \le 1.85$, $0 < t < 2000$	68	70.0
$1.5 \le \operatorname{Re} s \le 1.85$, $0 < t < 5000$	12	12.0
$1.6 \le \operatorname{Re} s \le 1.85$, $0 < t < 5000$	0	1.2

Each counted zero was then located by Newton's method (Figure 2) to $|P| < 10^{-9}$ (floating point). The real parts of the 68 zeros below height 2000, grouped by bins of $\operatorname{Re} s$ (the totals are certified, the assignment to bins is numerical):

bin	$[1.1, 1.2)$	$[1.2, 1.3)$	$[1.3, 1.4)$	$[1.4, 1.5)$	$[1.5, 1.6)$	$[1.6, 1.7)$	$[1.7, \sigma_P)$
observed	21	18	13	11	5	0	0
predicted	21.8	18.3	14.9	10.2	4.3	0.5	0.0

The first zeros with $\operatorname{Re} s \ge 1.1$ are

ρ	ρ
$1.44721094 + 38.10821211i$	$1.17973345 + 147.71901017i$
$1.16156144 + 52.71994968i$	$1.24638989 + 162.76815655i$
$1.28212460 + 71.68062565i$	$1.15753462 + 193.87635806i$
$1.27478463 + 132.76939611i$	$1.33721055 + 223.47522197i$

Below $\operatorname{Re} s = 1.1$ there are further zeros, for example $1.0619 + 23.7173i$ and $1.0706 + 85.9974i$. The largest real part below height 5000 is attained at $1.58696272 + 1402.15683346i$.

The 2018 clusters. The zeros with $1 < \operatorname{Re} s < 1.6$ found in [C0] are genuine. Those with real part below 1 lie in the region where P is multivalued, so they depend on the branch of $\log \zeta$ used by the software. The clusters with $\operatorname{Re} s > 10^9$ are artifacts: there $P(s) \approx 2^{-s}$, which is smaller than any floating-point tolerance, so a root-finder accepts every point. Those with $\operatorname{Im} s > 10^{13}$ are also artifacts, because the phases $t \log p$ exceed double precision; genuine zeros with $\operatorname{Re} s < \sigma_P$ do exist at such heights by Theorem 2.1(b), but not at the computed locations. Theorem 2.1(a) excludes all of these with $\operatorname{Re} s \ge \sigma_P$ rigorously.

5. Inside the critical strip

For $0 < \operatorname{Re} s < 1$ the continuation $\sum_k \mu(k) k^{-1} \log \zeta(ks)$ has no poles. It has logarithmic branch points:

- at $s = 1/k$ for squarefree k, where $P(s) \approx -\frac{\mu(k)}{k} \log(s - 1/k)$;
- at $s = \rho/k$ for every zero ρ of ζ, where $P(s) \approx \frac{\mu(k)}{k}\, m_\rho \log(s - \rho/k)$ and m_ρ is the multiplicity of ρ.

A zero of P in this region is only defined once a single-valued branch on a specified domain is fixed. We separate three statements: which domains carry a single-valued branch, the relation with RH, and the branch we compute.

Proposition 5.1. Let $\Omega = \{\operatorname{Re} s > \frac{1}{2}\} \setminus (\frac{1}{2}, 1]$, a simply connected domain, and $A(s) = \sum_{m \ge 2} P(ms)/m$, which is analytic in $\operatorname{Re} s > \frac{1}{2}$.

(a) RH holds if and only if P extends from $\operatorname{Re} s > 1$ to a single-valued analytic function on Ω.

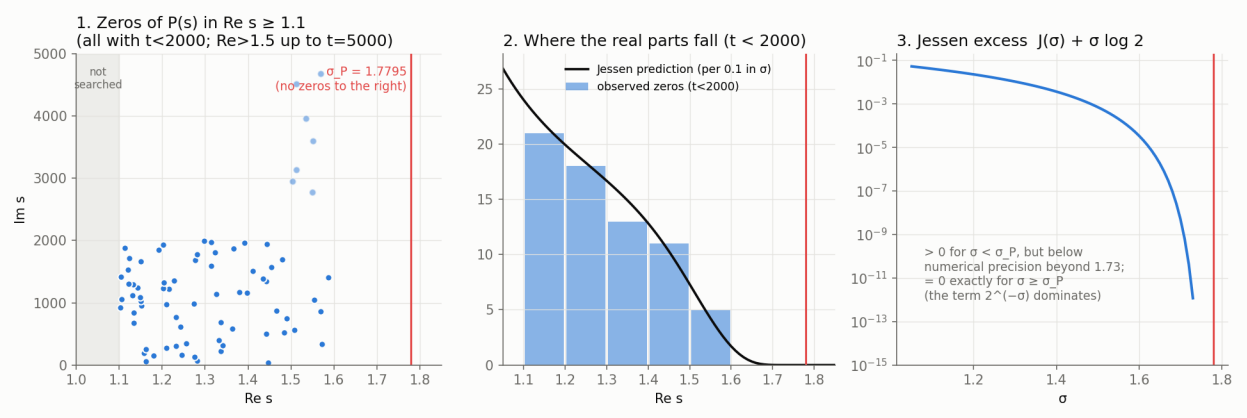


FIGURE 2. Left: zeros of P with $\operatorname{Re} s \geq 1.1$ (all below height 2000; those with $\operatorname{Re} s > 1.5$ up to height 5000; counts certified, locations floating point) and the edge σ_P. Centre: distribution of real parts below height 2000 against the Jessen prediction $TJ''(\sigma)/2\pi$. Right: the excess $J(\sigma) + \sigma \log 2$.

(b) Let $T_0 = 3 \cdot 10^{12}$. On the convex domain $U = \{\operatorname{Re} s > \frac{1}{2},\ 0 < \operatorname{Im} s < T_0\}$, ζ has no zeros [PT], so $\log \zeta$ has a unique branch on U that agrees with the series for $\operatorname{Re} s > 1$, and $P = \log \zeta - A$ is single-valued on U. Every continuation of P along a path inside U gives this branch, so the zeros of P in U are canonical.

