

Small algebraic integers and Frobenius prime sums

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The streamlined determinant argument in recent proofs of Landau–Siegel zero exclusion [3, 8] rules out the excess of inert primes that an exceptional real zero would require, because such an excess would force a positive integer of controlled size to be divisible by an integer larger than itself. Here we observe that the same comparison applies to every Frobenius class in a finite Galois extension of $\mathbb{Q}$, bounding its prime weight in terms of the supply of small algebraic integers.

The argument lifts directly. We collect the corresponding Frobenius congruences into the divisibility of a single nonzero integer. A determinant is well-suited to this task: adding multiples of rows reveals the congruences without changing its value, and factors dividing different rows multiply together. We build that determinant from small algebraic integers and bound its absolute norm.

Let $K/\mathbb{Q}$ be a finite Galois extension, $G = \text{Gal}(K/\mathbb{Q})$, and $n = |G|$. Write $\mathcal{O}_K$ for its ring of integers. At an unramified prime ideal $\mathfrak{p}$ above a rational prime p , arithmetic Frobenius satisfies $\text{Frob}_{\mathfrak{p}}(\alpha) \equiv \alpha^p \pmod{\mathfrak{p}}$ for $\alpha \in \mathcal{O}_K$. Its conjugacy class Frob_p depends only on p .

For a conjugacy class $C \subseteq G$ of size c , let $\mathcal{P}_C$ be the unramified rational primes with $\text{Frob}_p = C$. We study the weighted count

$$T_C(X) := \sum_{\substack{p \leq X \\ p \in \mathcal{P}_C}} \frac{\log p}{p}.$$

Theorem 1 (Height bound). *Suppose $X \geq 2$, $H \geq 1$, and K contains at least $2X$ distinct algebraic integers whose conjugates all have absolute value at most H . Then*

$$T_C(X) \leq c \log H + \log \log(3X) + O(c). \quad (1)$$

The implied constant is independent of K , n , and C .

A lattice-averaging argument turns Theorem 1 into a bound in terms of the discriminant, uniform in K . The theorem bounds every Frobenius class only from above, but the main terms of these upper bounds add up to the total weight of the primes. Subtracting the bounds for the other classes therefore gives a matching lower bound. For each fixed field, this yields a weighted form of Chebotarev’s theorem. For a quadratic field there are only two classes, so the subtraction remains uniform in the conductor; this recovers uniform Landau–Siegel zero exclusion.

Our argument extends Barreto’s Wronskian version [3] of Green’s determinant distillation [8] of the OpenAI proof [12]. Barreto [3, Theorem 1 and Corollary 2] already gives the inert-prime estimate and the asymptotic zero bound that appear in our quadratic application.

The height theorem includes the identity class and $K = \mathbb{Q}$; the chosen integers need not form a Galois-stable set. All logarithms are natural; unsubscripted implied constants are absolute and effective.

Elementary prime estimates. We need only the estimates

$$\vartheta(x) := \sum_{p \leq x} \log p \ll x, \quad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1) \quad (x \geq 2). \quad (2)$$

For completeness, we give the elementary proofs: For an integer $N \geq 1$, every prime $N < p \leq 2N$ divides $\binom{2N}{N}$, so $\vartheta(2N) - \vartheta(N) \leq 2N \log 2$. Summing over dyadic intervals then gives the first estimate in (2).

Next, define $\Lambda(p^a) = \log p$ for integers $a \geq 1$, and $\Lambda(m) = 0$ otherwise. The first estimate gives $\psi(x) := \sum_{m \leq x} \Lambda(m) = \vartheta(x) + O(\sqrt{x} \log x) \ll x$. The identity $\log m = \sum_{d|m} \Lambda(d)$ now yields

$$\log(N!) = \sum_{d \leq N} \Lambda(d) \lfloor N/d \rfloor = N \sum_{d \leq N} \frac{\Lambda(d)}{d} + O(N).$$

Integral comparison gives $\log(N!) = N \log N + O(N)$. Dividing by N , removing the convergent contribution $\sum_{p, a \geq 2} \log p/p^a$, and taking $N = \lfloor x \rfloor$ proves the second estimate.

