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Superfluid Stiffness and Nonclassical Rotational Inertia of the Dilute Hard-Sphere Bose Gas

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WARNING!

Although we have done extensive machine auditing of these proofs, including adversarial passes with multiple models, the proof is too complex for Lean autoformalization at the present time and it has not yet undergone verification by a human expert. Furthermore, the results rest extensively on unrefereed and unformalized work recently released by OpenAI.

Image: detail from Abraham Ortelius, *Theatrum orbis terrarum* (1570).

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AUTHORS' NOTE

With the exception of these notes, the entire document is machine-written. We have carefully supervised its preparation to ensure that the proofs are readable and that previous literature is appropriately cited. We are releasing this manuscript on Hexagon to disseminate the results as quickly as possible because they may be of interest to the mathematical community. We welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

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ABSTRACT

We study the three-dimensional Bose gas of hard spheres of diameter a at density ρ and temperature T , in the thermodynamic limit with a , ρ , and T fixed. When the gas parameter ρa^3 is sufficiently small and $0 < T < T_*$, we prove two equilibrium signatures of superfluidity. First, in a periodic box, the superfluid fraction, defined through the free-energy cost of twisted boundary conditions, has thermodynamic lower limit at least $c(\rho a^3)^{3/8}$, uniformly in T in this interval. Second, in dilations of an arbitrary nonempty bounded open container with hard walls, the nonclassical rotational inertia fraction $1 - I/I_{\text{cl}}$ has thermodynamic lower limit at least $c'(\rho a^3)^{3/8}$ for every fixed rotation axis, where I is the moment of inertia, I_{cl} its classical value, and $c' > 0$ depends only on the container and the axis. In both results the derivative of the free energy is taken at finite volume, before the thermodynamic limit. The temperature T_* is of order $(\rho a^3)^{1/3} / \log(1/(\rho a^3))$ times the critical temperature of the ideal Bose gas. We also bound the moment generating function of the winding vector of the Feynman–Kac paths by that of a centered Gaussian, at every density and positive temperature. Combining this upper bound with the first result, we determine the order of the winding fluctuations in the regime above. Further, for $0 < \varepsilon \leq 1$, $\rho a^3 \leq (\varepsilon/100)^3$, and $T \geq (1 + \varepsilon)4\pi\rho^{2/3}$, about $1.9(1 + \varepsilon)$ times the critical temperature of the ideal Bose gas, the superfluid fraction in a periodic box tends to zero exponentially in its side length.

By identities of Pollock and Ceperley and of Sindzingre, Klein, and Ceperley, we can express the two responses as multiples of the variances of the winding and of the signed projected area of the paths. We bound these variances from below with a variance inequality of Hammersley–Chapman–Robbins type, applied to random rewirings of the paths in a short imaginary-time interval along closed lattice routes. The rewirings adapt route comparisons developed in OpenAI’s preprints on Bose–Einstein condensation in dilute Bose gases, and we extend the thermal estimates of the positive-temperature hard-sphere preprint in that series to higher temperatures. We average the rewirings over randomly translated closed lattice walks whose independent copies rarely approach each other. This makes the changed law close to the original law in chi-square divergence. Using conditional independence between disjoint imaginary-time intervals, we obtain bounds uniform in temperature. We also transfer the thermal estimates to arbitrary containers.

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Introduction

Below a temperature of about two kelvin, liquid helium-4 flows through narrow channels without measurable friction, and part of it does not follow the rotation of its container. A fluid with these properties is called a superfluid. The goal of this paper is to prove two equilibrium forms of superfluidity for a basic model of an interacting Bose gas: hard spheres in three dimensions, at a fixed small density and a fixed positive temperature, in the thermodynamic limit. Recall that identical particles are called bosons if their wave functions are symmetric under permutations of the particle coordinates. The thermodynamic limit means that the particle number and the volume tend to infinity while their ratio tends to a fixed density. We show that the superfluid fraction, defined through the free-energy cost of a twist of the boundary conditions, has a positive lower bound, and that the moment of inertia of the gas in a slowly rotating container is smaller than its classical value by a positive fraction. We prove both statements through Feynman's representation of the gas as a system of interacting Brownian loops. We will define two geometric quantities of these loops, their winding and their stochastic area, and prove lower bounds on their fluctuations.

1.1 Superfluidity and its equilibrium signatures

In 1938, Kapitza [82] and Allen and Misener [5] independently reported that liquid helium-4, cooled below its transition temperature into the superfluid phase, flows through thin slits and capillaries with almost no viscosity. In the same year, London proposed that the transition is a form of Bose–Einstein condensation, in which a positive fraction of the particles occupies a single one-particle state [107]. Einstein had predicted this condensation for an ideal (noninteracting) gas of bosons [46], on the basis of the quantum statistics of Bose [28]. Tisza then suggested that the liquid behaves as a mixture of two components, a normal one that carries viscosity and a superfluid one that does not [142]. Landau developed a hydrodynamics of two interpenetrating fluids in which the normal component is a gas of collective disturbances of the liquid [90, 91]. Andronikashvili measured the normal fraction with closely spaced disks that oscillate about an axis [6]. Viscosity makes the normal component move with the disks, and the effective moment of inertia decreases as the liquid is cooled. The two components describe different responses of the same liquid, not two permanently labeled populations of atoms. Bogoliubov

showed that in a weakly interacting Bose gas, the energy of an excitation is approximately proportional to its momentum at small momentum, as for sound waves [27]. Feynman gave an atomic theory of the transition and of the two-fluid model [48–51]. We refer to [95, 96, 123] for modern accounts.

Motion without friction concerns the dynamics of the fluid, and a circulating current can last for a long time without being an equilibrium state. We study two equilibrium criteria for superfluidity: the responses of the equilibrium state to a twist and to a slow rotation, two perturbations whose effect on a classical fluid in equilibrium is trivial. We describe them for N particles of mass m at temperature T in a region of side length L , and we write ρ for the number density and $\hbar$ for the reduced Planck constant; Boltzmann's constant is set equal to one. For a Hamiltonian H (the energy operator) at temperature T , we define the free energy by $F = -T \log Z$, where $Z = \text{Tr} e^{-H/T}$ is the partition function.

The first perturbation is a twist. Let the particles move on the torus $\Lambda_L = (\mathbb{R}/L\mathbb{Z})^3$, and require the many-body wave function ψ to satisfy

$$\psi(x_1, \dots, x_j + Le_1, \dots, x_N) = e^{i\theta} \psi(x_1, \dots, x_N) \quad (1 \leq j \leq N), \quad (1.1)$$

where $e_1 = (1, 0, 0)$ and the real number θ is the twist angle. Equivalently, the particles obey periodic boundary conditions and feel a constant vector potential of magnitude $\hbar\theta/L$ in the direction e_1 . Let $F(\theta)$ denote the free energy with the boundary condition (1.1). In classical statistical mechanics we can remove a constant vector potential by shifting the momentum variables. Then F does not depend on θ . In a superfluid, a gradient of the phase of the wave function corresponds to a velocity of the superfluid component. The twist imposes the phase gradient θ/L . The superfluid component then moves with velocity $v_s = \hbar\theta/(mL)$ relative to the box while the normal component stays at rest, and the free energy increases by the kinetic energy of this motion:

$$F(\theta) - F(0) = \frac{1}{2} m \rho_s L^3 v_s^2 + o(\theta^2). \quad (1.2)$$

Here ρ_s is the number density of the superfluid component. Fisher, Barber, and Jasnow defined the helicity modulus through the second-order free-energy response to a slowly varying phase, and related it to the superfluid density in this way [52]. Solving (1.2) for ρ_s , we obtain the superfluid fraction

$$f_s = \frac{\rho_s}{\rho} = \frac{mL^2}{\hbar^2 N} F''(0). \quad (1.3)$$

The second perturbation is a slow rotation. Place the particles in a container and rotate it with a small angular velocity ω about an axis with direction n . In the frame of the container, the Hamiltonian H becomes $H - \omega \mathcal{L}_n$, where $\mathcal{L}_n$ is the total angular momentum about the axis; its formula for our model is given in Section 1.2. Let $F^{\text{rot}}(\omega)$ be the free energy of $H - \omega \mathcal{L}_n$. The moment of inertia of the equilibrium state is

$I = -\partial_{\omega}^2 F^{\text{rot}}(0)$. A classical fluid in equilibrium rotates rigidly with its container, and for a classical system I equals

$$I_{\text{cl}} = m \left\langle \sum_{j=1}^N r_j^2 \right\rangle, \quad (1.4)$$

where r_j is the distance of particle j from the axis and $\langle \cdot \rangle$ is the equilibrium average. In an axisymmetric vessel and in the absence of vortices (circulation around line defects), the superfluid component can remain at rest under slow rotation. In this case we have $I < I_{\text{cl}}$. Hess and Fairbank observed this effect in helium cooled through the transition in a slowly rotating cylinder [70]. Leggett proposed the nonclassical rotational inertia fraction $1 - I/I_{\text{cl}}$ as a criterion for superfluid behavior, in the context of solids [93]; see also [136] for the moment of inertia of a trapped Bose gas.

Superfluidity is different from Bose–Einstein condensation. For a state Γ of N particles, that is, a positive operator of trace one, the one-particle density matrix is $\gamma^{(1)} = N \text{Tr}_{2,\dots,N} \Gamma$, where the partial trace integrates out the coordinates of the other $N - 1$ particles. It is an operator on one-particle wave functions with trace N . Penrose and Onsager defined condensation in an interacting system as the existence of an eigenvalue of $\gamma^{(1)}$ of order N [120]; the corresponding eigenvector is the condensate orbital. Yang placed this definition in the framework of off-diagonal long-range order [146]. In a homogeneous gas, this concerns the persistence of the kernel $\gamma^{(1)}(x, y)$ at large separation $|x - y|$. Penrose and Onsager already pointed out that the condensate fraction and the superfluid fraction need not be equal. For the ideal Bose gas in the thermodynamic limit, Barber showed that the helicity modulus is proportional to the condensate density, so that $f_s = 1 - (T/T_c)^{3/2}$ below the critical temperature T_c [10, Eq. (3.27) and Eqs. (4.2)–(4.3)]. Yet the Landau critical velocity of the ideal gas is zero: by Landau’s energetic criterion, a moving obstacle can create excitations in it at any velocity [123, Section 6.1, pp. 65–67]. At zero temperature, where the state is a ground state, that is, a state of minimum energy, Leggett derived conditions for a unit superfluid fraction, involving translation and time-reversal invariance and an additional condition related to off-diagonal long-range order [94, pp. 935–936, conditions (C1)–(C2)]. Interactions, on the other hand, deplete the condensate; the depleted fraction is the fraction of particles outside the condensate orbital. At zero temperature, Lee, Huang, and Yang predicted the leading depleted fraction $8(\rho a^3)^{1/2}/(3\sqrt{\pi})$ for hard spheres of diameter a [92]. This asymptotic is proved in [113, Corollary 1.2], uniformly over density operators supported on the lowest eigenspace of the Hamiltonian, with the thermodynamic limit taken before the dilute limit $\rho a^3 \rightarrow 0$. In a one-dimensional ground-state model with a random potential, Könenberg, Moser, Seiringer, and Yngvason proved that complete condensation in a Gross–Pitaevskii limit can coexist with an arbitrarily small superfluid fraction [87]. In that one-dimensional model the pair-interaction coupling is scaled by N^{-1} in a fixed region. In the three-dimensional Gross–Pitaevskii limits discussed below, it is the scattering length that is

of order N^{-1} relative to the fixed spatial scale; see Section 1.3. The microscopic relation between condensation and superfluidity was analyzed by Hohenberg and Martin [73]; see also [17, 80, 96].

Feynman developed a sum-over-paths formulation of quantum mechanics [47]. In imaginary time, we can describe both responses probabilistically through this formulation [48]. We use units with $\hbar = 1$ and $m = 1/2$, so that the kinetic energy operator is $-\sum_j \Delta_j$, and with inverse temperature $\beta = 1/T$. The paths are parametrized by the time variable $t \in [0, \beta]$ of the heat semigroup e^{-tH} , which physicists call imaginary time; it is not the physical time of the dynamics. Let $p_t(x, y)$ be the heat kernel of the Laplacian on $\mathbb{R}^3$, or the periodic heat kernel on the torus. Let $\mathbb{P}_{X,Y}$ be the law of N independent Brownian bridges with generator Δ in the time interval $[0, \beta]$, on $\mathbb{R}^3$ or on the torus, the j -th going from x_j to y_j . Since the underlying Brownian motion has generator Δ , each of its coordinates has variance $2t$ at time t . For hard spheres of diameter a , let $\mathcal{C}$ be the event that $|x_i(t) - x_j(t)| > a$ for all $t \in [0, \beta]$ and all $i \neq j$, where on the torus $|x_i(t) - x_j(t)|$ denotes the torus distance. In a container we use the free Euclidean heat kernel and free Euclidean bridges, and $\mathcal{C}$ also requires the paths to stay inside the container. The Feynman–Kac formula expresses the bosonic partition function as

$$Z = \text{Tr } e^{-\beta H} = \frac{1}{N!} \sum_{\pi \in S_N} \int \prod_{j=1}^N p_\beta(x_j, x_{\pi(j)}) \mathbb{P}_{X, \pi X}(\mathcal{C}) dX, \quad (1.5)$$

where $X = (x_1, \dots, x_N)$ and $\pi X = (x_{\pi(1)}, \dots, x_{\pi(N)})$; see Proposition 3.5 and Lemma 3.6 for the representation and the trace normalization, and [48, 61–63, 133] for background. The sum over permutations expresses the symmetry of bosonic wave functions. After normalization, the integrand in (1.5) defines a probability measure μ on permutations and paths, which we call the Bose path measure. Since the path of particle j ends where the path of particle $\pi(j)$ begins, the N paths join into closed loops; a cycle of π of length k gives one loop of duration $k\beta$. In this way the Bose gas becomes a gas of Brownian loops with a hard-core interaction at equal times.

On the torus, we lift each loop to $\mathbb{R}^3$. After one full turn, its displacement is L times an integer vector, which we call its winding vector. Let $W \in \mathbb{Z}^3$ be the sum of the winding vectors of all loops. Pollock and Ceperley observed that the twist (1.1) multiplies the weight of each configuration by the phase $e^{i\theta W_1}$, the phase acquired by closed loops in a constant vector potential, so that $Z(\theta)/Z(0) = \mathbb{E}_\mu e^{i\theta W_1}$ [124]. Since $\mathbb{E}_\mu W_1 = 0$ by symmetry, differentiating twice at $\theta = 0$ yields $F''(0) = T \text{Var}_\mu(W_1)$. In our units, (1.3) becomes

$$f_s = \frac{TL^2}{2N} \text{Var}_\mu(W_1). \quad (1.6)$$

This is the winding-number formula of Pollock and Ceperley [124]. For rotation, the Hamiltonian $H - \omega \mathcal{L}_n$ differs from the Hamiltonian of particles in a uniform magnetic

field along n by a centrifugal term proportional to $\omega^2 \sum_j r_j^2$, which accounts for I_{cl} . The magnetic field multiplies the weight of each configuration by $e^{i\omega \mathcal{A}_n}$, where the stochastic area $\mathcal{A}_n$ is the sum over loops of the signed areas of their projections onto the plane orthogonal to n . We obtain

$$I_{\text{cl}} - I = T \text{Var}_\mu(\mathcal{A}_n), \quad (1.7)$$

which is the area formula of Sindzingre, Klein, and Ceperley [135]; see Ceperley's review [33] for both formulas and their use in path-integral Monte Carlo simulations.

Superfluidity, in the two senses above, is the statement that the winding (or the area) of an interacting system of Brownian loops has macroscopic fluctuations. By (1.6), the superfluid fraction has a positive lower bound in the thermodynamic limit if and only if the normalized variance $\text{Var}_\mu(W_1)/(\beta\rho L)$ has a positive lower limit. In a container of linear size L , the classical moment of inertia is at most a constant times NL^2 . By (1.7), the nonclassical rotational inertia fraction then has a positive lower limit whenever $\text{Var}_\mu(\mathcal{A}_n)/(\beta\rho L^5)$ does. Bose statistics is essential here. For noninteracting distinguishable particles, only the identity permutation occurs, each loop has duration β , and the probability that a single loop winds around the torus is of order $e^{-L^2/(4\beta)}$; the superfluid fraction then tends to zero. At fixed inverse temperature, long permutation cycles provide a natural mechanism for winding on the scale of the container, since a cycle through k particles is a loop of duration $k\beta$. Feynman related the superfluid transition to the appearance of such cycles [48]; see also [33, Section III.F]. The present paper proves that, for the dilute hard-sphere gas at low positive temperature, the winding and the area have fluctuations of the required size.

1.2 The model and the main results

We use units in which $\hbar = 1$, the particle mass is $m = 1/2$, and Boltzmann's constant is one, so that $\hbar^2/(2m) = 1$ and temperature is measured in units of energy. The dictionary to general $\hbar$ and m is given in Remark 1.10. We fix the diameter $a > 0$ of the hard spheres, which is the smallest allowed distance between two particle centers. It is also the scattering length of the interaction: the zero-energy scattering solution outside the excluded ball is $1 - a/|x|$, and its coefficient a is called the scattering length. Throughout, $T > 0$ is the temperature and $\beta = 1/T$.

The gas on the torus. For $N \geq 1$ and $L > 0$, let d_L be the Euclidean distance on the torus $\Lambda_L = (\mathbb{R}/L\mathbb{Z})^3$, and let

$$\mathcal{X}_{N,L} = \{X = (x_1, \dots, x_N) \in \Lambda_L^N : d_L(x_i, x_j) > a \text{ for all } i \neq j\}.$$

We consider only N and L for which $\mathcal{X}_{N,L}$ is nonempty; along the sequences in our theorems, this holds for all large L . The Hamiltonian $H_{N,L}$ is the self-adjoint operator

associated with the quadratic form $\psi \mapsto \sum_{j=1}^N \int_{\mathcal{X}_{N,L}} |\nabla_j \psi|^2$ on the symmetric functions in $H_0^1(\mathcal{X}_{N,L})$. In other words, $H_{N,L}$ is the Dirichlet realization of $-\sum_j \Delta_j$ on bosonic wave functions that vanish when two particles touch. For $\theta \in \mathbb{R}$, the twisted Hamiltonian $H_{N,L}(\theta)$ is defined by the same form domain and the differential expression $\sum_j (-i\nabla_j + \theta e_1/L)^2$. It is unitarily equivalent to $-\sum_j \Delta_j$ with the boundary condition (1.1). The twisted free energy is

$$F_{N,L,T}(\theta) = -T \log \text{Tr} e^{-H_{N,L}(\theta)/T},$$

where the trace is over symmetric functions. We define the superfluid fraction at finite volume by

$$f_s(N, L, T) = \frac{L^2}{2N} \partial_\theta^2 F_{N,L,T}(0). \quad (1.8)$$

In Section 3, we show that $F_{N,L,T}$ is twice continuously differentiable, that (1.6) holds with the winding vector W of the Bose path measure μ on Λ_L , and that $0 \leq f_s(N, L, T) \leq 1$.

The gas in a container. Let $\Omega \subset \mathbb{R}^3$ be a nonempty bounded open set, the container, and let $|\Omega| > 0$ be its volume. For $L > 0$, consider N hard spheres with centers in the dilated container $L\Omega$, and let

$$\mathcal{X}_{N,L}^\Omega = \{X \in (L\Omega)^N : |x_i - x_j| > a \text{ for all } i \neq j\}.$$

As on the torus, we consider only N and L for which $\mathcal{X}_{N,L}^\Omega$ is nonempty. The Hamiltonian $H_{N,L}^\Omega$ is the Dirichlet realization of $-\sum_j \Delta_j$ on the symmetric functions in $H_0^1(\mathcal{X}_{N,L}^\Omega)$; the wave functions vanish when two particles touch and when a particle touches the wall. Fix a unit vector $n \in \mathbb{R}^3$ and a point $x_0 \in \mathbb{R}^3$, and consider rotations about the axis $\{Lx_0 + sn : s \in \mathbb{R}\}$. The angular momentum about this axis is

$$\mathcal{L}_n = \sum_{j=1}^N n \cdot ((x_j - Lx_0) \times (-i\nabla_j)).$$

The operator $H_{N,L}^\Omega - \omega \mathcal{L}_n$ is defined through its closed quadratic form on the form domain of $H_{N,L}^\Omega$; the first-order term is infinitesimally form bounded because the container is bounded. For $\omega \in \mathbb{R}$, let $F_n^{\text{rot}}(\omega) = -T \log \text{Tr} e^{-(H_{N,L}^\Omega - \omega \mathcal{L}_n)/T}$, again with the trace over symmetric functions. We define the moment of inertia and the classical moment of inertia by

$$I_n = -\partial_\omega^2 F_n^{\text{rot}}(0), \quad I_{\text{cl},n} = \frac{1}{2} \left\langle \sum_{j=1}^N r_j^2 \right\rangle, \quad (1.9)$$

where r_j is the distance from x_j to the axis and $\langle \cdot \rangle$ denotes expectation in the normalized state $e^{-H_{N,L}^\Omega/T} / \text{Tr} e^{-H_{N,L}^\Omega/T}$, the Gibbs state at temperature T . The number $1 - I_n/I_{\text{cl},n}$ is the nonclassical rotational inertia fraction. The circle in the integral below denotes the

Stratonovich convention, obtained by midpoint evaluation in stochastic Riemann sums. In Section 3, we show that I_n is well defined and that

$$I_{\text{cl},n} - I_n = T \text{Var}_\mu(\mathcal{A}_n), \quad \mathcal{A} = \frac{1}{2} \sum_{j=1}^N \int_0^\beta x_j(t) \times \circ dx_j(t), \quad \mathcal{A}_n = n \cdot \mathcal{A}, \quad (1.10)$$

where μ is now the Bose path measure in the container, and $x_j(t)$ is the path of particle j . We call $\mathcal{A} \in \mathbb{R}^3$ the stochastic area vector. Because the paths join into closed loops, $\mathcal{A}$ does not depend on the choice of origin. In particular, the deficit $I_{\text{cl},n} - I_n$ is the same for all parallel axes. The identity (1.10) implies $I_n \leq I_{\text{cl},n}$. Since F_n^{rot} is concave (Proposition 3.16), we also have $I_n \geq 0$. We do not assume any symmetry of Ω or any regularity of its boundary.

Main results. Let $\alpha_0 \geq 2$ be the absolute constant fixed in Step 1 of Section 10.3, large enough for the scaled setting of [112, Section 2] and for Appendices A and B; see Section 8.1. Let κ_T be the numerical constant fixed in Appendix B. For $a, \rho > 0$ with $\rho a^3 < \alpha_0^3$, we set $D_\rho = \alpha_0^{3/2}(\rho a^3)^{-1/2}$ and define

$$T_*(a, \rho) = \frac{\kappa_T \rho a}{\log D_\rho}. \quad (1.11)$$

Under the change of units of [112, Section 10], every temperature in the open interval $(0, T_*(a, \rho))$ maps for all large L into the scaled temperature window $0 < T \leq \tau_0 / \log D$ of Section 8, where $\tau_0 = \kappa_T \alpha_0$ and D is the number of particles per unit cell in scaled units; see Section 10. Once D is large, this window contains the window $0 < T \leq D^{-1}$ of [112, Theorem 2.1], and we extend the thermal estimates of [112] to it in Appendix B. In the following theorems, $f_s(N, L, T)$, I_n , and $I_{\text{cl},n}$ are the finite-volume quantities defined in (1.8) and (1.9).

Theorem 1.1. *There exist absolute constants $c_0 > 0$ and $c_1 > 0$ such that the following holds. Let $a > 0$ and $\rho > 0$ satisfy $\rho a^3 \leq c_0$, and let $0 < T < T_*(a, \rho)$. Then*

$$\liminf_{\substack{N, L \rightarrow \infty \\ N/L^3 \rightarrow \rho}} f_s(N, L, T) \geq c_1 (\rho a^3)^{3/8}.$$

The lower limit is over all sequences of N and L tending to infinity with $N/L^3 \rightarrow \rho$, at fixed a , ρ , and T . The lower bound is the same for every temperature in the interval $(0, T_*)$. By (1.6), Theorem 1.1 states that $\text{Var}_\mu(W_1) \geq (2c_1 - \varepsilon)(\rho a^3)^{3/8} \beta \rho L$ for every $\varepsilon > 0$ and all sufficiently large L in any such sequence.

For comparison, the ideal Bose gas of density ρ condenses below the temperature $T_c(\rho) = 4\pi(\rho/\zeta(3/2))^{2/3}$, where ζ is the Riemann zeta function, and

$$\frac{T_*(a, \rho)}{T_c(\rho)} = \frac{\kappa_T \zeta(3/2)^{2/3}}{4\pi} \frac{(\rho a^3)^{1/3}}{\log D_\rho}. \quad (1.12)$$

Numerical simulations place the transition of the dilute hard-sphere gas close to T_c [65, 122]. Seiringer and Ueltschi proved an upper bound on the critical temperature for Bose–Einstein condensation, by proving exponential decay, with spatial separation, of the kernel of the one-particle density matrix above an explicit temperature [131]. The normal regime of Theorem 1.5 below starts at $(1 + \varepsilon)\zeta(3/2)^{2/3}T_c \approx 1.897(1 + \varepsilon)T_c$, whereas their bound for the absence of condensation is much closer to T_c , namely $T_c(1 + 5.09(\rho a^3)^{1/6}(1 + o(1)))$ as $\rho a^3 \rightarrow 0$, for a different quantity, in the grand-canonical ensemble on domains with Dirichlet boundary conditions [131]. The interval $(0, T_*)$ is a small fraction, of order $(\rho a^3)^{1/3} / \log(1/(\rho a^3))$, of the expected range of superfluidity. It lies a logarithmic factor below the energy scale ρa of the excitations of the gas. For momentum $p \in \mathbb{R}^3$, writing $p^2 = |p|^2$, Bogoliubov’s excitation spectrum in our units is $\epsilon(p) = (p^2(p^2 + 16\pi a\rho))^{1/2}$ [27]. It is approximately linear for $p^2 \ll a\rho$, where the excitations are phonons, that is, low-momentum collective excitations, and approximately quadratic for $p^2 \gg a\rho$; the crossover energy is of order ρa . We have $T_*/(\rho a) = \kappa_T / \log D_\rho$. In this regime the theories of Landau and Bogoliubov predict that the normal fraction $1 - f_s$ is very small [27, 91, 123]. Our lower bound $c_1(\rho a^3)^{3/8}$ is far from this prediction. Theorem 1.1 implies that the gas is superfluid in the sense of (1.2), but it does not determine its expected near-unit value.

Our second result concerns rotation. The covariance matrix $\text{Cov}_\mu(\mathcal{A})$ of the stochastic area vector has smallest eigenvalue $\lambda_{\min}(\text{Cov}_\mu(\mathcal{A})) = \inf_{|n|=1} \text{Var}_\mu(\mathcal{A}_n)$. For $x_0 \in \mathbb{R}^3$, let $R_\Omega(x_0) = \sup_{y \in \Omega} |y - x_0|$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a nonempty bounded open set. There exist constants $c_0 > 0$ and $c_1 > 0$, depending only on Ω , such that the following holds. Let $a > 0$ and $\rho > 0$ satisfy $\rho a^3 \leq c_0$, let $0 < T < T_*(a, \rho)$, and consider the hard-sphere gas in $L\Omega$ along sequences with $N, L \rightarrow \infty$ and $N/(L^3|\Omega|) \rightarrow \rho$. Then*

$$\liminf_{\substack{N, L \rightarrow \infty \\ N/(L^3|\Omega|) \rightarrow \rho}} \frac{\lambda_{\min}(\text{Cov}_\mu(\mathcal{A}))}{\beta \rho L^5} \geq c_1(\rho a^3)^{3/8}. \quad (1.13)$$

Further, for every unit vector $n \in \mathbb{R}^3$ and every $x_0 \in \mathbb{R}^3$,

$$\liminf_{\substack{N, L \rightarrow \infty \\ N/(L^3|\Omega|) \rightarrow \rho}} \left(1 - \frac{I_n}{I_{\text{cl}, n}}\right) \geq \frac{2c_1}{|\Omega|R_\Omega(x_0)^2} (\rho a^3)^{3/8}. \quad (1.14)$$

The bound (1.13) says that $\text{Cov}_\mu(\mathcal{A}) \geq (1 - \varepsilon)c_1(\rho a^3)^{3/8}\beta\rho L^5 \text{Id}$ in the sense of quadratic forms, for every $\varepsilon > 0$ and all sufficiently large L . By (1.10), the inertia deficit $I_{\text{cl}, n} - I_n$ equals $T n \cdot \text{Cov}_\mu(\mathcal{A}) n$ for every axis with direction n . Then (1.13) bounds the deficit from below for all axes at once. For a general direction n , the proof rotates the coordinates so that n becomes a coordinate direction; no symmetry of Ω is used. The

constant in (1.14) depends on Ω and on the axis point x_0 , and not on n . Combining (1.13), the identity (1.10), and the inequality $I_{\text{cl},n} \leq NL^2 R_\Omega(x_0)^2/2$, we obtain the bound (1.14). When $\Omega = (-1/2, 1/2)^3$ and $x_0 = 0$, the symmetries of the cube imply that I_n and $I_{\text{cl},n}$ do not depend on n , at every finite volume. In this case the nonclassical rotational inertia fraction is the same for all axes through the center of the cube.

Our third result bounds the moment generating function of the winding vector by that of a centered Gaussian (a sub-Gaussian bound), without any restriction on the density or the temperature. Let μ be the Bose path measure on Λ_L and W its winding vector, as in (1.6).

Theorem 1.3. *Let $N \geq 1$, $L > 0$, $a > 0$, and $\beta > 0$, and suppose that $\mathcal{X}_{N,L}$ is nonempty. Then for every $t \in \mathbb{R}^3$,*

$$\mathbb{E}_\mu \exp(t \cdot W) \leq \exp(\beta N |t|^2 / L^2).$$

Expanding both sides to second order in t , we find that $\text{Var}_\mu(u \cdot W) \leq 2\beta N |u|^2 / L^2$ for $u \in \mathbb{R}^3$; by (1.6), this is the upper bound $f_s \leq 1$. Combining this with Theorem 1.1, we obtain the order of magnitude of the winding vector.

Corollary 1.4. *Assume the hypotheses of Theorem 1.1, let c_1 be the constant there, and set $\hat{W} = W/(\beta \rho L)^{1/2}$. Then $\mathbb{E}_\mu \exp(t \cdot \hat{W}) \leq \exp(N |t|^2 / (\rho L^3))$ for every $t \in \mathbb{R}^3$, and $\sup_{N,L} \mathbb{E}_\mu |\hat{W}|^p < \infty$ for every $p \geq 1$, where the supremum is over N and L in any sequence with $N/L^3 \rightarrow \rho$. Further, every subsequential limit in distribution $\hat{W}_\infty$ of $\hat{W}$ along such a sequence satisfies $\mathbb{E} \hat{W}_\infty = 0$ and*

$$2c_1(\rho a^3)^{3/8} \text{Id} \leq \text{Cov}(\hat{W}_\infty) \leq 2 \text{Id}.$$

The area satisfies a two-sided bound of the same kind. Since $I_n \geq 0$, we have $\text{Var}_\mu(\mathcal{A}_n) \leq \beta I_{\text{cl},n}$ by (1.10), and $\beta I_{\text{cl},n} \leq (1 + o(1)) |\Omega| R_\Omega(x_0)^2 \beta \rho L^5 / 2$. Then, by Theorem 1.2, we find that the area variance has order of magnitude $\beta \rho L^5$. We do not prove the analog of Corollary 1.4 for the area, because we have no uniform bound on its higher moments.

Theorem 1.5. *Let $\varepsilon \in (0, 1]$, let $a, \rho > 0$ satisfy $\rho a^3 \leq (\varepsilon/100)^3$, and consider a sequence with $N, L \rightarrow \infty$ and $N/L^3 \rightarrow \rho$. Then, for all sufficiently large L in the sequence and all $T \geq (1 + \varepsilon) 4\pi \rho^{2/3}$,*

$$f_s(N, L, T) \leq 57 \left(\frac{L^2 T}{4\pi} \right)^{3/4} \exp \left(-\frac{L}{4} \sqrt{T \log(1 + \varepsilon)} \right), \quad (1.15)$$

and $\mathbb{P}_\mu(W \neq 0) \leq 190 (L^2 T / 4\pi)^{3/2} \exp(-\frac{L}{2} \sqrt{T \log(1 + \varepsilon)})$. In particular, $f_s(N, L, T) \rightarrow 0$ and $\text{Var}_\mu(W_1) \rightarrow 0$, uniformly in $T \geq (1 + \varepsilon) 4\pi \rho^{2/3}$.

With $\varepsilon = 1$ and $\rho a^3 \leq \min\{c_0, 10^{-6}\}$, Theorems 1.1 and 1.5 give a superfluid interval $(0, T_*(a, \rho))$ and a normal interval $[8\pi \rho^{2/3}, \infty)$, which are disjoint; we prove nothing about the temperatures in between.

We conclude this section with several remarks on the scope of these results.

Remark 1.6. In Theorems 1.1 and 1.2, the derivatives in θ and ω are taken at $\theta = 0$ and $\omega = 0$ at each finite N and L , and the thermodynamic limit is taken afterward. We do not prove that $f_s(N, L, T)$ or $I_n/I_{cl,n}$ converges, and we make no statement about the free energy at a fixed nonzero twist or angular velocity. A twist of angle θ changes the free energy by $f_s N \theta^2 / L^2 + o(\theta^2)$, which is of order L , while the free energy itself is of order L^3 . For this reason, asymptotic formulas for the free energy per unit volume, such as those discussed in Section 1.3, do not by themselves control this curvature; one needs estimates that resolve the twist-dependent part of the free energy, which is smaller than the volume, together with control of its derivatives. Both of our quantities are responses of the equilibrium state to static perturbations. Fisher, Barber, and Jasnow emphasized that a nonzero equilibrium helicity modulus does not by itself establish the dynamical stability of finite superflow [52, p. 1113]. Prokof'ev and Svistunov discuss how such definitions relate to dynamical ones [125].

Remark 1.7. The proofs give the following finite-volume form of Theorem 1.1, uniform in the density and the temperature. Let c_0 and c_1 be the constants of that theorem, let $a > 0$, and let $0 < \rho_1 \leq \rho_2$ with $\rho_2 a^3 \leq 2c_0$. There exists L_0 such that $f_s(N, L, T) \geq c_1(\rho_L a^3)^{3/8}$, where $\rho_L = N/L^3$, for all $L \geq L_0$, all N with $\rho_L \in [\rho_1, \rho_2]$, and all $0 < T \leq T_*(a, \rho_L)$. The analogous statement holds for (1.13) and (1.14), with the lower limits omitted, ρ replaced throughout by $\rho_L = N/(L^3|\Omega|)$, and the constants of Theorem 1.2. Each finite-volume quantity is computed at a positive temperature; the remark gives no statement about ground states.

Remark 1.8. Several properties expected of the dilute Bose gas are not addressed here. We do not prove that f_s tends to one as T or ρa^3 tends to zero, and we do not determine the leading low-temperature behavior of $1 - f_s$. We do not prove a positive critical velocity, indefinitely circulating currents, or vortices with quantized circulation. For an excitation-spectrum formulation of Landau's criterion and the role of the order of limits, see [36, Sections 2.3, 2.5, and 2.6 in arXiv version 4]. We do not prove a sharp phase transition, or identify the temperature at which superfluidity disappears. We do not prove a central limit theorem for the winding vector, or that the limit in distribution of $\hat{W}$ in Corollary 1.4 is unique. Finally, our results do not imply Bose–Einstein condensation; condensation for this model is the subject of [112, 115].

Remark 1.9. The two results use different boundary conditions for a reason. In a box with hard walls, a constant vector potential is removed by multiplying the wave function by a phase, so the free energy does not depend on a constant twist; the twist must be imposed through periodic boundary conditions. For the rotational response we use a physical container. The nonclassical rotational inertia fraction depends on the shape of the

container, and it is not expected to equal f_s . In a vessel without rotational symmetry about the axis, such as a cube, the walls set the superfluid component into an irrotational (curl-free) motion that carries angular momentum, and even a fully superfluid fluid need not have $I_n = 0$; see [38] and [123, Section 14.2]. For the same reason, the angular momentum $\mathcal{L}_n$ does not commute with $H_{N,L}^\Omega$ in general, and I_n is a Duhamel correlation of $\mathcal{L}_n$, that is, an average over imaginary time of covariances, as in quantum linear response theory [89, Eq. (3.16)], rather than $\beta \text{Var}(\mathcal{L}_n)$. See [33, Section III.E, Eqs. (3.18)–(3.25)] for this rotational response without cylindrical symmetry.

Remark 1.10. In general units, write $\kappa = \hbar^2/(2m)$ and measure temperature in units of energy. The superfluid fraction is $f_s = mL^2 F''(0)/(\hbar^2 N) = mL^2 \text{Var}_\mu(W_1)/(\beta \hbar^2 N)$, which agrees with (1.3). The identity (1.10) becomes $I_{\text{cl},n} - I_n = 4m^2 \text{Var}_\mu(\mathcal{A}_n)/(\beta \hbar^2)$, with $I_{\text{cl},n}$ as in (1.4). The bound of Theorem 1.3 becomes $\mathbb{E}_\mu \exp(t \cdot W) \leq \exp(\beta \kappa N |t|^2/L^2)$. The threshold temperature is $\kappa \kappa_T \rho a / \log D_\rho$, and the ideal-gas critical temperature is $4\pi \kappa (\rho/\zeta(3/2))^{2/3}$; the ratio (1.12) is unchanged. The fractions f_s and $1 - I_n/I_{\text{cl},n}$ are dimensionless and do not depend on the choice of units. In the Brownian representation, imaginary time then runs over $[0, \beta]$ and the paths have generator $\kappa \Delta$.

1.3 Previous work

Physics. The dilute hard-sphere Bose gas is a standard model in the physics literature. Following the work of Bogoliubov [27], Lee, Huang, and Yang computed its ground-state energy to second order in the gas parameter and described its low-temperature properties [92]. Hugenholtz and Pines derived a relation between the chemical potential and the self-energies that yields a gapless excitation spectrum for an interacting condensed Bose system at zero temperature [75, Section 6, Eq. (6.2)]. Path-integral Monte Carlo simulations of the hard-sphere gas locate the superfluid transition close to the ideal-gas temperature T_c [65, 122]; both use the winding-number formula (1.6). Rotational properties of trapped dilute gases are discussed in [123, 136]. These physical analyses motivate superfluidity at low temperature, with a fraction close to one in the phonon regime, while the simulations locate the transition. They do not constitute proofs of our lower bounds on the response.

Lattice models. Rigorous results on long-range order and condensation are available for lattice spin systems and lattice Bose gases [4, 45, 58, 83]. Several of them concern lattice gases of hard-core bosons, whose nearest-neighbor hopping corresponds to spin-1/2 XY couplings through the correspondence of Matsubara and Matsuda [108, Sections 2–3]. For the superfluid response, Paramekanti, Trivedi, and Randeria obtained variational upper bounds on the superfluid stiffness of disordered lattice systems [118]. These results do not give positive lower bounds on the response of a continuum gas. In two dimensions,

Fröhlich and Spencer proved the Kosterlitz–Thouless transition of classical rotator models, a transition at which the decay of correlations changes from exponential to a power law [57].