Proof. For $m \geq 2$ and $\operatorname{Re} s > \frac{1}{2}$, $\operatorname{Re} ms > 1$, so A is analytic there, and $P = \log \zeta - A$ on $\operatorname{Re} s > 1$. (a) If RH holds, ζ has no zeros on Ω (it has none on $(\frac{1}{2}, 1)$ in any case, and its pole $s = 1$ is not in Ω); since Ω is simply connected, $\log \zeta$ has an analytic branch on Ω extending the series, and $\log \zeta - A$ extends P. Conversely, if F is analytic on Ω and equals P on $\operatorname{Re} s > 1$, then $\exp(F + A)$ is analytic on Ω and equals ζ on $\operatorname{Re} s > 1$, hence on Ω by the identity theorem; so ζ has no zeros in Ω, and therefore none with $\operatorname{Re} s > \frac{1}{2}$, which is RH. (b) U is convex, hence simply connected, and $\zeta \neq 0, \infty$ on U. □

Statement (a) is a reformulation of the classical fact that RH is equivalent to the existence of an analytic branch of $\log \zeta$ in $\operatorname{Re} s > \frac{1}{2}$ minus the segment $(\frac{1}{2}, 1]$; it is not progress towards RH. If RH failed, a zero ρ with $\operatorname{Re} \rho > \frac{1}{2}$ would be a logarithmic branch point of P, and continuation around it would change P by $2\pi i\, m_\rho$.

Zeros of P and values of ζ. On U, $P(s) = L(s) - A(s)$, where L is the branch of $\log \zeta$ of Proposition 5.1(b). So

$$P(s) = 0 \iff L(s) = A(s) \implies \zeta(s) = e^{A(s)},$$

and the converse of the last implication holds only up to the branch: $\zeta(s) = e^{A(s)}$ if and only if $P(s) \in 2\pi i\, \mathbb{Z}$. So the zeros of P are the solutions of $\zeta(s) = e^{A(s)}$ on which, in addition, $|P(s)| < 2\pi$. For $\operatorname{Re} s \geq 1.1$ the extra condition is automatic, since $|P(s)| \leq P(1.1) < 2\pi$. This links the zeros of P with the value distribution of ζ: they are points where ζ takes the prescribed value $e^{A(s)}$, whose modulus $e^{\operatorname{Re} A(s)}$ lies between $e^{-A(\sigma)}$ and $e^{A(\sigma)}$ for $\operatorname{Re} s = \sigma$.

Numerics. On rectangles $[0.55, 1.1] \times [a, a + 5]$ we continue $\log \zeta$ along the boundary, starting from the principal value at $\operatorname{Re} s = 1.1$. The total change of $\arg \zeta$ around every rectangle was 0, which confirms that no zero of ζ lies inside, so the computed values are those of the branch of Proposition 5.1(b). The terms $m \geq 2$ are computed as in Section 4. These computations are in floating point and are not certified.

Up to height 1000 there are 240 zeros of P with $0.55 \leq \operatorname{Re} s \leq 1.1$. We located the 128 below height 600 by Newton's method.

The Bohr–Jessen random model still describes the values of P in this range. The mean of $\log|P(\sigma+it)|$ over $500<t<1000$ agrees with $J(\sigma)=\mathbb{E}\log|\sum_p p^{-\sigma}e^{i\theta_p}|$ to within 0.01 for $\sigma=0.6,\dots,1.0$; the random series converges almost surely for $\sigma>\frac12$.

The counts agree with the model only away from $\frac12$:

band of $\operatorname{Re} s$	$[0.55,0.6)$	$[0.6,0.7)$	$[0.7,0.8)$	$[0.8,0.9)$	$[0.9,1.1)$
observed, $t<600$	16	46	27	17	22
$\frac{600}{2\pi}\Delta J'$	148	86	32	17	19

We separate what is proved, what is observed and what is only a proposed explanation. *Proved:* the asymptotic theory of Section 3 applies to any closed strip inside $\operatorname{Re} s>1$. Inside the critical strip Jessen's theorem does not apply directly, since P is not almost periodic there; Borchsenius and Jessen [BoJ] extended the theory to the values of ζ in $\frac12<\operatorname{Re} s<1$ by approximation in mean square, and their method should extend to $P=\log\zeta-A$, but we do not prove this. The existence of a limiting value distribution of $P(\sigma+it)$ for $\sigma>\frac12$ should likewise follow from the Bohr–Jessen method [BJ]; the agreement of the mean of $\log|P|$ with J and the zero counts are numerical. That P has $\gg T$ zeros below height T in every strip $\frac12<\sigma_1<\operatorname{Re} s<\sigma_2<1$ is the kind of statement that Voronin's universality theorem [Vo] gives for the a-points of $\log\zeta$; here the target $A(s+i\tau)$ moves with the shift τ, so the theorem does not apply verbatim, and we do not derive it. The second row of the table was recomputed on 10 November 2026. Version 1 printed 208 and 91 in the first two bands (203 and 91 with central differences): `jessen.py` replaced the tail $\sum_{p>X}p^{-2\sigma}$, $X=2\cdot10^6$, by $X^{1-2\sigma}/((2\sigma-1)\log X)$, which is 50% too large at $\sigma=0.55$ (0.1615 against 0.1076). With the tail $E_1((2\sigma-1)\log X)$ (`jessen_fix.py`) one gets, by central differences with step 0.004, $J'(0.55)=-4.07$, $J'(0.6)=-2.52$ and the entries 147.5 and 86; the Gaussian approximation $J'\approx P'(2\sigma)/P(2\sigma)$ gives -4.21 and -2.65. For $\sigma\ge1.1$ the two tails differ by less than 10^{-9}, so Sections 3 and 4 are not affected. The right panel of Figure 3 was redrawn with the new values. *Observed:* for $\operatorname{Re} s\ge0.7$ the counts below height 600 agree with $\frac{T}{2\pi}\Delta J'$; nearer to $\frac12$ they fall far below it, and the density in the band $[0.6,0.7)$ doubles between $t<500$ and $500<t<1000$. *Heuristic explanation (not proved):* the random model's variance $\sum_p p^{-2\sigma}$ diverges as $\sigma\to\frac12$, but at height t only primes up to about t are expected to act independently, so the asymptotic density would be reached only when $\log\log t$ is large compared with $\log\frac{1}{\sigma-1/2}$, which is out of numerical reach near $\sigma=\frac12$. We have no quantitative estimate for this.