Proof of Theorem 1. If $H \geq X$, then $c \log H \geq \log X$, and (2) already gives (1); if $X < e^3$, then $T_C(X) = O(1)$. We thus assume $X \geq e^3$ and $1 \leq H < X$. Put $k = \log X$, and choose $M = \lceil 2X \rceil$ distinct algebraic integers $\alpha_1, \dots, \alpha_M$ satisfying the height hypothesis $|\tau(\alpha_i)| \leq H$ for every embedding $\tau: K \hookrightarrow \mathbb{C}$. Fix $\sigma \in C$. We use the pairs $(\alpha_i, \sigma\alpha_i)$, whose coordinate lists are both pairwise distinct.

The determinant. Take the monomials $z^r w^s$ indexed by

$$\mathcal{T} = \{(r, s) \in \mathbb{Z}_{\geq 0}^2 : r + ks < X\}, \quad m = |\mathcal{T}|, \quad U = \sum_{(r,s) \in \mathcal{T}} r.$$

The larger weight on w limits its contribution to the size estimate. It also ensures that replacing a factor z^p by w stays within $\mathcal{T}$ whenever $p \geq k$, as we shall use for divisibility.

The triangle $\{r, s \geq 0 : r + ks < X\}$ lies in the rectangle $[0, X] \times [0, X/k]$, which contains $O(X^2/k)$ lattice points. On the other hand, the rectangle $[X/3, 2X/3] \times [0, X/(3k)]$ lies inside the triangle and contains $\gg X^2/k$ lattice points, each with $r \geq X/3$. Consequently, we have

$$m \ll X^2/k, \quad U \gg X^3/k, \quad Xm \ll U, \quad m \log m \ll U. \quad (3)$$

We use the bounds in (3) to absorb the comparison errors in the sequel.

Putting $J = \lceil 2X/k \rceil$, we construct a matrix A with rows indexed by $\mathcal{T}$ and columns indexed by (i, j) , where $1 \leq i \leq M$ and $0 \leq j < J$, by setting

$$A_{(r,s),(i,j)} = (\partial_z^j(z^r w^s))(\alpha_i, \sigma\alpha_i).$$

We show next that these derivatives make the rows of A linearly independent, so that some $m \times m$ minor is nonzero.

Independence. Suppose, for the sake of seeking a contradiction, that the rows are dependent. The dependence gives a nonzero polynomial $F \in \mathbb{C}[z, w]$ supported on $\mathcal{T}$, with all the corresponding derivative values equal to 0. In particular, $\deg_z F < X$ and $\deg_w F < X/k$.

At most $\deg_w F$ of the polynomials $F(z, \sigma\alpha_i)$ can vanish identically: each such $\sigma\alpha_i$ must be a root of every coefficient of F viewed as a polynomial in z . Thus more than X nonzero specializations remain. Writing $F(z, w) = \sum_s f_s(z) w^s$ shows that these specializations all lie in the span of the f_s . Their span therefore has dimension $\ell \leq \deg_w F + 1 < X/k + 1$. Since $2\ell - 2 < 2X/k$, we have $J \geq 2\ell - 1$.

We choose a monic basis $g_1, \dots, g_\ell$ with strictly increasing degrees $d_1 < \dots < d_\ell$ and consider the Wronskian

$$W(z) = \det(g_b^{(a)}(z))_{0 \leq a < \ell, 1 \leq b \leq \ell}.$$

The coefficient of $z^{\sum_b d_b - \binom{\ell}{2}}$ in W is the Vandermonde product $\prod_{a < b} (d_b - d_a)$ —which is nonzero because the d_b are distinct. Thus we have

$$W \neq 0, \quad \deg W = \sum_{b=1}^{\ell} d_b - \binom{\ell}{2} < \ell X.$$

Now fix an index i for which $h_i(z) := F(z, \sigma\alpha_i)$ is not identically zero. The assumed row dependence gives $h_i^{(j)}(\alpha_i) = 0$ for $0 \leq j < J$, so h_i is divisible by $(z - \alpha_i)^J$. Extend h_i to a basis of the same span. In the Wronskian matrix for this basis, the column corresponding to h_i has entries $h_i^{(a)}$, $0 \leq a < \ell$. Every such entry is divisible by $(z - \alpha_i)^{J-\ell+1}$, and $J - \ell + 1 \geq \ell$. Hence, the Wronskian is divisible by $(z - \alpha_i)^\ell$.