Energy and free energy of the continuum gas. For the continuum gas at fixed density, most rigorous results concern the energy and the free energy. Dyson proved upper and lower bounds on the ground-state energy of hard spheres [44]. Lieb and Yngvason proved that the ground-state energy per unit volume is $4\pi a\rho^2(1 + o(1))$ as $\rho a^3 \rightarrow 0$, for repulsive interactions including hard cores [101]. The second-order term of Lee, Huang, and Yang was proved as an upper bound for sufficiently weak, smooth, rapidly decaying repulsive potentials in [147, Theorem 2.1], and for nonnegative radial compactly supported $L^3(\mathbb{R}^3)$ potentials, without a smallness assumption, in [12, Theorem 1.1], and as a lower bound by Fournais and Solovej [54, 55], whose second paper treats interactions with hard cores; see also [31]. For hard spheres, an upper bound with an error of the order of the second-order term was proved in [13]. The sharp second-order upper bound is stated for hard spheres in [15] and for general potentials in [16]. At positive temperature, Seiringer [130, Theorem 1] proved a lower bound for the leading interaction correction to the free energy, allowing hard cores and uniformly for temperatures up to the order of T_c , down to zero. Yin [148, Theorem 1 and Corollary 1] proved the matching upper bound for bounded, radial, compactly supported repulsive potentials when T/T_c tends to a fixed positive value. A leading-order free-energy upper bound for nonnegative radial compactly supported L^3 potentials is proposed in [14]. At low temperatures, the free energy to second order was proved as a lower bound in [66], and as an upper bound for nonnegative radial compactly supported $L^2(\mathbb{R}^3)$ potentials, at temperatures up to a fixed multiple of ρa , in [67]. The lower bound was extended to nonnegative radial non-increasing potentials whose range is bounded by a fixed multiple of their scattering length, including hard cores, in [53]. As explained in Remark 1.6, these bulk results do not by themselves control the superfluid fraction.

Condensation in scaling limits. For continuum gases with short-range interactions, Bose–Einstein condensation was proved in scaling limits in which the gas parameter tends to zero as $N \rightarrow \infty$. Lieb and Seiringer proved complete condensation of the ground state of a trapped gas in the Gross–Pitaevskii limit [99]; we refer to [105] for an account. Bogoliubov theory and optimal rates of condensation in Gross–Pitaevskii limits were obtained for periodic boxes [25, 26] and trapped gases [29, 30, 110], and scalings beyond the Gross–Pitaevskii regime were treated in [3]. At positive temperature, condensation was proved for a trapped gas by Deuchert, Seiringer, and Yngvason [40], and for a homogeneous gas in the Gross–Pitaevskii limit by Deuchert and Seiringer [39]. For the grand-canonical mean-field gas on a three-dimensional unit torus, Deuchert, Nam, and Napiórkowski justified a Bogoliubov description of the Gibbs state at temperatures

comparable to the condensation transition [41]; they also related condensation and spontaneous $U(1)$ symmetry breaking through ordered quasi-averages [42]. These mean-field results concern an interaction scaled with the particle number. For nonnegative radial decreasing finite-range L^1 potentials, condensation has also been propagated from small boxes to larger ones for low-energy states in a dilute scaling regime [35, Theorem 2]. A Neumann localization inequality for this strategy is given in [81, Theorem 3]. Sütő has proposed a proof of condensation for a class of potentials with nonnegative Fourier transform at high density [140].

Condensation at fixed density. In a series of preprints, OpenAI proved Bose–Einstein condensation for the dilute gas at fixed density in the thermodynamic limit. In [115], it is proved that every ground state of the hard-sphere gas has a constant one-particle orbital whose occupation is proportional to N when ρa^3 is small. In [111], a condensate fraction bounded below uniformly in small densities is obtained for fixed bounded, nonnegative, radial, finite-range, nonzero potentials, at temperatures $0 \leq T \leq \rho^2$. In [112], condensation is proved for the hard-sphere gas at a positive temperature. In the main theorem [112, Theorem 1.1], one positive temperature is given for each small density. Its scaled form [112, Theorem 2.1] holds uniformly in the scaled window $0 < T \leq D^{-1}$. Under the change of units of [112, Section 10], our interval $(0, T_*)$ maps into a larger window, for which we extend the thermal estimates of [112], not its condensation theorem, in Appendix B. In the companion preprints [113, 114], the leading fraction outside the condensate at zero temperature, called the quantum depletion, is determined for hard spheres and for fixed bounded, nonnegative, radial, finite-range potentials with positive scattering length, respectively. The thermodynamic limit is taken before the dilute limit $\rho a^3 \rightarrow 0$. Our proofs adapt the thermal estimates of [112] and several of the methods of [111, 112, 115]; Section 1.4 summarizes these debts, and the technical sections give the precise attributions.

Superfluidity. Rigorous results on the superfluid response of interacting continuum gases are few. Lieb, Seiringer, and Yngvason proved complete superfluidity of the ground state of a dilute Bose gas in a Gross–Pitaevskii limit, defined through the response to an imposed velocity field [103]; we compare their result with ours below. Könenberg, Moser, Seiringer, and Yngvason studied the superfluid fraction of a one-dimensional condensate in a random potential, in a Gross–Pitaevskii limit [87], as mentioned in Section 1.1. For rotating gases, Lieb and Seiringer derived the Gross–Pitaevskii equation from the many-body problem [100], and the rotating Gross–Pitaevskii functional has been analyzed in detail; see, for example, [37, 129]. These works concern rotation in the ground state and Gross–Pitaevskii theory, including regimes in which the angular velocity scales with the coupling. Here we study the susceptibility at positive temperature and $\omega = 0$. For the ideal gas, the helicity modulus was computed by Barber [10].

Cycles and loops. The loop representation (1.5) has been studied in its own right. Sütő introduced cycle percolation in the Bose gas, in which a nonzero fraction of particles belongs to permutation cycles whose lengths diverge in the thermodynamic limit, and related it to condensation in the ideal gas [137]. He later proved that the two are equivalent for the ideal gas and for the imperfect Bose gas, whose interaction depends only on the particle number [138]. Ueltschi studied Feynman cycles and their relation to off-diagonal long-range order [143, 144]. Betz and Ueltschi proved the existence of infinite cycles and identified their length distribution in models of spatial random permutations [21–23]. Adams, Collevecchio, and König obtained variational bounds from the loop representation, and an equality identifying the free energy in specified high-temperature or low-density regimes [2]. König developed a description in which infinitely long paths appear alongside the finite loops [88]. Random loop representations of quantum spin systems are studied in [24, 145], and loop ensembles of lattice Bose gases in [59]. We do not know of a general argument that deduces winding fluctuations of order $\beta\rho L$ from the existence of long cycles.

Comparison with the closest results. We now compare our results with the two closest ones. Lieb, Seiringer, and Yngvason [103, Theorem 1] proved the asymptotics of the ground-state energy under a twist that correspond to complete superfluidity, in a periodic cube of fixed side length L , as $N \rightarrow \infty$ with Na/L fixed. In this limit the density grows and $\rho a^3 = O(N^{-2})$. Their result controls the twist-dependent energy in this scaling limit, which is stronger than computing a zero-temperature curvature at each fixed finite volume. They also treated an annular cylinder with an imposed azimuthal velocity field [103, Theorem 2]. In Theorem 1.1, we instead bound the curvature of the canonical free energy, with particle number fixed, at finite volume. We fix a , the density, and the positive temperature before taking the ordinary thermodynamic limit. The strength of the conclusions also differs: the superfluid fraction is one in the limit of [103], while in Theorem 1.1 we obtain only the lower bound $c_1(\rho a^3)^{3/8}$. The Gross–Pitaevskii superfluidity proof of [103] uses generalized Poincaré inequalities with a vector potential [104, Theorem 3 and Corollary 1].

The condensation theorem of [112] holds for the same model, and, in its scaled form, at temperatures within our interval $(0, T_*)$. It concerns a different quantity. Condensation is a property of the one-particle density matrix, whose kernel $\gamma^{(1)}(x, y)$ is represented by configurations in which one path is open and runs from x to y ; the superfluid fraction is a property of the windings of closed loops. Neither property is known to imply the other in general, and the examples above show that the two fractions can differ substantially. The proofs also differ in an essential way. Both use a randomized local change of a short piece of the paths along a random route, and a second-moment computation over two independent choices of the route. In [112], as in [111, 115], a bounded second moment of the resulting likelihood suffices, because the conclusion is a positive lower bound on the

occupation of the constant orbital. To bound a variance from below by a quantity of order L , we need a second moment that is $1 + O(1/L)$, and this requires new constructions; see Section 1.4.

To our knowledge, Theorems 1.1 and 1.2 are the first positive lower bounds, in the thermodynamic limit, on the superfluid fraction defined by a twist and on the nonclassical rotational inertia fraction $1 - I_n/I_{cl,n}$ of the three-dimensional dilute hard-sphere gas, at fixed exclusion distance, fixed positive density, and fixed positive temperature, with the boundary conditions stated above. Earlier results include the Gross–Pitaevskii ground-state theorem of [103], the ideal-gas calculation of [10], and the one-dimensional disordered model of [87]. The exact winding bound of Theorem 1.3 is an elementary consequence of the Gaussian shift and common-translation formulas underlying [33, 124]; related center-of-mass formulas are given in [139, Theorem III.1, Eqs. (52)–(60) in arXiv version 2].

1.4 Methods and their sources

The identities of Pollock and Ceperley and of Sindzingre, Klein, and Ceperley reduce our two response bounds to lower bounds on the variances of the winding and of the stochastic area; see Section 3. We obtain these bounds by changing the Brownian paths in one short interval of imaginary time, a slab. The change cyclically reassigns endpoints along a closed lattice route and resamples the selected paths from their conditional law. For open routes, this construction is the one of [111, Section 9]. Closed routes allow us to increase the winding, or the mean projected area, while preserving the configurations at the two ends of the slab. The mean area gain uses moments of the stochastic area of Brownian bridges, which can be computed from Lévy’s conditional characteristic function [97, Equation (1.3.4)]; see also [149, Equation (2)]. Theorem 1.3 has a separate and short proof. Shifting all paths by a common linear drift turns the weight $e^{t \cdot W}$ into a translation of the endpoint configuration, and the Cauchy–Schwarz inequality at imaginary time $\beta/2$ bounds the resulting overlap by the partition function.

We give an explicit canonical normal-phase estimate for the periodic dilute hard-sphere gas (Theorem 1.5). The argument combines the canonical cycle representation with a hard-core particle-insertion bound obtained from the normalized Fourier frame. Related canonical ratio estimates for integrable repulsive potentials were proved by Adams, Collevecchio, and König [2]; negative-chemical-potential cycle suppression and high-temperature absence of condensation were established by Ueltschi [143] and by Seiringer and Ueltschi [131].

The variance inequality of Hammersley and of Chapman and Robbins [34, 69] compares the squared mean change of an observable with the chi-square divergence of the changed law. We need to produce a macroscopic change at a small cost in divergence. We

average the change over a random skeleton, a closed lattice walk drawn independently of the configuration. From this skeleton we build the route by detours around the cells where the local estimates fail. Squaring the averaged likelihood then produces two independent skeletons, as in the standard second-moment method for averaged likelihoods; see, for example, [7, 20, 121] and the source discussion after Lemma 5.3. As explained in Section 1.3, a variance lower bound of order L in one slab requires a divergence of order L^{-1} . We obtain it from closed skeletons translated uniformly at random, for which two independent copies rarely come close, together with exact cancellation of the signed terms when the two routes have no encounters and their indicators agree in the compared matchings. Since disjoint slabs are conditionally independent, we obtain a factor proportional to β , which makes the response bounds uniform in the temperature.

The path modifications and their comparisons adapt the methods of [111, 112, 115]. We state the thermal estimates adapted from [112] in Section 8, give precise attribution at each technical step, and prove the enlarged torus window and the container versions in Appendices A and B. The overview in Section 2 explains the route construction, the cancellation in the divergence estimate, and the differences between winding and area. As far as we know, the following parts of the argument are new relative to these sources.

- The use of surgeries along closed routes, combined with the Hammersley–Chapman–Robbins inequality, to bound the variance of the winding and of the stochastic area from below, and the signed estimate of order L^{-1} for the chi-square divergence that this requires (Sections 5 and 7). Likelihoods with their conditional means subtracted, and the cancellation they allow, already appear in [113, Lemmas 6.6–6.8] and [114, Lemma 5.2]. Averaging closed-route surgeries over many retained groups, with cross-group invariance of rates and route indicators and an encounter bound for the signed cross terms, improves the power from $1/2$ to $3/8$. The disjoint-group selection is adapted from [112, Section 7.2], following [115, Section 6].
- The comparison of conditional normalizing constants for four endpoint matchings, which adapts [115, Lemma 8.3], [112, Lemma 9.3], and [111, Section 10] to changes of the endpoint matching (Section 7), and the adaptations of the local estimates of [112] to the retained endpoint matching (Section 9).
- Random closed skeletons that wind once around the torus, and random polygons whose projections have area of order L^2 , both translated uniformly at random, for which the number J of nearby pairs of vertices of two independent copies satisfies $\mathbb{E} \exp(cJ) \leq 1 + O(1/L)$. Here we use the ordered expansion in the proof of [111, Lemma 8.1] with its first moment retained, as in Khas'minskii's continuous-time exponential estimate [85, Section 3, Lemma 4, p. 315]. A related predecessor is the endpoint-averaged encounter argument for open routes in [113, Sections 7 and 8, especially Proposition 7.1 and Lemma 7.3], restated in [114, Proposition 6.1]; here closed skeletons translated as a whole give the full exponential-moment excess $O(1/L)$.

Further new parts are the selection of a cycle with positive winding after detours, detours anchored at fixed points that preserve a positive fraction of the area, and a version for closed routes of the route-stability argument of [111, Lemmas 10.2 and 10.3, Section 10.4] (Section 6).

- The amplification over slabs through the law of total variance, which makes the bounds uniform in the temperature (Section 5).
- The lower bound on the conditional mean of the area gain of a surgery (Section 11).
- The extension of the thermal estimates of [112] from scaled temperatures $T \leq D^{-1}$ to $T \leq \tau_0 / \log D$, by their proofs with each use of the temperature checked, which raises the temperature threshold from order $(\rho a^3)^{1/2} \rho a$ to order $\rho a / \log(1/(\rho a^3))$ (Appendix B, and Appendix A in a container).
- The transfer of the thermal estimates of [112] to every nonempty bounded open container with hard walls (Appendix A).
- The finite-volume moment-generating-function formulation of Theorem 1.3 and its scaling consequences in Corollary 1.4, and the area-variance lower bounds uniform over all rotation axes. Combined with the area identity and nonnegative inertia, the latter give two-sided bounds on the area variance.
- The canonical high-temperature bound $f_s \rightarrow 0$ of Theorem 1.5, obtained from a hard-core insertion-ratio estimate and cycle winding bounds (Section 4).

The elementary ingredients of these steps, such as the Cauchy–Schwarz inequality, the law of total variance, and the decomposition of a closed walk into simple cycles, are classical.

1.5 Organization of the paper

In Section 2, we give an overview of the proofs. Part I consists of Sections 3 and 4. In the first, we define the Bose path measure, prove the exact identities (1.6) and (1.10), prove Theorem 1.3, and show that the slabs are conditionally independent; in the second, we prove Theorem 1.5. Part II develops the abstract variance, lattice-route, and matching arguments. Section 5 proves the variance inequality for surgeries in one slab. Section 6 constructs the random routes and estimates their encounters, on the lattice alone. Section 7 studies surgeries of endpoint matchings under explicit hypotheses (H1)–(H5) and bounds the chi-square divergence of surgeries averaged over groups of paths. Part III applies these results to the hard-sphere gas. Section 8 states the thermal estimates adapted from [112] that we use. Section 9 verifies (H1)–(H4) for one slab of the hard-sphere gas, including the adaptations of the estimates of [112] to the retained endpoint matching. Section 10 verifies (H5) for the winding routes and proves Theorem 1.1 and Corollary 1.4, and Section 11 does the same for the area routes and proves Theorem 1.2. Appendix A

proves the thermal estimates in a nonempty bounded open container with hard walls, and Appendix B proves their extension to higher temperatures on the torus.

Overview of the proofs

In this section we explain the proofs of Theorems 1.1 and 1.2 and indicate where each step is carried out. The two proofs follow the same scheme, and we describe it first for the superfluid fraction. No statement in this section is used later.

It is convenient to work in the scaled units of [112]. We choose a length unit s such that the box has side $K = L/s$, an integer, and the hard-core diameter becomes $r = \alpha/D$, where $D = N/K^3$ is the number of particles per unit cell and α is a large fixed constant. The dilute regime corresponds to large D , and the temperature range of our theorems corresponds to scaled temperatures $0 < T \leq \tau_0/\log D$. In this overview, $T = s^2 T_{\text{phys}}$ and $\beta = 1/T$ denote the scaled temperature and inverse temperature, where T_{phys} is the temperature of the introduction. In these units a unit cell contains many particles, the particles are far apart compared with r , and a Brownian path of duration of order one moves a distance of order one. Section 10 returns to physical units.

2.1 From response to fluctuations

The first step is classical. By the Feynman–Kac formula, the canonical partition function of the gas is an integral over configurations of N Brownian paths in the imaginary-time interval $[0, \beta]$, which avoid each other at distance r and are closed up by a permutation of their endpoints. On the torus, the closed paths have a winding vector $W \in \mathbb{Z}^3$, the sum of their displacements divided by K . Under the twisted Hamiltonian, the weight of a path configuration is multiplied by $e^{i\theta W_1}$, which leads to

$$f_s = \frac{TK^2}{2N} \text{Var}(W_1). \quad (2.1)$$

This identity is due to Pollock and Ceperley [124]. In the same way, rotation about an axis with direction n multiplies the weight by a phase proportional to the stochastic area $n \cdot \mathcal{A}$ swept by the paths, and

$$I_{\text{cl}} - I = T \text{Var}(n \cdot \mathcal{A}), \quad (2.2)$$

which is the identity of Sindzingre, Klein, and Ceperley [135]. We prove both identities for the hard-core gas in Section 3, together with the bounds $0 \leq f_s \leq 1$ and $0 \leq I \leq I_{\text{cl}}$.

To prove Theorems 1.1 and 1.2, we need lower bounds on two variances. Since $N = DK^3$ and $I_{\text{cl}} \leq CNK^2$, positive lower bounds on f_s and on $1 - I/I_{\text{cl}}$ would follow from

$$\text{Var}(W_1) \geq c\beta DK, \quad \text{Var}(n \cdot \mathcal{A}) \geq c\beta DK^5. \quad (2.3)$$

Our bounds have this form with D replaced by $D^{1/4}$, which gives the factor $D^{-3/4}$, of order $(\rho a^3)^{3/8}$, in the theorems.

The difficulty is that the winding vector is a topological quantity. A modification of the paths near a point, or inside a ball of radius smaller than $K/2$, cannot change W . To change W_1 by one, the paths must be reorganized along an entire loop around the torus. To bound $\text{Var}(W_1)$ from below, we need a mechanism that acts on a macroscopic scale, and the problem is to find one whose cost can be controlled.

2.2 A variance inequality

We use the following elementary inequality, which goes back to Hammersley [69] and Chapman and Robbins [34]. Let μ be a probability measure on the configurations, and let Q be a Markov kernel, that is, a random transformation of a configuration; μQ is the law of the transformed configuration. Let O be an observable, that is, a real function of the configuration, with $O \in L^2(\mu)$, and suppose that μQ has a density $L = d(\mu Q)/d\mu$ in $L^2(\mu)$. If the divergence $\chi^2(\mu Q \parallel \mu) = \int (L - 1)^2 d\mu$ is positive, then

$$\text{Var}_\mu(O) \geq \frac{(\mathbb{E}_{\mu Q} O - \mathbb{E}_\mu O)^2}{\chi^2(\mu Q \parallel \mu)}. \quad (2.4)$$

Indeed, the numerator is the square of $\int (O - \mathbb{E}_\mu O)(L - 1) d\mu$, and we may apply the Cauchy–Schwarz inequality; if the divergence is zero, then so is the numerator. We apply (2.4) with $O = W_1$ and with a kernel Q that we call a surgery: with probability at least $1/2$ it rewires the paths so that W_1 increases by a positive integer, and otherwise it does nothing. The numerator is then at least $1/4$, and a bound

$$\chi^2(\mu Q \parallel \mu) \leq C_\# D^{-1/4}/K \quad (2.5)$$

implies $\text{Var}(W_1) \geq KD^{1/4}/(4C_\#)$. We prove this bound in a single short slab of imaginary time. The inequality (2.4) and its use are formalized in Section 5.

A rewiring along one fixed macroscopic loop generally changes the law too much for this estimate. We obtain the small divergence by averaging over random choices of the loop. We sample a random lattice loop Γ , which we call a skeleton, independently of the configuration. The kernel Q_Γ builds a route from Γ that depends on the configuration (see Section 2.6), and we set $Q = \mathbb{E}_\Gamma Q_\Gamma$. Then

$$\chi^2(\mu Q \parallel \mu) = \mathbb{E}_{\Gamma, \Gamma'} \int L_\Gamma L_{\Gamma'} d\mu - 1, \quad (2.6)$$

where Γ' is an independent copy of Γ and L_Γ is the density of μQ_Γ . Averaged likelihoods occur in minimax hypothesis testing; see [77, Section 1.1, equations (1.1)–(1.4) in the preprint version]. The independent-copy second-moment expansion appears explicitly for random paths in [7, Section 2.1.2, equation (2.6) in arXiv version 2] and [20, proof of Lemma 2.4, equation (2.8) in arXiv version 1], and for point reassignments in perturbed lattices in [121, proof of Proposition 2.1, equation (2.3) in arXiv version 1]. In our setting, if surgeries along two routes that never come close did not interact at all, then $\int L_\Gamma L_{\Gamma'} d\mu$ would equal one for such routes, and (2.6) would be controlled by the event that the two routes meet. Two independent skeletons of length of order K in the torus, each translated uniformly in the two transverse directions, meet with probability of order $K \cdot K^{-2} = K^{-1}$. This is the source of the factor $1/K$ in (2.5). Making this heuristic rigorous is the main work of the paper. There are three main difficulties: the paths interact through the hard core, so that surgeries along distant routes do interact, through normalizing constants; the routes must avoid regions where the gas is atypical; and two routes that do meet must separate quickly.

2.3 One slab, and uniformity in temperature

We cut the imaginary-time interval $[0, \beta]$ into $n_\beta = \lfloor \beta/\delta \rfloor$ slabs of a fixed small length δ , together with a remainder. We record the positions of all particles at the times $0, \delta, 2\delta, \dots$, which we call the cut configurations, and we relabel the particles independently at each cut. Given all cut configurations, the paths and endpoint matchings in distinct slabs are independent. Before conditioning, every full slab, together with its two cut configurations, has the same law. The winding vector and the stochastic area are sums of contributions of the slabs. By the law of total variance, we have

$$\text{Var}(W_1) \geq n_\beta \mathbb{E} \text{Var}(W_1^{(1)} \mid \text{endpoints of the first slab}), \quad (2.7)$$

where $W_1^{(1)}$ is the contribution of the first slab. We prove these facts in Section 3. Applying (2.4) to the joint law with a surgery that acts in one slab and fixes its endpoints, we obtain a lower bound of order $KD^{1/4}$ on the expected conditional variance (Section 5). Then we have $\text{Var}(W_1) \geq n_\beta KD^{1/4}/(4C_\sharp)$, and by (2.1), we obtain

$$f_s \geq \frac{T n_\beta}{8C_\sharp D^{3/4}} \geq \frac{1}{16C_\sharp \delta D^{3/4}} \quad (2.8)$$

throughout the scaled temperature window $0 < T \leq \tau_0/\log D$, where $T < \delta^{-1}$ for D sufficiently large. The factor T in the Pollock–Ceperley identity is compensated by the number of independent slabs, which is why our lower bound does not deteriorate as T decreases. We fix the slab length δ and the constant $C_\sharp$ before D , so the bound in (2.8) is of order $D^{-3/4}$.

2.4 The surgery in one slab

Fix one slab and condition on its initial configuration $Y = (Y_1, \dots, Y_N)$ and its final configuration $Z = (Z_1, \dots, Z_N)$. What remains random is a matching σ , which tells us that the path starting at Y_i ends at $Z_{\sigma(i)}$, and the paths themselves. Their conditional law is proportional to

$$\prod_{i=1}^N \mathbb{W}_{Y_i \rightarrow Z_{\sigma(i)}}^\delta(d\omega_i) \mathbf{1}\{\text{the paths stay at distance greater than } r\}, \quad (2.9)$$

where $\mathbb{W}_{y \rightarrow z}^\delta$ is the unnormalized Brownian bridge measure. The matching is far from uniform: its weight is a product of heat kernels, corrected by the hard-core interaction.

The thermal estimates of [112] describe the typical configuration of a short slab. At most cells, there are many paths that stay near their cell and can be moved through any neighboring cell, with probability close to one of avoiding all other paths. A cell where this fails is called bad. The key input is a joint bound: for every prescribed set A of cells, the probability that all cells of A are bad is at most $p^{|A|}$, where p can be made small by choosing the parameters. This bound is uniform in the scaled temperature window and in all sufficiently large volumes. Section 8 states these inputs, and Appendix A proves their versions in a container.

The surgery acts on a retained family of paths. In each cell with at least mM eligible paths, we retain mM of them in a uniformly random order and divide them into m groups of M ; in other cells we retain none. We reveal everything except the matching of the retained paths and the retained paths themselves (Section 9). The surgery chooses a group uniformly at random. Given a closed lattice route $\gamma = (v_0, v_1, \dots, v_{\ell-1})$ of cells that are good for this group, with positive winding number in the first coordinate direction, it chooses one retained path of the group in each cell v_k , reconnects the path from cell v_k to the terminal point of the path chosen in cell v_{k+1} (indices modulo ℓ), and resamples the chosen paths from their exact conditional law given everything else; see Figure 2.1. The changes of the displacements of the chosen paths add up to K times the homology vector $\text{hom}(\gamma)$ of the route, so W_1 increases by $\text{hom}_1(\gamma) \geq 1$ (Lemma 10.2). The closed route is what makes this change topological. A cyclic shift of endpoints along an open route, followed by exact resampling, was introduced in [111, Section 9] to compare a closed path configuration with an opened one in a proof of Bose–Einstein condensation.

2.5 Why the chi-square divergence is small

Fix a group a and a labeled route of this group, that is, a route together with the paths of the group chosen along it, and write π for the corresponding cyclic reassignment and S for the chosen rows. The accepted surgery along it is a subprobability kernel whose

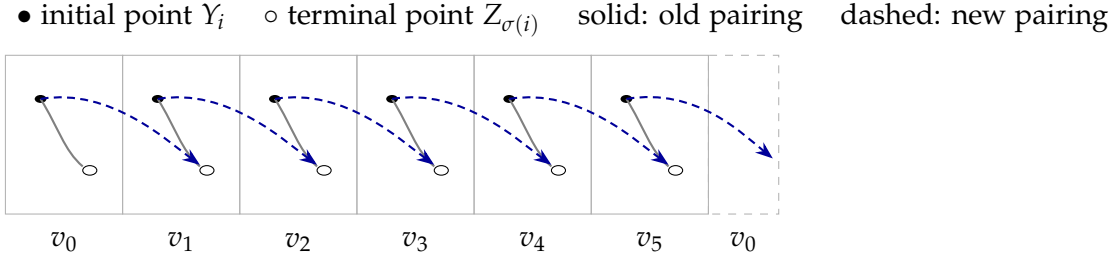


Figure 2.1. The surgery along a route $\gamma = (v_0, \dots, v_5)$ that winds once around the torus in the first coordinate direction; the dashed cell is v_0 again. In each cell one retained path is chosen. Its initial point is reconnected to the terminal point of the path chosen in the next cell, and the chosen paths are resampled from their exact conditional law. All other paths, and all endpoints, are unchanged. The changes of the displacements add up to $K \text{hom}(\gamma)$, so in the example shown the first coordinate of the winding vector increases by one.

outgoing mass at a matching σ is the probability $R_\pi(\sigma)$ of these choices. Its incoming density at a configuration reached by the surgery is explicit (Lemma 7.6):

$$c_\pi = R_\pi(\sigma\pi^{-1}) \frac{Z_S(\sigma\pi^{-1}; \omega_{S^c})}{Z_S(\sigma; \omega_{S^c})},$$

the probability of the choices at the matching before the surgery, multiplied by the conditional partition function of the chosen paths at that matching, and divided by their conditional partition function at the matching after the surgery. After the completion by the identity, the likelihood of the surgeries of group a for a skeleton Γ is $L_{a,\Gamma} = 1 + \sum_\pi (c_\pi - R_\pi)$, where the sum is over the labeled routes of group a . The kernel Q is the mixture of the kernels $Q^{(a)}$ of the m groups, each with weight $1/m$, so the density L_Γ in (2.6) is $m^{-1} \sum_a L_{a,\Gamma}$, and $\chi^2(\mu Q \parallel \mu)$ is the average over the m^2 pairs of groups (a, b) of $\mathbb{E}_{\Gamma, \Gamma'} \int (L_{a,\Gamma} - 1)(L_{b,\Gamma'} - 1) d\mu$. For the m diagonal terms, $a = b$, and the $m(m-1)$ cross terms, $a \neq b$, we must bound the sums, over pairs of labeled routes of groups a and b built from two independent skeletons, of

$$\int (c_\pi - R_\pi)(c_{\pi'} - R_{\pi'}) d\mu. \quad (2.10)$$

Each such term involves four matchings of the retained paths: the current one, the two matchings before each of the two surgeries, and the matching before both; the last enters through the product $c_\pi c_{\pi'}$.

First let $a = b$, and suppose that the four matchings produce the same pair of routes from the two skeletons. If, in addition, no cell of one route is within a fixed distance R of the other, then the hard-core interaction between the two sets of chosen paths vanishes, the normalizing constants factor, neither surgery changes the choices available to the other, and the four products in (2.10) cancel exactly (Lemma 7.9). When the routes do

come within distance R of each other, we only need an upper bound. Each nearby chosen path costs a factor $(1 - \eta_{\text{loc}})^{-1}$, where η_{loc} bounds the probability that a resampled path collides with one of the other chosen paths near it, and a path chosen by both routes costs a fixed factor G . The first cost is small because η_{loc} tends to zero as $D \rightarrow \infty$, by the collision estimates of [112]. The second is rare, because each route chooses among at least m_* paths of the group in each cell, and two routes choose the same path with probability at most $1/m_*$. By Lemmas 7.8 and 7.12, when the routes agree in the four matchings, we find that a pair of routes contributes zero if $J_R(\gamma, \gamma') = 0$ and at most $Ce^{\varepsilon J_R(\gamma, \gamma')}$ otherwise, where $J_R(\gamma, \gamma')$ counts pairs of cells of the two routes at distance at most R and ε is small. Since $e^x \leq C(e^{2x} - 1)$ for $x \geq \varepsilon$, if the routes did not depend on the matching we would obtain

$$\chi^2(\mu Q^{(a)} \parallel \mu) \leq C \mathbb{E}[e^{2\varepsilon J_R(\gamma, \gamma')} - 1]. \quad (2.11)$$

Section 2.6 explains the additional term needed when they do.

Now let $a \neq b$. The two surgeries move rows of different groups, and a move of group b changes none of the counts, bad sets, rates, and route indicators of group a . After reindexing the matchings, three of the four products in (2.10) have the same integral, and the comparison of the four normalizing constants bounds the signed sum from above by a loss confined to the encounters of the two routes, which vanishes if $J_R(\gamma, \gamma') = 0$ (Lemma 7.13). This is an upper bound on an integrated signed sum, not a pointwise cancellation: the hard core couples paths of different groups, and a move of one group changes the conditional law of the paths of another. Averaged over the skeletons and the configuration, each cross term is at most $C_\times \eta_{\text{loc}}/K$.

This comparison adapts two arguments of [111, 115]. The first is the cancellation of conditional normalizing constants away from the overlap of two changes [115, Lemma 8.3], reproduced in [112, Lemma 9.3]. The second is the comparison of four endpoint assignments, with the costs concentrated on shared paths and their predecessors, in [111, Section 10]. There is one essential difference. In [111, 112, 115], a bound of order one on the second moment of a density suffices, while here we need the divergence to be small. We must keep the subtracted outgoing rates in (2.10) until we have used the cancellation for route pairs without encounters. This is why we work with the signed expansion rather than with positive bounds alone.

2.6 Random routes

The bound (2.11) is useful only if two routes built from independent skeletons rarely meet and, when they do meet, separate quickly. A route built from a simple random walk would not do: two independent walks in the two transverse directions return to each other with probability of order k^{-1} after k steps, which is not summable. We use instead

the dyadic random skeleton of [111, Section 8.1], also used in [112, 115]. It advances by one cell in the first coordinate direction at each step, and its transverse displacement over k steps has a conditional density bounded by $Ck^{-3/2}$, given its past. The exponent $3/2$ exceeds one, so that the expected number of future encounters with a fixed opposing route is bounded. This is the mechanism of unpredictable paths, due to Benjamini, Pemantle, and Peres [19] and Häggström and Mossel [68]. To close the loop we read the transverse displacement forward and then backward, and to make encounters rare we translate the loop by a uniform transverse vector. By an ordered-moment argument, we obtain

$$\mathbb{E}e^{\kappa J_R(\Gamma, \Gamma')} \leq 1 + C/K \quad (2.12)$$

for two independent skeletons Γ, Γ' (Proposition 6.6). The excess over one, rather than a bounded exponential moment, is the new requirement here; it uses the random translation in an essential way.

The skeleton ignores the bad cells. We replace each visit of the skeleton to a connected cluster of bad cells by a detour along the outer boundary of the cluster, which consists of good cells and is connected by a theorem of Kesten and Timár [84, 141]. The joint bound $p^{|A|}$ makes large clusters rare, so the detoured route is accepted with probability at least $1/2$ and has few additional encounters. These detours are the construction of [115, Section 7] and [112, Section 8] (Proposition 6.15).

There is an apparent circularity here, since the bad cells depend on the matching of the retained paths, which the surgery changes. Precisely, a cell is bad for a group when it contains fewer than m_* retained paths of the group that can be moved, and it is vulnerable when it contains fewer than $m_* + 2$ of them. The four matchings in (2.10) may produce different routes from the same skeleton, and the exact cancellation described above can fail. The way out is that a surgery changes the number of usable paths only in the cells it visits, and only by a bounded amount. All four bad sets lie in the vulnerable set of a common matching, and the routes can differ only if a component of the vulnerable set is approached by both skeletons. Let A_* be the total number of cells in such components. If $A_* = 0$, the route indicators agree in all four matchings. Contributions of route pairs without encounters then cancel exactly, and contributions with encounters are bounded by the positive estimate above. In general, the number of possible pairs of routes is at most 4^{A_*} (Lemma 6.16). This follows the stability and path-list argument of [111, Lemmas 10.2–10.3]. Averaging over the environment and using the joint bound for the vulnerable set, we obtain $\mathbb{E}[e^{2\varepsilon J_R(\Gamma, \Gamma') + 2c_{\text{rec}} A_*}] - 1 \leq C/K$ for a constant c_{rec} (Corollary 6.19), which bounds the divergence of the surgeries of one group by C_*/K . Combining the diagonal and the cross terms, Theorem 7.14 bounds $\chi^2(\mu_Q \| \mu)$ by $C_*/(mK) + C_{\times} \eta_{\text{loc}}/K$. The one-opponent collision bound of Lemma 9.9 applies to the normalized one-path trial law before conditioning on avoidance of the other retained paths. The resulting union bound over all relevant retained paths is at

most $1/2$ for the choice of m in (10.7), with m comparable to $D^{1/4}$. Since η_{loc} is of order $D^{-1/4}$, we obtain (2.5).

2.7 Rotation

For Theorem 1.2 we replace the winding loops by polygons of large area. Inside the container we place a cube of side H comparable to K , at distance comparable to K from the walls, and in it a diamond-shaped lattice polygon of side of order H , with the same dyadic transverse fluctuations and a uniform random translation. Detours are made separately on each of its four sides, so that the area of the routed polygon remains at least cH^2 .

The polygon is not planar, since its transverse fluctuations also move it out of the plane, but its projection onto the plane orthogonal to the axis has signed area of order H^2 . The surgery along a polygon γ changes the component of the stochastic area along the axis by this projected area, up to two errors. The chords between neighboring endpoints contribute an error that is bounded pathwise by a constant times the length of γ . The stochastic areas of the original and resampled Brownian bridges are not controlled pathwise, since a confined Brownian path can sweep a large signed area through small loops. We control them in mean, averaging over both the original chosen paths and their replacements, conditionally on everything else (Proposition 11.7). The mean gain of the surgeries of each group is at least cH^2 , and so is that of their mixture, whose χ^2 divergence is at most $C_{\sharp}D^{-1/4}/H$. Combining (2.4) with the slab decomposition, we obtain $\text{Var}(n \cdot \mathcal{A}) \geq c n_{\beta} D^{1/4} K^5$. Since $I_{\text{cl}} \leq CNK^2$, we conclude from the identity (2.2) that $1 - I/I_{\text{cl}}$ has the claimed lower bound. Rotating the container reduces every axis direction to the vertical one.

The thermal estimates of [112] are proved on the torus, and in Appendix B we extend them to higher temperatures. In Appendix A we prove them for cells at a fixed distance from the walls of an arbitrary bounded open container. The routes never approach the walls, so the remaining arguments are the same as on the torus.

2.8 The order of choices

The constants are chosen in the following order (Section 10). We first fix the spatial constants of the thermal estimates and the temperature coefficient κ_T , then the geometry of the routes and the encounter coefficients, including the exponent $\theta_{\times}$ for routes of different groups, then the upper bound p_{max} on the base for bad cells. From the thermal estimates we then obtain the slab length δ and the local constants, including the factor G . Next we choose the number M of retained paths per cell in each group, which makes the cost of shared paths small, and then a density threshold D_{min} , which makes the collision

probability η_{loc} small, and, for each $D \geq D_{\min}$, the number $m(D)$ of groups. The constant $C_{\sharp}$ depends only on the choices made before $D_{\min}$. Only after these choices do we let K tend to infinity. In particular, no constant depends on the volume or on the temperature, so we have the uniformity stated in Theorems 1.1 and 1.2.

PART I

Exact identities

The Bose path measure and exact identities

This section collects the exact response identities. All statements are at finite volume and require no dilution assumption; the configuration domain is assumed nonempty. All derivatives are taken at finite volume.