The zeros of P are also drawn towards the zeros of ζ. Their median distance to the nearest zero of ζ is 0.31 for $0.55\le\operatorname{Re} s<0.8$. For random points with the same real parts it is 0.45–0.51. This attraction is consistent with the identity above: near a zero of ζ the argument of ζ turns quickly, so the condition $\zeta(s)=e^{A(s)}$ is met more often there. A sharper empirical rule is that the zeros of P follow the 1-points of ζ (where A is replaced by 0): below $t=100$ each of the 18 zeros of P located in [CE8, Sect. 9] lies within 0.08 to 0.59 of a 1-point, and the 1-points themselves fall, typically one per gap, between consecutive zeros of ζ [CE8, Rem. 17]. The first-order "satellite" prediction $s\approx\rho+e^{A(\rho)}/\zeta'(\rho)$ is not accurate, because the displacement is not small.

Below $\frac12$. For $\operatorname{Re} s<\frac12$ the continuation of $\log\zeta(s)$ from the right has to pass around the zeros of ζ. One convenient single-valued branch of P is obtained by introducing horizontal cuts extending leftward from the logarithmic branch points ρ and ρ/k; across such a cut it jumps by $2\pi i\,m_\rho\mu(k)/k$. Other compatible systems of cuts are possible, and the zeros of P in this region depend on the choice of branch; they are zeros of a particular branch, not of the multivalued function. The 2018 zeros with real part below $\frac12$ [C0] belong to this regime.

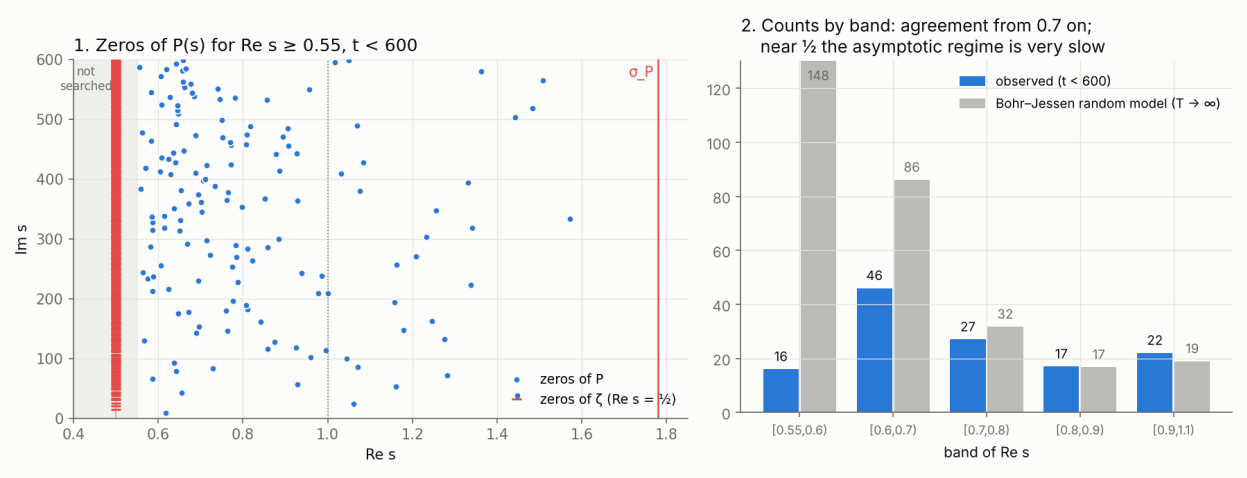


FIGURE 3. Left: zeros of P with $\operatorname{Re} s \geq 0.55$ below height 600 (branch of Proposition 5.1(b); floating point), with the zeros of ζ on the critical line. Right: counts by band of $\operatorname{Re} s$ against the asymptotic Bohr–Jessen prediction, which is not proved to apply inside the critical strip.

6. On the critical line: the partial prime zeta function and the smooth slices

This section is new in version 2 (2 November 2026). It compares two ways of reading the primes at a height T on the critical line. Part 37 [CE37] and Part 3 [CE3] read them through smooth Gaussian windows. Here we read them through P itself. The two instruments carry the same information about RH. They differ in how they weight the zeros: P_x gives a zero the weight $1/((\rho - s)\log x)$, which decays algebraically with the distance to the height examined, and a slice gives it a Gaussian weight whose width $1/\varepsilon$ can be chosen.

6.1. **The two instruments.** Throughout, $s = \frac{1}{2} + iT$ with T large, and $\varepsilon > 0$ is a resolution. For a nontrivial zero ρ we write $t_\rho := (\rho - \frac{1}{2})/i$, so that $t_\rho = \gamma_\rho$ for a zero $\rho = \frac{1}{2} + i\gamma_\rho$ on the line and $t_\rho = \gamma - i\delta$ for a zero $\rho = \beta + i\gamma$, $\gamma > 0$, off the line, with $\delta = \beta - \frac{1}{2}$.

The partial prime zeta function. On the line the series $\sum_p p^{-s}$ does not converge absolutely. The primes larger than x contribute, on average, $\int_x^\infty t^{-s}\,\frac{dt}{\log t} = E_1\big((s-1)\log x\big)$, where E_1 is the exponential integral. So we put

$$P_x(s) = \sum_{p \leq x} p^{-s} + E_1\big((s-1)\log x\big).$$

For $\operatorname{Re} s > 1$, $P_x(s) \to P(s)$ as $x \to \infty$.

The smooth slice (the "texture" of the primes). For $u = \log x$,

$$\mathcal{S}_\varepsilon(x, T) = \frac{1}{2\sqrt{\pi}\,\varepsilon} \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-(\log x - \log n)^2/4\varepsilon^2} \left(\frac{x}{n}\right)^{iT}.$$

It keeps the prime powers with $|\log n - \log x| \lesssim 2\varepsilon$, about $4\varepsilon x$ integers around x, each with its phase n^{-iT}. It is the content of $-\zeta'/\zeta(\frac{1}{2} + iT)$, or of $-P'(\frac{1}{2} + iT)$ up to the prime powers, at the single scale x. Summing the slices over all scales gives back the whole series: up to the prime powers and the weight $1/\log n$ (the difference between ζ'/ζ and $\log\zeta$), $P_x(s)$ is the accumulation of the slices $\mathcal{S}_\varepsilon(e^v, T)$ over $v \leq \log x$.

Sharp and smooth weights. P_x cuts the primes sharply at x; $\mathcal{S}_\varepsilon$ weights them with a Gaussian. By Fourier duality, a weight on the primes is a weight on the zeros. The sharp cut gives each zero

a weight decaying only like $1/|\gamma - T|$. The Gaussian gives $e^{-\varepsilon^2(\gamma-T)^2}$: with $\varepsilon = 0.02$, a zero at distance 200 from T keeps weight $1/200 = 0.005$ in the first case and $1.1 \cdot 10^{-7}$ in the second.