Changing the basis multiplies the Wronskian by a nonzero constant, so $(z - \alpha_i)^\ell$ also divides W . There are more than X such indices i , with distinct α_i . Thus W has more than ℓX zeros counted with multiplicity, contradicting $\deg W < \ell X$.

Size. The matrix entries are algebraic integers. Choose m columns giving a nonzero determinant $\delta \in \mathcal{O}_K$, and put $\Delta = |\mathrm{N}_{K/\mathbb{Q}} \delta| \in \mathbb{Z}_{>0}$. At every embedding, an entry in row (r, s) has modulus at most

$$X^j H^{r+s} \leq X^{2X/k} H^{r+s} \leq H^r e^{3X}.$$

Indeed, differentiating costs at most X^j , and $j < J$ implies that $j < 2X/k$. Meanwhile, $s < X/k$ and $\log H < k$ give $H^s \leq e^X$. Hadamard's inequality and (3) now give

$$\log \Delta \leq n(U \log H + 3Xm + \tfrac{1}{2}m \log m) \leq nU(\log H + O(1)). \quad (4)$$

Divisibility. Fix $p \in \mathcal{P}_C$ with $p \geq k$. At a prime ideal $\mathfrak{p}$ with $\mathrm{Frob}_{\mathfrak{p}} = \sigma$, the congruence $\alpha_i^p \equiv \sigma\alpha_i \pmod{\mathfrak{p}}$ means that $z^p - w$ vanishes modulo $\mathfrak{p}$ at every chosen pair. We build as many of these factors as possible into each row.

For row (r, s) , put $h = \lfloor r/p \rfloor$ and replace $z^r w^s$ by $z^{r-ph} (z^p - w)^h w^s$. Expanding gives an integer combination of the monomials $z^{r-pa} w^{s+a}$ with $0 \leq a \leq h$. The term with $a = 0$ is $z^r w^s$, and each term with $a > 0$ has weight $r + ks - (p - k)a < X$. All those terms are still indexed by $\mathcal{T}$ and have smaller r . Ordering the rows by r therefore makes this an integer unitriangular row change, which leaves δ unchanged.

The derivatives retain the divisibility we have introduced. Indeed, the ideal $I = (p, z^p - w) \subset \mathbb{Z}[z, w]$ is preserved by ∂_z , since $\partial_z(z^p - w) = pz^{p-1} \in I$. The product rule then shows that every power I^h is preserved too. Evaluation at a chosen pair sends I into $\mathfrak{p}$, so every transformed entry in row (r, s) lies in $\mathfrak{p}^h$. Expanding the determinant gives $\delta \in \mathfrak{p}^{\sum_{\mathcal{T}} \lfloor r/p \rfloor}$.

To pass to the norm, we count the primes with the particular Frobenius element σ . Let f be the order of σ . There are n/f primes above p , all with residue degree f . The transitive action of G on these primes conjugates their Frobenius elements, so each of the c elements of C occurs $n/(fc)$ times. Each unit of divisibility at such a prime contributes f to the exponent in the norm. Writing $v_p(\Delta)$ for the exponent of p in Δ , we obtain

$$v_p(\Delta) \geq \frac{n}{c} \sum_{\mathcal{T}} \lfloor r/p \rfloor \geq \frac{n}{c} \left(\frac{U}{p} - m \right). \quad (5)$$

We sum (5) against $\log p$ over $k \leq p \leq X$ in $\mathcal{P}_C$. Since $\vartheta(X) \ll X$, the total error is $O(nXm/c) = O(nU/c)$. Comparing with (4) and dividing by nU/c gives

$$\sum_{\substack{k \leq p \leq X \\ p \in \mathcal{P}_C}} \frac{\log p}{p} \leq c \log H + O(c).$$