In Section 3.1, we define the hard-core Hamiltonians on the torus and in a bounded container, the twisted and rotating free energies, the superfluid fraction, and the moment of inertia. Section 3.2 recalls the Feynman–Kac representation, defines the Bose path measure, and introduces the winding vector and the stochastic area. Section 3.3 bounds the moment generating function of the winding vector by that of a centered Gaussian and records the consequences for its scaling. In Section 3.4, we express the superfluid fraction through the variance of the winding vector and through the fluctuations of the total momentum. In Section 3.5, we prove that the moment of inertia is nonnegative and that its deficit from the classical value is the variance of the stochastic area. Finally, Section 3.6 decomposes imaginary time into slabs, proves that the slabs are conditionally independent given the configurations at the cut times, and deduces the variance inequality used to amplify single-slab estimates.

3.1 The model

We use the units of Section 1.2 and Remark 1.10: $\hbar = 1$, $m = 1/2$, and Boltzmann's constant one. The one-particle kinetic energy is $-\Delta$, the temperature is $T > 0$, and $\beta = 1/T$.

Containers and configuration domains. The torus of side $L > 0$ is $\Lambda_L = (\mathbb{R}/L\mathbb{Z})^3$ with the distance $d_L(x, y) = \min_{k \in \mathbb{Z}^3} |x - y + Lk|$. A container is a nonempty bounded open set $U \subset \mathbb{R}^3$, equipped with the Euclidean distance. Fix an exclusion distance $a > 0$ and $N \geq 1$. The configuration domains are

$$\begin{aligned}\mathcal{X}_{N,L} &= \{X \in \Lambda_L^N : d_L(x_i, x_j) > a \text{ for all } i \neq j\}, \\ \mathcal{X}_N(U) &= \{X \in U^N : |x_i - x_j| > a \text{ for all } i \neq j\},\end{aligned}\tag{3.1}$$

where $X = (x_1, \dots, x_N)$. With the periodic distance, $\mathcal{X}_N(\Lambda_L) = \mathcal{X}_{N,L}$. In applications the container is $U = L\Omega$ in physical units or $U = K\Omega$ in scaled units; in particular, $\mathcal{X}_N(L\Omega) = \mathcal{X}_{N,L}^\Omega$ in the notation of Section 1. We write $\mathcal{X}_N$ for either $\mathcal{X}_{N,L}$ or $\mathcal{X}_N(U)$.

when a statement applies to both, and $\mathbb{X}$ for the corresponding one-particle space, which is Λ_L on the torus and $\mathbb{R}^3$ for a container. The set $\mathcal{X}_N$ is open, and it is invariant under the permutations $(\pi X)_i = x_{\pi(i)}$, $\pi \in S_N$. We assume throughout that $\mathcal{X}_N$ is nonempty.

Hamiltonians. Let $L^2_{\text{sym}}(\mathcal{X}_N)$ be the space of square-integrable functions on $\mathcal{X}_N$ that are invariant under all permutations of the particle coordinates. Let $H_0^1(\mathcal{X}_N)$ be the closure in H^1 of the smooth functions with compact support in $\mathcal{X}_N$ (on the torus, smooth functions on Λ_L^N with compact support in $\mathcal{X}_{N,L}$), and set $\mathcal{Q}_N = H_0^1(\mathcal{X}_N) \cap L^2_{\text{sym}}(\mathcal{X}_N)$. The form

$$q[\psi] = \sum_{i=1}^N \int_{\mathcal{X}_N} |\nabla_i \psi|^2 dX, \quad \psi \in \mathcal{Q}_N, \quad (3.2)$$

is closed and nonnegative. We denote the associated self-adjoint operator on $L^2_{\text{sym}}(\mathcal{X}_N)$ by $H_{N,L}$ on the torus and by $H_N(U)$ for a container, and write H for either. It is the Dirichlet realization of $-\sum_i \Delta_i$ on symmetric functions: the boundary condition is imposed on the collision set $\{|x_i - x_j| = a\}$ and, for a container, on the walls ∂U . The same form on $H_0^1(\mathcal{X}_N)$, without the symmetry constraint, defines the labeled operator H^{lab} on $L^2(\mathcal{X}_N)$. Since $\mathcal{X}_N$ is invariant under permutations, H^{lab} commutes with them, and H is the restriction of H^{lab} to the reducing subspace $L^2_{\text{sym}}(\mathcal{X}_N)$.

Lemma 3.1. *The operator H has compact resolvent, and for every $t > 0$ and $k \geq 0$ we have $\text{tr}(H^k e^{-tH}) < \infty$. The partition function $Z = \text{tr} e^{-\beta H}$ satisfies $0 < Z < \infty$.*

Proof. Extension by zero maps $H_0^1(\mathcal{X}_N)$ isometrically into $H^1(\Lambda_L^N)$ on the torus, and into $H_0^1(B^N)$ for a container, where B is an open cube containing U ; the form (3.2) is preserved. By the min-max principle [127, Theorem XIII.2], the k -th eigenvalue of H , counted with multiplicity, is at least the k -th eigenvalue of the Laplacian on Λ_L^N , respectively of the Dirichlet Laplacian on B^N . These comparison operators have explicit spectra and finite heat traces at every positive time. This proves that H has compact resolvent and that $\text{tr} e^{-tH} < \infty$. Since $\lambda^k e^{-t\lambda} \leq C_{k,t} e^{-t\lambda/2}$ for $\lambda \geq 0$, we also have $\text{tr}(H^k e^{-tH}) < \infty$. Since $\mathcal{X}_N$ is open and nonempty, it contains the support of a nonnegative, nonzero smooth function ϕ , and $\sum_{\pi} \phi(\pi X)$ is a nonzero element of $\mathcal{Q}_N$. In particular $L^2_{\text{sym}}(\mathcal{X}_N) \neq \{0\}$, and $Z > 0$. $\square$

We write $\langle B \rangle = Z^{-1} \text{tr}(B e^{-\beta H})$ for the Gibbs expectation of a bounded operator B . For a bounded measurable function f on $\mathcal{X}_N$ that is invariant under permutations, $\langle f \rangle$ denotes the expectation of the corresponding multiplication operator.

Twisted boundary conditions. On the torus, let $\vartheta \in \mathbb{R}^3$. The form

$$q_{\vartheta}[\psi] = \sum_{i=1}^N \int_{\mathcal{X}_{N,L}} |(-i\nabla_i + \vartheta/L)\psi|^2 dX, \quad \psi \in \mathcal{Q}_N, \quad (3.3)$$

is closed and bounded below, since $q_\theta \leq 2q + 2N|\theta|^2 L^{-2}$ and $q \leq 2q_\theta + 2N|\theta|^2 L^{-2}$ in the sense of forms. We denote the associated self-adjoint operator by $H_{N,L}(\theta)$, and for $\theta \in \mathbb{R}$ we write $H_{N,L}(\theta) = H_{N,L}(\theta e_1)$, so that

$$H_{N,L}(\theta) = \sum_{i=1}^N (-i\nabla_i + \theta e_1/L)^2.$$

The twist convention is that of Section 1.2: $H_{N,L}(\theta)$ is unitarily equivalent to $-\sum_i \Delta_i$ with phase $e^{i\theta}$ when one particle is translated by Le_1 . By the min-max comparison of q_θ with q and Lemma 3.1, the operator $e^{-tH_{N,L}(\theta)}$ is trace class for every $t > 0$. The twisted free energy is

$$F_{N,L,T}(\theta) = -T \log \operatorname{tr} e^{-\beta H_{N,L}(\theta)}, \quad F_{N,L,T}(\theta) = F_{N,L,T}(\theta e_1). \quad (3.4)$$

Definition 3.2. The superfluid fraction of the canonical hard-core Bose gas on Λ_L is

$$f_s(N, L, T) = \frac{L^2}{2N} \partial_\theta^2 F_{N,L,T}(\theta) \Big|_{\theta=0}. \quad (3.5)$$

By Lemma 3.13(3), $\theta \mapsto F_{N,L,T}(\theta)$ is twice continuously differentiable. The derivative in (3.5) exists. The physical interpretation and the order of limits are given in Section 1.2 and Remark 1.10.

Rotation. Let U be a container, let $n \in \mathbb{R}^3$ be a unit vector, and let $x_0 \in \mathbb{R}^3$. The axis of rotation is the line through x_0 with direction n . We write $y_i = x_i - x_0$ and $R_0 = \sup_{x \in U} |x - x_0| < \infty$. For $\psi \in \mathcal{Q}_N$, define the components of the angular momentum about x_0 by

$$\mathcal{L}_b \psi = \sum_{i=1}^N e_b \cdot (y_i \times (-i\nabla_i)) \psi = -i \sum_{i=1}^N (e_b \times y_i) \cdot \nabla_i \psi, \quad b = 1, 2, 3, \quad (3.6)$$

and set $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ and $\mathcal{L}_n = n \cdot \mathcal{L}$. For $\omega \in \mathbb{R}^3$, the operator $\omega \cdot \mathcal{L}$ is given by (3.6) with e_b replaced by ω .

Lemma 3.3. Let $\omega \in \mathbb{R}^3$. For all $\phi, \psi \in \mathcal{Q}_N$, we have $\omega \cdot \mathcal{L} \psi \in L_{\text{sym}}^2(\mathcal{X}_N(U))$,

$$\|\omega \cdot \mathcal{L} \psi\|^2 \leq |\omega|^2 N R_0^2 q[\psi], \quad \langle \phi, \omega \cdot \mathcal{L} \psi \rangle = \langle \omega \cdot \mathcal{L} \phi, \psi \rangle. \quad (3.7)$$

The form $q[\psi] - \langle \psi, \omega \cdot \mathcal{L} \psi \rangle$ on $\mathcal{Q}_N$ is real, closed, and bounded below. Its self-adjoint operator $H - \omega \cdot \mathcal{L}$ satisfies $\operatorname{tr} e^{-t(H - \omega \cdot \mathcal{L})} < \infty$ for every $t > 0$.

Proof. The vector field $v(x) = \omega \times (x - x_0)$ satisfies $|v| \leq |\omega| R_0$ on U and $\operatorname{div} v = 0$. By the Cauchy-Schwarz inequality in $\mathbb{R}^{3N}$, we have pointwise

$$\left| \sum_i v(x_i) \cdot \nabla_i \psi \right|^2 \leq \left(\sum_i |v(x_i)|^2 \right) \sum_i |\nabla_i \psi|^2 \leq |\omega|^2 N R_0^2 \sum_i |\nabla_i \psi|^2,$$

and integrating, we obtain the first inequality in (3.7). Since the differential expression is invariant under permutations of the particles, it maps symmetric functions to symmetric functions. For smooth ϕ, ψ with compact support in $\mathcal{X}_N(U)$, we obtain the symmetry relation in (3.7) by integration by parts, using $\operatorname{div} v = 0$. Since both sides are continuous on $\mathcal{Q}_N$ with the H^1 norm, the relation extends to $\mathcal{Q}_N$; in particular $\langle \psi, \omega \cdot \mathcal{L}\psi \rangle$ is real. The first inequality in (3.7) implies

$$|\langle \psi, \omega \cdot \mathcal{L}\psi \rangle| \leq \frac{1}{2}q[\psi] + \frac{1}{2}|\omega|^2 NR_0^2 \|\psi\|^2, \quad (3.8)$$

so the perturbation has relative form bound less than one. By the theorem on relatively form-bounded perturbations of closed quadratic forms [126, Theorem X.17], we conclude that the perturbed form is closed and bounded below on $\mathcal{Q}_N$. By (3.8) and the min-max principle, the k -th eigenvalue of $H - \omega \cdot \mathcal{L}$ is at least half the k -th eigenvalue of H minus $|\omega|^2 NR_0^2/2$, and the trace bound follows from Lemma 3.1. $\square$

The rotating-frame free energy is

$$F^{\text{rot}}(\omega) = -T \log \operatorname{tr} e^{-\beta(H - \omega \cdot \mathcal{L})}, \quad \omega \in \mathbb{R}^3, \quad (3.9)$$

and for the fixed axis we write $F^{\text{rot}}(\omega) = F^{\text{rot}}(\omega n)$, $\omega \in \mathbb{R}$. Both depend on x_0 through $\mathcal{L}$.

Definition 3.4. The moment of inertia tensor about x_0 and the moment of inertia about the axis (n, x_0) are

$$I_{bc}(x_0) = -\partial_{\omega_b} \partial_{\omega_c} F^{\text{rot}}(\omega) \Big|_{\omega=0}, \quad I(n, x_0) = -\partial_{\omega}^2 F^{\text{rot}}(\omega n) \Big|_{\omega=0} = \sum_{b,c} n_b n_c I_{bc}(x_0). \quad (3.10)$$

The classical moment of inertia tensor and the classical moment of inertia are

$$I_{\text{cl},bc}(x_0) = m \left\langle \sum_{i=1}^N (|y_i|^2 \delta_{bc} - y_{i,b} y_{i,c}) \right\rangle, \quad I_{\text{cl}}(n, x_0) = m \left\langle \sum_{i=1}^N |n \times y_i|^2 \right\rangle, \quad (3.11)$$

with $m = 1/2$ and $y_i = x_i - x_0$.

The function F^{rot} is twice continuously differentiable by Proposition 3.16, which justifies (3.10); the second identity in (3.10) is the chain rule. The identity $|y|^2 - (n \cdot y)^2 = |n \times y|^2$ shows that $I_{\text{cl}}(n, x_0) = \sum_{b,c} n_b n_c I_{\text{cl},bc}(x_0)$. The ratio $1 - I(n, x_0)/I_{\text{cl}}(n, x_0)$ is the nonclassical rotational inertia fraction. For $U = L\Omega$, the axis point in this definition is the physical point Lx_0 used in Section 1.2.

3.2 The Feynman–Kac representation and the Bose path measure

Brownian bridges. Let $p_t(x) = (4\pi t)^{-3/2} \exp(-|x|^2/(4t))$ be the heat kernel of Δ on $\mathbb{R}^3$. Brownian motion with generator Δ has independent coordinates, each of variance $2s$ at time s . For $x, y \in \mathbb{R}^3$ and $t > 0$, let $\mathbb{W}_{x,y}^t$ be the finite measure on $C([0, t]; \mathbb{R}^3)$ with total mass $p_t(y - x)$ whose normalization is the law of the Brownian bridge from x to y of duration t . Every continuous path $\gamma: [0, t] \rightarrow \Lambda_L$ has a continuous lift $\tilde{\gamma}$ to $\mathbb{R}^3$, unique up to a translation by $L\mathbb{Z}^3$, and we call

$$d(\gamma) = \tilde{\gamma}(t) - \tilde{\gamma}(0) \in \mathbb{R}^3$$

the lifted displacement of γ . For $x, y \in \Lambda_L$ with representatives $\tilde{x}, \tilde{y} \in \mathbb{R}^3$, the torus bridge measure is

$$\mathbb{W}_{x,y}^t = \sum_{k \in \mathbb{Z}^3} (\text{pr}_L)_* \mathbb{W}_{\tilde{x}, \tilde{y} + Lk}^t$$

where $\text{pr}_L: \mathbb{R}^3 \rightarrow \Lambda_L$ is the projection; it does not depend on the choice of representatives, and the paths in the k -th term are exactly those with $d(\gamma) = \tilde{y} + Lk - \tilde{x}$. For configurations $X, Y \in \mathbb{X}^N$, we set $\mathbb{W}_{X,Y}^t = \bigotimes_{i=1}^N \mathbb{W}_{x_i, y_i}^t$, a measure on paths $X(\cdot) = (x_1(\cdot), \dots, x_N(\cdot))$ in $C([0, t]; \mathbb{X}^N)$. We use two standard properties of these measures. First, they are invariant under time reversal, in the sense that the image of $\mathbb{W}_{X,Y}^t$ under $X(\cdot) \mapsto X(t - \cdot)$ is $\mathbb{W}_{Y,X}^t$. Second, they satisfy the concatenation identity

$$\mathbb{W}_{X,Z}^{s+t} = \int_{\mathbb{X}^N} \mathbb{W}_{X,Y}^s \circledast \mathbb{W}_{Y,Z}^t dY, \quad s, t > 0, \quad (3.12)$$

where $\circledast$ denotes the image of the product measure under concatenation of paths at time s . Identity (3.12) is the Markov property of Brownian motion.

For $t > 0$, let $\mathcal{E}_t = \{X(\cdot) : X(s) \in \mathcal{X}_N \text{ for all } s \in [0, t]\}$ be the event that the configuration path stays in $\mathcal{X}_N$; on this event no two particles come within distance a and, for a container, no particle leaves U . For $X, Y \in \mathcal{X}_N$, the killed kernel is

$$K_t(X, Y) = \mathbb{W}_{X,Y}^t(\mathcal{E}_t), \quad K_t^s(X, Y) = \frac{1}{N!} \sum_{\pi \in S_N} K_t(X, \pi Y). \quad (3.13)$$

Proposition 3.5. *Let $t > 0$.*

1. *The operator $e^{-tH^{\text{lab}}}$ on $L^2(\mathcal{X}_N)$ is an integral operator with kernel K_t .*
2. *For a container U , let $a(x) = Ax + c$ be an affine real vector field with $\text{tr } A = 0$, let V be a bounded measurable real function on $\mathcal{X}_N(U)$ satisfying $V(\pi X) = V(X)$ for every $\pi \in S_N$ (almost everywhere in X), and let $H_{a,V}^{\text{lab}}$ be the operator associated with the closed form $\sum_i \|(-i\nabla_i - a(x_i))\psi\|^2 + \langle \psi, V\psi \rangle$ on $H_0^1(\mathcal{X}_N(U))$. Then $e^{-tH_{a,V}^{\text{lab}}}$ has the kernel*

$$\int_{\mathcal{E}_t} \exp\left(-i \sum_{i=1}^N \int_0^t a(x_i(s)) \cdot \circ dx_i(s) - \int_0^t V(X(s)) ds\right) \mathbb{W}_{X,Y}^t(dX(\cdot)), \quad (3.14)$$

where $\circ$ denotes the Stratonovich integral, which agrees with the Itô integral because $\operatorname{div} \mathbf{a} = 0$.

The same statements hold for the restrictions of these operators to $L^2_{\text{sym}}(\mathcal{X}_N)$, with the kernels symmetrized in the second variable as in (3.13).

The path-integral description of the Bose gas through permutation cycles of Brownian paths goes back to Feynman [48]. Ginibre studied reduced density matrices of quantum gases, including hard-core potentials [61, 62]; see also his account of functional integration [63]. For general background on functional integration, see [133, Sections 6 and 15]. For arbitrary open domains, see [32, Proposition 2.9 and Theorem 6.1], with time rescaled from generator $\Delta/2$ to Δ . For a container, let $\mathcal{O}_m \uparrow \mathcal{X}_N$ be an increasing smooth relatively compact exhaustion. After extension by zero, $H_0^1(\mathcal{O}_m) \subset H_0^1(\mathcal{X}_N)$, and their union is dense in $H_0^1(\mathcal{X}_N)$ in the form norm. Then the Dirichlet semigroups on $\mathcal{O}_m$, extended by zero, converge strongly to the semigroup on $\mathcal{X}_N$. The same conclusion holds for the magnetic forms, since the vector potential is bounded on the configuration domain and the scalar potential is bounded. A continuous path whose image on $[0, t]$ lies in $\mathcal{X}_N$ lies in some $\mathcal{O}_m$. The survival indicators increase to $\mathbf{1}_{\mathcal{E}_t}$. The nonnegative path integrals converge monotonically; the magnetic ones converge by dominated convergence, since their integrands have modulus at most $e^{t\|V_-\|_\infty}$. For the torus case of (1), see [18, Proposition 4.4 in arXiv version 4], applied to the H_0^1 realization on the open configuration domain; conditioning on the endpoint gives the kernel above. The sign conventions for the phases in part (2) play no role below, since all the partition functions we consider are real.

Lemma 3.6. *The kernels have the following properties.*

1. For all $X, Y \in \mathcal{X}_N$ and $\pi \in S_N$, we have $K_t(X, Y) = K_t(Y, X) \geq 0$ and $K_t(\pi X, \pi Y) = K_t(X, Y)$. The kernel K_t^s is symmetric and invariant under permutations of either argument.
2. For $s, t > 0$ and $X, Z \in \mathcal{X}_N$,

$$\int_{\mathcal{X}_N} K_s(X, Y) K_t(Y, Z) dY = K_{s+t}(X, Z), \quad \int_{\mathcal{X}_N} K_s^s(X, Y) K_t^s(Y, Z) dY = K_{s+t}^s(X, Z). \quad (3.15)$$

3. We have

$$Z = \operatorname{tr} e^{-\beta H} = \int_{\mathcal{X}_N} K_\beta^s(X, X) dX = \frac{1}{N!} \sum_{\pi \in S_N} \int_{\mathcal{X}_N} K_\beta(X, \pi X) dX, \quad (3.16)$$

and $\langle f \rangle = Z^{-1} \int_{\mathcal{X}_N} f(X) K_\beta^s(X, X) dX$ for every bounded measurable symmetric function f . The analogous formula holds for the operators of Proposition 3.5(2).

Proof. Part (1) follows from the time-reversal invariance of the bridge measures, the invariance of $\mathcal{X}_N$ under permutations, and the definition of K_t^s . For part (2), we apply

(3.12) and note that a concatenated path lies in $\mathcal{E}_{s+t}$ if and only if both pieces lie in $\mathcal{E}_s$ and $\mathcal{E}_t$, respectively; the requirement $Y \in \mathcal{X}_N$ is part of this event. This proves the first identity. For the second, we insert the definition of K^s , change variables $Y \mapsto \pi^{-1}Y$ in the term indexed by (π, σ) , and use part (1) and the first identity; the double sum over $S_N \times S_N$ then reduces to $N!$ copies of the sum defining K_{s+t}^s .

For part (3), let P_{sym} be the orthogonal projection of $L^2(\mathcal{X}_N)$ onto $L^2_{\text{sym}}(\mathcal{X}_N)$. It commutes with H^{lab} , and by Proposition 3.5(1) the operator $A = e^{-\beta H^{\text{lab}}/2} P_{\text{sym}}$ has the kernel $K_{\beta/2}^s$. It is self-adjoint and Hilbert–Schmidt by Lemma 3.1, and $\text{tr}_{L^2_{\text{sym}}} e^{-\beta H} = \text{tr} A^2 = \iint |K_{\beta/2}^s(X, Y)|^2 dX dY$. By part (1) and (3.15), the last integral equals $\int K_{\beta/2}^s(X, X) dX$. The formula for $\langle f \rangle$ follows in the same way from $\text{tr}(f e^{-\beta H}) = \text{tr}(A f e^{-\beta H^{\text{lab}}/2} P_{\text{sym}})$. For the operators of Proposition 3.5(2), write k_t for the kernel (3.14) and k_t^s for its symmetrization. Apply [32, Theorem 6.1] to the globally defined affine configuration-space vector field $(a(x_1), \dots, a(x_N))$, with scalar potential $V/2$ extended by zero and time $2t$. It gives joint continuity of k_t on $\mathcal{X}_N(U)^2$. The bound $|k_t(X, Y)| \leq e^{t\|V\|_\infty} (4\pi t)^{-3N/2}$ and the finite volume of $\mathcal{X}_N(U)$ show that the convolution of two symmetrized kernels is continuous. Thus the semigroup identity, initially valid almost everywhere on $\mathcal{X}_N(U)^2$, holds everywhere. Self-adjointness likewise gives $k_t^s(Y, X) = \overline{k_t^s(X, Y)}$ everywhere. In particular,

$$k_{\beta/2}^s(X, X) = \int_{\mathcal{X}_N(U)} |k_{\beta/2}^s(X, Y)|^2 dY.$$

Since the vector field and the scalar potential are bounded on the configuration domain, their form satisfies $q_{a,V}[\psi] \geq \frac{1}{2}q[\psi] - C\|\psi\|^2$. The min–max argument of Lemma 3.1 therefore makes the half-time symmetrized semigroup Hilbert–Schmidt. The same argument as above proves both trace formulas. $\square$

Definition 3.7. We denote the Bose path measure by μ , or by $\mu_{N,L,\beta}$ on the torus and $\mu_{N,U,\beta}$ in a container, suppressing the fixed exclusion distance. It is the probability measure on $S_N \times C([0, \beta]; \mathbb{X}^N)$ given by

$$\mu(\{\pi\} \times dX(\cdot)) = \frac{1}{Z N!} \int_{\mathcal{X}_N} \mathbf{1}_{\mathcal{E}_\beta}(X(\cdot)) W_{X_0, \pi X_0}^\beta(dX(\cdot)) dX_0. \quad (3.17)$$

We write $\mathbb{E}$, Var , and Cov for expectation, variance, and covariance under μ .

By (3.13) and (3.16), μ has total mass one, and its normalizing constant is $Z = \int_{\mathcal{X}_N} K_{\beta/2}^s(X, X) dX$. This expression is positive and finite without reference to H : $K_{\beta/2}^s(X, \pi X)$ is bounded by $\prod_i \sup_z \sum_{k \in \mathbb{Z}^3} p_\beta(z + Lk)$ on the torus and by $(4\pi\beta)^{-3N/2}$ in a container, $\mathcal{X}_N$ has finite volume, and, since $\mathcal{X}_N$ is open, the bridge from $X \in \mathcal{X}_N$ to itself stays in $\mathcal{X}_N$ with positive probability. The proof of Theorem 3.10 uses only this description of μ . Under μ , the random path satisfies $X(0) = X_0$ and $x_i(\beta) = x_{\pi(i)}(0)$ for every i . If $(i_1, \dots, i_k)$ is a cycle of π , with $\pi(i_j) = i_{j+1}$ and $i_{k+1} = i_1$, then the concatenation

of $x_{i_1}(\cdot), \dots, x_{i_k}(\cdot)$ is a continuous closed loop of duration $k\beta$. We call these loops the worldlines. They are closed by the endpoint permutation alone; no segments are added to close them. The labels are bookkeeping: all observables below are functions of the unlabeled collection of worldlines.

Lemma 3.8. *For every $\tau \in [0, \beta]$ and every bounded measurable function f on $\mathcal{X}_N$ that is invariant under permutations, we have $\mathbb{E}f(X(\tau)) = \langle f \rangle$.*

Proof. For $\tau = 0$, the claim follows from (3.17) and Lemma 3.6(3), and for $\tau = \beta$ it follows from the case $\tau = 0$ because $X(\beta) = \pi X(0)$ and f is symmetric. For $0 < \tau < \beta$, we split the bridge measure at time τ using (3.12), as in the proof of Lemma 3.6(2). This gives

$$\begin{aligned} \mathbb{E}f(X(\tau)) &= \frac{1}{Z N!} \sum_{\pi} \iint K_{\tau}(X, Y) f(Y) K_{\beta-\tau}(Y, \pi X) dX dY \\ &= \frac{1}{Z} \iint K_{\beta-\tau}^s(Y, X) K_{\tau}(X, Y) f(Y) dX dY. \end{aligned}$$

Using $K_{\tau}(\pi^{-1}X', Y) = K_{\tau}(X', \pi Y)$ and (3.15), the integral over X equals $K_{\beta}^s(Y, Y)$, and Lemma 3.6(3) completes the proof. $\square$

Winding vector and stochastic area. On the torus, the winding vector of $(\pi, X(\cdot))$ is

$$W = \frac{1}{L} \sum_{i=1}^N d(x_i(\cdot)) \in \mathbb{R}^3. \quad (3.18)$$

Let $\tilde{x}_i \in [0, L)^3$ be representatives of $x_i(0)$. Since $x_i(\beta) = x_{\pi(i)}(0)$, we have $d(x_i(\cdot)) = \tilde{x}_{\pi(i)} - \tilde{x}_i + Lk_i$ for some $k_i \in \mathbb{Z}^3$. Summing over i , we find that $W = \sum_i k_i \in \mathbb{Z}^3$. The vector W is the total homology class of the worldlines. We write $W = (W_1, W_2, W_3)$.

For a container, the stochastic area vector of $(\pi, X(\cdot))$ is

$$\mathcal{A} = \frac{1}{2} \sum_{i=1}^N \int_0^{\beta} (x_i(s) - x_0) \times \circ dx_i(s). \quad (3.19)$$

For a path of any duration $s > 0$ and reference point a , we use the vector limit of the dyadic left-point sums $\frac{1}{2} \sum_{k=0}^{2^{\ell}-1} (\omega(ks2^{-\ell}) - a) \times (\omega((k+1)s2^{-\ell}) - \omega(ks2^{-\ell}))$, set to zero where that vector limit does not exist. This is a Borel function of the path and agrees $\mathbb{W}_{y,z}^s$ -almost everywhere, for every y, z , with the Itô and Stratonovich area integrals, whose coordinate integrands and integrators have zero mutual quadratic covariation. To justify the asserted almost-sure limit, let A_{ℓ} be the dyadic sum under a normalized bridge of duration s . Given the level- ℓ vertices, each midpoint of an interval with endpoints P, Q and duration h is $(P + Q)/2 + \xi$, with independent centered Gaussian ξ of covariance $(h/2)I$. Its contribution to $A_{\ell+1} - A_{\ell}$ is $\frac{1}{2}\xi \times (Q - P)$. Thus (A_{ℓ}) is a martingale and $\mathbb{E}|A_{\ell+1} - A_{\ell}|^2 \leq C_{s,y,z} 2^{-\ell}$. The summable increment variances give convergence almost

surely and in L^2 ; convergence in probability of the left-point sums to the Itô integral identifies the limit. All stochastic areas below use this convention, and configuration areas are sums of the single-path areas. We write $\mathcal{A}_n = n \cdot \mathcal{A}$. Replacing x_0 by x'_0 changes the right side of (3.19) by

$$-\frac{1}{2}(x'_0 - x_0) \times \sum_{i=1}^N (x_i(\beta) - x_i(0)) = -\frac{1}{2}(x'_0 - x_0) \times \sum_{i=1}^N (x_{\pi(i)}(0) - x_i(0)) = 0,$$

so $\mathcal{A}$ does not depend on x_0 . Here we used the fact that the worldlines are closed; the area of an individual path $x_i(\cdot)$, or of a piece of the worldlines on a subinterval of $[0, \beta]$, depends on the origin in general. The vector $\mathcal{A}$ is the stochastic area estimator of Sindzingre, Klein, and Ceperley [135].

Lemma 3.9. *For every nonnegative measurable function G of $(\pi, X(\cdot))$, we have*

$$\mathbb{Z} \mathbb{E} G \leq \frac{1}{N!} \sum_{\pi \in S_N} \int_{\mathcal{X}_N} \int G(\pi, X(\cdot)) \mathbb{W}_{X_0, \pi X_0}^\beta(dX(\cdot)) dX_0. \quad (3.20)$$

On the torus, $\mathbb{E} e^{r|W|} < \infty$ for every $r \geq 0$. For a container, $\mathbb{E} |\mathcal{A}|^2 < \infty$.

Proof. Inequality (3.20) follows from (3.17) by dropping the indicator of $\mathcal{E}_\beta$. On the torus, we use $e^{r|W|} \leq \prod_i e^{r|d(x_i(\cdot))|/L}$ and decompose each torus bridge measure into its lifts as above. On the term in which $d(x_i(\cdot)) = d_i$, with $d_i = \tilde{x}_{\pi(i)} + Lk_i - \tilde{x}_i$, the integrand is constant, and the mass of that term is $\prod_i p_\beta(d_i)$. Since $r|d|/L \leq |d|^2/(8\beta) + 2\beta r^2/L^2$, we have

$$p_\beta(d) e^{r|d|/L} \leq e^{2\beta r^2/L^2} (4\pi\beta)^{-3/2} e^{-|d|^2/(8\beta)},$$

and $\sup_{z \in \mathbb{R}^3} \sum_{k \in \mathbb{Z}^3} e^{-|z+Lk|^2/(8\beta)} < \infty$. After inserting these bounds into (3.20), the integral over $X_0 \in [0, L)^{3N}$ is finite.

For a container, write $\mathcal{A} = \sum_i \mathcal{A}^{(i)}$, where $\mathcal{A}^{(i)}$ is the contribution of particle i in (3.19), so that $|\mathcal{A}|^2 \leq N \sum_i |\mathcal{A}^{(i)}|^2$. Under $\mathbb{W}_{X,Y}^\beta$, the paths of different particles are independent, and the normalized law of the path of particle i is that of $s \mapsto x_i + s\beta^{-1}(y_i - x_i) + b(s)$, where b is a Brownian bridge from 0 to 0 of duration β . Expanding (3.19) in terms of b , we find that $\mathcal{A}^{(i)}$ is the sum of a deterministic vector, linear functionals of b , and the stochastic area $(1/2) \int_0^\beta b(s) \times \circ db(s)$. Each of these terms has a finite second moment, bounded uniformly for $x_i, y_i \in \bar{U}$. Since $p_\beta \leq (4\pi\beta)^{-3/2}$ and U is bounded, we conclude from (3.20) that $\mathbb{E} |\mathcal{A}|^2 < \infty$. $\square$

Used in: the square-integrability assertions in Propositions 3.17 and 5.6.

3.3 An exact sub-Gaussian bound for the winding vector

In this section and the next, we work on the torus Λ_L . For $w \in \mathbb{R}^3$ and $X \in \Lambda_L^N$, we write $\tau_w X = (x_1 + w, \dots, x_N + w)$. The map τ_w preserves $\mathcal{X}_{N,L}$ and Lebesgue measure and commutes with permutations, and $K_s(\tau_w X, \tau_w Y) = K_s(X, Y)$ for $s > 0$, since the torus bridge measures and the events $\mathcal{E}_s$ are invariant under a common translation of all particles. The identity below is a path-space form of the common-translation relation behind the winding and momentum formulas of Pollock and Ceperley [33, 124]. The normalized overlap $\Phi(v)/Z$ is the characteristic function of the total momentum and equals the center-of-mass density-matrix kernel divided by the density in [139, Theorem III.1, Eqs. (52)–(60) in arXiv version 2]. We prove the winding moment-generating-function identity directly by shifting the bridges.

Theorem 3.10. *Let $N \geq 1$, $L > 0$, $\beta > 0$, and $a > 0$ be such that $\mathcal{X}_{N,L}$ is nonempty. For $v \in \mathbb{R}^3$, let*

$$\Phi(v) = \int_{\mathcal{X}_{N,L}} K_\beta^s(X, \tau_{-v} X) dX. \quad (3.21)$$

Then $0 \leq \Phi(v) \leq \Phi(0) = Z$ for every $v \in \mathbb{R}^3$, with $\Phi(v) = Z$ if $v \in L\mathbb{Z}^3$, and for every $t \in \mathbb{R}^3$,

$$\mathbb{E} e^{t \cdot W} = \exp\left(\frac{\beta N |t|^2}{L^2}\right) \frac{\Phi(2\beta t/L)}{Z}. \quad (3.22)$$

In particular,

$$\mathbb{E} e^{t \cdot W} \leq \exp\left(\frac{\beta N |t|^2}{L^2}\right), \quad (3.23)$$

with equality for $t \in (L^2/2\beta)\mathbb{Z}^3$. The law of W is invariant under $W \mapsto SW$ for every signed permutation matrix S ; in particular $\mathbb{E} W = 0$, $\text{Cov}(W) = \text{Var}(W_1) \text{Id}$, and

$$\text{Var}(u \cdot W) \leq \frac{2\beta N}{L^2} |u|^2, \quad u \in \mathbb{R}^3. \quad (3.24)$$

Proof. Let $t \in \mathbb{R}^3$ and $v = 2\beta t/L$. Fix $\pi \in S_N$ and $X_0 \in \mathcal{X}_{N,L}$, and let $\tilde{x}_1, \dots, \tilde{x}_N \in \mathbb{R}^3$ be representatives of the coordinates of X_0 . For $k = (k_1, \dots, k_N) \in (\mathbb{Z}^3)^N$, put $d_i = \tilde{x}_{\pi(i)} + Lk_i - \tilde{x}_i$ and $W_k = \bigotimes_i (\text{pr}_L)_* W_{\tilde{x}_i, \tilde{x}_i + d_i}^\beta$. By the definition of the torus bridge measures, $W_{X_0, \pi X_0}^\beta = \sum_k W_k$, and the paths in the term indexed by k satisfy $d(x_i(\cdot)) = d_i$. Since $\sum_i (\tilde{x}_{\pi(i)} - \tilde{x}_i) = 0$, we have $W = \sum_i k_i$ and $t \cdot W = L^{-1} \sum_i t \cdot d_i$ on these paths. Completing the square in the exponent of p_β gives

$$p_\beta(d) e^{t \cdot d/L} = e^{\beta |t|^2/L^2} p_\beta(d - v), \quad d \in \mathbb{R}^3.$$

The normalized Brownian bridge from $\tilde{x}$ to $\tilde{x} + d$ of duration β is the law of $s \mapsto \tilde{x} + (s/\beta)d + b(s)$, where b is a Brownian bridge from 0 to 0. Hence the shear $\omega(s) \mapsto \omega(s) -$

$(s/\beta)v$ maps it onto the normalized bridge from $\tilde{x}$ to $\tilde{x} + d - v$. Since $W_{\tilde{x}, \tilde{x}+d}^\beta$ has mass $p_\beta(d)$, the last display shows that the shear maps $e^{t \cdot d/L} W_{\tilde{x}, \tilde{x}+d}^\beta$ onto $e^{\beta|t|^2/L^2} W_{\tilde{x}, \tilde{x}+d-v}^\beta$. We apply this shear to every particle, on $\mathbb{R}^3$ and then on Λ_L , where it maps $X(s)$ to $\tau_{-sv/\beta} X(s)$. It translates all particles by the same vector at each time, and therefore it preserves $\mathcal{E}_\beta$. Hence

$$\int \mathbf{1}_{\mathcal{E}_\beta} e^{t \cdot W} dW_k = e^{\beta N|t|^2/L^2} W'_k(\mathcal{E}_\beta), \quad W'_k = \bigotimes_i (\text{pr}_L)_* W_{\tilde{x}_i, \tilde{x}_i+d_i-v}^\beta.$$

As k ranges over $(\mathbb{Z}^3)^N$, the endpoints $\tilde{x}_i + d_i - v = \tilde{x}_{\pi(i)} - v + Lk_i$ range over all lifts of $x_{\pi(i)} - v$, which is the i -th coordinate of $\pi\tau_{-v}X_0$. Summing over k , we find

$$\int_{\mathcal{E}_\beta} e^{t \cdot W} W_{X_0, \pi X_0}^\beta(dX(\cdot)) = e^{\beta N|t|^2/L^2} K_\beta(X_0, \pi\tau_{-v}X_0).$$

We sum over π , divide by $Z N!$, and integrate over X_0 ; all integrands are nonnegative. By (3.17) and (3.13), this gives (3.22).

Next, let $A = K_{\beta/2}^s \geq 0$. Since τ_w commutes with permutations, $K_s^s(\tau_w X, \tau_w Y) = K_s^s(X, Y)$. By Lemma 3.6(1)–(2) and this invariance,

$$\Phi(v) = \iint A(X, Y) A(Y, \tau_{-v} X) dY dX = \iint A(X, Y) A(X, \tau_v Y) dY dX.$$

Since $Y \mapsto \tau_v Y$ preserves $\mathcal{X}_{N,L}$ and Lebesgue measure, the Cauchy–Schwarz inequality in $L^2(dX dY)$ gives $0 \leq \Phi(v) \leq \iint A(X, Y)^2 dX dY$. By Lemma 3.6(1)–(2), the last integral equals $\int K_\beta^s(X, X) dX = \Phi(0)$, which is Z by the description of μ after Definition 3.7. If $v \in LZ^3$, then τ_{-v} is the identity of Λ_L^N , and $\Phi(v) = \Phi(0)$. Together with (3.22), this proves (3.23) and the equality case.