6.2. **The two explicit formulas. Proposition 6.1.** Let $s = \frac{1}{2} + iT$, $\varepsilon > 0$, $x > 1$, $u = \log x$, and let $\rho = \frac{1}{2} + it_\rho$ run over the nontrivial zeros of ζ, with multiplicity. Put

$$h(r) = e^{-\varepsilon^2(r-T)^2} e^{iru}, \qquad g(v) = \frac{1}{2\pi} \int h(r) e^{-irv}\, dr = \frac{1}{2\sqrt{\pi}\,\varepsilon}\, e^{-(u-v)^2/4\varepsilon^2} e^{iT(u-v)}.$$

(a) *(Slices; exact and unconditional.)*

$$\sum_\rho e^{-\varepsilon^2(t_\rho - T)^2}\, x^{\rho - \frac{1}{2}} = \mathcal{H}_\varepsilon(x,T) + \Gamma_\varepsilon(x,T) - \mathcal{S}_\varepsilon(x,T) - \mathcal{M}_\varepsilon(x,T),$$

where $\mathcal{S}_\varepsilon = \sum_n \Lambda(n) n^{-1/2} g(\log n)$ is the slice; $\mathcal{H}_\varepsilon = h(\frac{i}{2}) + h(-\frac{i}{2})$ is the pole term, of size $e^{-\varepsilon^2(T^2 - \frac{1}{4})} x^{1/2}$; Γ_ε is the archimedean term,

$$\Gamma_\varepsilon(x,T) = \frac{1}{2\pi} \int e^{-\varepsilon^2(r-T)^2} e^{iru} \Big(\operatorname{Re} \frac{\Gamma'}{\Gamma}\big(\tfrac{1}{4} + \tfrac{ir}{2}\big) - \log \pi \Big) dr,$$

which decays faster than any power of u; and $\mathcal{M}_\varepsilon = \sum_n \Lambda(n) n^{-1/2} g(-\log n)$ is the mirror term, with $|\mathcal{M}_\varepsilon| \le C_\varepsilon e^{-(\log 2x)^2/4\varepsilon^2}$.

(b) *(Consequences.)* If RH holds, then for all $x > 1$

$$\big|\mathcal{S}_\varepsilon(x,T) - \mathcal{H}_\varepsilon(x,T)\big| \le \sum_\gamma e^{-\varepsilon^2(\gamma-T)^2} + |\Gamma_\varepsilon| + |\mathcal{M}_\varepsilon|, \qquad \sum_\gamma e^{-\varepsilon^2(\gamma-T)^2} \approx \frac{\sqrt{\pi}}{\varepsilon} \cdot \frac{1}{2\pi} \log \frac{T}{2\pi},$$

and $|\Gamma_\varepsilon| + |\mathcal{M}_\varepsilon| \to 0$ as $x \to \infty$. Conversely, let $\frac{1}{2} \le \theta < 1$. If $\mathcal{S}_\varepsilon(x,T) - \mathcal{H}_\varepsilon(x,T) = O(x^{\theta - 1/2})$ as $x \to \infty$ for one T and one ε, then ζ has no zero with real part $> \theta$.

(c) *(Partial prime zeta.)* Let x not be a prime power. For $\operatorname{Re} s > 1$,

$$P_x(s) = P(s) + \sum_\rho E_1\big((s-\rho)\log x\big) + R_x(s), \qquad R_x(s) = \sum_{m \ge 2} \frac{1}{m} \sum_{p > x^{1/m}} p^{-ms} + \sum_{k \ge 1} E_1\big((s+2k)\log x\big),$$

where the sum over ρ is taken symmetrically in $|\gamma|$. On the line $\operatorname{Re} s = \frac{1}{2}$ the term $m = 2$ of R_x and the value of P are understood by continuation. The identity continues analytically in s to every point that can be joined to $\operatorname{Re} s > 1$ without crossing a horizontal half-line going left from a zero, with P the branch of Section 5; in particular it holds for $s = \frac{1}{2} + iT$ when T is not an ordinate and no zero lies to the right of the line at height T. Under RH, on the line,

$$\Big| \sum_\rho E_1\big((s-\rho)\log x\big) + \sum_\rho \frac{x^{\rho - s}}{(\rho - s)\log x} \Big| \le \frac{1}{\log^2 x} \sum_\rho \frac{1}{|\rho - s|^2}.$$

Proof. (a) Put $k(w) = h((w - \frac{1}{2})/i) = e^{\varepsilon^2(w - \frac{1}{2} - iT)^2} x^{w - \frac{1}{2}}$. It is entire and has Gaussian decay in $\operatorname{Im} w$, uniformly in vertical strips. Let $1 < c < 3$ and $I = \frac{1}{2\pi i} \int_{(c)} -\frac{\zeta'}{\zeta}(w) k(w)\, dw$. Expanding the Dirichlet series and moving each integral $\frac{1}{2\pi i} \int_{(c)} n^{-w} k(w) dw$ to the line $\operatorname{Re} w = \frac{1}{2}$, where it equals $n^{-1/2} g(\log n)$, gives $I = \mathcal{S}_\varepsilon$. Now move the line of I to $\operatorname{Re} w = 1 - c$; the horizontal segments vanish because ζ'/ζ is $O(\log^2 |t|)$ on suitable heights [T, Ch. IX] and k is Gaussian. The pole at $w = 1$ gives $k(1)$ and the zeros give $-\sum_\rho k(\rho)$. On the new line use the functional equation in the form $-\frac{\zeta'}{\zeta}(w) = \frac{\zeta'}{\zeta}(1-w) - \log \pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{w}{2}) + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1-w}{2})$. The first term, after $w \mapsto 1 - w$, is $-\sum_n \Lambda(n) n^{-1/2} g(-\log n) = -\mathcal{M}_\varepsilon$. For the rest, move the line back to $\operatorname{Re} w = \frac{1}{2}$; the only pole crossed is that of $\frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{w}{2})$ at $w = 0$, with residue -1, which gives $+k(0)$, and on $\operatorname{Re} w = \frac{1}{2}$ the bracket is $\operatorname{Re} \frac{\Gamma'}{\Gamma}(\frac{1}{4} + \frac{ir}{2}) - \log \pi$, which gives Γ_ε. So $\mathcal{S}_\varepsilon = k(1) + k(0) - \sum_\rho k(\rho) - \mathcal{M}_\varepsilon + \Gamma_\varepsilon$, and

$k(1)+k(0)=h(-\frac{i}{2})+h(\frac{i}{2})$, $k(\rho)=e^{-\varepsilon^2(t_\rho-T)^2}x^{\rho-1/2}$. Γ_ε is the Fourier transform of a smooth function with Gaussian decay, and the bound for $\mathcal{M}_\varepsilon$ follows from $|g(-\log n)|=\frac{1}{2\sqrt{\pi}\varepsilon}e^{-(\log nx)^2/4\varepsilon^2}$. (This is Weil's explicit formula for the test function h; the same computation is Proposition 11.3 of Part 37 [CE37].)