Since $k \geq 3$, (2) shows that the omitted primes $p < k$ contribute at most $\log k + O(1) = \log \log X + O(1) \leq \log \log(3X) + O(1)$, which proves (1). $\square$

Corollary 2 (Discriminant bound and weighted Chebotarev). *Write $D = |\text{disc } K|$. Uniformly for $X \geq 2$, we have*

$$T_C(X) \leq \frac{c}{n} \left(\log X + \frac{1}{2} \log D \right) + \log \log(3X) + O(c), \quad (6)$$

with an absolute effective implied constant. In particular, for each fixed finite Galois extension $K/\mathbb{Q}$,

$$T_C(X) = \frac{c}{n} \log X + O_K(\log \log(3X)) \quad (X \geq 2). \quad (7)$$

The set $\mathcal{P}_C$ has Dirichlet density c/n .

Proof. Upper bound. We obtain the required small algebraic integers by finding a translate of a box that contains enough lattice points. Suppose K has r_1 real embeddings and r_2 conjugate pairs of complex embeddings, so $n = r_1 + 2r_2$. Embedding $\mathcal{O}_K$ in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ gives a lattice of covolume $2^{-r_2}\sqrt{D}$, the volume of a fundamental domain.

For $H \geq 1$, take a closed box of side H in each real-embedding coordinate, and side $H/\sqrt{2}$ in each real and imaginary coordinate at a complex embedding. Its volume is $2^{-r_2}H^n$. Two points in this box differ by at most H at every real embedding and by complex modulus at most H at every complex embedding.

Over a fundamental domain of the lattice, the average number of lattice points in a translate of the box is its volume divided by the covolume, namely $H^n/\sqrt{D}$. Thus some translate contains at least $\lceil H^n/\sqrt{D} \rceil$ lattice points. Subtract one selected point from every selected point. The resulting points represent that many distinct algebraic integers, all of whose conjugates have absolute value at most H .

Taking $H = (2X\sqrt{D})^{1/n}$ supplies the integers required by Theorem 1 and proves (6). For fixed K , this gives the upper bound in (7).

Lower bound by complementation. The upper bound holds for every conjugacy class, and the main terms exhaust the total prime mass. Indeed, the unramified primes are partitioned by their Frobenius classes, and only finitely many primes ramify, so by (2) we have

$$\sum_{C'} T_{C'}(X) = \log X + O_K(1), \quad \sum_{C'} \frac{c'}{n} = 1,$$

where C' runs over the conjugacy classes of G and $c' = |C'|$. Subtracting the upper bounds for the classes $C' \neq C$ therefore gives

$$T_C(X) \geq \left(1 - \sum_{C' \neq C} \frac{c'}{n} \right) \log X - O_K(\log \log(3X)) = \frac{c}{n} \log X - O_K(\log \log(3X)).$$

Each class contributes its own error term, and the other classes also contribute their discriminant terms; so, unlike (6), this lower bound depends on K . Combined with the upper bound, this proves (7).

Dirichlet density. To obtain the Dirichlet density, we smooth the weighted estimate and then remove the factor $\log p$. For $1 \leq X < 2$ we have $T_C(X) = 0$ and $(c/n)\log X < \log 2$, while $\log \log(3X) \geq \log \log 3 > 0$, so (7) also holds on this range. For $0 < t \leq 1$, partial summation gives

$$\sum_{p \in \mathcal{P}_C} \frac{\log p}{p^{1+t}} = t \int_0^\infty T_C(e^u) e^{-tu} du = \frac{c}{nt} + O_K(\log(2 + t^{-1})).$$

For the error estimate, substitute $v = tu$ and use $\log(2 + v/t) \leq \log(2 + t^{-1}) + \log(1 + v)$; the latter term is integrable against $e^{-v} dv$.