Let S be a 3×3 signed permutation matrix. The map $x \mapsto Sx$ is an isometry of Λ_L , it maps Brownian bridges to Brownian bridges, and it preserves $\mathcal{X}_{N,L}$. Then the measure μ is invariant under $(\pi, X(\cdot)) \mapsto (\pi, SX(\cdot))$, and this map sends W to SW . With $S = -\text{Id}$, and since W is integrable by Lemma 3.9, we obtain $\mathbb{E}W = 0$. Further, $\text{Cov}(W) = S \text{Cov}(W) S^\top$ for all such S . Invariance under the diagonal sign changes forces the off-diagonal entries of $\text{Cov}(W)$ to vanish, and invariance under coordinate permutations makes the diagonal entries equal. Finally, let $u \in \mathbb{R}^3$. By Lemma 3.9, the function $r \mapsto \mathbb{E}e^{ru \cdot W}$ is smooth, with derivatives computed under the expectation. By (3.23), it is bounded by $e^{\beta N|u|^2 r^2/L^2}$. The two functions are equal to one and have derivative zero at $r = 0$, since $\mathbb{E}W = 0$. Their difference is nonpositive, so its second derivative at $r = 0$ is nonpositive, which gives $\mathbb{E}(u \cdot W)^2 \leq 2\beta N|u|^2/L^2$. This is (3.24). $\square$

Used in: Theorem 1.3, Corollary 1.4, and Lemma 4.4; see Section 10.6.

The bound (3.23) states that every linear functional $u \cdot W$ with $|u| = 1$ is sub-Gaussian with variance proxy

$$\sigma^2 = \frac{2\beta N}{L^2} = 2\beta\rho_L L, \quad \rho_L = \frac{N}{L^3}. \quad (3.25)$$

Here a variance proxy means that $\mathbb{E}e^{tu \cdot W} \leq e^{\sigma^2 t^2/2}$ for all real t ; it bounds the variance from above without asserting equality. By (3.32), σ^2 is also the value of $\text{Var}(W_1)$ that corresponds to $f_s = 1$. We record the standard consequences of (3.23).

Corollary 3.11. *Let $u \in \mathbb{R}^3$ be a unit vector. For all $r \geq 0$ and $p > 0$,*

$$\mathbb{P}(|u \cdot W| \geq r) \leq 2e^{-r^2/(2\sigma^2)}, \quad \mathbb{E}|u \cdot W|^p \leq p(2\sigma^2)^{p/2} \Gamma(p/2). \quad (3.26)$$

In particular $\mathbb{E}(u \cdot W)^4 \leq 16\sigma^4$. If $\text{Var}(W_1) \geq v > 0$, then

$$\mathbb{P}(|u \cdot W| \geq (v/2)^{1/2}) \geq \frac{v^2}{64\sigma^4}. \quad (3.27)$$

Proof. For $s > 0$, Markov's inequality and (3.23) imply that $\mathbb{P}(u \cdot W \geq r) \leq e^{-sr + \sigma^2 s^2/2}$, and the choice $s = r/\sigma^2$ gives $e^{-r^2/(2\sigma^2)}$. The same bound holds for $-u \cdot W$, which proves the first inequality in (3.26). Then

$$\mathbb{E}|u \cdot W|^p = \int_0^\infty pr^{p-1} \mathbb{P}(|u \cdot W| > r) dr \leq 2p \int_0^\infty r^{p-1} e^{-r^2/(2\sigma^2)} dr = p(2\sigma^2)^{p/2} \Gamma(p/2).$$

For (3.27), set $Y = (u \cdot W)^2$. By Theorem 3.10, we have $\mathbb{E}Y = \text{Var}(W_1) \geq v$. Since $\mathbb{E}[Y\mathbf{1}_{\{Y < v/2\}}] \leq v/2$, we have by the Cauchy–Schwarz inequality

$$\mathbb{E}Y - \frac{v}{2} \leq \mathbb{E}[Y\mathbf{1}_{\{Y \geq v/2\}}] \leq (\mathbb{E}Y^2)^{1/2} \mathbb{P}(Y \geq v/2)^{1/2}.$$

Using $\mathbb{E}Y - v/2 \geq v/2$ and $\mathbb{E}Y^2 \leq 16\sigma^4$, we obtain (3.27). This is the Paley–Zygmund argument; for its original Rademacher-series forms, see [116, p. 192, Lemma 19] and [117, p. 269, Lemma γ]. $\square$

The next corollary concerns sequences of parameters. It uses a lower bound on $\text{Var}(W_1)$ of order L , which is the content of Theorem 1.1 in the dilute regime; here it is an assumption.

Corollary 3.12. *Let $(N_k, L_k, \beta_k)_{k \geq 1}$ be parameters for which $\mathcal{X}_{N_k, L_k}$ is nonempty, let $W^{(k)}$ be the corresponding winding vectors, and set $X_k = L_k^{-1/2} W^{(k)}$. Suppose that $\beta_k N_k / L_k^3 \leq \lambda$ for all k , with $\lambda \in (0, \infty)$. Then for all k , all $t \in \mathbb{R}^3$, all unit vectors u , and all $r \geq 0$,*

$$\mathbb{E}e^{t \cdot X_k} \leq e^{\lambda |t|^2}, \quad \mathbb{P}(|u \cdot X_k| \geq r) \leq 2e^{-r^2/(4\lambda)}. \quad (3.28)$$

In particular the laws of (X_k) are tight, and every subsequential weak limit X_∞ has a law invariant under $X_\infty \mapsto -X_\infty$ and satisfies $\mathbb{E}e^{t \cdot X_\infty} \leq e^{\lambda|t|^2}$ for all $t \in \mathbb{R}^3$. Suppose in addition that $\text{Var}(W_1^{(k)}) \geq cL_k$ for all k , with $c > 0$. Then for all k and all unit vectors u ,

$$\mathbb{P}(|u \cdot W^{(k)}| \geq (cL_k/2)^{1/2}) \geq \frac{c^2}{256\lambda^2}, \quad (3.29)$$

and every subsequential weak limit X_∞ satisfies $\text{Cov}(X_\infty) = v_\infty \text{Id}$ with $c \leq v_\infty \leq 2\lambda$.

The scope of these fluctuation bounds is discussed after Corollary 1.4; the hypotheses here do not imply a unique limiting law or a central limit theorem.

Proof. Let $\sigma_k^2 = 2\beta_k N_k / L_k^2$, so that $\sigma_k^2 \leq 2\lambda L_k$. Then (3.23) implies that $\mathbb{E}e^{t \cdot X_k} \leq \exp(\sigma_k^2 |t|^2 / (2L_k)) \leq e^{\lambda|t|^2}$, and the tail bound in (3.28) follows from (3.26). The tail bound implies that the laws are tight. Since $\mathbb{E}e^{2t \cdot X_k} \leq e^{4\lambda|t|^2}$, the variables $e^{t \cdot X_k}$ are uniformly integrable, and $\mathbb{E}e^{t \cdot X_k}$ converges to $\mathbb{E}e^{t \cdot X_\infty}$ along a weakly convergent subsequence. This proves the bound on the limit. The law of X_k is symmetric by Theorem 3.10, and so is any weak limit.

Under the variance assumption, (3.27) with $v = cL_k$ and $\sigma_k^4 \leq 4\lambda^2 L_k^2$ implies (3.29). By Theorem 3.10, $\text{Cov}(X_k) = v_k \text{Id}$ with $v_k = \text{Var}(W_1^{(k)}) / L_k$, and $c \leq v_k \leq \sigma_k^2 / L_k \leq 2\lambda$ by (3.24). By (3.26), the fourth moments of X_k are bounded, and the second moments converge along a weakly convergent subsequence. Then $\text{Cov}(X_\infty)$ is the limit of $v_k \text{Id}$ along this subsequence, and its value lies in $[c, 2\lambda]$. $\square$

3.4 Winding and the superfluid fraction

In this section, we abbreviate $H = H_{N,L}$, $H(\vartheta) = H_{N,L}(\vartheta)$, and $Z(\vartheta) = \text{tr } e^{-\beta H(\vartheta)}$, so that $Z = Z(0)$. Pollock and Ceperley related the superfluid density to winding fluctuations and momentum-density correlations, through the representation $Z(\vartheta)/Z = \mathbb{E}e^{i\vartheta \cdot W}$ of the twisted partition function [33, 124]. We verify the second-order formulas (3.32) and (3.33) for the present hard-core model from the untwisted Feynman–Kac representation and Theorem 3.10, without using this representation. For $z \in \mathbb{R}^3$, let $(\mathcal{T}_z \psi)(X) = \psi(\tau_{-z} X)$ on $L_{\text{sym}}^2(\mathcal{X}_{N,L})$. For an eigenvalue λ of H , let E_λ be its eigenspace, which is finite-dimensional by Lemma 3.1.

Lemma 3.13. *The following statements hold.*

1. For every $z \in \mathbb{R}^3$, the operator $\mathcal{T}_z$ is unitary, commutes with H , and maps every eigenspace E_λ onto itself.
2. For every eigenvalue λ , there are commuting Hermitian operators $P_{\lambda,1}, P_{\lambda,2}, P_{\lambda,3}$ on E_λ such that $\mathcal{T}_z|_{E_\lambda} = e^{-iz \cdot P_\lambda}$ for $z \in \mathbb{R}^3$, and $P_{\lambda,b} \psi = -i \sum_j \partial_{x_{j,b}} \psi$ for $\psi \in E_\lambda$. Moreover $\sum_b \|P_{\lambda,b} \psi\|^2 \leq N\lambda \|\psi\|^2$.

3. For every $\vartheta \in \mathbb{R}^3$, the operator $H(\vartheta)$ maps E_λ into itself and acts there as $\lambda + 2L^{-1}\vartheta \cdot P_\lambda + NL^{-2}|\vartheta|^2$. Consequently

$$Z(\vartheta) = e^{-\beta N|\vartheta|^2/L^2} \sum_{\lambda} e^{-\beta\lambda} \operatorname{tr} e^{-2\beta\vartheta \cdot P_\lambda/L}, \quad (3.30)$$

where the series converges uniformly on compact sets together with all partial derivatives of order at most two, and $F_{N,L,T}$ is twice continuously differentiable on $\mathbb{R}^3$.

Proof. For part (1), the map τ_{-z} preserves $\mathcal{X}_{N,L}$ and Lebesgue measure and commutes with permutations, so $\mathcal{T}_z$ is unitary. It maps smooth functions with compact support in $\mathcal{X}_{N,L}$ to such functions and preserves the H^1 norm and the form q . Hence it preserves $\mathcal{Q}_N$, and $q(\varphi, \mathcal{T}_z\psi) = q(\mathcal{T}_{-z}\varphi, \psi)$ for $\varphi, \psi \in \mathcal{Q}_N$. For ψ in the domain of H , this gives $q(\varphi, \mathcal{T}_z\psi) = \langle \mathcal{T}_{-z}\varphi, H\psi \rangle = \langle \varphi, \mathcal{T}_z H\psi \rangle$ for every $\varphi \in \mathcal{Q}_N$. Then $\mathcal{T}_z\psi$ lies in the domain of H , and $H\mathcal{T}_z\psi = \mathcal{T}_z H\psi$. In particular $\mathcal{T}_z$ maps E_λ into itself, and since $\mathcal{T}_{-z}$ does too, onto itself.

For part (2), the translations are strongly continuous, so $z \mapsto \mathcal{T}_z|_{E_\lambda}$ is a continuous unitary representation of $\mathbb{R}^3$ on the finite-dimensional space E_λ . For each b , the continuous one-parameter group $s \mapsto \mathcal{T}_{se_b}|_{E_\lambda}$ of unitary matrices is $e^{-isP_{\lambda,b}}$ for a Hermitian matrix $P_{\lambda,b}$. These matrices commute, since the groups do, and $\mathcal{T}_z|_{E_\lambda} = e^{-iz \cdot P_\lambda}$. Let $\psi \in E_\lambda$. After extension by zero, $\psi \in H^1(\Lambda_L^N)$, and the difference quotients $s^{-1}(\mathcal{T}_{se_b}\psi - \psi)$ converge in L^2 to $-\sum_j \partial_{x_{j,b}}\psi$. Their limit is also $-iP_{\lambda,b}\psi$, which proves the formula for $P_{\lambda,b}$. The pointwise bound $|\sum_j \nabla_j \psi|^2 \leq N \sum_j |\nabla_j \psi|^2$ gives $\sum_b \|P_{\lambda,b}\psi\|^2 \leq Nq[\psi] = N\lambda\|\psi\|^2$.

For part (3), let $\psi \in E_\lambda$ and $\varphi \in \mathcal{Q}_N$. By integration by parts for smooth functions with compact support in $\mathcal{X}_{N,L}$, and by density, $\langle \varphi, -i \sum_j \partial_{x_{j,b}}\psi \rangle = \langle -i \sum_j \partial_{x_{j,b}}\varphi, \psi \rangle$. Expanding the square in (3.3) and using part (2), we find

$$q_\vartheta(\varphi, \psi) = q(\varphi, \psi) + \frac{2}{L} \langle \varphi, \vartheta \cdot P_\lambda \psi \rangle + \frac{N|\vartheta|^2}{L^2} \langle \varphi, \psi \rangle = \left\langle \varphi, \left(\lambda + \frac{2}{L} \vartheta \cdot P_\lambda + \frac{N|\vartheta|^2}{L^2} \right) \psi \right\rangle.$$

Hence ψ lies in the domain of $H(\vartheta)$, and $H(\vartheta)$ acts on E_λ as stated. Let $B_\lambda(\vartheta)$ be this Hermitian operator on E_λ . Since H has compact resolvent, $L_{\text{sym}}^2(\mathcal{X}_{N,L})$ is the orthogonal sum of the spaces E_λ . The orthogonal sum of the operators $B_\lambda(\vartheta)$ is self-adjoint, and its restriction to finite sums of eigenvectors of H is a core. This restriction is contained in $H(\vartheta)$. Since $H(\vartheta)$ is closed, it also contains the closure of the restriction, which is the orthogonal sum. A self-adjoint operator has no proper symmetric extension, so the orthogonal sum is $H(\vartheta)$. This gives (3.30), since $e^{-\beta B_\lambda(\vartheta)} = e^{-\beta\lambda - \beta N|\vartheta|^2/L^2} e^{-2\beta\vartheta \cdot P_\lambda/L}$.

Let $r > 0$, and let γ be a multi-index with $|\gamma| \leq 2$. By part (2) and the Cauchy-Schwarz inequality, $\|\vartheta \cdot P_\lambda\| \leq |\vartheta| \sqrt{N\lambda}$ for $\vartheta \in \mathbb{R}^3$. Since the matrices $P_{\lambda,b}$ commute, we obtain for $|\vartheta| \leq r$

$$\left| \partial_\vartheta^\gamma \operatorname{tr} e^{-2\beta\vartheta \cdot P_\lambda/L} \right| \leq \dim E_\lambda \left(\frac{2\beta\sqrt{N\lambda}}{L} \right)^{|\gamma|} e^{2\beta r \sqrt{N\lambda}/L} \leq \dim E_\lambda \left(\frac{2\beta\sqrt{N\lambda}}{L} \right)^{|\gamma|} e^{\beta\lambda/2 + 2\beta N r^2/L^2}.$$

Since $\sum_{\lambda} (1 + \lambda) \dim E_{\lambda} e^{-\beta\lambda/2} = \text{tr}((1 + H)e^{-\beta H/2}) < \infty$ by Lemma 3.1, the series in (3.30) and its partial derivatives of order at most two converge uniformly on compact subsets of $\mathbb{R}^3$. Hence Z is twice continuously differentiable, and since $Z(\vartheta) > 0$, so is $F_{N,L,T} = -T \log Z$. $\square$

Proposition 3.14. *Let $\langle P_1^2 \rangle = Z^{-1} \sum_{\lambda} e^{-\beta\lambda} \text{tr} P_{\lambda,1}^2$, with P_{λ} as in Lemma 3.13. Then*

$$\text{Var}(W_1) = \frac{2\beta N}{L^2} - \frac{4\beta^2}{L^2} \langle P_1^2 \rangle, \quad (3.31)$$

$$f_s(N, L, T) = \frac{TL^2}{2N} \text{Var}(W_1), \quad (3.32)$$

$$f_s(N, L, T) = 1 - \frac{2}{NT} \langle P_1^2 \rangle, \quad (3.33)$$

and $0 \leq f_s(N, L, T) \leq 1$.

Since $\langle P_1 \rangle = Z^{-1} \sum_{\lambda} e^{-\beta\lambda} \text{tr} P_{\lambda,1} = 0$ (see the proof), $\langle P_1^2 \rangle$ is the variance of the first component of the total momentum in the Gibbs state, and (3.33) reads $f_s = 1 - 2 \text{Var}(P_1)/(NT)$. In physical units, it states that the normal fraction $1 - f_s$ equals $\text{Var}(P_1)/(mNT)$, where P_1 is the physical total momentum. This is the total-momentum fluctuation formula for the normal fraction; see [124].

Proof. Let $v \in \mathbb{R}^3$. As in the proof of Lemma 3.6(3), let $A = e^{-\beta H^{\text{lab}}/2} P_{\text{sym}}$, and extend $\mathcal{T}_{-v}$ to $L^2(\mathcal{X}_{N,L})$ by the same formula. Since $(\mathcal{T}_{-v}\psi)(Y) = \psi(\tau_v Y)$ and τ_{-v} preserves $\mathcal{X}_{N,L}$ and Lebesgue measure, Proposition 3.5(1) shows that the Hilbert–Schmidt operators A and $A\mathcal{T}_{-v}$ have the kernels $K_{\beta/2}^s(X, Y)$ and $K_{\beta/2}^s(X, \tau_{-v}Y)$. Since $\mathcal{T}_{-v}$ commutes with P_{sym} , Lemma 3.6(2) gives

$$\text{tr}(e^{-\beta H} \mathcal{T}_{-v}) = \text{tr}(A \cdot A\mathcal{T}_{-v}) = \iint K_{\beta/2}^s(X, Y) K_{\beta/2}^s(Y, \tau_{-v}X) dY dX = \Phi(v),$$

with Φ as in (3.21). On the other hand, Lemma 3.13(1)–(2) give $\text{tr}(e^{-\beta H} \mathcal{T}_{-v}) = \sum_{\lambda} e^{-\beta\lambda} \text{tr} e^{iv \cdot P_{\lambda}}$. With $t = se_1$ and $v = 2\beta se_1/L$, (3.22) becomes

$$Z \mathbb{E} e^{sW_1} = e^{\beta N s^2 / L^2} \sum_{\lambda} e^{-\beta\lambda} \text{tr} e^{2i\beta s P_{\lambda,1} / L}, \quad s \in \mathbb{R}.$$

We differentiate twice at $s = 0$. On the left, this is justified by Lemma 3.9. On the right, the j -th derivative of the term indexed by λ is bounded by $\dim E_{\lambda} e^{-\beta\lambda} (2\beta\sqrt{N\lambda}/L)^j$ by Lemma 3.13(2), and these bounds are summable by Lemma 3.1. The first derivatives give $\mathbb{E}W_1 = (2i\beta/L) \langle P_1 \rangle$, where $\langle P_1 \rangle = Z^{-1} \sum_{\lambda} e^{-\beta\lambda} \text{tr} P_{\lambda,1}$. Both $\mathbb{E}W_1$ and $\langle P_1 \rangle$ are real, so $\langle P_1 \rangle = 0$. The second derivatives then give $\mathbb{E}W_1^2 = 2\beta N L^{-2} - 4\beta^2 L^{-2} \langle P_1^2 \rangle$, which is (3.31), since $\mathbb{E}W_1 = 0$.

On the operator side, (3.30) with $\vartheta = \theta e_1$ may be differentiated term by term by Lemma 3.13(3). Since $\langle P_1 \rangle = 0$, the second derivative of $\log Z(\theta e_1)$ at $\theta = 0$ equals $-2\beta N L^{-2} + 4\beta^2 L^{-2} \langle P_1^2 \rangle$. Multiplying by $-T$, we obtain $\partial_\theta^2 F_{N,L,T}(0) = 2NL^{-2} - 4\beta L^{-2} \langle P_1^2 \rangle = T \text{Var}(W_1)$, and (3.32) follows from (3.5). Then (3.33) follows from (3.32) and (3.31). Finally, $f_s \geq 0$ by (3.32), and $f_s \leq 1$ by (3.33), since $\text{tr } P_{\lambda,1}^2 \geq 0$. $\square$

Remark 3.15. In general units, with mass m , the reduced Planck constant $\hbar$, and $\kappa = \hbar^2/(2m)$, the Brownian motions have generator $\kappa\Delta$ on the time interval $[0, \beta]$, and the identities of this section read

$$f_s = \frac{mL^2}{\hbar^2 N} F''_{N,L,T}(0) = \frac{mL^2}{\beta \hbar^2 N} \text{Var}(W_1) = 1 - \frac{\beta}{mN} \text{Var}(\hbar P_1), \quad \text{Var}(u \cdot W) \leq \frac{2\beta \kappa N}{L^2} |u|^2.$$

The physical interpretation of the twist response is given in Section 1.1; the conversion to general units is recorded in Remark 1.10. For the rotational identities of Section 3.5, the factor T in front of $\text{Cov}(\mathcal{A})$ becomes $4m^2/(\beta \hbar^2)$.

3.5 Rotation and the stochastic area

In this section, U is a container, n is a unit vector, $x_0 \in \mathbb{R}^3$, and $H = H_N(U)$. We do not assume that U has any rotational symmetry.

For a container, the area functional and the angular momentum are related by the magnetic completion of the square. Let $\omega \in \mathbb{R}^3$, and let $R(X)$ be the symmetric matrix with entries $R_{bc}(X) = m \sum_i (|y_i|^2 \delta_{bc} - y_{i,b} y_{i,c})$, where $y_i = x_i - x_0$ and $m = 1/2$, so that $I_{\text{cl},bc}(x_0) = \langle R_{bc} \rangle$. Expanding the square and using Lemma 3.3, we find for $\psi \in \mathcal{Q}_N$ that

$$\sum_{i=1}^N \left\| \left(-i\nabla_i - \frac{1}{2} \omega \times y_i \right) \psi \right\|^2 = q[\psi] - \langle \psi, \omega \cdot \mathcal{L} \psi \rangle + \frac{1}{2} \langle \psi, (\omega \cdot R \omega) \psi \rangle, \quad (3.34)$$

because $|\omega \times y|^2 = |\omega|^2 |y|^2 - (\omega \cdot y)^2$. We denote the operator associated with the left side of (3.34) by $H^{\text{mag}}(\omega)$, so that $H^{\text{mag}}(\omega) = H - \omega \cdot \mathcal{L} + (1/2) \omega \cdot R \omega$. It is the Hamiltonian of the particles in the vector potential $\omega \times (x - x_0)/2$, which corresponds to the uniform magnetic field ω . We write $G(\omega) = -T \log \text{tr } e^{-\beta H^{\text{mag}}(\omega)}$.

Proposition 3.16. *Let U be a container, n a unit vector, and $x_0 \in \mathbb{R}^3$. Then the following statements hold.*

1. For every $\omega \in \mathbb{R}^3$,

$$\begin{aligned} \frac{\text{tr } e^{-\beta(H - \omega \cdot \mathcal{L})}}{Z} &= \mathbb{E} \exp \left(-i\omega \cdot \mathcal{A} + \frac{1}{2} \int_0^\beta \omega \cdot R(X(s)) \omega \, ds \right), \\ \frac{\text{tr } e^{-\beta H^{\text{mag}}(\omega)}}{Z} &= \mathbb{E} e^{-i\omega \cdot \mathcal{A}}. \end{aligned} \quad (3.35)$$

The functions F^{rot} and G are twice continuously differentiable on $\mathbb{R}^3$.

2. The function F^{rot} is concave on $\mathbb{R}^3$. In particular, $I(x_0) \geq 0$ in the sense of symmetric matrices, and $I(n, x_0) \geq 0$.
3. We have $\mathbb{E}\mathcal{A} = 0$ and

$$\begin{aligned} I_{\text{cl},bc}(x_0) - I_{bc}(x_0) &= \partial_{\omega_b} \partial_{\omega_c} G(0) = T \text{Cov}(\mathcal{A})_{bc}, \\ I_{\text{cl}}(n, x_0) - I(n, x_0) &= T \text{Var}(\mathcal{A}_n). \end{aligned} \quad (3.36)$$

In particular, $I_{\text{cl}} - I$ does not depend on x_0 .

4. We have $0 \leq I(n, x_0) \leq I_{\text{cl}}(n, x_0)$, and $0 \leq I(x_0) \leq I_{\text{cl}}(x_0)$ in the sense of symmetric matrices. Further, $\text{tr} e^{-\beta H^{\text{mag}}(\omega)} \leq Z$, so that $G(\omega) \geq G(0)$, for every $\omega \in \mathbb{R}^3$.

Part (2) is the standard concavity of a free energy in a parameter that enters the Hamiltonian linearly. Formula (3.36) is the stochastic area estimator of the moment of inertia introduced by Sindzingre, Klein, and Ceperley [135]; the proof below verifies it, with its constant, for the hard-core gas with hard walls. The inequality $\text{tr} e^{-\beta H^{\text{mag}}(\omega)} \leq Z$ is a diamagnetic trace bound, proved below from the Feynman–Kac representation; see [132] for ground-state diamagnetism of spinless bosons and [8, Theorems 2.3–2.5] for magnetic semigroup domination.

Proof. For part (1), by Lemma 3.3 and (3.34), we have $H - \omega \cdot \mathcal{L} = H^{\text{mag}}(\omega) + V_\omega$ with the bounded symmetric potential $V_\omega(X) = -(1/2)\omega \cdot R(X)\omega$. We apply Proposition 3.5(2) with the affine, divergence-free vector field $a(x) = \omega \times (x - x_0)/2$. Since $(\omega \times y) \cdot \circ dy = \omega \cdot (y \times \circ dy)$, the phase in (3.14) is

$$-\frac{i}{2} \sum_{i=1}^N \int_0^\beta (\omega \times y_i(s)) \cdot \circ dy_i(s) = -i\omega \cdot \mathcal{A}.$$

Inserting this into the trace formula of Lemma 3.6(3) and comparing with (3.17) yields (3.35). Let $g(\omega)$ denote the right side of the first identity in (3.35), and write $M = \int_0^\beta R(X(s)) ds$. The matrix M is positive semidefinite and bounded by $\beta m N R_0^2$ in operator norm, and $\mathbb{E}|\mathcal{A}|^2 < \infty$ by Lemma 3.9. The second derivatives in ω of the integrand are bounded by $((|\mathcal{A}| + \|M\|\|\omega\|)^2 + \|M\|)e^{\|M\|\|\omega\|^2/2}$, which is integrable uniformly for ω in compact sets. Then g is twice continuously differentiable, and we may differentiate under the expectation. The function g is positive, since it is a ratio of traces of positive operators, and $F^{\text{rot}} = F^{\text{rot}}(0) - T \log g$ is twice continuously differentiable. The same argument applies to G .

For part (3), we first note that $\partial_{\omega_b} g(0) = -i\mathbb{E}\mathcal{A}_b$. This number is real because g is real, and $\mathbb{E}\mathcal{A}_b$ is real, so $\mathbb{E}\mathcal{A}_b = 0$. Then

$$\partial_{\omega_b} \partial_{\omega_c} \log g(0) = \partial_{\omega_b} \partial_{\omega_c} g(0) = -\mathbb{E}[\mathcal{A}_b \mathcal{A}_c] + \mathbb{E}M_{bc}.$$

By Lemma 3.8 and Fubini's theorem, $\mathbb{E}M_{bc} = \beta \langle R_{bc} \rangle = \beta I_{\text{cl},bc}(x_0)$. Multiplying by $-T$ gives $\partial_{\omega_b} \partial_{\omega_c} F^{\text{rot}}(0) = T \text{Cov}(\mathcal{A})_{bc} - I_{\text{cl},bc}(x_0)$, which is the first claim of (3.36) after

recalling (3.10). The same computation applied to the second identity in (3.35) yields $\partial_{\omega_b} \partial_{\omega_c} G(0) = T \text{Cov}(\mathcal{A})_{bc}$. Contracting with n gives the second identity in (3.36). The vector $\mathcal{A}$ does not depend on x_0 , as shown below (3.19), and neither does $I_{\text{cl}} - I$.

For part (2), let $\sigma = \sum_k \lambda_k |f_k\rangle\langle f_k|$ be a symmetric density matrix of finite kinetic energy: $f_k \in \mathcal{Q}_N$ when $\lambda_k > 0$, and $\sum_k \lambda_k q[f_k] < \infty$. By (3.7), $|\langle f_k, \omega \cdot \mathcal{L} f_k \rangle| \leq |\omega| \sqrt{N} R_0 q[f_k]^{1/2}$, and $\sum_k \lambda_k q[f_k]^{1/2} \leq (\sum_k \lambda_k q[f_k])^{1/2}$, so the series $\sum_k \lambda_k \langle f_k, \omega \cdot \mathcal{L} f_k \rangle$ converges absolutely. We use the Gibbs variational principle [56, Lemma 3.1] for the self-adjoint operators of the forms, with energies given by the forms. For H , it gives $S(\sigma) \leq \beta \sum_k \lambda_k q[f_k] + \log Z < \infty$, where $S(\sigma) = -\text{tr } \sigma \log \sigma$. For $H - \omega \cdot \mathcal{L}$, it gives

$$F^{\text{rot}}(\omega) \leq \mathcal{F}_\omega(\sigma) = \sum_k \lambda_k (q[f_k] - \langle f_k, \omega \cdot \mathcal{L} f_k \rangle) - TS(\sigma), \quad \omega \in \mathbb{R}^3,$$

with equality for the Gibbs state Γ_ω of $H - \omega \cdot \mathcal{L}$. This state has finite kinetic energy. Indeed, by Lemma 3.3, $H - \omega \cdot \mathcal{L}$ has an orthonormal basis of eigenvectors $\phi_k \in \mathcal{Q}_N$, with eigenvalues $E_k(\omega)$ that are bounded below and satisfy $\sum_k |E_k(\omega)| e^{-\beta E_k(\omega)} < \infty$, and (3.8) gives $q[\phi_k] \leq 2E_k(\omega) + |\omega|^2 N R_0^2$. Thus $F^{\text{rot}}(\omega)$ is the minimum of $\mathcal{F}_\omega(\sigma)$ over the symmetric density matrices of finite kinetic energy, a set that does not depend on ω . Each function $\omega \mapsto \mathcal{F}_\omega(\sigma)$ is affine, and F^{rot} , as their infimum, is concave. By part (1), it is twice continuously differentiable, and its Hessian at $\omega = 0$ is negative semidefinite. By (3.10), this means that $I(x_0) \geq 0$, and then $I(n, x_0) \geq 0$.

For part (4), the inequality $I \geq 0$ follows from part (2), and $I \leq I_{\text{cl}}$ follows from part (3), since covariance matrices are positive semidefinite. By the second identity in (3.35), we have $\text{tr } e^{-\beta H^{\text{mag}}(\omega)} \leq Z \mathbb{E} |e^{-i\omega \cdot \mathcal{A}}| = Z$. $\square$

3.6 Imaginary-time slabs

In this section, $\mathcal{X}_N$ is either $\mathcal{X}_{N,L}$ or $\mathcal{X}_N(U)$. Fix a slab length $\delta \in (0, \beta]$, and let

$$n_\beta = \lfloor \beta/\delta \rfloor \geq 1, \quad r_\beta = \beta - n_\beta \delta \in [0, \delta).$$

We cut the interval $[0, \beta]$ at the times $\tau_j = j\delta$, $0 \leq j \leq n_\beta$. This produces n_β full slabs $[\tau_{j-1}, \tau_j]$ of length δ and, if $r_\beta > 0$, a remainder slab $[n_\beta \delta, \beta]$ of length r_β . Let J be the number of slabs, so that $J = n_\beta$ if $r_\beta = 0$ and $J = n_\beta + 1$ otherwise, and let $s_1, \dots, s_J$ be their lengths.

The marked slab measure. The construction uses the semigroup and Bose symmetrization identities of Lemma 3.6; for functional-integration background, see [63, 133]. Its one-slab marginal is obtained from [112, Proposition 5.1] by integrating out the auxiliary configuration variable; we give the multiple-cut representation needed here. The slab decomposition is described on an extended space. Its coordinates are cut configurations $Y_0, \dots, Y_{J-1} \in \mathcal{X}_N$, with the convention $Y_J = Y_0$, and, for each slab $1 \leq j \leq J$, a

permutation $\pi_j \in S_N$ and a configuration path $X^{(j)}(\cdot) \in C([0, s_j]; \mathbb{X}^N)$. We define the measure

$$\bar{\mu} = \frac{1}{Z(N!)^J} \prod_{j=1}^J \left(\mathbf{1}_{\mathcal{E}_{s_j}}(X^{(j)}(\cdot)) \mathbb{W}_{Y_{j-1}, \pi_j Y_j}^{s_j}(\mathrm{d}X^{(j)}(\cdot)) \right) \prod_{j=0}^{J-1} \mathrm{d}Y_j, \quad (3.37)$$

summed over $(\pi_1, \dots, \pi_J)$ with counting measure. Under $\bar{\mu}$, the path of slab j runs from the cut Y_{j-1} to the cut $\pi_j Y_j$, so that particle i of slab j ends at particle $\pi_j(i)$ of the cut Y_j . The concatenation map Cat sends the slab data to a pair $(\pi, X(\cdot))$ as follows. Set $\sigma_0 = \text{id}$ and $\sigma_j = \pi_j \circ \dots \circ \pi_1$, and define

$$x_i(\tau) = X_{\sigma_{j-1}(i)}^{(j)}(\tau - \tau_{j-1}) \quad \text{for } \tau \in [\tau_{j-1}, \tau_{j-1} + s_j], \quad \pi = \sigma_J, \quad (3.38)$$

where $\tau_j = \beta$. Then particle i follows the path of slab 1 that starts at the i -th particle of Y_0 , and at every cut it continues with the path of the next slab that starts where it arrived. Let $\mathcal{F}_{\text{cut}}$ be the σ -algebra generated by $Y_0, \dots, Y_{J-1}$.

Proposition 3.17. *The following statements hold.*

1. *The measure $\bar{\mu}$ is a probability measure. The cut configurations $(Y_0, \dots, Y_{J-1}) \in \mathcal{X}_N^J$ have the density*

$$\rho_{\text{cut}}(Y_0, \dots, Y_{J-1}) = \frac{1}{Z} \prod_{j=1}^J K_{s_j}^s(Y_{j-1}, Y_j). \quad (3.39)$$

2. *Conditionally on $\mathcal{F}_{\text{cut}}$, the slab data $(\pi_j, X^{(j)}(\cdot))$, $1 \leq j \leq J$, are independent, and the conditional law of $(\pi_j, X^{(j)}(\cdot))$ is*

$$\nu_{s_j}(\cdot \mid Y_{j-1}, Y_j) = \frac{1}{N! K_{s_j}^s(Y_{j-1}, Y_j)} \sum_{\pi_j \in S_N} \mathbf{1}_{\mathcal{E}_{s_j}} \mathbb{W}_{Y_{j-1}, \pi_j Y_j}^{s_j} \quad (3.40)$$

which depends only on s_j , Y_{j-1} , and Y_j . This formula is used where $K_{s_j}^s(Y_{j-1}, Y_j) > 0$; elsewhere choose an arbitrary probability law. The exceptional cut tuples have ρ_{cut} -measure zero.

3. *The image of $\bar{\mu}$ under Cat is the Bose path measure μ .*
4. *If $\beta > \delta$, then for every $1 \leq j \leq n_\beta$, the joint law of $(Y_{j-1}, Y_j, \pi_j, X^{(j)}(\cdot))$ is*

$$\frac{1}{Z N!} K_{\beta-\delta}^s(Y_j, Y_{j-1}) \mathbf{1}_{\mathcal{E}_\delta}(X^{(j)}(\cdot)) \mathbb{W}_{Y_{j-1}, \pi_j Y_j}^\delta(\mathrm{d}X^{(j)}(\cdot)) \mathrm{d}Y_{j-1} \mathrm{d}Y_j, \quad (3.41)$$

which does not depend on j .

5. *On the torus, let $O_j^W = L^{-1} \sum_i d(X_i^{(j)}(\cdot)) \in \mathbb{R}^3$. For a container, let $O_j^A = (1/2) \sum_i \int_0^{s_j} (X_i^{(j)}(s) - x_0) \times \circ \mathrm{d}X_i^{(j)}(s)$. Then $\bar{\mu}$ -almost surely,*

$$W \circ \text{Cat} = \sum_{j=1}^J O_j^W, \quad \mathcal{A} \circ \text{Cat} = \sum_{j=1}^J O_j^A, \quad (3.42)$$

and every O_j^W and O_j^A has finite second moment under $\bar{\mu}$.

Proof. For fixed cuts, the total mass of the slab- j factor in (3.37), summed over π_j , is $N! K_{s_j}^s(Y_{j-1}, Y_j)$ by (3.13). Integrating out the slab data in (3.37) gives (3.39), and the product form of (3.37) implies part (2). Integrating (3.39) over $Y_1, \dots, Y_{j-1}$ with (3.15) leaves $Z^{-1} K_{\beta}^s(Y_0, Y_0)$, whose integral is one by (3.16). This proves part (1).

For part (3), fix $(\pi_1, \dots, \pi_J)$ and push the corresponding term of (3.37) forward under Cat . Consider first the two slabs $j = 1, 2$. By (3.38), particle i follows $X_i^{(1)}$ from $y_{0,i}$ to $y_{1,\pi_1(i)}$ and then $X_{\pi_1(i)}^{(2)}$ from $y_{1,\pi_1(i)}$ to $y_{2,\sigma_2(i)}$. The change of variables $z_i = y_{1,\pi_1(i)}$ preserves Lebesgue measure on $\mathcal{X}_N$, and (3.12) applied to each particle shows that integrating over Y_1 produces the bridge measure of duration $s_1 + s_2$ from Y_0 to $\sigma_2 Y_2$. Since $\mathcal{X}_N$ is invariant under permutations, the configuration path of slab 2 stays in $\mathcal{X}_N$ if and only if its relabeling by π_1 does; the killing indicators combine into $\mathbf{1}_{\mathcal{E}_{s_1+s_2}}$ of the concatenated path. Repeating this for the remaining cuts, we conclude that the image of the term indexed by $(\pi_1, \dots, \pi_J)$ is

$$\frac{1}{Z(N!)^J} \mathbf{1}_{\mathcal{E}_{\beta}}(X(\cdot)) \mathbb{W}_{Y_0, \sigma_J Y_0}^{\beta}(dX(\cdot)) dY_0$$

at $\pi = \sigma_J$. For each $\pi \in S_N$, exactly $(N!)^{J-1}$ tuples $(\pi_1, \dots, \pi_J)$ satisfy $\pi_J \circ \dots \circ \pi_1 = \pi$, since $\pi_1, \dots, \pi_{J-1}$ are arbitrary and π_J is then determined. Summing over the tuples gives (3.17).