(b) Under RH, $|x^{\rho-1/2}|=1$ and t_ρ is real, so (a) gives the inequality; the approximate value of the sum is the density $\frac{1}{2\pi}\log\frac{T}{2\pi}$ of the zeros times $\int e^{-\varepsilon^2r^2}dr$. For the converse, put $D(x)=\sum_\rho c_\rho x^{\rho-1/2}$ with $c_\rho=e^{-\varepsilon^2(t_\rho-T)^2}\neq 0$. By (a) and the hypothesis, $D(x)=O(x^{\theta-1/2})$ for $x\geq 1$, because Γ_ε and $\mathcal{M}_\varepsilon$ are bounded, $\theta\geq\frac12$, and D is bounded on compact subsets of $[1,\infty)$. Hence $F(w)=\int_1^\infty D(x)x^{-w-1}dx$ is analytic in $\operatorname{Re}w>\theta-\frac12$. Since $\sum_\rho|c_\rho|<\infty$, for $\operatorname{Re}w>\frac12$ one can integrate term by term: $F(w)=\sum_\rho c_\rho/(w-\rho+\frac12)$, and this series is meromorphic in the plane with a pole at each point $\rho-\frac12$, of residue $m_\rho c_\rho\neq 0$ (different zeros give different points, so nothing cancels). A zero with $\operatorname{Re}\rho>\theta$ would be a pole of F in $\operatorname{Re}w>\theta-\frac12$, a contradiction. No positivity is needed, because the hypothesis is a two-sided bound.

(c) For $\operatorname{Re}s>1$, $\log\zeta(s)-\sum_{n\leq x}\frac{\Lambda(n)}{\log n}n^{-s}=\int_x^\infty\frac{t^{-s}}{\log t}\,d\psi(t)$. Insert $\psi(t)=t-\sum_\rho\frac{t^\rho}{\rho}-\log 2\pi+\sum_{k\geq1}\frac{t^{-2k}}{2k}$ and use $\int_x^\infty t^{w-s-1}\frac{dt}{\log t}=E_1((s-w)\log x)$ for $\operatorname{Re}(s-w)>0$: the term t gives $E_1((s-1)\log x)$, which is the term added in the definition of P_x, and each zero, trivial or not, gives $-E_1((s-\rho)\log x)$. Passing from $\log\zeta$ and prime powers to P and primes gives the first sum of R_x. The termwise integration of the conditionally convergent sum over ρ is justified as in the proof of Riemann's formula for $\Pi(x)$, of which this is the analogue with the weight t^{-s}; we do not reproduce it. For the continuation: with $u=\log x$, $E_1((s-\rho)u)=-\frac{x^{\rho-s}}{(\rho-s)u}+O_x(|\gamma|^{-2})$ and $\frac{1}{\rho-s}=\frac{1}{\rho}+\frac{s}{\rho(\rho-s)}$, so $\sum_\rho E_1((s-\rho)u)=-\frac{x^{-s}}{u}\sum_\rho\frac{x^\rho}{\rho}+$ a series that converges absolutely and locally uniformly in s; the sum $\sum_\rho x^\rho/\rho$ converges symmetrically for $x>1$ (von Mangoldt) and does not depend on s. Hence the sum over ρ is analytic in s away from its singularities, and both sides have the same logarithmic branch points at the zeros. For the last inequality, $E_1(z)=\frac{e^{-z}}{z}-\int_z^\infty\frac{e^{-w}}{w^2}dw$ along the horizontal ray, and $|z+t|\geq|z|$ for $\operatorname{Re}z\geq0$, $t\geq0$, so $|E_1(z)-e^{-z}/z|\leq|e^{-z}|/|z|^2$; under RH $z=(s-\rho)\log x$ is purely imaginary. The right-hand side converges, unlike $\sum_\rho|\rho-s|^{-1}$. Status: (a) and (b) proved; (c) classical, with the termwise integration cited and not proved here. Whether $\sum_\rho E_1((s-\rho)\log x)\to0$ as $x\to\infty$ on the line is not known.

Under RH, the bound for $P_x(s)-P(s)$ that follows from $\pi(x)-\operatorname{li}(x)=O(\sqrt{x}\log x)$ by partial summation is only $O(|s|\log^2 x)$. Convergence of $\sum_p p^{-1/2-iT}$ is not known even under RH. In practice, the first-order form of (c), $P_x(s)\approx P(s)-\sum_\rho\frac{x^{\rho-s}}{(\rho-s)\log x}$, shows that the deviation decays like $1/\log x$ and is driven by the zeros nearest to T, each contributing a slow rotation $x^{i(\gamma-T)}/(i(\gamma-T)\log x)$.

6.3. **Numerics at height** 1000**.** We computed $P(\frac12+iT)$ by Möbius inversion,

$$P(s)=\sum_k\frac{\mu(k)}{k}\log\zeta(ks).$$

For $k=1$ we used

$$\log\zeta(\tfrac12+iT)=\log|\zeta(\tfrac12+iT)|+i\pi S(T),\qquad S(T)=N(T)-1-\vartheta(T)/\pi,$$

with $N(T)$ from the zeros. For $k=2,3$ we used Euler–Maclaurin, and for $k\geq5$ the Euler product. This gives $P(\frac12+1000i)=0.26584+1.04958i$, of modulus 1.0827. The partial sums P_x use the 6.8 million prime powers up to $1.2\cdot10^8$.

- *The slices stay of size one* (Figure 4(a)). At $T=1000$, $2\sqrt{\pi}\varepsilon|\mathcal{S}_\varepsilon(x,T)|$ stays between 0 and 1.3 for $2\leq x\leq10^8$. Its root mean square over 40 heights in $[500,1500]$ is 0.491

for $x > 10^4$. This matches the value $2\sqrt{\pi}\varepsilon(\sqrt{\pi/2}\,d(T)/\varepsilon)^{1/2} = 0.504$ predicted by the zeros, $d(T) = \frac{1}{2\pi}\log\frac{T}{2\pi}$, and lies below the random-phase (Cramér) model, which grows like $\sqrt{\log x}$.