Integrating from ε to 1 removes $\log p$ on the left. The error is integrable at 0, and $\sum_p p^{-2} < \infty$, so

$$\sum_{p \in \mathcal{P}_C} p^{-1-\varepsilon} = \frac{c}{n} \log(1/\varepsilon) + O_K(1) \quad (\varepsilon \downarrow 0);$$

dividing by $\log(1/\varepsilon)$ gives Dirichlet density c/n in the sense of [5, Definition 2.1 and Lemma 2.2]. $\square$

For a quadratic field, the absolute discriminant is the conductor of its associated primitive character. The uniform estimate (6) therefore gives the prime-sum bound needed for Landau–Siegel zero exclusion.

Corollary 3 (Landau–Siegel zero exclusion). *Let χ be a primitive nonprincipal real Dirichlet character of conductor $q \geq 3$. Then $(\log q)^{-1} \ll L(1, \chi) \ll \log q$, and every real zero $\beta \in (0, 1)$ of $L(s, \chi)$ satisfies*

$$(1 - \beta) \log q \geq 1 - O\left(\frac{\log \log(3q)}{\log q}\right). \quad (8)$$

In particular, there is an absolute effective $c_0 > 0$ such that $L(s, \chi)$ has no real zero in $(1 - c_0/\log q, 1]$. All implied constants are absolute and effective.

Proof. Let K be the quadratic field corresponding to χ , so $n = 2$ and $D = q$. The two conjugacy classes have size 1 and correspond to $\chi(p) = 1$ and $\chi(p) = -1$. Applying (6) gives

$$T_{\pm}(X) := \sum_{\substack{p \leq X \\ \chi(p) = \pm 1}} \frac{\log p}{p} \leq \frac{1}{2} \log X + \frac{1}{4} \log q + \log \log(3X) + O(1).$$

The ramified primes divide q . With $y = \max(2, \log q)$, splitting at y gives

$$\sum_{p|q} \frac{\log p}{p} \leq \log y + O(1) + \frac{\log q}{y}.$$

Thus $T_+ + T_- = \log X + O(\log \log(3q))$ by (2).

We now apply the complementation step from the proof of Corollary 2. With only two classes, just one upper bound is subtracted, and every term in it is explicit in q , so the result remains uniform:

$$T_+ - T_- = 2T_+ - (T_+ + T_-) \leq \frac{1}{2} \log q + O(\log \log(3X) + \log \log(3q)),$$

and symmetrically for $T_- - T_+$. For $X \geq \sqrt{q}$ we have $\log \log(3q) \ll \log \log(3X)$, and so

$$S_{\chi}(X) := \sum_{p \leq X} \chi(p) \frac{\log p}{p}, \quad |S_{\chi}(X)| \leq \frac{1}{2} \log q + O(\log \log(3X)). \quad (9)$$

For $X < \sqrt{q}$, the elementary bound $|S_{\chi}(X)| \leq \log X + O(1)$ gives the same estimate.

The prime-sum bound controls the logarithmic derivative near 1. Put $\lambda = \log q$. For $0 < t \leq 1$, logarithmically differentiating the absolutely convergent Euler product gives

$$-\frac{L'}{L}(1+t, \chi) = \sum_p \sum_{a \geq 1} \frac{\chi(p)^a \log p}{p^{a(1+t)}} = \sum_p \frac{\chi(p) \log p}{p^{1+t}} + O(1),$$

since $\sum_p \sum_{a \geq 2} (\log p)/p^a$ converges. We may then express the remaining prime sum in terms of S_χ via summation by parts:

$$\begin{aligned} -\frac{L'}{L}(1+t, \chi) &= t \int_0^\infty S_\chi(e^u) e^{-tu} du + O(1), \\ \left| \frac{L'}{L}(1+t, \chi) \right| &\leq \frac{1}{2} \lambda + O(\log(2+t^{-1})). \end{aligned} \tag{10}$$

Here $S_\chi(X) = 0$ for $1 \leq X < 2$. To obtain the inequality in (10), we apply (9), noting that $t \int_0^\infty e^{-tu} du = 1$ and

$$\begin{aligned} t \int_0^\infty \log(2+u) e^{-tu} du &= \int_0^\infty \log(2+v/t) e^{-v} dv \\ &\leq \log(2+t^{-1}) + \int_0^\infty \log(1+v) e^{-v} dv \\ &= \log(2+t^{-1}) + O(1). \end{aligned}$$

The bound in (10) is integrable at 0, so $\log L(1+t, \chi)$ has a finite limit as $t \downarrow 0$. Bounded character sums give continuity of L at 1, hence $L(1, \chi) > 0$.