For part (4), note that $\beta > \delta$ implies $J \geq 2$. In (3.37), we integrate out the slab data of all slabs other than j , which produces the factors $K_{s_i}^s(Y_{i-1}, Y_i)$ for $i \neq j$, and then the cuts other than Y_{j-1} and Y_j . By (3.15), applied around the cycle of cuts from Y_j back to Y_{j-1} , the product of these kernels integrates to $K_{\beta-\delta}^s(Y_j, Y_{j-1})$. The result is (3.41).

For part (5), the lifted displacement is additive under concatenation of paths, and for each j the map $i \mapsto \sigma_{j-1}(i)$ is a bijection. Then

$$\sum_{i=1}^N d(x_i(\cdot)) = \sum_{i=1}^N \sum_{j=1}^J d(X_{\sigma_{j-1}(i)}^{(j)}(\cdot)) = \sum_{j=1}^J \sum_{i=1}^N d(X_i^{(j)}(\cdot)).$$

The Stratonovich integral in (3.19) is additive over the subintervals $[\tau_{j-1}, \tau_{j-1} + s_j]$, and the same reindexing proves the second identity in (3.42). For the second moments, we argue as in the proof of Lemma 3.9. If $J = 1$, then $O_1^W = W \circ \text{Cat}$ and $O_1^A = \mathcal{A} \circ \text{Cat}$, and the claim is Lemma 3.9. If $J \geq 2$, then the law of slab j with its cuts is given by the analog of (3.41) with δ replaced by s_j . In this formula, $K_{\beta-s_j}^s$ is bounded by the supremum of the free heat kernel at time $\beta - s_j > 0$, which is finite, and dropping the killing indicator leaves a free Gaussian integral of the same type as in Lemma 3.9, which is finite. $\square$

Used in: Lemma 8.2 and Proposition 5.6.

We record three comments on Proposition 3.17. First, the labels of the cuts in (3.37) are assigned independently at each cut: the density (3.39), which contains a symmetrized kernel for every slab, is the law of the cut configurations after an independent uniform relabeling of the particles at each cut. It is not the joint law of $X(\tau_0), \dots, X(\tau_{J-1})$ under μ with the labels followed continuously from time 0; in that labeling, the internal factors would be the labeled kernels K_{s_j} and only the closing factor would be symmetrized. Part (3) shows that the relabeled description has the correct image, and the observables W and $\mathcal{A}$ do not depend on the labels. Second, conditioning on $\mathcal{F}_{\text{cut}}$ fixes the cut configurations but not the permutations π_j , the cycle structure of π , or the winding vector. Third, O_j^W is not an integer in general: with representatives $\tilde{y}$ in a fundamental cell, O_j^W equals $L^{-1} \sum_i (\tilde{y}_{j,i} - \tilde{y}_{j-1,i})$ plus an integer vector, and the noninteger part is a function of the cuts. Similarly, replacing x_0 by x'_0 in O_j^A adds $-(1/2)(x'_0 - x_0) \times \sum_i (y_{j,i} - y_{j-1,i})$, a function of the cuts that cancels in the sum (3.42). Neither change affects conditional variances given $\mathcal{F}_{\text{cut}}$.

Using conditional independence, additivity, and the law of total variance, we obtain the following bound. The kernel form used below is proved in Lemma 5.5 and Proposition 5.6.

Lemma 3.18. *Let g be a real measurable function of $(Y, Y', \pi', X'(\cdot)) \in \mathcal{X}_N \times \mathcal{X}_N \times S_N \times C([0, \delta]; \mathbb{X}^N)$, and set $O_j = g(Y_{j-1}, Y_j, \pi_j, X^{(j)}(\cdot))$ for $1 \leq j \leq n_\beta$. Let O_{rem} be a function of the cuts and, if $r_\beta > 0$, of the remainder slab. Suppose that $O_1, \dots, O_{n_\beta}$ and O_{rem} have finite second moments under $\bar{\mu}$, and let $O = \sum_{j=1}^{n_\beta} O_j + O_{\text{rem}}$. Then*

$$\text{Var}_{\bar{\mu}}(O) \geq \sum_{j=1}^{n_\beta} \mathbb{E}_{\bar{\mu}} \text{Var}_{\bar{\mu}}(O_j \mid \mathcal{F}_{\text{cut}}) = n_\beta \mathbb{E}_{\bar{\mu}} \text{Var}_{\bar{\mu}}(O_1 \mid Y_0, Y_1), \quad (3.43)$$

where $\text{Var}_{\bar{\mu}}(O_1 \mid Y_0, Y_1)$ is the variance of $g(Y_0, Y_1, \cdot)$ under $\nu_\delta(\cdot \mid Y_0, Y_1)$.

Proof. By Proposition 3.17(2), conditional variances add given $\mathcal{F}_{\text{cut}}$. Dropping the nonnegative remainder variance and applying the law of total variance, we obtain the inequality in (3.43). Each full-slab conditional law depends only on its two cuts. If $n_\beta \geq 2$, then $\beta > \delta$, and Proposition 3.17(4) implies that the expected conditional variance is the same for every full slab. For $n_\beta = 1$, the equality is immediate. $\square$

For winding and area, the additivity and square integrability required here are given in Proposition 3.17(5). Their one-slab kernel bounds are combined in Proposition 5.6.

A normal phase at high temperature

Lemma 4.1 adapts the insertion argument in the proof of [112, Proposition 3.5], which combines the normalized insertion orbital of [112, Lemma 3.4] and [115, Lemma 3.3], a refinement of the nearest-neighbor construction of [102, Section 3.2 in arXiv version 1], with a Fourier frame. That proof applies Parseval's identity to all frequencies and then restricts the sum to low ones; we keep all frequencies. Canonical ratio estimates for integrable repulsive pair potentials are proved in [2, Lemma 3.4 in arXiv version 3]; their bound contains the integral of the pair potential, which is infinite for a hard core. Lemma 4.2 is the inequality form of Ueltschi's canonical cycle formula [143, Eq. (2.8) in arXiv version 2]. The Gaussian winding-sector weights of Lemma 4.3 appear for negative chemical potential in the proof of [143, Theorem 2 in arXiv version 2], which establishes the absence of infinite cycles and of off-diagonal long-range order; for the canonical free gas, cycle winding weights are computed in [74, equations (4)–(7) in the preprint version]. Sütő related cycle percolation to condensation in the ideal and mean-field gases [137, 138]. Seiringer and Ueltschi bounded the density in terms of the activity and deduced the decay of off-diagonal correlations above an explicit temperature close to T_c , in the grand-canonical ensemble on domains with Dirichlet boundary conditions [131]. The mechanism, suppression of long cycles by a fugacity below one, is that of the grand-canonical results of Ueltschi and of Seiringer and Ueltschi.

Throughout this chapter, we work on the torus Λ_L , write $V = L^3$, fix $a, \beta > 0$, and set $\lambda_\beta = (4\pi\beta)^{1/2}$. For $j \geq 1$ such that the domain $\mathcal{X}_{j,L}$ of (3.1) is nonempty, let $Z_j = \text{tr } e^{-\beta H_{j,L}}$ be the partition function of j particles at the same L, a , and β . We regard $\mathcal{X}_{0,L}$ as a point and set $Z_0 = 1$. We write q_j and $\mathcal{Q}_j$ for the form (3.2) and its domain for j particles, and H_j^{lab} for the labeled operator. Let $N \geq 1$ be such that $\mathcal{X}_{N,L}$ is nonempty, and let $\rho_L = N/V$ as in (3.25). We set $\epsilon_{\text{ex}} = 36\pi\rho_L a^3$ and, if $\epsilon_{\text{ex}} < 1$,

$$\mathfrak{z} = \frac{\rho_L \lambda_\beta^3}{1 - \epsilon_{\text{ex}}} \exp\left(\frac{152\pi}{3} \frac{\beta \rho_L a}{1 - \epsilon_{\text{ex}}}\right). \quad (4.1)$$

If $\mathfrak{z} < 1$, we set $\gamma_* = (\log(1/\mathfrak{z})/(2\beta))^{1/2}$. The Bose path measure $\mu = \mu_{N,L,\beta}$ and the winding vector W are those of Definition 3.7 and (3.18), and $\mathbb{E}, \mathbb{P}$, and Var refer to μ .

4.1 A canonical insertion bound

Lemma 4.1. *Let $j \geq 1$ be such that $\mathcal{X}_{j-1,L}$ is nonempty, and suppose that $\epsilon_j = 36\pi(j-1)a^3/V < 1$. Then $\mathcal{X}_{j,L}$ is nonempty, and*

$$Z_{j-1} \leq \frac{j\lambda_\beta^3}{V(1-\epsilon_j)} \exp\left(\frac{152\pi}{3} \frac{\beta(j-1)a}{(1-\epsilon_j)V}\right) Z_j. \quad (4.2)$$

Consequently, if $\epsilon_{\text{ex}} < 1$, then $Z_{N-m} \leq \mathfrak{z}^m Z_N$ for $0 \leq m \leq N$.

Proof. The consequence follows by applying (4.2) for $j = N - m + 1, \dots, N$; the domains $\mathcal{X}_{j-1,L}$ are nonempty because $\mathcal{X}_{N,L}$ is. The factor in front of Z_j in (4.2) is increasing in j/V , in $(j-1)/V$, and in ϵ_j . For $j \leq N$, we have $(j-1)/V \leq j/V \leq \rho_L$ and $\epsilon_j \leq \epsilon_{\text{ex}} < 1$, so this factor is at most $\mathfrak{z}$.

We prove (4.2). Write points of Λ_L^j as (X, y) , with $X = (x_1, \dots, x_{j-1})$ and $y = x_j$. For $j = 1$, the variable X is absent, $\epsilon_1 = 0$, and the orbital below is the constant $V^{-1/2}$; the steps below apply with these conventions.

Step 1: the orbital. Let $h(r) = \min\{1, (r-2a)_+/a\}$ for $r \geq 0$. It vanishes on $[0, 2a]$, equals one on $[3a, \infty)$, and is a^{-1} -Lipschitz. For $X \in \Lambda_L^{j-1}$ and $y \in \Lambda_L$, let

$$b(X, y) = V^{-1/2} \min_{i < j} h(d_L(y, x_i)), \quad \mathfrak{m}(X) = \int_{\Lambda_L} b(X, y)^2 dy, \quad \varphi_X(y) = \frac{b(X, y)}{\mathfrak{m}(X)^{1/2}}.$$

For $x \in \Lambda_L$, the map $z \mapsto x + z$ from the centered cube $C_L = [-L/2, L/2]^3$ onto Λ_L is a measure-preserving bijection, and $d_L(x + z, x) = |z|$ for $z \in C_L$. Hence the ball $\{y : d_L(y, x) < 3a\}$ and the shell $\{y : 2a < d_L(y, x) < 3a\}$ correspond to the intersections of C_L with the Euclidean ball and shell, and their volumes are at most $36\pi a^3$ and $\frac{4\pi}{3}(27-8)a^3 = \frac{76\pi}{3}a^3$, respectively. The function $b(X, \cdot)^2$ equals V^{-1} outside the union of the balls of radius $3a$ about the points x_i , and this union has volume at most $\epsilon_j V < V$. We obtain

$$1 \geq \mathfrak{m}(X) \geq 1 - \epsilon_j \quad \text{for every } X \in \Lambda_L^{j-1}. \quad (4.3)$$

For $X \in \mathcal{X}_{j-1,L}$ and y outside this union, we have $(X, y) \in \mathcal{X}_{j,L}$; hence $\mathcal{X}_{j,L}$ is nonempty. The function φ_X is a symmetric function of X , it vanishes if $d_L(y, x_i) \leq 2a$ for some i , and $0 \leq \varphi_X \leq (V(1-\epsilon_j))^{-1/2}$.

Step 2: the gradient of the orbital. Each function $f_i(X, y) = h(d_L(y, x_i))$ is Lipschitz. Since d_L is 1-Lipschitz in each argument, we have $|\nabla_y f_i|^2 + |\nabla_{x_i} f_i|^2 \leq 2a^{-2}$ almost everywhere, and $\nabla f_i = 0$ almost everywhere outside the shell $S_i = \{2a < d_L(y, x_i) < 3a\}$, because the gradient of a Lipschitz function vanishes almost everywhere on each of its level sets. For Lipschitz functions u and v , we have $\nabla u = \nabla v$ almost everywhere on $\{u = v\}$. Hence, almost everywhere, the gradient of $\min_i f_i$ equals ∇f_i for an index i attaining the minimum, and

$$|\nabla_y b|^2 + \sum_{i < j} |\nabla_{x_i} b|^2 \leq \frac{2}{Va^2} \sum_{i < j} \mathbf{1}_{S_i}(y).$$

The functions b and m are Lipschitz, and $m \geq 1 - \epsilon_j > 0$, so φ is bounded and Lipschitz on $\Lambda_L^{j-1} \times \Lambda_L$. If u is Lipschitz on $\Lambda_L^{j-1} \times \Lambda_L$, then $X \mapsto \int u(X, y) dy$ is Lipschitz, with gradient $\int \nabla_X u(X, y) dy$ almost everywhere. To see this, we pair with derivatives of a smooth function of X , integrate by parts in X for fixed y , and use Fubini's theorem; by Rademacher's theorem and Fubini's theorem, for almost every y the gradient of $u(\cdot, y)$ agrees almost everywhere with $\nabla_X u(\cdot, y)$. Since $\int \varphi_X^2 dy = 1$ for every X , applying this to $u = \varphi^2$ and $u = b^2$ gives, for almost every X ,

$$\int \varphi_X \nabla_{x_i} \varphi_X dy = 0, \quad \nabla_{x_i} m(X) = 2 \int b \nabla_{x_i} b dy.$$

Then $\nabla_{x_i} \varphi_X = m^{-1/2}(\nabla_{x_i} b - \varphi_X \int \varphi_X \nabla_{x_i} b dy)$. Thus $m^{1/2} \nabla_{x_i} \varphi_X$ is the component of $\nabla_{x_i} b$ orthogonal to the unit vector φ_X in $L^2(dy)$, and $\int |\nabla_{x_i} \varphi_X|^2 dy \leq m^{-1} \int |\nabla_{x_i} b|^2 dy$. Also $\nabla_y \varphi_X = m^{-1/2} \nabla_y b$. By the shell volume of Step 1 and (4.3), we obtain for almost every X

$$\begin{aligned} G(X) &:= \int \left(|\nabla_y \varphi_X|^2 + \sum_{i < j} |\nabla_{x_i} \varphi_X|^2 \right) dy \\ &\leq \frac{1}{1 - \epsilon_j} \cdot \frac{2(j-1)}{Va^2} \cdot \frac{76\pi}{3} a^3 = \frac{152\pi}{3} \frac{(j-1)a}{(1 - \epsilon_j)V} =: G_*. \end{aligned}$$

Step 3: trial vectors. Let $(E_k, f_k)_{k \geq 1}$ be an orthonormal eigenbasis of $H_{j-1,L}$ in $L^2_{\text{sym}}(\mathcal{X}_{j-1,L})$, which exists by Lemma 3.1; for $j = 1$, it consists of $f_1 = 1$ with $E_1 = 0$. For $p \in (2\pi/L)\mathbb{Z}^3$, let

$$\psi_{k,p}(X, y) = f_k(X) \varphi_X(y) e^{ip \cdot y}, \quad (X, y) \in \mathcal{X}_{j,L}.$$

Let $P' = ((j-1)!)^{-1} \sum_{\sigma \in S_{j-1}} U_\sigma$, where $(U_\sigma \psi)(X, y) = \psi(\sigma X, y)$, be the orthogonal projection of $L^2(\mathcal{X}_{j,L})$ onto the functions that are symmetric in X . It commutes with H_j^{lab} . Let H_{dist} be the restriction of H_j^{lab} to $\text{ran } P'$; its form domain is $H_0^1(\mathcal{X}_{j,L}) \cap \text{ran } P'$, and its form is given by (3.2), which we again denote by q_j . We show that $\psi_{k,p}$ lies in this form domain. If $f \in C_c^\infty(\mathcal{X}_{j-1,L})$, then $f(X) \varphi_X(y) e^{ip \cdot y}$ is Lipschitz on Λ_L^j and supported in $\{(X, y) : X \in \text{supp } f, d_L(y, x_i) \geq 2a \text{ for all } i\}$. This set is compact and contained in $\mathcal{X}_{j,L}$, since $2a > a$; by mollification, the function lies in $H_0^1(\mathcal{X}_{j,L})$. For $f \in \mathcal{Q}_{j-1}$, we approximate f in H^1 by such functions. Multiplication by the bounded Lipschitz function $\varphi_X(y) e^{ip \cdot y}$ is bounded in H^1 , and $H_0^1(\mathcal{X}_{j,L})$ is closed. Since f_k and φ_X are symmetric in X , this proves the claim for every p . For $X \in \mathcal{X}_{j-1,L}$, the function φ_X vanishes at every y with $(X, y) \notin \mathcal{X}_{j,L}$, so $\|\psi_{k,p}\| = 1$ since $\int \varphi_X^2 dy = 1$. Almost everywhere,

$$\nabla_{x_i} \psi_{k,p} = (\varphi_X \nabla_i f_k + f_k \nabla_{x_i} \varphi_X) e^{ip \cdot y}, \quad \nabla_y \psi_{k,p} = f_k (\nabla_y \varphi_X + ip \varphi_X) e^{ip \cdot y}.$$

Since φ_X is real, $|\nabla_y \psi_{k,p}|^2 = |f_k|^2 (|\nabla_y \varphi_X|^2 + |p|^2 \varphi_X^2)$. We expand $|\nabla_{x_i} \psi_{k,p}|^2$ and integrate first in y . By Step 2, the cross terms vanish, and

$$q_j[\psi_{k,p}] = q_{j-1}[f_k] + |p|^2 + \int |f_k(X)|^2 G(X) dX \leq E_k + |p|^2 + G_*. \quad (4.4)$$

Step 4: a frame bound for all frequencies. Let $u \in \text{ran } P'$, extended by zero to Λ_L^j , and let $g_p(X) = \int \varphi_X(y) e^{-ip \cdot y} u(X, y) dy$. Then g_p is symmetric, it vanishes outside $\mathcal{X}_{j-1, L}$, and $|g_p(X)|^2 \leq \int |u(X, y)|^2 dy$. Hence $g_p \in L_{\text{sym}}^2(\mathcal{X}_{j-1, L})$, and $\langle \psi_{k,p}, u \rangle = \langle f_k, g_p \rangle$. Parseval's identity for the basis (f_k) , Parseval's identity in $L^2(\Lambda_L)$ for the orthonormal basis $(V^{-1/2} e^{ip \cdot y})_p$, Tonelli's theorem, and the bound on φ_X in Step 1 give

$$\sum_{k,p} |\langle \psi_{k,p}, u \rangle|^2 = \int \sum_p |g_p(X)|^2 dX = V \iint \varphi_X(y)^2 |u(X, y)|^2 dy dX \leq \frac{\|u\|^2}{1 - \epsilon_j}. \quad (4.5)$$

Step 5: partition functions. Let $Z_{\text{dist}} = \text{tr } e^{-\beta H_{\text{dist}}}$, and let (e_n) be an orthonormal basis of $\text{ran } P'$. Applying (4.5) to $u = e^{-\beta H_{\text{dist}}/2} e_n$ and summing over n , we obtain by Tonelli's theorem

$$\sum_{k,p} \langle \psi_{k,p}, e^{-\beta H_{\text{dist}}} \psi_{k,p} \rangle = \sum_n \sum_{k,p} |\langle \psi_{k,p}, e^{-\beta H_{\text{dist}}/2} e_n \rangle|^2 \leq \frac{Z_{\text{dist}}}{1 - \epsilon_j}.$$

For a unit vector ψ in the form domain of H_{dist} , Jensen's inequality for the spectral measure of ψ gives $\langle \psi, e^{-\beta H_{\text{dist}}} \psi \rangle \geq e^{-\beta q_j[\psi]}$. We apply this to each $\psi_{k,p}$ and use (4.4) and the Poisson summation formula. This gives

$$\frac{Z_{\text{dist}}}{1 - \epsilon_j} \geq e^{-\beta G_*} \sum_k e^{-\beta E_k} \sum_p e^{-\beta |p|^2} = e^{-\beta G_*} Z_{j-1} \frac{V}{\lambda_\beta^3} \sum_{n \in \mathbb{Z}^3} e^{-L^2 |n|^2 / (4\beta)} \geq e^{-\beta G_*} Z_{j-1} \frac{V}{\lambda_\beta^3}.$$

It remains to compare Z_{dist} with Z_j . The proof of Lemma 3.6(3) applies with P_{sym} replaced by P' . By Proposition 3.5(1), the operator $e^{-\beta H_j^{\text{lab}}/2} P'$ has the kernel $((j-1)!)^{-1} \sum_\sigma K_{\beta/2}((X, y), (\sigma X', y'))$. It is Hilbert-Schmidt by the min-max argument in the proof of Lemma 3.1, which does not use the symmetry. Then Lemma 3.6(1) and (2) give

$$Z_{\text{dist}} = \frac{1}{(j-1)!} \sum_{\sigma \in S_{j-1}} \int_{\mathcal{X}_{j,L}} K_\beta((X, y), (\sigma X, y)) dX dy.$$

In the formula of Lemma 3.6(3) for Z_j , the kernel K_β is nonnegative by Lemma 3.6(1). Retaining only the $(j-1)!$ permutations that fix the label j , we obtain $Z_j \geq Z_{\text{dist}}/j$. Together with the previous display, this proves (4.2). $\square$

4.2 Cycles and winding sectors

Let $(\pi, X(\cdot)) \in S_N \times C([0, \beta]; \Lambda_L^N)$ satisfy $x_i(\beta) = x_{\pi(i)}(0)$ for every i ; this holds μ -almost surely. We write $\text{cyc}(\pi)$ for the set of cycles of π . For a cycle $c = (i_1, \dots, i_m)$ of π with $i_1 = \min c$ and $\pi(i_l) = i_{l+1}$, where $i_{m+1} = i_1$, we write $|c| = m$, and we define the worldline $\omega_c \in C([0, m\beta]; \Lambda_L)$ by $\omega_c(s) = x_{i_l}(s - (l-1)\beta)$ for $s \in [(l-1)\beta, l\beta]$. It is a closed loop, and we set $w_c = d(\omega_c)/L$.

Lemma 4.2. *For each $m \geq 1$, let $F_m: C([0, m\beta]; \Lambda_L) \rightarrow [0, \infty]$ be measurable. Then*

$$\mathbb{E} \sum_{c \in \text{cyc}(\pi)} F_{|c|}(\omega_c) \leq \sum_{m=1}^N \frac{Z_{N-m}}{m Z_N} \int_{\Lambda_L} \int F_m dW_{x,x}^{m\beta} dx.$$

Proof. By (3.17),

$$Z_N \mathbb{E} \sum_{c \in \text{cyc}(\pi)} F_{|c|}(\omega_c) = \frac{1}{N!} \sum_{\pi \in S_N} \sum_{c \in \text{cyc}(\pi)} \int_{\Lambda_L^N} \int \mathbf{1}_{\mathcal{E}_\beta} F_{|c|}(\omega_c) dW_{X_0, \pi X_0}^\beta dX_0,$$

where we may integrate over Λ_L^N because $\mathbf{1}_{\mathcal{E}_\beta}$ vanishes unless $X_0 = X(0) \in \mathcal{X}_{N,L}$. The pairs (π, c) with $|c| = m$ correspond bijectively to the triples (S, c, π') , where $S = \text{supp } c$ has m elements, c is a cyclic permutation of S , and π' is a permutation of the complement S^c ; there are $\binom{N}{m}(m-1)!$ choices of (S, c) . Fix such a triple. We bound $\mathbf{1}_{\mathcal{E}_\beta}$ by the indicator of the event that the paths with labels in S^c stay in $\mathcal{X}_{N-m,L}$; this drops every exclusion that involves a label in S . The measure $W_{X_0, \pi X_0}^\beta$ is a product over the particles, and by Tonelli's theorem the integral is at most the product of two factors. The first is the integral of $F_m(\omega_c)$ over the initial positions and paths with labels in S . By $m-1$ applications of (3.12) for one particle, it equals $\int_{\Lambda_L} \int F_m dW_{x,x}^{m\beta} dx$. The second factor, summed over π' , equals $(N-m)! Z_{N-m}$ by Lemma 3.6(3) for $N-m$ particles, and it equals $1 = 0! Z_0$ if $m = N$. Since $\binom{N}{m}(m-1)!(N-m)!/N! = 1/m$, the lemma follows. $\square$

Lemma 4.3.

(i) *For $t > 0$ and $k \in \mathbb{Z}^3$, we have $\int_{\Lambda_L} W_{x,x}^t(\{\omega : d(\omega) = Lk\}) dx = V p_t(Lk)$.*

(ii) *For every $c \in \text{cyc}(\pi)$, we have $w_c \in \mathbb{Z}^3$, and $W = \sum_{c \in \text{cyc}(\pi)} w_c$.*

Proof. By the definition of the torus bridge measure in Section 3.2, $W_{x,x}^t$ is the sum over $k' \in \mathbb{Z}^3$ of the images of $W_{\tilde{x}, \tilde{x}+Lk'}^t$, and the k' -th term is carried by $\{d(\omega) = Lk'\}$ and has mass $p_t(Lk')$. Integrating over x , we obtain (i). The lifted displacement is additive under concatenation, so $d(\omega_c) = \sum_{i \in c} d(x_i(\cdot))$, and $d(\omega_c) \in L\mathbb{Z}^3$ because ω_c is closed. Summing over the cycles and using (3.18), we obtain $LW = \sum_i d(x_i(\cdot)) = \sum_c d(\omega_c)$, which proves (ii). $\square$

4.3 The normal-phase estimates

Lemma 4.4. *Let $N \geq 1$ and $L, a, \beta > 0$ be such that $\mathcal{X}_{N,L}$ is nonempty. Then*

$$\text{Var}(W_1) \leq \frac{8\beta N}{L^2} \mathbb{P}(W_1 \neq 0)^{1/2}, \quad \text{equivalently} \quad f_s(N, L, T) \leq 4 \mathbb{P}(W_1 \neq 0)^{1/2}.$$

Proof. Let $\sigma^2 = 2\beta N/L^2$ as in (3.25). By Theorem 3.10, we have $\mathbb{E}W_1 = 0$, and by Corollary 3.11, we have $\mathbb{E}W_1^4 \leq 16\sigma^4$. By the Cauchy–Schwarz inequality,

$$\text{Var}(W_1) = \mathbb{E}[W_1^2 \mathbf{1}_{\{W_1 \neq 0\}}] \leq (\mathbb{E}W_1^4)^{1/2} \mathbb{P}(W_1 \neq 0)^{1/2} \leq 4\sigma^2 \mathbb{P}(W_1 \neq 0)^{1/2}.$$

By (3.32), $f_s(N, L, T) = \text{Var}(W_1)/\sigma^2$, which gives the equivalent form. $\square$

Theorem 4.5. *Let $N \geq 1$ and $L, a, \beta > 0$ be such that $\mathcal{X}_{N,L}$ is nonempty, and suppose that $\epsilon_{\text{ex}} < 1$, $\mathfrak{z} < 1$, and $\gamma_* L \geq 1$. Then the following hold.*

(a) *We have*

$$\mathbb{P}(W \neq 0) \leq \mathbb{E} \#\{c \in \text{cyc}(\pi) : w_c \neq 0\} \leq 190 \left(\frac{L}{\lambda_\beta}\right)^3 e^{-\gamma_* L} =: P_*.$$

(b) *We have $0 \leq f_s(N, L, T) \leq 4 \mathbb{P}(W_1 \neq 0)^{1/2} \leq 4P_*^{1/2}$ and $\text{Var}(W_1) \leq (8\beta N/L^2)P_*^{1/2}$.*

Proof. If $W \neq 0$, then $w_c \neq 0$ for some cycle c by Lemma 4.3(ii), which proves the first inequality in (a). We apply Lemma 4.2 with $F_m(\omega) = \mathbf{1}_{\{d(\omega) \neq 0\}}$, and then Lemma 4.3(i) and Lemma 4.1. Since $Vp_{m\beta}(Lk) = (L/\lambda_\beta)^3 m^{-3/2} e^{-L^2|k|^2/(4m\beta)}$ and all terms are nonnegative, we obtain

$$\mathbb{E} \#\{c : w_c \neq 0\} \leq \sum_{m=1}^N \frac{\mathfrak{z}^m}{m} \sum_{k \neq 0} Vp_{m\beta}(Lk) \leq \left(\frac{L}{\lambda_\beta}\right)^3 \sum_{m \geq 1} \frac{\mathfrak{z}^m}{m^{5/2}} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} e^{-L^2|k|^2/(4m\beta)}.$$

By the inequality between the arithmetic and geometric means,

$$\frac{m}{2} \log \frac{1}{\mathfrak{z}} + \frac{L^2|k|^2}{4m\beta} \geq L|k| \left(\frac{\log(1/\mathfrak{z})}{2\beta}\right)^{1/2} = \gamma_* L|k|,$$

and therefore $\mathfrak{z}^m e^{-L^2|k|^2/(4m\beta)} \leq \mathfrak{z}^{m/2} e^{-\gamma_* L|k|}$. Now $\sum_{m \geq 1} m^{-5/2} \mathfrak{z}^{m/2} \leq \zeta(5/2) < 1.342$. There are $24n^2 + 2 \leq 26n^2$ points $k \in \mathbb{Z}^3$ with $|k|_\infty = n \geq 1$, and $|k| \geq |k|_\infty$. Since $\sum_{n \geq 1} n^2 q^n = q(1+q)/(1-q)^3$ for $q = e^{-\gamma_* L} \leq e^{-1}$, we have

$$\sum_{k \neq 0} e^{-\gamma_* L|k|} \leq 26 \frac{1 + e^{-1}}{(1 - e^{-1})^3} e^{-\gamma_* L} \leq 140.9 e^{-\gamma_* L}.$$

Since $1.342 \cdot 140.9 < 190$, this proves (a). By Proposition 3.14, we have $f_s(N, L, T) \geq 0$. The remaining bounds in (b) follow from Lemma 4.4, the inequality $\mathbb{P}(W_1 \neq 0) \leq \mathbb{P}(W \neq 0)$, and (a). $\square$

Proof of Theorem 1.5. Let $\varepsilon \in (0, 1]$ and $a, \rho > 0$ with $g = \rho a^3 \leq (\varepsilon/100)^3$, and set $T_{\text{hi}} = (1 + \varepsilon)4\pi\rho^{2/3}$ and $\eta = \log(1 + \varepsilon)$. Consider a sequence of N and L with $L \rightarrow \infty$ and $\rho_L = N/L^3 \rightarrow \rho$. Choose L_0 such that, for all $L \geq L_0$ in the sequence,

$$\rho_L \leq (1 + \varepsilon)^{1/2} \rho \quad \text{and} \quad L\sqrt{T_{\text{hi}}} \geq 6\eta^{-1/2}.$$

Neither condition depends on T . Let $L \geq L_0$ and $T \geq T_{\text{hi}}$. Since $\rho(4\pi/T_{\text{hi}})^{3/2} = (1 + \varepsilon)^{-3/2}$, we have

$$\rho_L \lambda_\beta^3 \leq (1 + \varepsilon)^{-1}, \quad 36\pi\rho_L a^3 \leq 36\pi\sqrt{2}g < 160g \leq 1.6 \cdot 10^{-4}\varepsilon^3.$$

In particular $\epsilon_j \leq 36\pi\rho_L a^3 < 1$ for $1 \leq j \leq N$, and Lemma 4.1, applied for $j = 1, \dots, N$, shows that $\mathcal{X}_{N,L}$ is nonempty. Since $\beta\rho_L a = (\rho_L \lambda_\beta^3)^{2/3} (\rho_L a^3)^{1/3} / (4\pi) \leq 2^{1/6} g^{1/3} / (4\pi)$, the exponent in (4.1) satisfies

$$\frac{152\pi}{3} \frac{\beta\rho_L a}{1 - \epsilon_{\text{ex}}} \leq \frac{38}{3} \frac{2^{1/6}}{1 - 1.6 \cdot 10^{-4}} g^{1/3} \leq 14.3 g^{1/3} \leq 0.143 \epsilon.$$

We use $(1 - \epsilon_{\text{ex}})^{-1} \leq 1 + 0.00017\epsilon$, $e^{0.143\epsilon} \leq 1 + 0.143e^{0.143}\epsilon \leq 1 + 0.165\epsilon$, and $(1 + \epsilon)^{1/2} \geq 1 + (\sqrt{2} - 1)\epsilon$ for $\epsilon \in [0, 1]$. We obtain

$$\mathfrak{z} \leq (1 + \epsilon)^{-1} (1 + 0.00017\epsilon) (1 + 0.165\epsilon) \leq (1 + \epsilon)^{-1} (1 + 0.17\epsilon) \leq (1 + \epsilon)^{-1/2}.$$

Hence $\mathfrak{z} < 1$, $\gamma_* \geq \frac{1}{2}(T\eta)^{1/2}$, and $\gamma_* L \geq \frac{1}{2}L(T_{\text{hi}}\eta)^{1/2} \geq 3$. Since $(L/\lambda_\beta)^3 = (L^2 T / 4\pi)^{3/2}$ and $4\sqrt{190} < 57$, Theorem 4.5 gives (1.15) and the bound on $\mathbb{P}(W \neq 0)$, for all $L \geq L_0$ in the sequence and all $T \geq T_{\text{hi}}$.

For the uniformity, let $b = \frac{1}{2}\eta^{1/2}$ and $s = L\sqrt{T}$. The bounds just proved are $57(4\pi)^{-3/4} s^{3/2} e^{-bs/2}$ for f_s and $190(4\pi)^{-3/2} s^3 e^{-bs}$ for $\mathbb{P}(W \neq 0)$. Both decrease in s for $s \geq 3/b$, and $L\sqrt{T} \geq L\sqrt{T_{\text{hi}}} \geq 3/b$ for $L \geq L_0$. Therefore their suprema over $T \geq T_{\text{hi}}$ are attained at $T = T_{\text{hi}}$, and these tend to zero as $L \rightarrow \infty$. By (3.32),

$$\text{Var}(W_1) = \frac{2\rho_L L}{T} f_s(N, L, T) \leq 114(4\pi)^{-3/4} \rho_L L^{5/2} T^{-1/4} e^{-bL\sqrt{T}/2},$$

and the right side decreases in T . At $T = T_{\text{hi}}$, it tends to zero as $L \rightarrow \infty$, since ρ_L is bounded. This proves the uniform convergence. $\square$

PART II

Surgery in one slab

A variance inequality for surgeries

We first complete accepted moves by the identity and prove the Hammersley–Chapman–Robbins variance inequality. We then combine it with conditional independence of the time slabs. In Proposition 5.6, we reduce each response bound to the mean gain and chi-square divergence of a kernel acting on one slab.

5.1 Completed kernels and the chi-square divergence

Throughout this section, $(X, \mathcal{F})$ is a standard Borel space and μ is a probability measure on X . A subprobability kernel on X is a map $q: X \times \mathcal{F} \rightarrow [0, 1]$ such that $q(x, \cdot)$ is a measure of total mass at most one for every $x \in X$, and $x \mapsto q(x, B)$ is measurable for every $B \in \mathcal{F}$. It is a probability kernel if every $q(x, \cdot)$ is a probability measure. We write μq for the measure $B \mapsto \int q(x, B) \mu(dx)$, and $qf(x) = \int f(y) q(x, dy)$ for measurable $f \geq 0$. For probability measures ν and μ on X , the chi-square divergence of ν from μ is

$$\chi^2(\nu \parallel \mu) = \int \left(\frac{d\nu}{d\mu} - 1 \right)^2 d\mu$$

if ν is absolutely continuous with respect to μ , and $\chi^2(\nu \parallel \mu) = \infty$ otherwise. We write $\langle f, g \rangle_\mu = \int fg d\mu$.

Definition 5.1. Let q be a subprobability kernel on X , and set $\bar{q}(x) = q(x, X)$. The completion of q is the probability kernel

$$Q(x, dy) = q(x, dy) + (1 - \bar{q}(x))\delta_x(dy).$$

We call $\bar{q}$ the outgoing mass of q . If μq is absolutely continuous with respect to μ , we call its density $c = d(\mu q)/d\mu$ the incoming density of q .

A sample of μQ is obtained from a sample x of μ by moving according to $q(x, \cdot)$ with probability $\bar{q}(x)$, and by leaving x unchanged otherwise. In our application, q describes the accepted surgeries, $\bar{q}(x)$ is the probability that a surgery is accepted at x , and a rejected surgery leaves the configuration unchanged. The kernel q is not normalized by its acceptance probability.

Lemma 5.2. Let q be a subprobability kernel on X with outgoing mass $\bar{q}$ and completion Q , and suppose that μq is absolutely continuous with respect to μ , with incoming density c .

1. The measure μQ is absolutely continuous with respect to μ , and

$$L = \frac{d(\mu Q)}{d\mu} = 1 - \bar{q} + c, \quad \chi^2(\mu Q \parallel \mu) = \int (c - \bar{q})^2 d\mu. \quad (5.1)$$

2. Let $O \in L^2(\mu)$, and suppose that $L \in L^2(\mu)$. Then all integrals in (5.2) converge absolutely,

$$\mathbb{E}_{\mu Q} O - \mathbb{E}_{\mu} O = \iint (O(y) - O(x)) q(x, dy) \mu(dx) = \int (O - \mathbb{E}_{\mu} O)(L - 1) d\mu, \quad (5.2)$$

and

$$(\mathbb{E}_{\mu Q} O - \mathbb{E}_{\mu} O)^2 \leq \text{Var}_{\mu}(O) \chi^2(\mu Q \parallel \mu). \quad (5.3)$$

In particular, if $\chi^2(\mu Q \parallel \mu) = 0$, then $\mathbb{E}_{\mu Q} O = \mathbb{E}_{\mu} O$.

Proof. For $B \in \mathcal{F}$, we have

$$\mu Q(B) = \mu q(B) + \int_B (1 - \bar{q}) d\mu = \int_B (c + 1 - \bar{q}) d\mu,$$

which proves the first identity in (5.1). The second follows from $L - 1 = c - \bar{q}$.

For part (2), we note that $0 \leq c \leq L$, since $0 \leq \bar{q} \leq 1$ and $c \geq 0$; in particular, $c \in L^2(\mu)$. By the Cauchy–Schwarz inequality, we have

$$\iint |O(y)| q(x, dy) \mu(dx) = \int |O| c d\mu \leq \|O\|_{L^2(\mu)} \|c\|_{L^2(\mu)} < \infty,$$

and $\iint |O(x)| q(x, dy) \mu(dx) = \int |O| \bar{q} d\mu \leq \|O\|_{L^1(\mu)} < \infty$. In the same way, $\int |O| d(\mu Q) = \int |O| L d\mu < \infty$. Then the double integral in (5.2) converges absolutely, and it equals $\int O(c - \bar{q}) d\mu = \int O(L - 1) d\mu$, which is also $\mathbb{E}_{\mu Q} O - \mathbb{E}_{\mu} O$. Since $\int (L - 1) d\mu = 0$, we may replace O by $O - \mathbb{E}_{\mu} O$ in the last integral, which proves (5.2). The Cauchy–Schwarz inequality applied to the last expression in (5.2) implies (5.3). $\square$

Used in: Lemma 5.5, Section 7, and the gain computations in Sections 10 and 11.