- *The partial prime zeta wanders* (Figure 4(a)). $|P_x(s)|$ wanders between 1.1 and 1.5 for $10^3 \le x \le 10^8$ and is still 0.30 away from $P(s)$ at $x = 10^8$.
- *P sees single zeros, a slice does not* (Figure 4(b)). Along $960 \le T \le 1040$, $|P(\frac{1}{2}+iT)|$ has a logarithmic peak at each zero of ζ, the branch points $s = \rho$ of Section 5. A slice, of resolution $1/\varepsilon = 50$ in T, averages about 40 zeros (those in a window of length $1/\varepsilon$; about 70 with the full Gaussian weight $\sqrt{\pi}/\varepsilon$) and varies smoothly.
- *The wandering is the zeros* (Figure 5). Proposition 6.1(c) with the 1517 zeros below 2000 reproduces $|P_x(s)|$ from $x \approx 10^6$ on: 1.387 against 1.388 at $x = 10^8$ (moduli only; the arguments were not compared). The slow oscillations are explained by the five zeros within 3 of T (997.51, 998.83, 999.79, 1001.35, 1002.40). The zero at 999.79 has $|\gamma - T| = 0.21$, and its rotation $x^{0.21i}$ has period 30 in $\log x$: it has not completed one turn at $x = 10^8$. For $10^2 < x < 10^5$ the curves differ by 0.1 to 0.2, the size of the omitted $1/\log^2 x$ terms.

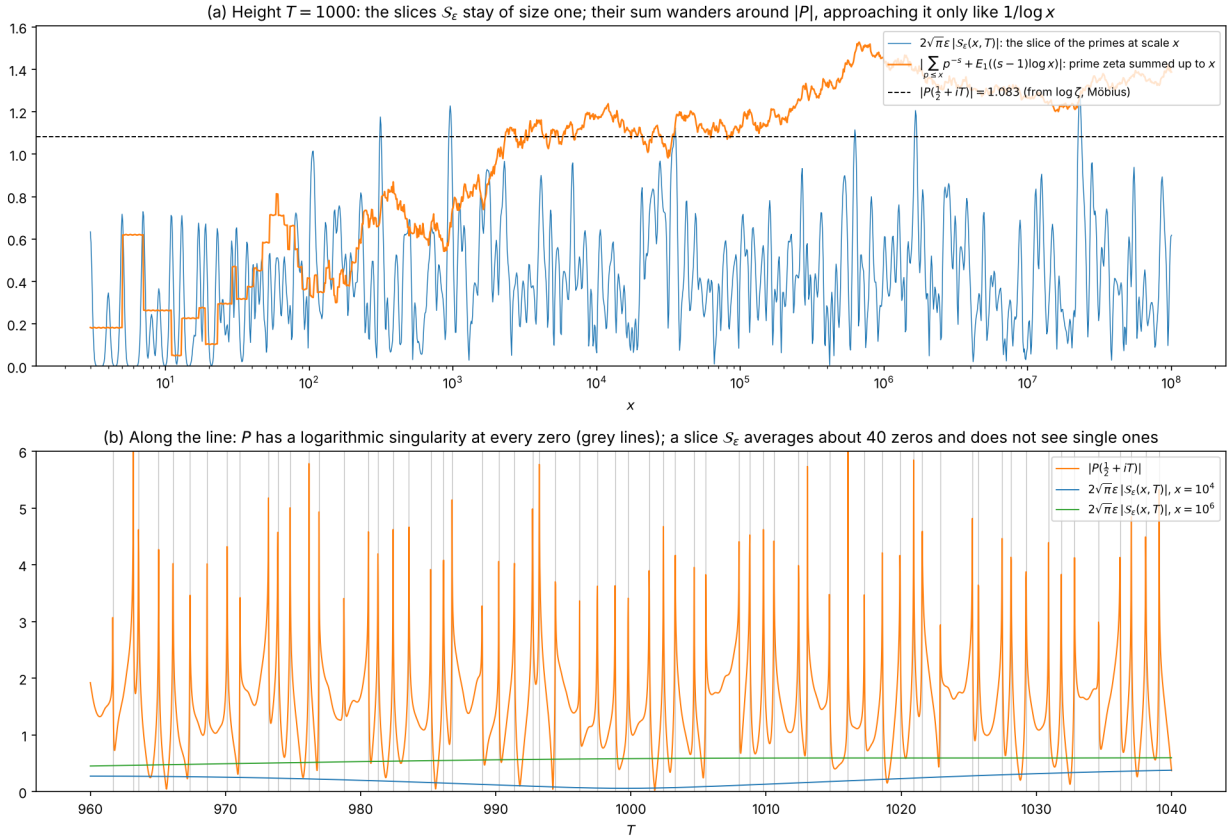


FIGURE 4. (a) At $T = 1000$: the slice $2\sqrt{\pi}\varepsilon|\mathcal{S}_\varepsilon(x,T)|$ ($\varepsilon = 0.02$) and the partial prime zeta function $|P_x(\frac{1}{2}+iT)|$ against x; dashed, $|P(\frac{1}{2}+iT)|$ from $\log\zeta$ and Möbius inversion. (b) Along $960 \le T \le 1040$: $|P(\frac{1}{2}+iT)|$, with a logarithmic peak at every zero of ζ (grey lines), and two slices.

6.4. **Two detectors: the magnifying glass and the scanner.** Which instrument sees an off-line zero sooner? We compare two worlds with the zeros of ζ below 2000. In the reference world, the