Integrating to $h = \lambda^{-1} < 1$ costs $O(1)$, so $L(1, \chi) \asymp L(1+h, \chi)$. The Euler-factor bounds $h \ll L(1+h, \chi) \ll h^{-1}$ [11] prove the estimates for $L(1, \chi)$.

A real zero near 1 would force a large logarithmic derivative. In Hadamard's identity [7, equation (3) and Exercise 1], the gamma-factor term is uniformly bounded for $0 < t \leq 1$. Discarding the other zeros' nonnegative real contributions therefore gives

$$\frac{1}{t+1-\beta} \leq \frac{L'}{L}(1+t, \chi) + \frac{1}{2} \lambda + O(1) \leq \lambda + O(\log(2+t^{-1})).$$

Taking $t = \lambda^{-2}$ gives, for an absolute effective constant $A > 0$,

$$(1-\beta)\lambda \geq \frac{\lambda}{\lambda + A \log(2 + \lambda^2)} - \frac{1}{\lambda} = 1 - O\left(\frac{\log \log(3q)}{\log q}\right).$$

This proves (8) and gives a uniform positive lower bound for $(1-\beta) \log q$ when q is sufficiently large.

To handle bounded conductors effectively, put $B(x) = \sum_{m \leq x} \chi(m)$, so $|B(x)| \leq q$. Differentiating $L(s, \chi) = s \int_1^\infty B(x) x^{-s-1} dx$ gives $|L'(s, \chi)| \leq 2q/s \leq 4q$ for $1/2 \leq s \leq 1$.

If $\beta \geq 1/2$, then by the mean value theorem $L(1, \chi) = L(1, \chi) - L(\beta, \chi) \leq 4q(1-\beta)$, so the lower bound for $L(1, \chi)$ gives $(1-\beta) \log q \gg 1/q$. If $\beta < 1/2$, then $(1-\beta) \log q > \frac{1}{2} \log 3$. On the remaining effectively bounded range of conductors, these estimates give an absolute effective positive lower bound. Together with the large-conductor estimate above, this proves the existence of c_0 . $\square$

Sources and related arguments. Other recent treatments of Landau–Siegel zero nonexistence include Banks's detailed exposition with explicit constants [2], Kerr's argument using the pigeonhole principle and a theorem of Masser [9], Kimura's quantitative treatment and application to class numbers [10], and Tao's exposition of Green's argument [14], which also records refinements by Barreto [3] and by Pescador. van Dijk [15], building on [3, 8, 9], uses the Frobenius congruence for both the divisibility and the nonvanishing of the determinant: reduction modulo the largest inert prime below the cutoff turns nonvanishing into a polynomial root count, replacing resultants and Wronskians. The same determinant gives an explicit bound for a hypothetical tenth imaginary quadratic field of class number 1, and a finite residue sieve then recovers the complete list.

There are also earlier precedents for this style of comparison of size and divisibility. Selberg [13, §3] obtains the weighted split-prime asymptotic for a fixed quadratic character by factoring a product of quadratic norms. Dobrowolski’s height argument [6] and Cantor–Straus’s determinant version [4] use prime congruences and size estimates to obtain height lower bounds. Here the supply of small algebraic integers is the input, and the prime sum is the output. For a related use of Frobenius congruences in height bounds over abelian extensions, see Amoroso–Zannier [1].

Use of Generative AI. This note was developed in the process of reading and exploring the Green and Barreto arguments with the assistance of GPT-6 Astra (accessed in part via Poe with the support of Quora, where I am an advisor). Claude Opus 5.5 helped identify the earlier precedents discussed above. The specific problem considered and final written form are my own, and of course any errors remain my responsibility.

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