Inequality (5.3) is the Cauchy–Schwarz inequality for the functions $O - \mathbb{E}_{\mu} O$ and $L - 1$ in $L^2(\mu)$. In this form, which compares the means of one observable under two probability measures, it is due to Hammersley [69, p. 197, equation (31)] and to Chapman and Robbins [34, Section 2, equations (3)–(5)], who used it to bound the variance of an estimator without regularity assumptions; see also Barankin [9, Theorem 4]. The middle expression in (5.2) shows that the gain is computed from the accepted moves alone: a rejected move does not change O . The integrability assumption $L \in L^2(\mu)$ is needed; for a general positive measure on pairs of states, the gain can fail to be defined even when both marginals have the same density.

The kernels of Section 7 are averages over an auxiliary random parameter, the skeleton of a route, which is sampled independently of the state. The next lemma expresses the divergence of such an average through two independent copies of the parameter.

Lemma 5.3. *Let $(\mathfrak{G}, \mathcal{G})$ be a standard Borel space with a probability measure $\mathfrak{p}$. For every $\Gamma \in \mathfrak{G}$, let Q_Γ be a probability kernel on X , such that $(\Gamma, x) \mapsto Q_\Gamma(x, B)$ is measurable for every $B \in \mathcal{F}$. Suppose that, for $\mathfrak{p}$ -almost every Γ , the measure μQ_Γ is absolutely continuous with respect to μ , with a nonnegative density $L_\Gamma(x)$ that is jointly measurable in (Γ, x) . Let $Q = \int Q_\Gamma \mathfrak{p}(\mathrm{d}\Gamma)$. Then μQ is absolutely continuous with respect to μ , with density $L = \int L_\Gamma \mathfrak{p}(\mathrm{d}\Gamma)$, and*

$$1 + \chi^2(\mu Q \parallel \mu) = \iint \langle L_\Gamma, L_{\Gamma'} \rangle_\mu \mathfrak{p}(\mathrm{d}\Gamma) \mathfrak{p}(\mathrm{d}\Gamma'), \quad (5.4)$$

where both sides may be infinite. If moreover $\int \|L_\Gamma\|_{L^2(\mu)}^2 \mathfrak{p}(\mathrm{d}\Gamma) < \infty$, then

$$\chi^2(\mu Q \parallel \mu) = \iint \langle L_\Gamma - 1, L_{\Gamma'} - 1 \rangle_\mu \mathfrak{p}(\mathrm{d}\Gamma) \mathfrak{p}(\mathrm{d}\Gamma'), \quad (5.5)$$

and the integrand in (5.5) is integrable with respect to $\mathfrak{p} \otimes \mathfrak{p}$.

Proof. For $B \in \mathcal{F}$, we have by Tonelli's theorem

$$\mu Q(B) = \int \mu Q_\Gamma(B) \mathfrak{p}(\mathrm{d}\Gamma) = \iint_B L_\Gamma \mathrm{d}\mu \mathfrak{p}(\mathrm{d}\Gamma) = \int_B L \mathrm{d}\mu,$$

which shows that L is the density of μQ . Another application of Tonelli's theorem yields

$$\int L^2 \mathrm{d}\mu = \int \left(\int L_\Gamma \mathfrak{p}(\mathrm{d}\Gamma) \right) \left(\int L_{\Gamma'} \mathfrak{p}(\mathrm{d}\Gamma') \right) \mathrm{d}\mu = \iint \langle L_\Gamma, L_{\Gamma'} \rangle_\mu \mathfrak{p}(\mathrm{d}\Gamma) \mathfrak{p}(\mathrm{d}\Gamma').$$

Here we used the fact that the integrands are nonnegative. If $L \in L^2(\mu)$, then $\chi^2(\mu Q \parallel \mu) = \int L^2 \mathrm{d}\mu - 1$, because $\int L \mathrm{d}\mu = 1$. If $L \notin L^2(\mu)$, both sides of (5.4) are infinite. This proves (5.4).

Under the additional assumption, by the Cauchy–Schwarz inequality, we have $|\langle L_\Gamma - 1, L_{\Gamma'} - 1 \rangle_\mu| \leq (1 + \|L_\Gamma\|_{L^2(\mu)})(1 + \|L_{\Gamma'}\|_{L^2(\mu)})$, and the right side is integrable with respect to $\mathfrak{p} \otimes \mathfrak{p}$. Since $\int L_\Gamma \mathrm{d}\mu = 1$ for $\mathfrak{p}$ -almost every Γ , we have $\langle L_\Gamma - 1, L_{\Gamma'} - 1 \rangle_\mu = \langle L_\Gamma, L_{\Gamma'} \rangle_\mu - 1$. Integrating this identity and using (5.4) proves (5.5). $\square$

Used in: Theorem 7.14.

Identity (5.4) is the elementary independent-copy expansion of the second moment of an averaged likelihood. Mixture likelihoods in minimax hypothesis testing are discussed in [77, Section 1.1, equations (1.1)–(1.4) in the preprint version]. The independent-copy expansion in the Gaussian model is given in [78, Section 3.3.1, equation (3.66), pp. 97–98]; see also [76] for related minimax testing background. Explicit forms of the independent-path expansion appear in [7, Section 2.1.2, equation (2.6) in arXiv version 2], [20, proof of Lemma 2.4, equation (2.8) in arXiv version 1], and [121, proof of Proposition 2.1, equation (2.3) in arXiv version 1]. Its Bose-gas applications in [111, Section 9] and [112, Section 9] average over random skeletons, as we do here. It is important that the parameter is averaged out in Q . If Γ were kept as a coordinate of the state, the divergence

would be $\iint (L_\Gamma - 1)^2 d\mu p(d\Gamma)$, which corresponds to two copies of the same route, and the decay that comes from the rarity of encounters between two independent routes would be lost.

The kernels of Section 7 also preserve part of the state, namely the exposed data of a slab, and the slab kernels of Section 5.2 preserve the cut configurations. The next lemma shows that such kernels can be analyzed on each fiber.

Lemma 5.4. *Let $(Y, \mathcal{Y})$ be a standard Borel space, let $\Phi: X \rightarrow Y$ be measurable, write μ_Φ for the image of μ under Φ , and let $(\mu_y)_{y \in Y}$ be a disintegration of μ along Φ : a probability kernel from Y to X such that $\mu_y(\Phi^{-1}(y)) = 1$ for μ_Φ -almost every y and $\mu = \int \mu_y \mu_\Phi(dy)$. Let Q be a probability kernel on X such that, for μ -almost every x , we have $\Phi(x') = \Phi(x)$ for $Q(x, \cdot)$ -almost every x' . Suppose that μQ is absolutely continuous with respect to μ , with density L . Then, for μ_Φ -almost every y , the measure $\mu_y Q$ is absolutely continuous with respect to μ_y with density L , and $\int L d\mu_y = 1$. Moreover,*

$$\chi^2(\mu Q \parallel \mu) = \int \chi^2(\mu_y Q \parallel \mu_y) \mu_\Phi(dy). \quad (5.6)$$

Proof. Let f be a bounded measurable function on X , and let φ be a bounded measurable function on Y . Since Q preserves Φ , we have

$$\begin{aligned} \int \varphi(\Phi(x)) f(x) L(x) \mu(dx) &= \iint \varphi(\Phi(x')) f(x') Q(x, dx') \mu(dx) \\ &= \int \varphi(\Phi(x)) Qf(x) \mu(dx). \end{aligned}$$

Disintegrating both sides along Φ gives

$$\int \varphi(y) \left(\int f L d\mu_y \right) \mu_\Phi(dy) = \int \varphi(y) \left(\int Qf d\mu_y \right) \mu_\Phi(dy).$$

Since φ is arbitrary, $\int f L d\mu_y = \int f d(\mu_y Q)$ for μ_Φ -almost every y , for each fixed f . Since X is standard Borel, $\mathcal{F}$ is generated by a countable collection of sets that is closed under finite intersections and contains X . Applying the last identity to the indicators of these sets, we conclude that the finite measures $L\mu_y$ and $\mu_y Q$ agree on this collection for μ_Φ -almost every y , and they are equal. Taking $f = 1$ yields $\int L d\mu_y = 1$. Finally, (5.6) follows from $\int (L - 1)^2 d\mu = \int (\int (L - 1)^2 d\mu_y) \mu_\Phi(dy)$. $\square$

Used in: Proposition 5.6.

5.2 Reduction to one slab

The next lemma combines the law of total variance with Lemma 5.2. It applies to an observable that is a sum of contributions of conditionally independent pieces, and to a kernel that changes one piece at a time.

Lemma 5.5. *Let $C, S_1, \dots, S_m, R$ be random elements of standard Borel spaces, defined on a probability space with probability measure $\mathbb{P}$. Suppose that, conditionally on C , the variables $S_1, \dots, S_m, R$ are independent, and let*

$$O = \sum_{j=1}^m O_j(C, S_j) + O_R(C, R), \quad O_1(C, S_1), \dots, O_m(C, S_m), O_R(C, R) \in L^2(\mathbb{P}).$$

Let $\nu_{j,c}$ be a regular conditional law of S_j given $C = c$. For each j , let $Q_{j,c}$ be a probability kernel on the space of S_j such that $(c, s) \mapsto Q_{j,c}(s, B)$ is jointly measurable for every measurable set B , and suppose that, for almost every value c of C , the measure $\nu_{j,c} Q_{j,c}$ is absolutely continuous with respect to $\nu_{j,c}$, with a density $L_j(c, \cdot)$ that is jointly measurable in its two arguments. Set

$$h_j = \mathbb{E} \int (L_j(C, s) - 1)^2 \nu_{j,C}(ds), \quad g_j = \mathbb{E} \left[\int O_j(C, s') Q_{j,C}(S_j, ds') - O_j(C, S_j) \right],$$

and suppose that $h_j < \infty$ for every j . Then g_j is well defined, $h_j = 0$ implies $g_j = 0$, and, with the convention $0/0 = 0$,

$$\text{Var}(O) \geq \sum_{j=1}^m \mathbb{E} \text{Var}(O_j | C) \geq \sum_{j=1}^m \frac{g_j^2}{h_j}. \quad (5.7)$$

In particular, if $g_j \geq g > 0$ and $h_j \leq h$ for every j , then $\text{Var}(O) \geq mg^2/h$.

Proof. By the law of total variance,

$$\text{Var}(O) = \mathbb{E} \text{Var}(O | C) + \text{Var}(\mathbb{E}[O | C]) \geq \mathbb{E} \text{Var}(O | C).$$

Conditionally on C , the variables $O_1(C, S_1), \dots, O_m(C, S_m), O_R(C, R)$ are independent and have finite second moments almost surely. Then $\text{Var}(O | C) = \sum_j \text{Var}(O_j | C) + \text{Var}(O_R | C) \geq \sum_j \text{Var}(O_j | C)$, which proves the first inequality in (5.7).

Fix j , and write $\bar{O}_j(c) = \int O_j(c, s) \nu_{j,c}(ds)$ for the conditional mean of O_j . Since $\int L_j(c, s) \nu_{j,c}(ds) = 1$ for almost every c , we have $\mathbb{E} \int L_j(C, s)^2 \nu_{j,C}(ds) = 1 + h_j < \infty$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \mathbb{E} \iint |O_j(C, s')| Q_{j,C}(s, ds') \nu_{j,C}(ds) &= \mathbb{E} \int |O_j(C, s)| L_j(C, s) \nu_{j,C}(ds) \\ &\leq \|O_j\|_{L^2(\mathbb{P})} (1 + h_j)^{1/2}, \end{aligned}$$

so g_j is well defined. Conditioning on C , we obtain

$$g_j = \mathbb{E} \int O_j(C, s) (L_j(C, s) - 1) \nu_{j,C}(ds) = \mathbb{E} \int (O_j(C, s) - \bar{O}_j(C)) (L_j(C, s) - 1) \nu_{j,C}(ds),$$

where in the second equality we used $\int (L_j(c, s) - 1) \nu_{j,c}(ds) = 0$. The Cauchy–Schwarz inequality applied to the joint law of C and s gives $g_j^2 \leq h_j \mathbb{E} \text{Var}(O_j | C)$. If $h_j = 0$, this forces $g_j = 0$. This proves the second inequality in (5.7). $\square$

Used in: Proposition 5.6.

We record two comments on Lemma 5.5. First, extend $Q_{j,c}$ to a kernel Q_j on the joint state by keeping C , the variables S_k with $k \neq j$, and R fixed. Since S_j is conditionally independent of the other variables given C , the density of $\mathbb{P}Q_j$ with respect to $\mathbb{P}$ is $L_j(C, S_j)$, and we have $h_j = \chi^2(\mathbb{P}Q_j \| \mathbb{P})$. Second, g_j and h_j are averages over C ; neither needs to be bounded on every fiber of C , and the conditional gain may even be negative on some fibers. This matters for us, because the thermal estimates of Section 8 are statements about the unconditional Gibbs law. On the other hand, conditioning on C must not fix the variable that the kernel changes. In our application, C consists of the cut configurations, and conditioning on them does not fix the endpoint matchings of the slabs, which the surgeries of Section 7 change.

We now apply Lemma 5.5 to the slab decomposition of Section 3.6. We use the notation of Proposition 3.17: $\delta \in (0, \beta]$ is the slab length, $n_\beta = \lfloor \beta/\delta \rfloor$, the cut configurations are $Y_0, \dots, Y_{J-1}$, and the data of slab j are $(\pi_j, X^{(j)}(\cdot))$. A one-slab variable is a quadruple $s = (Y, Y', \pi', X'(\cdot))$ with $Y, Y' \in \mathcal{X}_N$, $\pi' \in S_N$, and $X'(\cdot) \in C([0, \delta]; \mathbb{X}^N)$, and we call (Y, Y') its cuts. If $\beta > \delta$, we let μ_δ be the law (3.41), which by Proposition 3.17(4) is the law of $(Y_{j-1}, Y_j, \pi_j, X^{(j)}(\cdot))$ for every $1 \leq j \leq n_\beta$. The surgery of Section 7 acts on a slab that carries additional random marks. For this reason, we allow a probability kernel $\mathfrak{m}$ from one-slab variables to a standard Borel space $\mathbf{M}$ of marks, and we let

$$\mu_\delta^\mathfrak{m}(\mathrm{d}s, \mathrm{d}\theta) = \mu_\delta(\mathrm{d}s) \mathfrak{m}(s, \mathrm{d}\theta)$$

be the law of a marked one-slab variable (s, θ) . For a unit vector $u \in \mathbb{R}^3$ on the torus, and for a unit vector $n \in \mathbb{R}^3$ in a container, let

$$\varphi_u^W(s) = L^{-1} \sum_{i=1}^N u \cdot d(X'_i(\cdot)), \quad \varphi_n^A(s) = \frac{1}{2} \sum_{i=1}^N \int_0^\delta n \cdot ((X'_i(t) - x_0) \times \circ \mathrm{d}X'_i(t)),$$

so that $\varphi_u^W(Y_{j-1}, Y_j, \pi_j, X^{(j)}(\cdot)) = u \cdot O_j^W$ and $\varphi_n^A(Y_{j-1}, Y_j, \pi_j, X^{(j)}(\cdot)) = n \cdot O_j^A$ in the notation of Proposition 3.17(5).

Proposition 5.6. *Let $0 < \delta < \beta$ and $n_\beta = \lfloor \beta/\delta \rfloor$. Let φ be φ_u^W on the torus or φ_n^A in a container, and let O be $u \cdot W$, respectively $\mathcal{A}_n = n \cdot \mathcal{A}$. Let $\mathfrak{m}$ be a marking kernel as above, and let Q be a probability kernel on the space of marked one-slab variables such that, for $\mu_\delta^\mathfrak{m}$ -almost every (s, θ) , the measure $Q((s, \theta), \cdot)$ is concentrated on marked one-slab variables with the same cuts as s . Suppose that $\mu_\delta^\mathfrak{m}Q$ is absolutely continuous with respect to $\mu_\delta^\mathfrak{m}$, and that, for some $g > 0$ and $h < \infty$,*

$$\mathbb{E}_{\mu_\delta^\mathfrak{m}Q} \varphi - \mathbb{E}_{\mu_\delta^\mathfrak{m}} \varphi \geq g, \quad \chi^2(\mu_\delta^\mathfrak{m}Q \| \mu_\delta^\mathfrak{m}) \leq h,$$

where φ is regarded as a function of (s, θ) that does not depend on θ . Then

$$\mathrm{Var}_\mu(O) \geq \frac{n_\beta g^2}{h}. \tag{5.8}$$

Proof. We extend the marked slab measure $\bar{\mu}$ of (3.37) by marks $\Theta_1, \dots, \Theta_{n_\beta}$ that are sampled, given all cuts and all slab data, independently with laws $\mathfrak{m}((Y_{j-1}, Y_j, \pi_j, X^{(j)}(\cdot)), \cdot)$. Let $\mathbb{P}$ be the extended measure. We apply Lemma 5.5 with $m = n_\beta$, with $C = (Y_0, \dots, Y_{J-1})$, with $S_j = (\pi_j, X^{(j)}(\cdot), \Theta_j)$ for $1 \leq j \leq n_\beta$, and with R the data of the remainder slab if $\beta - n_\beta \delta > 0$ (and a constant otherwise). By Proposition 3.17(2) and the construction of the marks, $S_1, \dots, S_{n_\beta}, R$ are conditionally independent given C , and the conditional law of S_j given C is $\nu^{\mathfrak{m}}(\cdot \mid Y_{j-1}, Y_j)$, where

$$\nu^{\mathfrak{m}}(d\pi', dX', d\theta \mid Y, Y') = \nu_\delta(d\pi', dX' \mid Y, Y') \mathfrak{m}((Y, Y', \pi', X'), d\theta).$$

By Proposition 3.17(5), we have $O \circ \text{Cat} = \sum_{j=1}^{n_\beta} O_j + O_R$ almost surely, where $O_j = \varphi(Y_{j-1}, Y_j, \pi_j, X^{(j)}(\cdot))$ and O_R is the contribution of the remainder slab, and all these terms are in $L^2(\mathbb{P})$. Since the marks do not change the law of the slab data, Proposition 3.17(3) implies that $\text{Var}_{\mathbb{P}}(O \circ \text{Cat}) = \text{Var}_{\mu}(O)$.

For each j , let $Q_{j,C}$ be the kernel that acts on S_j as Q acts on the marked one-slab variable (Y_{j-1}, Y_j, S_j) ; this is well defined because Q preserves the cuts. By Proposition 3.17(4), the pair (Y_{j-1}, Y_j) has the law of the cuts under μ_δ , and by (3.40) and (3.41), the measures $\delta_{(Y, Y')} \otimes \nu^{\mathfrak{m}}(\cdot \mid Y, Y')$ form a disintegration of $\mu_\delta^{\mathfrak{m}}$ along its cuts. Let L be the density of $\mu_\delta^{\mathfrak{m}} Q$ with respect to $\mu_\delta^{\mathfrak{m}}$. By Lemma 5.4, applied with Φ the map to the cuts, for almost every (Y, Y') the density of $\nu^{\mathfrak{m}}(\cdot \mid Y, Y') Q$ with respect to $\nu^{\mathfrak{m}}(\cdot \mid Y, Y')$ is the restriction of L , and the quantity h_j of Lemma 5.5 equals $\chi^2(\mu_\delta^{\mathfrak{m}} Q \parallel \mu_\delta^{\mathfrak{m}}) \leq h$. In the same way, $g_j = \mathbb{E}_{\mu_\delta^{\mathfrak{m}} Q} \varphi - \mathbb{E}_{\mu_\delta^{\mathfrak{m}}} \varphi \geq g$. Then Lemma 5.5 implies (5.8). $\square$

Used in: Section 10 (proof of Theorem 1.1) and Section 11 (proof of Theorem 1.2).

For winding, Lemma 10.2 computes the gain used with Proposition 5.6. For area, the conditional mean gain is proved in Proposition 11.7.

Random routes with few nearby pairs

We construct two families of random closed walks: winding walks on the torus and polygons with large projected area inside a cube. A deterministic detour construction turns either walk into a simple cycle avoiding a sparse set of bad cells. We bound encounters between independent walks, control changes of the resulting cycles when the bad set changes locally, and average these estimates over the random bad set. The final bound, Corollary 6.19, is the lattice input to Theorem 7.14. We give the source attributions at the individual constructions below.

6.1 Notation

Cells and distances. Throughout this section $\mathbb{L}$ denotes either $\mathbb{Z}^3$ or the discrete torus $\mathbb{L}_K = (\mathbb{Z}/K\mathbb{Z})^3$ with an integer $K \geq 3$; its elements are called cells. We write $\text{pr}: \mathbb{Z}^3 \rightarrow \mathbb{L}_K$ for the projection, and a lift of a cell or set of cells of $\mathbb{L}_K$ is a cell or set of cells of $\mathbb{Z}^3$ that pr maps onto it. The maximum distance d_∞ is $|x - y|_\infty$ on $\mathbb{Z}^3$ and $\min_{k \in \mathbb{Z}^3} |\tilde{x} - \tilde{y} + Kk|_\infty$ on $\mathbb{L}_K$, for lifts $\tilde{x}, \tilde{y}$ of x, y . For sets F, F' we write $d_\infty(x, F) = \min_{y \in F} d_\infty(x, y)$ and $d_\infty(F, F') = \min_{y \in F} d_\infty(y, F')$, a minimum over the empty set being $+\infty$, and $\mathcal{N}_r(F) = \{x \in \mathbb{L} : d_\infty(x, F) \leq r\}$ for $r \geq 0$; thus $\mathcal{N}_r(\emptyset) = \emptyset$. The level of a cell is its first coordinate, in $\mathbb{Z}$ or in $\mathbb{Z}/K\mathbb{Z}$.

Adjacency and components. Two cells are nearest neighbors if they differ by one of $\pm e_1, \pm e_2, \pm e_3$ (on $\mathbb{L}_K$, if they have lifts that do). For an integer $r \geq 1$, two distinct cells x, y are range- r adjacent if $d_\infty(x, y) \leq r$. Range-one adjacency is called $*$ -adjacency, and $*$ -adjacent cells are called $*$ -neighbors; every cell has at most 26 of them. A set $F \subset \mathbb{L}$ is range- r connected if any two of its cells are joined by a chain of cells of F in which consecutive cells are range- r adjacent. The range- r components of a set are its maximal range- r connected subsets, and its $*$ -components are its range-one components. A range- r connected set of m cells has d_∞ -diameter at most $r(m - 1)$. For $F \subset \mathbb{Z}^3$, the coordinate span of F is the largest of the lengths of its three coordinate projections, and it is at most $r(m - 1)$ for such a set.

Walks and routes. A walk is a finite sequence $P = (z_0, \dots, z_{|P|})$ of cells in which consecutive cells are nearest neighbors. The integer $|P| \geq 0$ is its length, and the pairs (z_k, z_{k+1}) are its steps. It is closed if $z_{|P|} = z_0$. The occurrence list of a closed walk is $(z_0, \dots, z_{|P|-1})$;

the closing cell is not repeated. A sum over $z \in P$ runs over the occurrence list and counts repeated cells with multiplicity, while $d_\infty(x, P)$, $\mathcal{N}_r(P)$, and the phrase “cell of P ” refer to the set of cells of P . A closed walk is simple if $z_0, \dots, z_{|P|-1}$ are distinct. A route is either a simple closed walk of length at least three, represented by starting at its smallest cell in lexicographic order and keeping its orientation (on $\mathbb{L}_K$, the order of the representatives in $\{0, \dots, K-1\}^3$), or the empty route $\dagger$, which stands for a rejection and has no cells.

Encounter statistics. For closed walks or routes P, P' , an integer $R \geq 0$, and an integer $R_c \geq 1$, the encounter count and the exponentially weighted distance sum are

$$J_R(P, P') = \sum_{z \in P} \sum_{z' \in P'} \mathbf{1}\{d_\infty(z, z') \leq R\}, \quad S_{R_c}(P, P') = \sum_{z \in P} e^{-d_\infty(z, P')/R_c}. \quad (6.1)$$

Both count occurrences, and both vanish if P or P' is $\dagger$; for two routes they are sums over cells. In applications R is the encounter radius and R_c the component range, which is the length scale of the exponentially weighted distance sum because that sum arises from the expansion over components in Lemma 6.18.

Homology and area. Let P be a closed walk on $\mathbb{L}_K$. Since $K \geq 3$, each step (z_k, z_{k+1}) has a unique displacement $\delta_k \in \{\pm e_1, \pm e_2, \pm e_3\}$ such that $\text{pr}(\tilde{z} + \delta_k) = z_{k+1}$ for every lift $\tilde{z}$ of z_k . The sum of the displacements lies in $K\mathbb{Z}^3$, and we define the homology vector and the first homology by

$$\text{hom}(P) = K^{-1} \sum_{k=0}^{|P|-1} \delta_k \in \mathbb{Z}^3, \quad \text{hom}_1(P) = e_1 \cdot \text{hom}(P).$$

Equivalently, the lift of P that starts at a lift $\tilde{z}_0$ of z_0 ends at $\tilde{z}_0 + K \text{hom}(P)$. For a closed walk P in $\mathbb{Z}^3$, write $x(z)$ and $y(z)$ for the first two coordinates of a cell z , and define the projected area

$$\text{area}(P) = \sum_{k=0}^{|P|-1} x(z_k)(y(z_{k+1}) - y(z_k)) = - \sum_{k=0}^{|P|-1} y(z_k)(x(z_{k+1}) - x(z_k)). \quad (6.2)$$

The two sums agree because at most one of x and y changes in a step, so that $x(z_k)(y(z_{k+1}) - y(z_k)) + y(z_k)(x(z_{k+1}) - x(z_k))$ is the increment of xy in the k -th step, and these increments sum to zero. The quantity $\text{area}(P)$ is the line integral of $x \, dy$ along the projection of P to the plane of the first two coordinates. It is invariant under translations, changes sign under reversal of orientation, and for a simple closed curve in that plane equals the enclosed area, positive for counterclockwise orientation.

Cycle decomposition. Every closed walk P is decomposed into simple closed walks by the following fixed rule. If $|P| = 0$, the decomposition is empty. Otherwise let $j \geq 1$ be the smallest index such that $z_j = z_i$ for some $i < j$; this i is unique. Record the simple

closed walk $(z_i, \dots, z_j)$, remove it to obtain the closed walk $(z_0, \dots, z_i, z_{j+1}, \dots, z_{|P|})$ of length $|P| - (j - i)$, and repeat. Every step of P belongs to exactly one recorded walk, since the new step (z_i, z_{j+1}) is the old step (z_j, z_{j+1}) . Lengths, homology vectors, and areas are sums over steps, so each of them is additive: its value for P is the sum of its values for the recorded walks. A recorded walk has length at least two; one of length two traverses a single pair of cells back and forth and has zero area and zero homology.

Fields and constants. A field is a subset $B \subset \mathbb{L}$. Its cells are called bad, and all other cells are good; a route is good if all its cells are good. Constants $c, C, C', \dots$, with or without subscripts, depend only on the quantities named in the statement in which they appear, and never on K , on the side K_{in} of the inner cube, on a field, or on a skeleton. A constant is absolute if it depends on nothing.

6.2 Exponential excess from a small first moment

For a nonnegative random variable X , its exponential excess at coefficient $\vartheta \geq 0$ is $\mathbb{E}e^{\vartheta X} - 1$. The following lemma converts a small first moment into a small exponential excess for sums of bounded adapted variables whose conditional future sums are bounded.

Lemma 6.1. *Let $\bar{u} \geq 0$ be an integer, let $(\mathcal{F}_u)_{0 \leq u \leq \bar{u}}$ be a filtration, and let $X_0, \dots, X_{\bar{u}}$ be random variables with values in $[0, 1]$ such that X_u is $\mathcal{F}_u$ -measurable for every u . Suppose that there is $B_0 \geq 0$ such that*

$$\mathbb{E} \left[\sum_{u'=u+1}^{\bar{u}} X_{u'} \mid \mathcal{F}_u \right] \leq B_0 \quad \text{almost surely, for every } 0 \leq u \leq \bar{u}. \quad (6.3)$$

Set $X = \sum_{u=0}^{\bar{u}} X_u$. Then for every $\vartheta \geq 0$ with $(e^\vartheta - 1)B_0 < 1$, we have

$$\mathbb{E}e^{\vartheta X} - 1 \leq \frac{e^\vartheta - 1}{1 - (e^\vartheta - 1)B_0} \mathbb{E}X. \quad (6.4)$$

Proof. For $r \geq 1$ let $M_r = \mathbb{E} \sum_{0 \leq u_1 < \dots < u_r \leq \bar{u}} X_{u_1} \cdots X_{u_r}$. We claim that $M_r \leq B_0^{r-1} \mathbb{E}X$. For $r = 1$ this is an equality. For $r \geq 2$, we write

$$\sum_{u_1 < \dots < u_r} X_{u_1} \cdots X_{u_r} = \sum_{u_1 < \dots < u_{r-1}} X_{u_1} \cdots X_{u_{r-1}} \sum_{u_r = u_{r-1} + 1}^{\bar{u}} X_{u_r}.$$

The product $X_{u_1} \cdots X_{u_{r-1}}$ is nonnegative and $\mathcal{F}_{u_{r-1}}$ -measurable. Conditioning each term on $\mathcal{F}_{u_{r-1}}$ and using (6.3), we obtain $M_r \leq B_0 M_{r-1}$, which proves the claim by induction.

By convexity, $e^{\vartheta t} - 1 \leq (e^{\vartheta} - 1)t$ for $t \in [0, 1]$. Expanding the product, we find

$$\begin{aligned} e^{\vartheta X} &= \prod_{u=0}^{\bar{u}} (1 + (e^{\vartheta X_u} - 1)) = 1 + \sum_{r=1}^{\bar{u}+1} \sum_{u_1 < \dots < u_r} \prod_{l=1}^r (e^{\vartheta X_{u_l}} - 1) \\ &\leq 1 + \sum_{r=1}^{\bar{u}+1} (e^{\vartheta} - 1)^r \sum_{u_1 < \dots < u_r} X_{u_1} \cdots X_{u_r}. \end{aligned}$$

Taking expectations and using the claim gives $\mathbb{E}e^{\vartheta X} - 1 \leq \sum_{r \geq 1} (e^{\vartheta} - 1)^r B_0^{r-1} \mathbb{E}X$, and summing the geometric series yields (6.4). $\square$

No independence between the X_u is assumed. The expansion is that of the proof of [111, Lemma 8.1], reproduced in the proof of [112, Lemma 8.2]. The ordered-product estimate of [113, Lemma 7.3] is a related predecessor. Khas'minskii's continuous-time estimate [85, Section 3, Lemmas 3–4, pp. 314–315] bounds an exponential excess by the first moment of the additive functional, using an ordered-moment argument. The bounded-adapted discrete version used here is proved above. In Proposition 6.6, averaging over a random root makes $\mathbb{E}X$ of order $1/K$, while (6.3) holds conditionally on the root.

Used in: Proposition 6.6.

Remark 6.2. The conditional bound (6.3) cannot be replaced by bounds on the first moment and on a fixed exponential moment of X . Fix $\vartheta > 0$, and for $K \geq 3$ let X_K equal $(\log K)/(2\vartheta)$ with probability $1/(K \log K)$ and zero otherwise. Then $\mathbb{E}X_K = 1/(2\vartheta K)$ and $\mathbb{E}e^{2\vartheta X_K} = 1 + (K-1)/(K \log K) \leq 2$, but

$$\mathbb{E}e^{\vartheta X_K} - 1 = \frac{K^{1/2} - 1}{K \log K},$$

which is much larger than $1/K$. The random root must therefore enter the ordered-moment argument itself, as in the proof of Proposition 6.6.

The dyadic velocity array. Let $c_{\text{vel}} = (1 - 2^{-1/4})/2$, so that $c_{\text{vel}} \sum_{l \geq 0} 2^{-l/4} = 1/2$. A dyadic array is a family $(U_{l,k})_{l,k \geq 0}$ of independent random variables, each uniform on $[-1, 1]$. Its velocities and displacements are

$$v_i = c_{\text{vel}} \sum_{l \geq 0} 2^{-l/4} U_{l, \lfloor (i-1)/2^l \rfloor} \quad (i \geq 1), \quad Z_s = \sum_{i=1}^s v_i \quad (s \geq 0). \quad (6.5)$$

The series converges absolutely for every sample, $|v_i| \leq 1/2$, and $|Z_s - Z_{s'}| \leq |s - s'|/2$. The variable $U_{l,k}$ is attached to the dyadic block $I_{l,k} = \{k2^l + 1, \dots, (k+1)2^l\}$: it enters v_i if and only if $i \in I_{l,k}$. For $u \geq 0$, the variables revealed by time u are the $U_{l,k}$ with $I_{l,k} \cap \{1, \dots, u\} \neq \emptyset$. They determine $Z_1, \dots, Z_u$, because v_i involves only variables whose blocks contain i .

Lemma 6.3. Let $U^{(2)}$ and $U^{(3)}$ be independent dyadic arrays with displacements $Z_s^{(2)}$ and $Z_s^{(3)}$, and write $Z_s = (Z_s^{(2)}, Z_s^{(3)})$. Let $u \in \mathbb{Z}_{\geq 0}$, let $\mathcal{G}$ be a σ -algebra independent of the two arrays, and let $\mathcal{F}$ be generated by $\mathcal{G}$ and by the variables of both arrays revealed by time u . Set $c_d = 4^{3/4}/(2c_{\text{vel}})$. Then for every integer $k \geq 1$ the conditional law of Z_{u+k} given $\mathcal{F}$ has a density bounded by $c_d^2 k^{-3/2}$. Moreover, for every $w \in \mathbb{Z}^2$, we have

$$\mathbb{P}(\lfloor Z_{u+k}/8 \rfloor = w \mid \mathcal{F}) \leq 64 c_d^2 k^{-3/2}, \quad (6.6)$$

and for every integer $K \geq k$, we have

$$\mathbb{P}(\lfloor Z_{u+k} \rfloor \in w + K\mathbb{Z}^2 \mid \mathcal{F}) \leq 4 c_d^2 k^{-3/2}. \quad (6.7)$$

Here the integer part is taken in each coordinate.

Proof. We first find an unrevealed block inside $\{u+1, \dots, u+k\}$. If $k=1$, take $l=0$ and $m=u$, so that $I_{l,m} = \{u+1\}$. If $k \geq 2$, take l with $k/4 < 2^l \leq k/2$ and let m be the smallest integer with $m2^l \geq u$. Then $m2^l < u+2^l$, so $I_{l,m} \subset \{u+1, \dots, u+2^{l+1}\} \subset \{u+1, \dots, u+k\}$. In both cases, note that $k/4 \leq 2^l \leq k$. Since $I_{l,m}$ does not meet $\{1, \dots, u\}$, the variables $U_{l,m}^{(2)}$ and $U_{l,m}^{(3)}$ are not revealed by time u .

Let $\hat{\mathcal{F}}$ be generated by $\mathcal{F}$ and by all variables of both arrays other than $U_{l,m}^{(2)}$ and $U_{l,m}^{(3)}$. By independence, the pair $(U_{l,m}^{(2)}, U_{l,m}^{(3)})$ is independent of $\hat{\mathcal{F}}$ and uniform on $[-1, 1]^2$. Since $I_{l,m} \subset \{1, \dots, u+k\}$, the variable $U_{l,m}^{(j)}$ enters $Z_{u+k}^{(j)}$ through 2^l velocities, each time with coefficient $c_{\text{vel}}2^{-l/4}$. Then

$$Z_{u+k}^{(j)} = c_{\text{vel}}2^{3l/4} U_{l,m}^{(j)} + Y^{(j)}, \quad j \in \{2, 3\},$$

where $Y^{(j)}$ is $\hat{\mathcal{F}}$ -measurable. Conditionally on $\hat{\mathcal{F}}$, the two coordinates of Z_{u+k} are independent and uniform on intervals of length

$$\lambda = 2c_{\text{vel}}2^{3l/4} \geq 2c_{\text{vel}}(k/4)^{3/4} = k^{3/4}/c_d.$$

The conditional density given $\hat{\mathcal{F}}$ is at most $\lambda^{-2} \leq c_d^2 k^{-3/2}$. For a Borel set $E \subset \mathbb{R}^2$ we have $\mathbb{P}(Z_{u+k} \in E \mid \mathcal{F}) = \mathbb{E}[\mathbb{P}(Z_{u+k} \in E \mid \hat{\mathcal{F}}) \mid \mathcal{F}] \leq c_d^2 k^{-3/2} |E|$, which is the density bound. The same averaging applies to the two lattice bounds. It suffices to prove them conditionally on $\hat{\mathcal{F}}$.

A random variable uniform on an interval of length λ lies in a given interval of length h with probability at most $\min\{1, h/\lambda\} \leq h \min\{1, 1/\lambda\}$ for $h \geq 1$. Further, $\min\{1, 1/\lambda\} \leq c_d k^{-3/4}$: if $\lambda \geq 1$, this follows from $\lambda \geq k^{3/4}/c_d$, and if $\lambda < 1$, we have $c_d k^{-3/4} > 1$ by the same inequality. The event $\lfloor Z_{u+k}^{(j)}/8 \rfloor = w_j$ is the event that $Z_{u+k}^{(j)}$ lies in an interval of length 8, and its conditional probability is at most $8c_d k^{-3/4}$. Taking the product over the two coordinates yields (6.6). For (6.7), the event $\lfloor Z_{u+k}^{(j)} \rfloor \in w_j + K\mathbb{Z}$ is

the event that $Z_{u+k}^{(j)}$ lies in the union of the intervals $[w_j + Kt, w_j + Kt + 1)$, $t \in \mathbb{Z}$. Since $2^l \leq k \leq K$ and $2c_{\text{vel}} < 1$, we have $\lambda \leq k^{3/4} \leq K$. Points of two of these intervals whose indices t differ by at least two are more than $2K - 1 \geq K$ apart, so an interval of length $\lambda \leq K$ meets at most two of them. The conditional probability is at most $2c_d k^{-3/4}$. Taking the product over the two coordinates gives (6.7). $\square$

The velocity array (6.5) is the one of [111, Section 8.1], reproduced in [112, Equation (8.4)], and the density bound is proved in the course of [112, Lemma 8.2] for the first half of a folded skeleton; we have added the two lattice consequences. The mechanism of independent aligned blocks on all dyadic scales goes back to Häggström and Mossel [68, Section 3, Proposition 3.1]. The use of an unpredictable path, whose future position has small conditional point probabilities, to control intersections with an independent path is due to Benjamini, Pemantle, and Peres [19, Theorem 1.3 and Lemma 3.1]; see Hoffman [72] for the limits of predictability profiles of nearest-neighbor processes. Stopping and reflecting an unpredictable path is used as a symmetrization device in [1, Section 6, Lemma 6.1] and in [60, Section 2, proof of Theorem 1.3]; the latter's Lemma 2.5 bounds the second moment of the resulting estimator. The lattice constructions below instead fold the transverse displacements at deterministic midpoints; their geometry and encounter bounds are proved here.