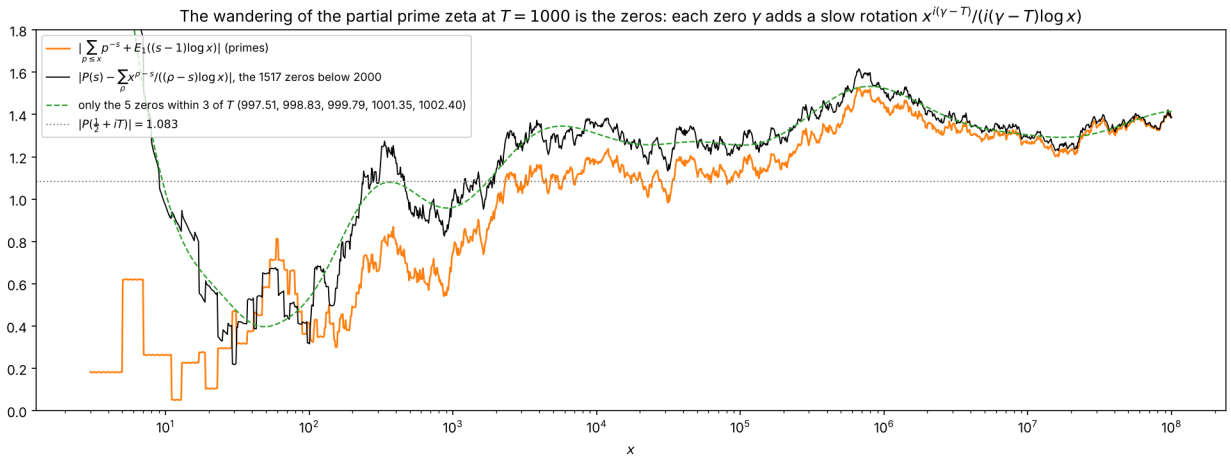


FIGURE 5. The partial prime zeta function at $T = 1000$ (primes) against the first-order formula of Proposition 6.1(c) with all 1517 zeros below 2000 and with the five zeros within 3 of T.

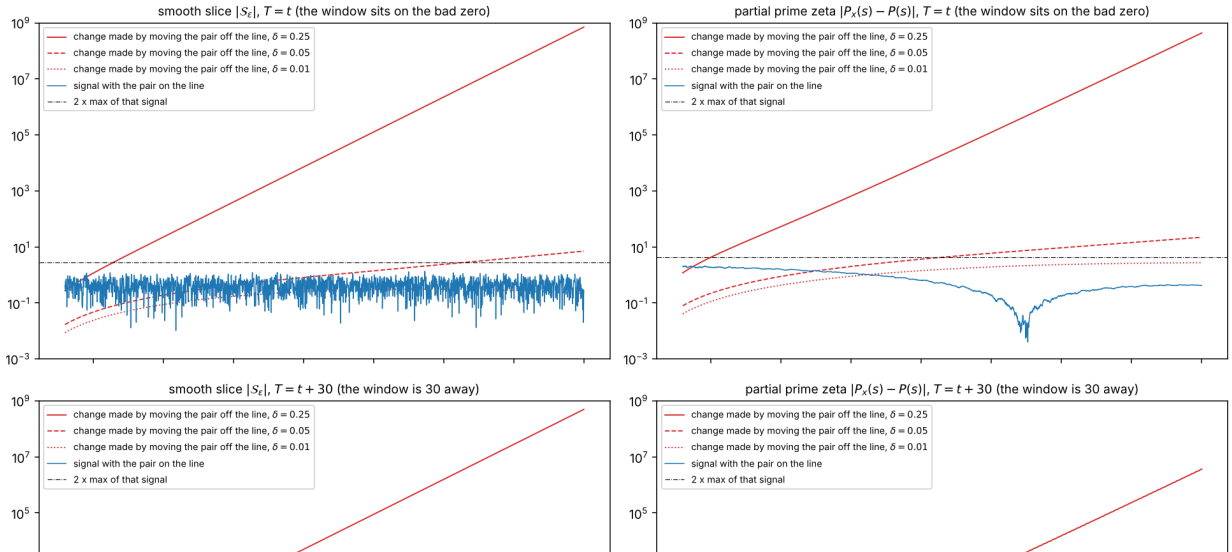
two zeros nearest 1000 are replaced by a pair on the line at their mean height $t = 999.31$ ($t \pm 0.05$). In the perturbed world, they are replaced by $\frac{1}{2} \pm \delta + it$. On the zero side, the two instruments are

$$\text{slice:}\quad 2\sqrt{\pi}\varepsilon \sum_\rho e^{-\varepsilon^2 (t_\rho - T)^2} x^{\rho - s}, \qquad \text{prime zeta:}\quad -\frac{1}{\log x} \sum_\rho \frac{x^{\rho-s}}{\rho - s}.$$

We record the first x at which moving the pair off the line changes the instrument by more than twice the largest value it takes in the reference world (Table 7, Figure 6).

	δ	slice ($\varepsilon = 0.02$)	slice ($\varepsilon = 1$)	partial prime zeta
aimed at the zero, $T = t$	0.25	$2.5 \cdot 10^6$	below 10^3	$9.4 \cdot 10^4$
	0.05	$2.0 \cdot 10^{31}$	$1.8 \cdot 10^{12}$	$2.3 \cdot 10^{21}$
	0.01	beyond 10^{40}	$9.2 \cdot 10^{22}$	beyond 10^{40}
30 away, $T = t + 30$	0.25	$8.2 \cdot 10^6$	beyond 10^{40}	$1.3 \cdot 10^{14}$
	0.05	$1.2 \cdot 10^{34}$	beyond 10^{40}	beyond 10^{40}
	0.01	beyond 10^{40}	beyond 10^{40}	beyond 10^{40}

Table 7: First $x \ge 10^3$ at which an off-line pair at height $t = 999.31$ is detected (numerical; `line_detectors.py` with $\varepsilon = 0.02$ and $\varepsilon = 1$). With $\varepsilon = 0.2$ the slice aimed at the zero detects at $2.4 \cdot 10^4$ and $3.1 \cdot 10^{19}$ for $\delta = 0.25, 0.05$.



Reading.

- *The outcome depends on the width of the window.* Aimed at the zero, P_x detects the pair sooner than the wide slice $\varepsilon = 0.02$ (27 times sooner for $\delta = 0.25$ and 10^{10} times sooner for $\delta = 0.05$), and later than the narrow slice $\varepsilon = 1$ (by a factor above 90 for $\delta = 0.25$ and 10^9 for $\delta = 0.05$; for $\delta = 0.01$ only the narrow slice detects the pair below 10^{40}). The reason is the background. A slice sees the pair with weight $2\sqrt{\pi}\varepsilon\, x^\delta$ against a background of $2\sqrt{\pi}\varepsilon$ times the weighted number of zeros in its window, $\frac{\sqrt{\pi}}{\varepsilon}\cdot\frac{1}{2\pi}\log\frac{T}{2\pi}$, about 70 for $\varepsilon = 0.02$ and between one and two for $\varepsilon = 1$. P_x sees it with weight $x^\delta/(\delta\log x)$ against a background $\frac{1}{\log x}\sum_\rho \frac{x^{\rho-s}}{\rho-s}$ to which every zero contributes, with the slowly decaying weight $1/|\gamma - T|$. So the factor $1/(\rho - s)$ does magnify the nearby zero, but a narrow Gaussian isolates it better.
- *Away from the zero.* At distance 30 the narrow slice is blind (its weight is e^{-900}), the wide slice, of width $1/\varepsilon = 50$, still sees the pair with weight $e^{-0.36} = 0.70$, and P_x, with weight $1/30$, detects only the largest displacement, and late.
- *What P_x offers.* It needs no choice of ε, its reference signal is smooth and decays like $1/\log x$, and it sees every height at once with an algebraic weight. It is not the sharpest instrument in either regime.