Used in: Proposition 6.6.

6.3 Two random skeletons

A skeleton is a random closed walk built from a finite family of independent random variables, which we call its seed. Two skeletons are independent if their seeds are independent. We use Γ for the full random input and for the walk it determines; statistics of Γ depend only on the walk. In Section 7, the seeds are drawn independently of the gas.

Definition 6.4. Let $K \geq 20$ be an integer. The seed of a winding skeleton consists of three independent components: a root ϱ uniform on $(\mathbb{Z}/K\mathbb{Z})^2$, and two dyadic arrays $U^{(2)}, U^{(3)}$ with displacements $Z_s = (Z_s^{(2)}, Z_s^{(3)})$. Identify ϱ with its representative in $\{0, \dots, K-1\}^2$. For $0 \leq s \leq K$, the lifted base vertices are

$$\tilde{b}_s = (s, \varrho + \lfloor Z_{\min(s, K-s)} \rfloor) \in \mathbb{Z}^3, \quad (6.8)$$

so that $\tilde{b}_K = \tilde{b}_0 + Ke_1$. For $0 \leq s < K$, the step from $\tilde{b}_s$ to $\tilde{b}_{s+1}$ is refined into the walk that first moves by e_1 , then moves in the second coordinate by the increment of the second coordinate, and then in the third coordinate by the increment of the third coordinate; zero increments are omitted. These increments lie in $\{-1, 0, 1\}$, because

$|\min(s+1, K-s-1) - \min(s, K-s)| \leq 1$ and hence the two displacements differ by at most $1/2$ in each coordinate. Concatenating the K refined steps gives a walk $\tilde{\Gamma}$ in $\mathbb{Z}^3$ from $\tilde{b}_0$ to $\tilde{b}_0 + Ke_1$, the lifted winding skeleton. Its projection $\Gamma = \text{pr}(\tilde{\Gamma})$ is a closed walk on $\mathbb{L}_K$ starting at $b_0 = \text{pr}(\tilde{b}_0)$, the winding skeleton.

The transverse displacement is read forward on the first half of the levels and backward on the second half, which closes the walk without conditioning the array on a future value. The same folding is used in [111, Section 8.1] and [112, Equation (8.5)] for a walk between two prescribed endpoints, with a linear drift; here the endpoints of the lift project to the same cell. There is no drift, and the random root replaces the prescribed endpoints. The levels s and $K-s$ carry the same random transverse position, which Proposition 6.6 exploits.

Definition 6.5. Let $K_{\text{in}} \geq 2^{14}$ be an integer, let $x_{\text{in}} \in \mathbb{Z}^3$, and let $Q_{\text{in}} = x_{\text{in}} + \{0, 1, \dots, K_{\text{in}} - 1\}^3$ be the inner cube. Set $n = \lfloor K_{\text{in}}/64 \rfloor$, so that $n \geq 256$. The seed of an area skeleton consists of nine independent components: a root $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ uniform on $x_{\text{in}} + \{\lceil K_{\text{in}}/3 \rceil, \dots, \lfloor K_{\text{in}}/2 \rfloor\}^3$, and eight dyadic arrays $U^{(\iota, j)}$, $\iota \in \{1, 2, 3, 4\}$, $j \in \{2, 3\}$, with displacements $Z_s^{(\iota, j)}$. For $0 \leq s \leq n$ set $d_s^{(\iota, j)} = \lfloor Z_{\min(s, n-s)}^{(\iota, j)} / 8 \rfloor$, and define the base vertices of the four segments by

$$\begin{aligned} b_s^1 &= (s, -2s + d_s^{(1,2)}, d_s^{(1,3)}), & b_s^2 &= (n+s, -2n+2s + d_s^{(2,2)}, d_s^{(2,3)}), \\ b_s^3 &= (2n-s, 2s + d_s^{(3,2)}, d_s^{(3,3)}), & b_s^4 &= (n-s, 2n-2s + d_s^{(4,2)}, d_s^{(4,3)}). \end{aligned} \quad (6.9)$$

Since $d_0^{(\iota, j)} = d_n^{(\iota, j)} = 0$, we have $b_n^\iota = b_0^{\iota+1}$ for $\iota \in \{1, 2, 3\}$ and $b_n^4 = b_0^1 = 0$. The anchors are

$$o_0 = \varrho, \quad o_1 = \varrho + (n, -2n, 0), \quad o_2 = \varrho + (2n, 0, 0), \quad o_3 = \varrho + (n, 2n, 0), \quad o_4 = o_0.$$

Refine each step from b_s^ι to b_{s+1}^ι as in Definition 6.4: first the step in the first coordinate, then the unit steps in the second coordinate, then those in the third, omitting zero steps. After translation by ϱ , segment ι is a walk from $o_{\iota-1}$ to o_ι , and the concatenation of the four segments is a closed walk Γ in $\mathbb{Z}^3$ starting at o_0 . We call Γ an area skeleton with inner cube Q_{in} . The union of segments 1 and 2 is its lower branch, and the union of segments 3 and 4 its upper branch.

Without perturbations, $\Gamma - \varrho$ is the diamond with corners $(0, 0)$, $(n, -2n)$, $(2n, 0)$, $(n, 2n)$ in the plane of the first two coordinates, of area $4n^2$. Each segment carries its own pair of arrays. The factor $1/8$ keeps the two branches apart, and the anchors do not depend on the arrays. In Section 11, Q_{in} is the cell-index set of the physical block Ω_K in the active region of the scaled container, with $x_{\text{in}} = v_\Omega$ and $K_{\text{in}} = H$ at least a fixed multiple of K ; the bounds below, stated in terms of K_{in} , become bounds in terms of K only through that comparison.

Proposition 6.6. *The following hold.*

- (a) *Let $K \geq 20$ and let Γ be a winding skeleton. Then Γ is a closed walk on $\mathbb{L}_K$ with $K \leq |\Gamma| \leq 3K$ and $\text{hom}(\Gamma) = e_1$. Every level of $\mathbb{L}_K$ carries at most three occurrences of Γ , and every cell of $\tilde{\Gamma}$ at level $t \in \{0, \dots, K\}$ lies within d_∞ -distance one of $\tilde{b}_t$.*
- (b) *Let $K_{\text{in}} \geq 2^{14}$, let $n = \lfloor K_{\text{in}}/64 \rfloor$, and let Γ be an area skeleton with inner cube Q_{in} . Then Γ is a simple closed walk in $\mathbb{Z}^3$ with $|\Gamma| \leq 20n$, which passes through o_0, o_1, o_2, o_3 in this order. Every level carries at most ten cells of Γ , and every cell created by the refinement of the step from b'_{s-1} to b'_s lies within d_∞ -distance three of $q + b'_s$. Moreover,*

$$\text{area}(\Gamma) \geq \frac{63}{16} n^2 - 4n, \quad d_\infty(\Gamma, \mathbb{Z}^3 \setminus Q_{\text{in}}) \geq \frac{K_{\text{in}}}{4}. \quad (6.10)$$

- (c) *For all integers $R \geq 0$ and $R_c \geq 1$ there exist $\kappa, \xi \in (0, 1]$ and $C < \infty$, depending only on R and R_c , with the following property. If Γ, Γ' are independent winding skeletons with $K \geq 20$, then*

$$\mathbb{E} e^{\kappa J_R(\Gamma, \Gamma')} \leq 1 + \frac{C}{K}, \quad \mathbb{E} e^{\xi S_{R_c}(\Gamma, \Gamma')} \leq 1 + \frac{C}{K}. \quad (6.11)$$

If Γ, Γ' are independent area skeletons with the same inner cube Q_{in} and $K_{\text{in}} \geq 2^{14}$, then (6.11) holds with K replaced by K_{in} .

Proof of Proposition 6.6(a) and (b). We begin with (a). The refinement of each of the K base steps consists of one, two, or three steps. Then $K \leq |\Gamma| \leq 3K$. The lift $\tilde{\Gamma}$ of Γ ends at $\tilde{b}_0 + Ke_1$. In particular, $\text{hom}(\Gamma) = e_1$. The cells created by refining the step from $\tilde{b}_{t-1}$ to $\tilde{b}_t$ lie at level t , and each differs from $\tilde{b}_t$ by at most one in the second and third coordinates. This proves the distance assertion, and it shows that each level $t \in \{1, \dots, K\}$ of $\tilde{\Gamma}$ carries at most three cells, while level 0 carries only $\tilde{b}_0$. The occurrence list of Γ omits the closing cell $\text{pr}(\tilde{b}_K) = b_0$. Then level 0 of $\mathbb{L}_K$ carries at most $1 + 2$ occurrences, and every other level at most three.

We turn to (b). Since $|Z_{s'} - Z_s| \leq 1/2$ for $|s' - s| \leq 1$, consecutive values of $d^{(\iota, j)}$ differ by at most one. Then the step from b'_s to b'_{s+1} changes the first coordinate by ± 1 , the second by an amount in $\{-3, -2, -1\}$ (for $\iota = 1, 4$) or $\{1, 2, 3\}$ (for $\iota = 2, 3$), and the third by at most one. Its refinement has at most five steps. Then $|\Gamma| \leq 20n$. Its cells lie at the level of b'_{s+1} , and they differ from $q + b'_{s+1}$ by at most three in the second coordinate and at most one in the third. The levels of the base vertices increase strictly along the lower branch, from that of o_0 to that of o_2 , and decrease strictly along the upper branch. Each level carries the cells created by at most one refined step of each branch, together with an anchor at the levels of o_0 and o_2 . The cells of one refined step are distinct because it moves monotonically in the second coordinate and then in the third. We conclude that there are at most ten cells per level and that each branch visits every cell at most once.

To see that the branches meet only at o_0 and o_2 , we use coordinates relative to q and write $m = \min(s, n - s)$. Since $|Z_m| \leq m/2$, we have $-m/16 - 1 \leq d_s^{(\iota, j)} \leq m/16$. By a

direct check of (6.9), we find that at the levels $q \in \{1, \dots, 2n-1\}$, the base vertices of the lower branch have second coordinate at most $-(31/16) \min(q, 2n-q) < -1$, and those of the upper branch at least $(31/16) \min(q, 2n-q) - 1 > 0$. Since these coordinates are integers, they are at most -2 and at least 1 , respectively. The cells created at a level by a refined step have second coordinates between those of the two base vertices of the step. For $2 \leq q \leq 2n-2$, the cells of the lower branch at level q have second coordinate at most -2 , and those of the upper branch at least 1 . At level 1 , the lower branch, whose step starts at o_0 , has second coordinates at most 0 , and the upper branch at least 1 ; level $2n-1$ is symmetric. At level 0 the lower branch consists of o_0 alone, and at level $2n$ the upper branch consists of o_2 alone. We conclude that Γ is simple and that it passes through the anchors in order.

For the area, note that in each refined step the only step changing the first coordinate is the first one, which starts at the base vertex. By the second formula in (6.2) and translation invariance, we have

$$\text{area}(\Gamma) = \sum_{s=0}^{n-1} \left[(2s - d_s^{(1,2)}) + (2n - 2s - d_s^{(2,2)}) + (2s + d_s^{(3,2)}) + (2n - 2s + d_s^{(4,2)}) \right].$$

The terms without d sum to $4n^2$. Since $|d_s^{(i,2)}| \leq \min(s, n-s)/16 + 1$ and $\sum_{s=0}^{n-1} \min(s, n-s) \leq n^2/4$, the remaining terms are at least $-4(n^2/64 + n)$. This proves the first bound in (6.10).

For the margin, the bounds on $d_s^{(i,j)}$ show that all base vertices, and hence all cells of $\Gamma - \varrho$, lie in $[0, 2n] \times [-2n-1, 2n+1] \times [-n/32-1, n/32+1]$. Since $2n+1 \leq K_{\text{in}}/32 + 1$ and $\varrho - x_{\text{in}} \in [K_{\text{in}}/3, K_{\text{in}}/2]^3$, every coordinate of $z - x_{\text{in}}$, for a cell z of Γ , lies in $[29K_{\text{in}}/96 - 1, 17K_{\text{in}}/32 + 1]$. A cell $v \notin Q_{\text{in}}$ has a coordinate of $v - x_{\text{in}}$ below 0 or at least K_{in} . The distance $d_{\infty}(z, v)$ is at least $\min\{29K_{\text{in}}/96, 15K_{\text{in}}/32 - 1\} \geq K_{\text{in}}/4$. $\square$

Proof of Proposition 6.6(c). Fix $R \geq 0$ and $R_c \geq 1$; in this proof C denotes a constant depending only on R and R_c , which may change from line to line. In both cases we construct an integer $\bar{u} \geq 0$, a filtration $(\mathcal{F}_u)_{0 \leq u \leq \bar{u}}$, and adapted random variables $X_u, X'_u \in [0, 1]$ such that, with $\bar{K} = K$ for winding skeletons and $\bar{K} = K_{\text{in}}$ for area skeletons,

$$J_R(\Gamma, \Gamma') \leq C_1 \sum_{u=0}^{\bar{u}} X_u, \quad S_{R_c}(\Gamma, \Gamma') \leq C_1 \sum_{u=0}^{\bar{u}} X'_u,$$

$$\mathbb{E}[X_{u'} \mid \mathcal{F}_u] + \mathbb{E}[X'_{u'} \mid \mathcal{F}_u] \leq C(u' - u)^{-3/2} \quad (0 \leq u < u' \leq \bar{u}), \quad \mathbb{E} \sum_{u=0}^{\bar{u}} (X_u + X'_u) \leq \frac{C}{\bar{K}},$$

(6.12)

where C_1 depends only on R and R_c . Given (6.12), both sequences satisfy (6.3) with $B_0 = C \sum_{k \geq 1} k^{-3/2}$. Let $\vartheta \in (0, 1]$ be such that $(e^\vartheta - 1)B_0 \leq 1/2$. Then Lemma 6.1 implies that $\mathbb{E}e^{\vartheta \sum_u X_u} - 1 \leq 2(e^\vartheta - 1)\mathbb{E} \sum_u X_u \leq C/\bar{K}$, and the same for (X'_u) . With $\kappa = \zeta = \vartheta/C_1$,

the first line of (6.12) implies (6.11). Since exponential moments are nondecreasing in their coefficients, the smaller of the values of κ and ξ and the larger value of C obtained in the two cases serve for both.

Winding skeletons. Set $\bar{u} = \lfloor K/2 \rfloor$, and for $0 \leq u \leq \bar{u}$ let $q_u = \varrho + \lfloor Z_u \rfloor \in (\mathbb{Z}/K\mathbb{Z})^2$ be the transverse position at unfolded time u . By (6.8), $\text{pr}(\tilde{b}_t) = (t, \varrho_{\min(t, K-t)})$ for $0 \leq t \leq K$. The levels $t = u$ and $t = K - u$ share the transverse position q_u . Fix Γ' , and for $t \in \mathbb{Z}/K\mathbb{Z}$ and $w \in (\mathbb{Z}/K\mathbb{Z})^2$ define

$$f_t(w) = \sum_{z' \in \Gamma'} \mathbf{1}\{d_\infty((t, w), z') \leq R + 1\}, \quad g_t(w) = \sum_{z' \in \Gamma'} e^{-d_\infty((t, w), z')/R_c}.$$

By (a) applied to Γ' , at most $3(2R + 3)$ occurrences of Γ' lie at levels within distance $R + 1$ of t . Then $0 \leq f_t \leq 3(2R + 3)$ and $\sum_w f_t(w) \leq 3(2R + 3)^3$. At most six occurrences lie at levels at distance exactly d from t , for each $d \geq 0$, so $g_t \leq 6 \sum_{d \geq 0} e^{-d/R_c} \leq 6(R_c + 1)$. If an occurrence z' lies at a level at distance d from t , then $d_\infty((t, w), z') \geq d$, and at most $(2r + 1)^2$ cells w satisfy $d_\infty((t, w), z') = r$; we have $\sum_w e^{-d_\infty((t, w), z')/R_c} \leq \sum_{r \geq d} (2r + 1)^2 e^{-r/R_c} \leq C(1 + d)^2 e^{-d/R_c}$. Summing over z' gives $\sum_w g_t(w) \leq C$.

Let z be an occurrence of Γ , and let $t \in \{0, \dots, K\}$ be the level of the corresponding cell of $\tilde{\Gamma}$. By (a), we have $d_\infty(z, \text{pr}(\tilde{b}_t)) \leq 1$, which implies that

$$\begin{aligned} \sum_{z' \in \Gamma'} \mathbf{1}\{d_\infty(z, z') \leq R\} &\leq f_t(q_{\min(t, K-t)}), \\ e^{-d_\infty(z, \Gamma')/R_c} &\leq \sum_{z' \in \Gamma'} e^{-d_\infty(z, z')/R_c} \leq e^{1/R_c} g_t(q_{\min(t, K-t)}). \end{aligned}$$

At most three occurrences of Γ correspond to each t . Then the variables

$$X_u = \frac{1}{6(2R + 3)} \sum_{t \in \{u, K-u\}} f_t(q_u), \quad X'_u = \frac{1}{12(R_c + 1)} \sum_{t \in \{u, K-u\}} g_t(q_u),$$

where $\{u, K - u\}$ has one or two elements and $f_K = f_0, g_K = g_0$, take values in $[0, 1]$ and satisfy the first line of (6.12) with $C_1 = \max\{18(2R + 3), 36e^{1/R_c}(R_c + 1)\}$.

Let $\mathcal{F}_u$ be generated by the seed of Γ' , by q , and by the variables of $U^{(2)}$ and $U^{(3)}$ revealed by time u . The tests are functions of Γ' and q_u is $\mathcal{F}_u$ -measurable. In particular, X_u and X'_u are $\mathcal{F}_u$ -measurable. For $u < u' \leq \bar{u}$ we apply Lemma 6.3 with $\mathcal{G}$ generated by the seed of Γ' and q , which is independent of the two arrays. Since q is $\mathcal{F}_u$ -measurable and takes finitely many values, and $u' - u \leq K$, we have $\mathbb{P}(q_{u'} = w \mid \mathcal{F}_u) \leq 4c_d^2(u' - u)^{-3/2}$ for every w by (6.7). Multiplying by the values of the tests and summing over w , we obtain the second line of (6.12). Finally, conditionally on the two arrays and the seed of Γ' , the root q is uniform on $(\mathbb{Z}/K\mathbb{Z})^2$, and so is q_u . We have $\mathbb{E}X_u \leq (6(2R + 3))^{-1} \cdot 2 \cdot 3(2R + 3)^3 K^{-2} = (2R + 3)^2 K^{-2}$, and likewise $\mathbb{E}X'_u \leq CK^{-2}$. Summing over the $\bar{u} + 1 \leq K$ values of u yields the last bound in (6.12).

Area skeletons. Set $\bar{u} = \lfloor n/2 \rfloor$. We write (x_s^t, y_s^t) for the deterministic parts of the first two coordinates of b_s^t in (6.9), namely $(s, -2s)$, $(n + s, -2n + 2s)$, $(2n - s, 2s)$, and $(n - s, 2n - 2s)$ for $\iota = 1, 2, 3, 4$. For each segment ι and $0 \leq u \leq \bar{u}$ define the transverse position

$$q_u^t = (q_2, q_3) + (\lfloor Z_u^{(\iota,2)}/8 \rfloor, \lfloor Z_u^{(\iota,3)}/8 \rfloor) \in \mathbb{Z}^2,$$

so that $q + b_s^t = (q_1 + x_s^t, q_{\min(s, n-s)}^t + (y_s^t, 0))$. Fix Γ' , an area skeleton with the same inner cube, and for $q \in \mathbb{Z}$ and $w \in \mathbb{Z}^2$ let

$$f(q, w) = \sum_{z' \in \Gamma'} \mathbf{1}\{d_\infty((q, w), z') \leq R + 3\}, \quad g(q, w) = \sum_{z' \in \Gamma'} e^{-d_\infty((q, w), z')/R_c},$$

with distances in $\mathbb{Z}^3$, and $f_s^t(w) = f(q_1 + x_s^t, w + (y_s^t, 0))$, $g_s^t(w) = g(q_1 + x_s^t, w + (y_s^t, 0))$. Since each level carries at most ten cells of Γ' by (b), we obtain $0 \leq f \leq 10(2R + 7)$ by the computation in the winding case, $\sum_w f(q, w) \leq 10(2R + 7)^3$, $g \leq 20(R_c + 1)$, and $\sum_w g(q, w) \leq C$. Every occurrence of Γ other than $o_0 = q + b_0^1$ is created by the refinement of a step ending at some b_s^t , at most five occurrences are created by each step, and each lies within d_∞ -distance three of $q + b_s^t$. Then the variables

$$X_u = \frac{1}{80(2R + 7)} \sum_{\iota=1}^4 \sum_{s \in \{u, n-u\}} f_s^t(q_u^t), \quad X'_u = \frac{1}{160(R_c + 1)} \sum_{\iota=1}^4 \sum_{s \in \{u, n-u\}} g_s^t(q_u^t)$$

take values in $[0, 1]$ and satisfy the first line of (6.12) with $C_1 = \max\{400(2R + 7), 800e^{3/R_c}(R_c + 1)\}$.

Let $\mathcal{F}_u$ be generated by the seed of Γ' , by q , and by the variables of all eight arrays revealed by time u ; the functions f_s^t, g_s^t depend only on q_1 and Γ' . For a fixed ι , we apply Lemma 6.3 to the pair $U^{(\iota,2)}, U^{(\iota,3)}$, with $\mathcal{G}$ generated by the seed of Γ' , by q , and by the variables of the other six arrays revealed by time u . This $\mathcal{G}$ is independent of the pair, and together with the variables of the pair revealed by time u it generates $\mathcal{F}_u$. By (6.6), $\mathbb{P}(q_{u'}^t = w \mid \mathcal{F}_u) \leq 64c_d^2(u' - u)^{-3/2}$, and the second line of (6.12) follows by linearity; no joint density for the four segments is needed. Finally, conditionally on q_1 , the eight arrays, and the seed of Γ' , the pair (q_2, q_3) is uniform on a set of M^2 points, where $M = \lfloor K_{\text{in}}/2 \rfloor - \lceil K_{\text{in}}/3 \rceil + 1 \geq K_{\text{in}}/7$, while the functions f_s^t, g_s^t are fixed. We have $\mathbb{P}(q_u^t = w \mid q_1, \text{arrays}, \Gamma') \leq 49K_{\text{in}}^{-2}$, $\mathbb{E}X_u \leq 49(2R + 7)^2 K_{\text{in}}^{-2}$, and likewise $\mathbb{E}X'_u \leq CK_{\text{in}}^{-2}$. Summing over the $\bar{u} + 1 \leq K_{\text{in}}$ values of u gives the last bound in (6.12). $\square$

Remark 6.7. The conditional estimate of Lemma 6.3 is used only in increasing unfolded time. The levels u and $K - u$ of the folded walk are tested together, at their common transverse position, by a test fixed once the opposing skeleton is revealed; no statement is made about the law of the walk at chronologically later levels after the fold, where the displacement is already determined. The root is revealed in $\mathcal{F}_0$ for the conditional bound and averaged for the first moment. The restriction to increasing unfolded time is already

present in the proofs of [111, Lemma 8.1] and [112, Lemma 8.2]. We use the excess bound of Proposition 6.6, with the random root averaged in its first-moment estimate.

Used in: Proposition 6.15, Corollary 6.19, and through them Theorem 7.14.

6.4 Exterior shells, periodic lifts, connected sets, and areas

Let $C \subset \mathbb{Z}^3$ be finite and nonempty. Outside a box containing C , all cells are connected by nearest-neighbor paths avoiding C , and every infinite nearest-neighbor component of $\mathbb{Z}^3 \setminus C$ meets this region. Then $\mathbb{Z}^3 \setminus C$ has exactly one infinite nearest-neighbor component, which we call the exterior of C . The fill of C , denoted $\text{fill}(C)$, is the complement of the exterior: the union of C and the finite nearest-neighbor components of $\mathbb{Z}^3 \setminus C$. The exterior shell of C is

$$\partial_*^{\text{ext}} C = \{x \in \mathbb{Z}^3 \setminus \text{fill}(C) : d_\infty(x, C) = 1\}.$$

Lemma 6.8. *Let $C \subset \mathbb{Z}^3$ be finite, nonempty, and $*$ -connected. Then $\text{fill}(C)$ lies in the coordinate bounding box of C , and $\partial_*^{\text{ext}} C$ is a nearest-neighbor connected set with at most $26|C|$ cells. If C is a $*$ -component of a field $B \subset \mathbb{Z}^3$, every cell of $\partial_*^{\text{ext}} C$ is good.*

The lemma is [112, Lemma 8.3], which reproduces [115, Lemma 7.2]; we recall the proof. The connectivity of the shell is Kesten’s boundary-connectivity theorem [84, pp. 138–139, Lemma 2.23], in the formulation of Timár [141, Theorem 4 in arXiv version 2]; see [43, Lemma 2.1] for $*$ -connectivity of exterior nearest-neighbor boundaries. The discussion immediately after that lemma also attributes the exterior- $*$ -boundary statement to [84, Lemma 2.23]. The other assertions are elementary: a cell outside the bounding box has a coordinate ray to infinity avoiding C , each cell has 26 $*$ -neighbors, and a bad shell cell would be $*$ -adjacent to C , hence in the same $*$ -component of B .

Lemma 6.9. *Let $r \geq 1$ and $K > 2r$ be integers, and let $C \subset \mathbb{L}_K$ be range- r connected with m cells, where $rm < K$. Then the range- r components of $\text{pr}^{-1}(C)$ are translates of one another by $K\mathbb{Z}^3$, each is finite and is mapped bijectively onto C by pr , and each has coordinate span at most $r(m - 1)$. If moreover $r = 1$ and C is a $*$ -component of a field $B \subset \mathbb{L}_K$, then each of these translates is a $*$ -component of $\text{pr}^{-1}(B)$.*

The first statement is [112, Lemma 8.4], which reproduces [115, Lemma 7.3]; we call the translates the lifts of C . To prove it, fix a spanning tree of C for range- r adjacency. Since $K > 2r$, if $x, y \in \mathbb{L}_K$ are range- r adjacent and $\tilde{x}$ is a lift of x , then y has a unique lift at d_∞ -distance at most r from $\tilde{x}$. Lifting the tree edge by edge from a lift of one of its cells produces a range- r connected set $\tilde{C} \subset \text{pr}^{-1}(C)$ of m cells, with coordinate span at most $r(m - 1) < K$, which pr maps bijectively onto C . The translates $\tilde{C} + K\mathbb{Z}^3$ cover $\text{pr}^{-1}(C)$. No cell of one of two distinct translates lies within d_∞ -distance r of a cell of the

other, since otherwise two cells of $\tilde{C}$ would differ by $Kk + w$ with $k \neq 0$ and $|w|_\infty \leq r$, and some coordinate difference would be at least $K - r > r(m - 1)$. Hence the translates are the range- r components of $\text{pr}^{-1}(C)$. For the second, let $\tilde{x} \in \text{pr}^{-1}(B)$ be $*$ -adjacent to a cell $\tilde{y}$ of a lift $\tilde{C}$. Since $K \geq 3$, the cells $\text{pr}(\tilde{x})$ and $\text{pr}(\tilde{y})$ are distinct and $*$ -adjacent, so $\text{pr}(\tilde{x}) \in C$ by maximality of C . Hence $\tilde{x} \in \text{pr}^{-1}(C)$ is range-one adjacent to $\tilde{y}$, and $\tilde{x} \in \tilde{C}$.

Lemma 6.10. *Let $r \geq 1$ and $m \geq 1$ be integers. On $\mathbb{Z}^3$ and on $\mathbb{L}_K$, the number of range- r connected sets of m cells containing a given cell is at most $(2r + 1)^{6(m-1)}$. If B is a random field with $\mathbb{P}(F \subset B) \leq p^{|F|}$ for every finite $F \subset \mathbb{L}$, then for every cell x ,*

$$\mathbb{P}(x \in B \text{ and the range-}r \text{ component of } B \text{ containing } x \text{ has at least } m \text{ cells}) \leq ((2r + 1)^6 p)^m. \quad (6.13)$$

Proof. Assign to each such set a spanning tree for range- r adjacency, rooted at the given cell, and traverse it depth first. The traversal has $2(m - 1)$ steps, each given by one of fewer than $(2r + 1)^3$ displacements, and the visited cells recover the set. This proves the first bound. If the range- r component of B containing x has at least m cells, then growing a spanning tree from x produces a range- r connected set $F \ni x$ of exactly m cells with $F \subset B$. The union bound and the first statement give at most $(2r + 1)^{6(m-1)} p^m$, which implies (6.13). $\square$

This counting bound and the depth-first argument are recorded in [112, Section 8.2]. Such estimates are instances of the classical Peierls counting principle [119, pp. 479–480]; see also [64, Section 1.4, pp. 16–18]. An early tree-encoding exponential bound for plane animals is given in [86, pp. 852–853, Lemma 3]. Only the joint bound on prescribed sets enters; no independence between cells is assumed.

Lemma 6.11. *Every closed walk P in $\mathbb{Z}^3$ satisfies $|\text{area}(P)| \leq |P|^2/16$. If $\text{area}(P) > 0$, then some walk c in the cycle decomposition of P satisfies*

$$\text{area}(c) \geq \frac{16 \text{area}(P)^2}{|P|^2}.$$

Proof. Let $t_x = \sum_k |x(z_{k+1}) - x(z_k)|$ and $t_y = \sum_k |y(z_{k+1}) - y(z_k)|$ be the total variations of the first two coordinates along P , so that $t_x + t_y \leq |P|$. Since P is closed, the values of x along P lie in an interval of length at most $t_x/2$; let $\bar{x}$ be its midpoint. Since the increments of y along P sum to zero, we obtain from (6.2)

$$|\text{area}(P)| = \left| \sum_k (x(z_k) - \bar{x})(y(z_{k+1}) - y(z_k)) \right| \leq \frac{t_x t_y}{4} \leq \frac{(t_x + t_y)^2}{16} \leq \frac{|P|^2}{16}.$$

For the second statement, let $c_1, \dots, c_J$ be the cycle decomposition of P and let $A_{\max} = \max_i \text{area}(c_i)$, which is positive since the areas sum to $\text{area}(P) > 0$. By additivity and the first statement applied to each c_i , we have

$$\text{area}(P) \leq \sum_{i: \text{area}(c_i) > 0} \text{area}(c_i) \leq A_{\max}^{1/2} \sum_{i: \text{area}(c_i) > 0} \text{area}(c_i)^{1/2} \leq A_{\max}^{1/2} \sum_i \frac{|c_i|}{4} = \frac{A_{\max}^{1/2} |P|}{4},$$

which is the claim. $\square$

Used in: Proposition 6.15 (the area bound after cycle selection).

6.5 Routing maps

A routing map turns a skeleton and a field into a route. In Section 7, the routing map of one surgery is evaluated at fields obtained from the current field by the other surgery, and Lemma 6.16 compares the outputs. This requires locality in a precise sense, which we formulate through the components of the field that meet the skeleton.

Definition 6.12. Let Γ be a closed walk on $\mathbb{L}$. For a field $B \subset \mathbb{L}$, let $\mathcal{C}(B, \Gamma)$ be the set of *-components of B that contain a cell of Γ , and let

$$\mathcal{Q}(B, \Gamma) = \Gamma \cup \mathcal{N}_1\left(\bigcup_{C \in \mathcal{C}(B, \Gamma)} C\right).$$

A routing map for Γ assigns to every field B either $\dagger$ or a route $\mathcal{R}(B, \Gamma)$. (For the skeletons of Section 6.3, the lift and the anchors are determined by Γ .) The routing map is component-local if the following hold for all fields B, B' .

- (a) If $\mathcal{C}(B, \Gamma) = \mathcal{C}(B', \Gamma)$, then $\mathcal{R}(B, \Gamma) = \mathcal{R}(B', \Gamma)$.
- (b) If $\mathcal{R}(B, \Gamma) \neq \dagger$, every cell of $\mathcal{R}(B, \Gamma)$ is a cell of Γ or lies within d_∞ -distance one of a component in $\mathcal{C}(B, \Gamma)$.

Lemma 6.13. Let Γ be a closed walk on $\mathbb{L}$, and let $B, B' \subset \mathbb{L}$ be fields that agree on $\mathcal{Q}(B, \Gamma)$. Then $\mathcal{C}(B', \Gamma) = \mathcal{C}(B, \Gamma)$ and $\mathcal{Q}(B', \Gamma) = \mathcal{Q}(B, \Gamma)$. In particular, $\mathcal{R}(B', \Gamma) = \mathcal{R}(B, \Gamma)$ for every component-local routing map for Γ .

Proof. Let $C \in \mathcal{C}(B, \Gamma)$. Its cells lie in $\mathcal{Q}(B, \Gamma) \cap B$, hence in B' . The cells at d_∞ -distance one from C lie in $\mathcal{Q}(B, \Gamma)$, and by maximality of C they lie outside B , hence outside B' . So C is a *-component of B' , and it contains a cell of Γ . Conversely, let $C' \in \mathcal{C}(B', \Gamma)$, and let z be a cell of C' and of Γ . Then $z \in B' \cap \mathcal{Q}(B, \Gamma) = B \cap \mathcal{Q}(B, \Gamma)$, so z lies in some $C \in \mathcal{C}(B, \Gamma)$. That C is a *-component of B' containing z , so $C' = C$. This proves $\mathcal{C}(B', \Gamma) = \mathcal{C}(B, \Gamma)$, which implies $\mathcal{Q}(B', \Gamma) = \mathcal{Q}(B, \Gamma)$, and the last assertion is Definition 6.12(a). $\square$

Definition 6.14. The two routing maps are defined as follows. In both, a walk produced in the last step is represented as a route.

Winding routing map. Let $K \geq 20$ and let Γ be a winding skeleton, with lift $\tilde{\Gamma}$ from $\tilde{b}_0$ to $\tilde{b}_0 + Ke_1$. For a field $B \subset \mathbb{L}_K$, the route $\mathcal{R}(B, \Gamma)$ is computed in four steps.

1. *Size rule.* Reject if some $C \in \mathcal{C}(B, \Gamma)$ has at least $\lfloor K/20 \rfloor$ cells. Otherwise every $C \in \mathcal{C}(B, \Gamma)$ has fewer than $K/20$ cells, and by Lemma 6.9 its lifts are finite $*$ -components of $\text{pr}^{-1}(B)$ mapped bijectively onto it.
2. *Fill rule.* Reject if $b_0 = \text{pr}(\tilde{b}_0)$ belongs to $\text{pr}(\text{fill}(\tilde{C}))$ for a lift $\tilde{C}$ of a component in $\mathcal{C}(B, \Gamma)$; this set does not depend on the choice of the lift.
3. *Detours.* Set $P = \tilde{\Gamma}$. As long as P has a cell in $\text{pr}^{-1}(B)$, let k be the smallest index with $P_k \in \text{pr}^{-1}(B)$, let $\tilde{C}$ be the lift of a component in $\mathcal{C}(B, \Gamma)$ that contains P_k , and let k' be the largest index with $P_{k'} \in \tilde{C}$. Replace the part $(P_{k-1}, \dots, P_{k'+1})$ of P by the shortest nearest-neighbor path from P_{k-1} to $P_{k'+1}$ in $\partial_*^{\text{ext}} \tilde{C}$ that comes first in lexicographic order.
4. *Selection.* Decompose the closed walk $\text{pr}(P)$ on $\mathbb{L}_K$ by the cycle decomposition, and output the first walk in it with positive first homology.

Area routing map. Let $K_{\text{in}} \geq 2^{14}$, let $n = \lfloor K_{\text{in}}/64 \rfloor$, and let Γ be an area skeleton with anchors $o_0, \dots, o_3$. For a field $B \subset \mathbb{Z}^3$, the route $\mathcal{R}(B, \Gamma)$ is computed in four steps.

1. *Budget rule.* Reject if $\sum_{C \in \mathcal{C}(B, \Gamma)} |C|^2 > n$, in particular if some $C \in \mathcal{C}(B, \Gamma)$ is infinite.
2. *Anchor rule.* Reject if one of o_0, o_1, o_2, o_3 belongs to $\text{fill}(C)$ for a component $C \in \mathcal{C}(B, \Gamma)$.
3. *Detours segment by segment.* For each $\iota \in \{1, 2, 3, 4\}$ separately, let P be segment ι of Γ , from $o_{\iota-1}$ to o_ι . As long as P has a bad cell, let k be the smallest index with $P_k \in B$, let C be the component in $\mathcal{C}(B, \Gamma)$ that contains P_k , and let k' be the largest index with $P_{k'} \in C$. Replace $(P_{k-1}, \dots, P_{k'+1})$ by the shortest nearest-neighbor path from P_{k-1} to $P_{k'+1}$ in $\partial_*^{\text{ext}} C$ that comes first in lexicographic order. Concatenate the four resulting walks into a closed walk P through $o_0, \dots, o_3$.
4. *Selection.* Decompose P by the cycle decomposition. Reject if no walk in it has positive area; otherwise output the first walk in the decomposition whose area is the largest.

The detours follow the exterior shell of a component from the cell immediately before the first visit of the walk to that component to the cell immediately after its last visit, as in [115, Section 7.3], reproduced in [112, Section 8.3]; related detours appear in [111, Lemma 8.2]. In these sources the size and fill tests are applied to the whole environment before the path is drawn [112, Equation (8.16)]. Here they are applied only to components met by the skeleton, which makes the maps component-local, and the loop erasure of an open path is replaced by the cycle decomposition, which preserves homology and

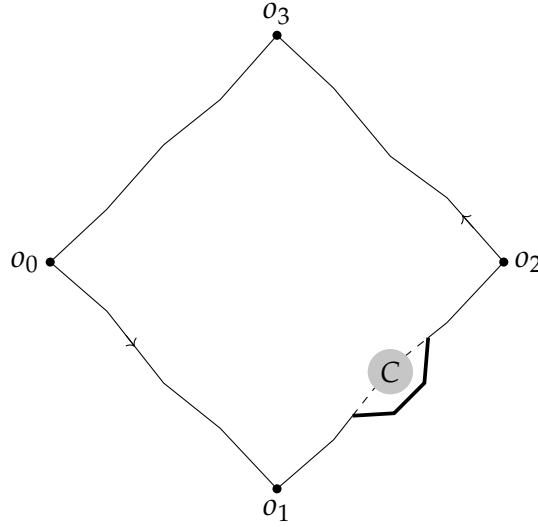


Figure 6.1. Schematic projection of an area skeleton (thin) to the plane of the first two coordinates, traversed counterclockwise through the anchors $o_0, \dots, o_3$. The second segment meets a bad $*$ -component C (gray). The area routing map replaces the part of this segment between its first and last visits to C (dashed) by a path in the exterior shell of C (thick). The other segments and the four anchors are unchanged. Perturbations, the third coordinate, and the lattice are not drawn to scale.

area. The size rule gives finite lifts, and the fill rule keeps the endpoints of the lifted walk outside the fills of the detoured components. The area map is anchored: each segment is corrected separately and its endpoints never move. A single first-to-last detour along the whole closed skeleton could remove the entire polygon, for instance when a small bad component near o_0 is met at the beginning and at the end of the walk. The bound on the sum of squared component sizes controls the area lost in the detours, since a detour around C changes the area by at most a constant times $|C|^2$, and it keeps the components that are used close to the skeleton.