The natural strategy combines the two. One scans the line with wide slices, in steps of order $1/\varepsilon$, and looks for windows whose size grows with x. Then one zooms on those heights with narrow slices, or with P_x. Neither instrument changes what a proof would require. A bound $O(x^{\theta-1/2})$ for the slice minus its pole term is a zero-free region for $\operatorname{Re} s > \theta$ (Proposition 6.1(b)), and for every $\theta > \frac{1}{2}$ it is RH; the corresponding statement for $P_x - P$ is expected from (c) and is not proved here. Status: Proposition 6.1(a)–(b) proved; the numbers are floating point.

7. Discussion

The zeros of P in its half-plane of absolute convergence are an interference phenomenon. The term 2^{-s} is cancelled by the joint contribution of the other primes, whose phases p^{-it} behave as independent random variables. The edge σ_P is where the other primes, perfectly aligned, just match $2^{-\sigma}$. The density near the edge is governed by the probability that independent phases nearly align. This is the prime-sum analogue of the interference between two Euler products that produces the off-line zeros of the Davenport–Heilbronn function [CE46, §4].

For ζ itself the analogous statement is trivial: an Euler product has no zeros where it converges absolutely. The contrast between $P = \sum_p p^{-s}$ and $\zeta = \prod_p (1 - p^{-s})^{-1}$ is the contrast between a sum and a product over the same primes. It gives no new restriction on the zeros of ζ.

status	results
proved	Theorem 2.1 (with [BCS] for $\operatorname{Re} s > \sigma_P$), Lemma 2.2, Proposition 3.1, Corollary 3.2, Lemma 3.3, Corollary 3.4, Proposition 5.1, the identity $P = 0 \iff L = A$, Proposition 6.1(a)–(b)
certified computation	the four counts of Section 4 (ball arithmetic)
numerical (floating point)	the Jessen values and predictions, the bin counts and locations of the zeros, everything in Section 5, the computations of Section 6
classical, first order	Proposition 6.1(c)
heuristic	the edge rate $\log(J + \sigma\log 2) \sim -I(\sigma)$ and the growth of I; the finite-height explanation near $\operatorname{Re} s = \frac{1}{2}$

Four questions remain open.

- **The rate at the edge.** Is $\log(J(\sigma) + \sigma\log 2) \sim -I(\sigma)$ as $\sigma \to \sigma_P^-$, and is $I(\sigma) \asymp (\sigma_P - \sigma)^{-1/(\sigma_P - 1)}$ up to logarithmic factors? (Decay faster than any power is Corollary 3.4.)

- **Near the critical line.** Does the Borchsenius–Jessen method give the density of zeros of P in $\frac{1}{2} < \operatorname{Re} s < 1$, and is there a finite-height model that predicts the counts for $\frac{1}{2} < \operatorname{Re} s < 0.7$? What is the precise law of the attraction between zeros of P and zeros of ζ?
- **On the critical line.** In the experiment of Section 6 the partial prime zeta function aimed at a height lies between a wide and a narrow Gaussian slice as a detector of an off-line zero. Is there an instrument that keeps the magnification $1/|\rho - s|$ of P and the localization of the Gaussian slice, for instance a smooth slice of $\log\zeta$ rather than of ζ'/ζ?
- **Values of ζ.** The zeros of P are solutions of $\zeta(s) = e^{A(s)}$ (Section 5). How are they distributed relative to the 1-points and a-points of ζ? This question is independent of RH.

Nothing in this part bears on the truth of RH beyond the reformulation in Proposition 5.1(a), which is classical. We do not prove the Riemann hypothesis.

Appendix A. Computations

All computations are reproduced by the ancillary scripts in `anc/` (see `anc/README.txt`).

Certified (ball arithmetic, Arb via python-flint). `certified/certify_counts.py` evaluates P by the formula of Section 4 with the stated truncation bound, bounds $|P'|$ by $-\zeta'/\zeta(s_0)$ on each side, walks along the sides with steps $h_j \le 0.45\, l_j/B$, and checks at each step that the arguments are resolved. To prevent the radius from growing with the number of steps (about 10^5 per side), the increments are grouped into chunks of length about 25: for each chunk the true change of argument lies both in the sum S of the increment balls and in $R + 2\pi\mathbb{Z}$, where $R = \operatorname{Arg}(P(\text{end})/P(\text{start}))$; when $\operatorname{rad} S + \operatorname{rad} R < 1$ the integer is unique and the chunk contributes $R + 2\pi k$. If R is not resolved (the ratio is near the negative real axis), or if $\operatorname{rad} S$ exceeds 0.5, the chunk is closed and contributes S itself, which is also rigorous. The script records the minimum of $|P|$ on each side and the number of steps (`run_left.log`, `run_right.log`); the windings are in `certify_counts.out`. Run time: a few minutes per side.

Floating point, not certified. σ_P (Brent's method on the Möbius-inverted real values, checked with a direct sum over the primes below $2\cdot 10^6$); the Jessen function and its derivatives (`jessen.py`, `density.py`; for Section 5, `jessen_fix.py` with the exact tail, and `strip_right_10nov.py` for the right panel of Figure 3); the edge rate and the extrapolations (`edge.py`, `fig_edge.py`); the location of the zeros (`locate.py`, `locate_edge.py`); and all of Section 5 (`strip.py`, `count_strip.py`, `locate_strip.py`, `vavg.py`, `trunc.py`). Section 6 (version 2), folder `line`: `line_slices.py` (Figure 4 and the value of $P(\frac{1}{2} + 1000i)$), `line_drift.py` (Figure 5), `line_detectors.py` (Table 7, Figure 6; the columns $\varepsilon = 1$ and $\varepsilon = 0.2$ are obtained by changing `eps` in its line 7), `textura_P.py` (the root mean square of the slices); data `ceros_zeta_2000.npy`.

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