Proposition 6.15. *Let $p_0 = 1/1458$. There is an absolute constant C_{acc} such that the following hold.*

- (a) *Let $K \geq 20$ and let Γ be a winding skeleton. The winding routing map is a component-local routing map for Γ . For every field $B \subset \mathbb{L}_K$, the route $\gamma = \mathcal{R}(B, \Gamma)$ is either $\dagger$ or a good route with $\text{hom}_1(\gamma) \geq 1$, every cell of which is a cell of Γ or the projection of a cell of $\partial_*^{\text{ext}} \tilde{C}$ for a lift $\tilde{C}$ of a $*$ -component of B containing a cell of Γ .*
- (b) *Let $K_{\text{in}} \geq 2^{14}$, let $n = \lfloor K_{\text{in}}/64 \rfloor$, and let Γ be an area skeleton with inner cube Q_{in} . The area routing map is a component-local routing map for Γ . For every field $B \subset \mathbb{Z}^3$, the route*

$\gamma = \mathcal{R}(B, \Gamma)$ is either $\dagger$ or a good route with

$$|\gamma| \leq 124n, \quad \text{area}(\gamma) \geq \frac{n^2}{250}, \quad (6.14)$$

every cell of which is a cell of Γ or a cell of $\partial_*^{\text{ext}}C$ for a $*$ -component C of B containing a cell of Γ . Every cell of γ lies in $\mathcal{N}_{\lfloor \sqrt{n} \rfloor + 1}(\Gamma) \subset Q_{\text{in}}$.

- (c) Let Γ be a random winding skeleton with $K \geq 20$, or a random area skeleton with $K_{\text{in}} \geq 2^{14}$. Let B be a random field, independent of the seed of Γ , such that $\mathbb{P}(F \subset B) \leq p^{|F|}$ for every finite $F \subset \mathbb{L}$, where $0 \leq p \leq p_0$. Then $\mathbb{P}(\mathcal{R}(B, \Gamma) = \dagger) \leq C_{\text{acc}} p$.

Proof. Component-locality. In both maps, the size and budget rules, the fill and anchor rules, and the exterior shells, with their lifts in the winding case, are functions of Γ and $\mathcal{C}(B, \Gamma)$, and the selection steps read only the walk P . In the detour steps, every cell of the current walk P is a cell of Γ (in the winding case, of $\tilde{\Gamma}$) or a cell of the exterior shell of some $C \in \mathcal{C}(B, \Gamma)$ (in the winding case, of a lift of such a C). A cell of $\partial_*^{\text{ext}}C$ is good, since otherwise it would be $*$ -adjacent to C and lie in C . In the winding case, a cell $\tilde{x}$ of $\partial_*^{\text{ext}}\tilde{C}$ satisfies $\text{pr}(\tilde{x}) \notin C$, since otherwise $\tilde{x}$ would be a cell of $\text{pr}^{-1}(C)$ that is $*$ -adjacent to $\tilde{C}$, hence a cell of $\tilde{C}$ by Lemma 6.9. So $\text{pr}(\tilde{x})$ is a $*$ -neighbor of C outside C , and it is good. A cell of Γ is bad exactly when it lies in a component in $\mathcal{C}(B, \Gamma)$. Hence a cell of P lies in B (in the winding case, in $\text{pr}^{-1}(B)$) if and only if it lies in a component in $\mathcal{C}(B, \Gamma)$ (in the winding case, in a lift of one). Every test in the detour steps, and the component used in each replacement, is therefore decided by $\mathcal{C}(B, \Gamma)$. By induction over the replacements, two fields B, B' with $\mathcal{C}(B, \Gamma) = \mathcal{C}(B', \Gamma)$ produce the same walks and the same route. This is Definition 6.12(a).

Proof of (a). Suppose that neither the size rule nor the fill rule rejects. The cell b_0 is good, since otherwise it would belong to a component $C \in \mathcal{C}(B, \Gamma)$ and to $C \subset \text{pr}(\text{fill}(\tilde{C}))$, and the fill rule would reject. We claim that throughout the detour step, P is a walk from $\tilde{b}_0$ to $\tilde{b}_0 + Ke_1$ whose cells are cells of $\tilde{\Gamma}$ or good cells, so that its cells in $\text{pr}^{-1}(B)$ are cells of $\tilde{\Gamma}$. This holds initially. Let $k, \tilde{C}$, and k' be as in the definition; the lift $\tilde{C}$ exists because $\text{pr}(P_k)$ is a bad cell of Γ , whose $*$ -component lies in $\mathcal{C}(B, \Gamma)$. Since the endpoints of P are good, $1 \leq k \leq k' < |P|$. The cells $P_0, \dots, P_{k-1}$ are good. They avoid $\tilde{C}$ and join $\tilde{b}_0$ to P_{k-1} . Since $\text{pr}(\tilde{b}_0) = b_0 \notin \text{pr}(\text{fill}(\tilde{C}))$, the cell $\tilde{b}_0$ lies in the exterior of $\tilde{C}$, and so does P_{k-1} . By the choice of k' , the cells after $P_{k'}$ avoid $\tilde{C}$ and join $P_{k'+1}$ to $\tilde{b}_0 + Ke_1$, which also lies in the exterior since it projects to b_0 . Both P_{k-1} and $P_{k'+1}$ are nearest neighbors of cells of $\tilde{C}$. They lie in $\partial_*^{\text{ext}}\tilde{C}$. By Lemmas 6.8 and 6.9, they are joined by a nearest-neighbor path in $\partial_*^{\text{ext}}\tilde{C}$, and all its cells are good. The replacement removes every visit of P to $\tilde{C}$ and adds only good cells. This proves the claim, and shows that the number of visits of P to $\text{pr}^{-1}(B)$ decreases strictly at each replacement. Then the detour step ends with a walk P from $\tilde{b}_0$ to $\tilde{b}_0 + Ke_1$ consisting of good cells.

The closed walk $\text{pr}(P)$ has homology e_1 , because its lift P ends at $\tilde{b}_0 + Ke_1$. By additivity, the first homologies of the walks in its cycle decomposition sum to one. Then one of them is positive, and the selection step does not reject. The selected walk γ is simple, consists of good cells, and has length at least $K \geq 3$, because a closed walk with nonzero first homology has at least K steps in the first coordinate. Its cells are cells of Γ or projections of shell cells of lifts of components in $\mathcal{C}(B, \Gamma)$, and the latter lie within distance one of the corresponding component. This proves (a) and Definition 6.12(b).

Proof of (b). On acceptance, $\sum_{C \in \mathcal{C}(B, \Gamma)} |C|^2 \leq n$. Each $C \in \mathcal{C}(B, \Gamma)$ is $*$ -connected, contains a cell of Γ , and has at most $\lfloor \sqrt{n} \rfloor$ cells, so C and its exterior shell lie in $\mathcal{N}_{\lfloor \sqrt{n} \rfloor}(\Gamma)$. All cells of an accepted route lie in $\mathcal{N}_{\lfloor \sqrt{n} \rfloor + 1}(\Gamma)$. Since $n \leq K_{\text{in}}/64$ and $K_{\text{in}} \geq 2^{14}$, we have $\sqrt{n} + 1 \leq \sqrt{K_{\text{in}}}/8 + 1 < K_{\text{in}}/4$, and the margin in (6.10) implies that $\mathcal{N}_{\lfloor \sqrt{n} \rfloor + 1}(\Gamma) \subset Q_{\text{in}}$.

Suppose that neither the budget rule nor the anchor rule rejects. The anchors are good, since a bad anchor would belong to a component in $\mathcal{C}(B, \Gamma)$ and to its fill. Fix a segment ι . Throughout the detour step for this segment, P is a walk from o_{l-1} to o_l which consists of an initial part of good cells, ending at a cell of the original segment, followed by the remaining part of the original segment. Indeed, each replacement keeps the good cells before P_k , inserts good cells, and ends the inserted path at the original cell $P_{k'+1}$. Let k , C , and k' be as in the definition. We use the argument for (a), with the anchors o_{l-1} and o_l in place of $\tilde{b}_0$ and $\tilde{b}_0 + Ke_1$ and the anchor rule in place of the fill rule. We find that P_{k-1} and $P_{k'+1}$ lie in $\partial_*^{\text{ext}} C$, that the replacement is well defined and adds only good cells, and that the procedure terminates. After the replacement, the walk no longer visits C , and later replacements add only good cells. Each component in $\mathcal{C}(B, \Gamma)$ is used at most once in each segment, and at most four times in total.

We now bound the length and the area. Let C be a component used in segment ι , with $m = |C|$ cells. The initial part of P is good and ends at a cell of the original segment. Then the first bad cell P_k lies in the remaining part, and the cells $P_{k-1}, \dots, P_{k'+1}$ form an arc of the original segment ι . By the monotonicity of levels along a segment, the cells $P_k, \dots, P_{k'}$ lie at levels between those of P_k and $P_{k'}$, which are cells of C ; these levels form an interval of at most m integers, since the coordinate span of C is at most $m - 1$. Each level carries at most five cells of the segment, by the proof of Proposition 6.6(b). The removed arc has at most $5m + 1 \leq 6m$ steps. The inserted path has at most $26m$ cells by Lemma 6.8, hence fewer than $26m$ steps. The removed arc followed by the reversed inserted path is a closed walk of length at most $32m$, and by additivity of the area, the replacement changes the area of the concatenation of the four current segments by the area of this closed walk. By Lemma 6.11, this change is at most $(32m)^2/16 = 64m^2$ in absolute value. On acceptance, the sum of $|C|^2$ over $C \in \mathcal{C}(B, \Gamma)$ is at most n , and $|C| \leq |C|^2$. Then the final closed walk P satisfies

$$|P| \leq |\Gamma| + 4 \cdot 26 \sum_C |C| \leq 20n + 104n = 124n,$$

$$\text{area}(P) \geq \text{area}(\Gamma) - 4 \cdot 64 \sum_C |C|^2 \geq \frac{63}{16} n^2 - 260n \geq 2n^2,$$

where we used Proposition 6.6(b), and $n \geq 256$ in the last inequality. In particular the selection step does not reject. By Lemma 6.11, the selected walk γ satisfies $|\gamma| \leq |P| \leq 124n$ and

$$\text{area}(\gamma) \geq \frac{16(2n^2)^2}{(124n)^2} = \frac{64}{15376} n^2 > \frac{n^2}{250}.$$

It is simple, its cells are good, and its length is at least four since its area is positive. Its cells are cells of Γ or shell cells of components in $\mathcal{C}(B, \Gamma)$. This proves (b) and Definition 6.12(b).

Proof of (c). Fix the seed of Γ . Since B is independent of the seed, the prescribed-set bound holds conditionally on it, and we apply Lemma 6.10 with $r = 1$, for which $(2r + 1)^6 = 729$; note that $729p \leq 1/2$. In the winding case, the size rule rejects only if some $*$ -connected set of $\lfloor K/20 \rfloor$ bad cells contains an occurrence of Γ , which has probability at most $3K(729p)^{\lfloor K/20 \rfloor}$. If the fill rule rejects because of a component $C \in \mathcal{C}(B, \Gamma)$ with m cells, then a lift of b_0 lies in the bounding box of a lift of C , whose coordinate span is at most $m - 1$ by Lemma 6.9. Every cell of C is then within distance $m - 1$ of b_0 , so the $*$ -component of B containing one of the at most $(2m - 1)^3$ cells within distance $m - 1$ of b_0 has at least m cells. Summing over m , we obtain

$$\begin{aligned} \mathbb{P}(\mathcal{R}(B, \Gamma) = \dagger) &\leq 3K(729p)^{\lfloor K/20 \rfloor} + \sum_{m \geq 1} (2m - 1)^3 (729p)^m \\ &\leq 729p \left(3K 2^{1 - \lfloor K/20 \rfloor} + \sum_{m \geq 1} (2m - 1)^3 2^{1 - m} \right), \end{aligned}$$

and the bracket is bounded by an absolute constant for $K \geq 20$.

In the area case, if the budget rule rejects, then the sum of $|C|^2$ over the $*$ -components C of B containing a cell of Γ exceeds n . Each such component contains an occurrence of Γ . Writing $C(z)$ for the $*$ -component of B containing z when $z \in B$, we have by Markov's inequality and Lemma 6.10

$$\begin{aligned} \mathbb{P}(\text{budget rule rejects}) &\leq \frac{1}{n} \sum_{z \in \Gamma} \mathbb{E}[\mathbf{1}\{z \in B\} |C(z)|^2] \\ &\leq \frac{|\Gamma|}{n} \sum_{m \geq 1} m^2 (729p)^m \leq 20 \cdot 729p \sum_{m \geq 1} m^2 2^{1 - m}. \end{aligned}$$

Here we used $\mathbb{E}Y^2 = \sum_{m \geq 1} (2m - 1) \mathbb{P}(Y \geq m)$ for $Y = \mathbf{1}\{z \in B\} |C(z)|$, which is almost surely finite by (6.13), and $|\Gamma| \leq 20n$. The anchor rule is bounded as the fill rule in the winding case, for each of the four anchors. This proves (c). $\square$

Used in: Theorem 7.14 (acceptance and component-locality), Lemma 10.2 (first homology), Proposition 11.7 (length and area), and Section 11 (containment in the inner cube).

6.6 Stability of routes under local changes of the field

In Section 7, the route of one surgery is evaluated at a field obtained from the current one by the other surgery. Such fields differ from a baseline field only within distance one of the other route, and all of them lie in a common vulnerable set. The next lemma shows that, for component-local routing maps, these changes matter only through the components of the vulnerable set that are approached by both skeletons.

Lemma 6.16. *Let $\mathbb{L}$ be $\mathbb{Z}^3$ or $\mathbb{L}_K$, let Γ, Γ' be closed walks of positive length on $\mathbb{L}$, and let $\mathcal{R}(\cdot, \Gamma)$ and $\mathcal{R}(\cdot, \Gamma')$ be component-local routing maps for Γ and Γ' . Let $R \geq 0$ and $R_c \geq R + 2$ be integers, let $V \subset \mathbb{L}$ be finite, and let $B_0 \subset V$. Let $\mathcal{H}$ be the set of range- R_c components H of V such that $d_\infty(H, \Gamma) \leq R_c$ and $d_\infty(H, \Gamma') \leq R_c$, and set*

$$U_* = \bigcup_{H \in \mathcal{H}} H, \quad A_* = |U_*|.$$

For $D \subset U_*$ put $B[D] = (B_0 \setminus U_*) \cup D$, and define the route lists

$$\mathcal{L} = \{\mathcal{R}(B[D], \Gamma) : D \subset U_*\}, \quad \mathcal{L}' = \{\mathcal{R}(B[D], \Gamma') : D \subset U_*\}.$$

(i) Let $B^1, B^2 \subset V$, let $\gamma = \mathcal{R}(B^1, \Gamma)$ and $\gamma' = \mathcal{R}(B^2, \Gamma')$, and suppose that

$$B^1 \triangle B_0 \subset \mathcal{N}_1(\gamma'), \quad B^2 \triangle B_0 \subset \mathcal{N}_1(\gamma). \quad (6.15)$$

Then $\mathcal{R}(B[B^1 \cap U_*], \Gamma) = \gamma$ and $\mathcal{R}(B[B^2 \cap U_*], \Gamma') = \gamma'$. In particular $(\gamma, \gamma') \in \mathcal{L} \times \mathcal{L}'$, and at most 4^{A_*} ordered pairs (γ, γ') arise in this way.

(ii) Let γ, γ' be routes, possibly $\dagger$, and let $B^1, B^2 \subset V$ satisfy (6.15) for this pair. If $A_* = 0$, then

$$\mathbf{1}\{\mathcal{R}(B^1, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B^2, \Gamma') = \gamma'\} = \mathbf{1}\{\mathcal{R}(B_0, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B_0, \Gamma') = \gamma'\}. \quad (6.16)$$

(iii) For all $B^1, B^2 \subset V$, the routes $\gamma = \mathcal{R}(B^1, \Gamma)$ and $\gamma' = \mathcal{R}(B^2, \Gamma')$ satisfy

$$J_R(\gamma, \gamma') \leq J_R(\Gamma, \Gamma') + C_{R,1} A_*, \quad C_{R,1} = 54(2R + 1)^3. \quad (6.17)$$

In particular (6.17) holds for the pairs in (i) and for every pair in $\mathcal{L} \times \mathcal{L}'$.

Proof. We first prove the following localization claim. Let $B, B' \subset V$, let $x \in \mathcal{Q}(B, \Gamma) \cap V$, and suppose that $d_\infty(x, \mathcal{R}(B', \Gamma')) \leq 1$. Then $x \in U_*$. The same holds with the roles of Γ and Γ' interchanged.

To prove the claim, let H be the range- R_c component of V containing x . Either x is a cell of Γ , or x lies within distance one of some $C \in \mathcal{C}(B, \Gamma)$. In the second case, $C \subset B \subset V$ is $*$ -connected and contains a cell of Γ , so it lies in a range- R_c component of V that contains a cell of Γ , and this component is H , since $x \in V$ and $d_\infty(x, C) \leq 1 \leq R_c$. In both cases H contains a cell of Γ . The route $\mathcal{R}(B', \Gamma')$ is not $\dagger$, since $\mathcal{N}_1(\dagger) = \emptyset$; let

y' be one of its cells with $d_\infty(x, y') \leq 1$. By Definition 6.12(b) for B' , either y' is a cell of Γ' , and then $d_\infty(H, \Gamma') \leq 1 \leq R_c$; or y' lies within distance one of a $*$ -component C' of B' containing a cell of Γ' . In the latter case $C' \subset V$ is $*$ -connected, so it lies in a single range- R_c component H' of V , which contains a cell of Γ' . Since $d_\infty(x, C') \leq 2 \leq R_c$, we have $H' = H$. In both cases $H \in \mathcal{H}$, and therefore $x \in U_*$.

We prove (i). Let $B' = B[B^1 \cap U_*]$. The fields B' and B^1 agree on U_* , and B' agrees with B_0 outside U_* . If B' and B^1 differed at a cell $x \in \mathcal{Q}(B^1, \Gamma)$, then $x \notin U_*$ and $x \in B^1 \triangle B_0 \subset V \cap \mathcal{N}_1(\gamma')$. The claim, applied with $B = B^1$ and $B' = B^2$, would then give $x \in U_*$, a contradiction. So B' and B^1 agree on $\mathcal{Q}(B^1, \Gamma)$, and Lemma 6.13 gives $\mathcal{R}(B', \Gamma) = \mathcal{R}(B^1, \Gamma) = \gamma$. The argument for Γ' is the same, with the roles of Γ and Γ' interchanged. Since $B^1 \cap U_*$ and $B^2 \cap U_*$ are subsets of U_* , we have $\gamma \in \mathcal{L}$ and $\gamma' \in \mathcal{L}'$, and each list has at most 2^{A_*} elements.

We prove (ii). Since $A_* = 0$, we have $U_* = \emptyset$ and $B[\emptyset] = B_0$. If the left side of (6.16) is one, then γ and γ' are the routes computed on B^1 and B^2 , and by (i) we have $\mathcal{R}(B_0, \Gamma) = \gamma$ and $\mathcal{R}(B_0, \Gamma') = \gamma'$, so the right side is one. Conversely, suppose that the right side is one, so that $\gamma = \mathcal{R}(B_0, \Gamma)$ and $\gamma' = \mathcal{R}(B_0, \Gamma')$. Let $x \in \mathcal{Q}(B_0, \Gamma)$ with $x \in B^1 \triangle B_0$. Then $x \in V \cap \mathcal{N}_1(\gamma')$, and the claim with $B = B' = B_0$ gives $x \in U_* = \emptyset$, which is impossible. Then B^1 and B_0 agree on $\mathcal{Q}(B_0, \Gamma)$, and $\mathcal{R}(B^1, \Gamma) = \gamma$ by Lemma 6.13. In the same way $\mathcal{R}(B^2, \Gamma') = \gamma'$, and the left side is one. In all other cases both sides vanish. The converse direction is what identifies all four products of route indicators in the signed sum of Section 7 with their baseline values.

We prove (iii). If γ or γ' is $\dagger$, the left side vanishes. Otherwise both are simple, and $J_R(\gamma, \gamma')$ counts the pairs of cells (x, x') with $x \in \gamma$, $x' \in \gamma'$, and $d_\infty(x, x') \leq R$. For pairs in which x is a cell of Γ and x' a cell of Γ' , choose an occurrence of x in Γ and one of x' in Γ' . Distinct pairs give distinct pairs of occurrences, and these are counted in $J_R(\Gamma, \Gamma')$. Suppose now that x is not a cell of Γ . By Definition 6.12(b), x lies within distance one of a $*$ -component C of B^1 containing a cell of Γ , and C lies in a range- R_c component H of V containing a cell of Γ . If x' is a cell of Γ' , then $d_\infty(H, \Gamma') \leq d_\infty(C, x') \leq R + 1 \leq R_c$. Otherwise x' lies within distance one of a $*$ -component C' of B^2 containing a cell of Γ' , which lies in a range- R_c component H' of V containing a cell of Γ' ; then $d_\infty(C, C') \leq R + 2 \leq R_c$, so $H' = H$. In both cases $H \in \mathcal{H}$, and $x \in \mathcal{N}_1(U_*)$. Symmetrically, if x' is not a cell of Γ' , then $x' \in \mathcal{N}_1(U_*)$. Since $|\mathcal{N}_1(U_*)| \leq 27A_*$, and at most $(2R + 1)^3$ cells of a simple route lie within distance R of a given cell, the pairs with $x \in \mathcal{N}_1(U_*)$ number at most $27(2R + 1)^3 A_*$, and so do the pairs with $x' \in \mathcal{N}_1(U_*)$. This proves (6.17). The routes in $\mathcal{L}$ and $\mathcal{L}'$ are computed from the fields $B[D] \subset V$. Then (6.17) applies to them. $\square$

Remark 6.17. In [111, Lemmas 10.2–10.3], all unusable sets arising from the permitted replacements lie in one enlarged set; the resulting pairs of paths are counted by an exponential in the total size of components approached by both paths. There are three

differences. First, our routes are closed cycles whose rejection rules read the components met by the skeleton, so the localization is formulated through the set $\mathcal{Q}(B, \Gamma)$ of Lemma 6.13; Definition 6.12(a) replaces the property in [111, Lemma 8.2] that bad components not hit by an accepted skeleton can be removed without changing the path. Second, the inclusion of the perturbed fields in V and the location (6.15) of their differences are hypotheses here; in Section 7 they follow from row-local accessibility, with B_0 the bad set and V the vulnerable set of the reindexed base matching. Third, part (ii), the identity in both directions including rejection, is needed because Theorem 7.14 bounds a signed sum, whereas [111] bounds a positive second moment, for which (i) and (iii) suffice. The fields $B[D]$ need not arise from any matching, which is why the routing maps are defined on every field; the lists count routes, not the rows selected along them.

Used in: Theorem 7.14.

6.7 Averaging over the environment

Lemma 6.18. *Let $\mathbb{L}$ be $\mathbb{Z}^3$ or $\mathbb{L}_K$, let $R_c \geq 1$ be an integer, and let $c_{\text{rec}} \geq 0$ and $p \geq 0$ satisfy $q = (2R_c + 1)^6 p e^{c_{\text{rec}}} \leq e^{-2}$. Let Γ, Γ' be random closed walks on $\mathbb{L}$, each of positive length almost surely, and let V be a random finite subset of $\mathbb{L}$, independent of (Γ, Γ') , such that $\mathbb{P}(F \subset V) \leq p^{|F|}$ for every finite $F \subset \mathbb{L}$. Define A_* from V, Γ, Γ' , and R_c as in Lemma 6.16. Then almost surely*

$$\mathbb{E}[e^{c_{\text{rec}} A_*} \mid \Gamma, \Gamma'] \leq \exp(\zeta(p, c_{\text{rec}}) S_{R_c}(\Gamma, \Gamma')), \quad \zeta(p, c_{\text{rec}}) = \frac{e(2R_c + 1)^3 \sqrt{q}}{1 - \sqrt{q}}. \quad (6.18)$$

In particular $\zeta(p, c_{\text{rec}}) \rightarrow 0$ as $p \downarrow 0$, for fixed R_c and c_{rec} .

Proof. Since V is independent of (Γ, Γ') , it suffices to prove (6.18) for fixed closed walks Γ, Γ' , with the expectation taken over V . Call a set $F \subset \mathbb{L}$ admissible if it is finite, nonempty, range- R_c connected, and satisfies $d_\infty(F, \Gamma) \leq R_c$ and $d_\infty(F, \Gamma') \leq R_c$. The elements of $\mathcal{H}$ are pairwise disjoint admissible sets contained in V , and $A_* = \sum_{H \in \mathcal{H}} |H|$. Expanding the product, we have

$$e^{c_{\text{rec}} A_*} = \prod_{H \in \mathcal{H}} (1 + (e^{c_{\text{rec}} |H|} - 1)) = \sum_{\mathcal{F} \subset \mathcal{H}} \prod_{H \in \mathcal{F}} (e^{c_{\text{rec}} |H|} - 1) \leq \sum_{\mathcal{F}} \mathbf{1}\left\{ \bigcup_{F \in \mathcal{F}} F \subset V \right\} \prod_{F \in \mathcal{F}} e^{c_{\text{rec}} |F|},$$

where on the right $\mathcal{F}$ ranges over the finite collections of pairwise disjoint admissible sets. In the last step we dropped maximality, since every subcollection of $\mathcal{H}$ is such a collection, and used $e^{c_{\text{rec}} |H|} - 1 \leq e^{c_{\text{rec}} |H|}$. By disjointness the union has $\sum_{F \in \mathcal{F}} |F|$ cells. Taking expectations, which is legitimate for the nonnegative countable sum by Tonelli's theorem, and applying the prescribed-set bound, we obtain

$$\mathbb{E} e^{c_{\text{rec}} A_*} \leq \sum_{\mathcal{F}} \prod_{F \in \mathcal{F}} (p e^{c_{\text{rec}}})^{|F|} \leq \prod_F (1 + (p e^{c_{\text{rec}}})^{|F|}) \leq \exp\left(\sum_F (p e^{c_{\text{rec}}})^{|F|}\right).$$

In the second inequality we dropped disjointness, and the product and the last sum run over all admissible sets F .

Let F be admissible with m cells. Let z be an occurrence of Γ and $x \in F$ a cell such that $d_\infty(z, x) \leq R_c$. Since F has diameter at most $R_c(m-1)$ and lies within distance R_c of Γ' , we have $d_\infty(z, \Gamma') \leq R_c + R_c(m-1) + R_c = R_c(m+1)$. Assigning F to the pair (z, x) , and using Lemma 6.10 for the number of range- R_c connected sets of m cells containing x , we find

$$\begin{aligned} \sum_F (pe^{c_{\text{rec}}})^{|F|} &\leq \sum_{z \in \Gamma} \sum_{x \in \mathcal{N}_{R_c}(z)} \sum_{m \geq 1} (2R_c + 1)^{6(m-1)} (pe^{c_{\text{rec}}})^m \mathbf{1}\{d_\infty(z, \Gamma') \leq R_c(m+1)\} \\ &\leq (2R_c + 1)^3 \sum_{z \in \Gamma} \sum_{m \geq 1} q^m \mathbf{1}\{d_\infty(z, \Gamma') \leq R_c(m+1)\}. \end{aligned}$$

Counting a set more than once only increases the bound. Let $d = d_\infty(z, \Gamma')$. If $d \leq R_c(m+1)$, then $m \geq d/R_c - 1$, and $q \leq e^{-2}$ gives $q^m = q^{m/2} q^{m/2} \leq q^{m/2} e^{-m} \leq e q^{m/2} e^{-d/R_c}$. Summing over $m \geq 1$ gives $\sum_m q^m \mathbf{1}\{d \leq R_c(m+1)\} \leq e\sqrt{q}(1 - \sqrt{q})^{-1} e^{-d/R_c}$. Inserting this bound into the previous display shows that the exponent is at most $\zeta(p, c_{\text{rec}}) S_{R_c}(\Gamma, \Gamma')$, which proves (6.18). On $\mathbb{L}_K$, admissible sets may wrap around the torus; the argument uses only the periodic distance and the counting bound of Lemma 6.10, both valid there. $\square$

This is the expansion over components in the proof of [115, Proposition 7.1] ([115, Section 7.4]), reproduced in [112, Section 8.4, Equations (8.18)–(8.20)]; see also [111, Section 10.5]. There it is applied to bad components approached by two paths in a common environment; here it is applied to the vulnerable set, with an exponent c_{rec} that in Theorem 7.14 includes the term $\log 4$ from the list count in Lemma 6.16. Only the prescribed-set bound under the law of V enters, with both walks fixed; no independence between cells and no conditioning on explored cells is used.

Used in: Corollary 6.19.

Corollary 6.19. *Let $R \geq 0$ and $R_c \geq 1$ be integers, and let κ, ξ, C be the constants of Proposition 6.6(c). Let Γ, Γ' be independent winding skeletons with $K \geq 20$, respectively independent area skeletons with a common inner cube and $K_{\text{in}} \geq 2^{14}$. Let V be a random finite subset of $\mathbb{L}_K$, respectively of $\mathbb{Z}^3$, independent of the two seeds, such that $\mathbb{P}(F \subset V) \leq p^{|F|}$ for every finite F . Let $\varepsilon \geq 0$ and $c_{\text{rec}} \geq 0$ satisfy*

$$2\varepsilon \leq \kappa, \quad (2R_c + 1)^6 pe^{c_{\text{rec}}} \leq e^{-2}, \quad 2\zeta(p, c_{\text{rec}}) \leq \xi,$$

with ζ as in Lemma 6.18, and define A_* as in Lemma 6.16. Then

$$\mathbb{E}[e^{\varepsilon J_R(\Gamma, \Gamma') + c_{\text{rec}} A_*}] - 1 \leq \frac{C}{K}, \quad \text{respectively} \quad \mathbb{E}[e^{\varepsilon J_R(\Gamma, \Gamma') + c_{\text{rec}} A_*}] - 1 \leq \frac{C}{K_{\text{in}}}. \quad (6.19)$$

Proof. We treat the winding case; the area case is identical with K_{in} in place of K . Conditionally on the two seeds, $J_R(\Gamma, \Gamma')$ is fixed, and by Lemma 6.18 we have

$$\mathbb{E} \left[e^{\varepsilon J_R(\Gamma, \Gamma') + c_{\text{rec}} A_*} \mid \Gamma, \Gamma' \right] \leq \exp(\varepsilon J_R(\Gamma, \Gamma') + \zeta(p, c_{\text{rec}}) S_{R_c}(\Gamma, \Gamma')).$$

By the Cauchy–Schwarz inequality, monotonicity of the exponential moments in their coefficients, and Proposition 6.6(c), we have

$$\mathbb{E} e^{\varepsilon J_R + \zeta S_{R_c}} \leq (\mathbb{E} e^{2\varepsilon J_R})^{1/2} (\mathbb{E} e^{2\zeta S_{R_c}})^{1/2} \leq (\mathbb{E} e^{\kappa J_R})^{1/2} (\mathbb{E} e^{\xi S_{R_c}})^{1/2} \leq 1 + \frac{C}{K'},$$

where J_R and S_{R_c} are evaluated at (Γ, Γ') . Taking expectations in the previous display proves (6.19). $\square$

In Theorem 7.14, we apply Corollary 6.19 with encounter coefficient $2\varepsilon_{\text{max}}$ and coefficient $2c_{\text{rec}}$ of A_* in the exponential. The parameter choices, including $4\varepsilon_{\text{max}} \leq \kappa$, $(2R_c + 1)^6 p e^{2c_{\text{rec}}} \leq e^{-2}$, and $2\zeta(p, 2c_{\text{rec}}) \leq \xi$, are given in Sections 10.3 and 11.5.

Used in: Theorem 7.14.

Matching systems and the surgery theorem

A row labels a path's initial endpoint, a slot labels a terminal endpoint, and a matching assigns one slot to each row. We call the data revealed before this assignment is resampled the exposed data. This section introduces matching systems, which include the law of one slab of the hard-sphere gas after suitable data have been exposed (Section 9). The rows are partitioned into m groups with at most M rows of each group in each cell, and a surgery moves rows of one group, chosen uniformly at random. Under hypotheses (H1)–(H5), Theorem 7.14 bounds the chi-square divergence of the completed surgery kernel, averaged over the group and over the random skeleton, by $C_*/(mK) + C_\times \eta_{\text{loc}}/K$, where the collision error η_{loc} does not depend on m , and its acceptance probability from below by $1 - Cp$. Paths belong to an abstract standard Borel space; the analytic inputs are the mass and insertion bounds (H2), (H3), and (H3₁), and the route properties (H5) come from Section 6. The applications in Sections 10 and 11 combine this theorem with the slab verification of (H1)–(H4) and (H4⁺).

The proof follows the second-moment method of Lemma 5.3. By (5.5), the divergence of the surgeries of one group is an average over two independent skeletons Γ, Γ' of the inner product $\langle L_\Gamma - 1, L_{\Gamma'} - 1 \rangle$ of two centered likelihoods. In Section 7.3, we compute L_Γ and write this inner product as a sum, over pairs of labeled routes, of four terms. Three of them are given by exact identities (Lemma 7.7). For the fourth, we compare the conditional normalizing constants of four matchings (Lemma 7.8) in Section 7.4. In Section 7.5, we sum over the selected rows (Lemma 7.12). In Section 7.7, we assemble the four terms. When the two routes have no encounters and their skeletons approach no common vulnerable component, route stability identifies their indicators, and the four terms cancel exactly (Lemma 7.9). Otherwise, only the two positive terms are kept; they are bounded by the cost of the encounters, and, when the skeletons approach a common vulnerable component, the routes are counted by route lists. Finally, an expansion over clusters of vulnerable cells averages over the configuration. The average over groups also produces inner products of the likelihoods of two different groups. A move of one group changes none of the counts, bad sets, and rates of another group, so three of the four terms coincide, and the comparison of four matchings bounds the signed sum from above by a loss confined to the encounters of the two routes (Lemma 7.13).

Most ingredients adapt arguments of [111, 112, 115]. The cyclic shift of endpoints along a route, combined with exact conditional resampling, is the construction of [111,

Section 9], where the routes are open; ours are closed, and the surgery changes the endpoint matching of a closed slab. The cancellation of conditional normalizing constants away from the places where two routes come close is the argument of [115, Lemma 8.3], reproduced as [112, Lemma 9.3], which we adapt to changes of the endpoint matching; Remark 7.10 explains why that lemma cannot be applied verbatim. The comparison of four endpoint states, including rows shared by the two routes and their neighbors on a route, adapts [111, Section 10]. The summation over selected rows adapts [112, Equation (9.16)] and [111, Section 10.5], and the route lists that control routes depending on the configuration adapt [111, Lemmas 10.2 and 10.3]. The signed estimate is proved in Theorem 7.14; see Remark 7.15 for its role in the variance bound.

7.1 Matching systems

Throughout this section, $\mathbb{L}$ is the cell lattice of Section 6, that is, $(\mathbb{Z}/K\mathbb{Z})^3$ on the torus and $\mathbb{Z}^3$ for a container, with the maximum distance d_∞ . Two cells are face neighbors if they differ by $\pm e_k$ for some $k \in \{1, 2, 3\}$; these are the nearest neighbors of Section 6.1. For $P \subset \mathbb{L}$ and $r \geq 0$, we write $\mathcal{N}_r(P) = \{x \in \mathbb{L} : d_\infty(x, P) \leq r\}$. As in Section 6.1, a route is either a simple closed walk $\gamma = (v_0, \dots, v_{r-1}, v_r = v_0)$ of length $r \geq 3$, or the empty route $\dagger$, which stands for a rejection. The cells $v_0, \dots, v_{r-1}$ of a route are distinct, consecutive cells v_k and v_{k+1} are face neighbors for $0 \leq k \leq r-1$, and the orientation is part of the route. We identify γ with its set of cells when we write $v \in \gamma$, $|\gamma|$, or $d_\infty(x, \gamma)$, and $\mathcal{N}_r(\dagger) = \emptyset$. For finite sequences P, P' of cells and $R \geq 0$, the encounter count of Section 6 is

$$J_R(P, P') = \sum_{z \in P} \sum_{w \in P'} \mathbf{1}\{d_\infty(z, w) \leq R\},$$

where occurrences are counted with multiplicity, and $J_R(\dagger, P') = J_R(P, \dagger) = 0$. For a finite measure ϱ , we write $\varrho(1)$ for its total mass.

Definition 7.1. A matching system consists of the following data.

1. A probability measure ν on a standard Borel space $\mathbf{E}$. Its elements E are called exposed data.
2. For each $E \in \mathbf{E}$, finite sets $I = I(E)$ of rows and $J = J(E)$ of slots with $|I| = |J|$, an anchor map $\text{anch} = \text{anch}_E: I \rightarrow \mathbb{L}$, and a partition $I = I^{(1)} \sqcup \dots \sqcup I^{(m)}$ of the rows into groups, where the integer $m \geq 1$ does not depend on E . We write $I_v = \{i \in I : \text{anch}(i) = v\}$ and $I_v^{(a)} = I_v \cap I^{(a)}$. There exists an integer $M \geq 1$ such that $|I_v^{(a)}| \leq M$ for all E, v , and a ; in particular, $|I_v| \leq mM$. The sets $I(E)$ and $J(E)$ are subsets of a fixed countable set, and they, the anchor map, and the groups depend measurably on E .

3. A standard Borel space P of paths. For each E , each $i \in I$, and each $j \in J$, a finite measure K_{ij}^E on P and a measurable set $\mathcal{P}_{ij}^E \subset P$ with $K_{ij}^E(P \setminus \mathcal{P}_{ij}^E) = 0$. For distinct rows $i, k \in I$, a measurable function $h_{ik}^E: P \times P \rightarrow \{0, 1\}$, the pair check, with $h_{ki}^E(\omega', \omega) = h_{ik}^E(\omega, \omega')$. The maps $(E, \omega) \mapsto \mathbf{1}\{\omega \in \mathcal{P}_{ij}^E\}$ and $(E, \omega, \omega') \mapsto h_{ik}^E(\omega, \omega')$ are Borel measurable on their index domains, and $E \mapsto K_{ij}^E(B)$ is measurable for every Borel set $B \subset P$.
4. Numbers $A_{ij}^E \in \{0, 1\}$ for $i \in I$ and $j \in J$, the accessibility indicators, depending measurably on E .
5. A finite set $\Lambda_{\text{act}} \subset \mathbb{L}$ of active cells, which may depend on K but does not depend on E , and an integer $m_* \geq 4$.

A matching is a bijection $\tau: I \rightarrow J$. For a matching τ , let

$$M_\tau^E(d\omega) = \prod_{i \in I} K_{i\tau(i)}^E(d\omega_i) \prod_{\{i, k\}} h_{ik}^E(\omega_i, \omega_k), \quad Z_\tau(E) = M_\tau^E(1), \quad \Xi(E) = \sum_\tau Z_\tau(E), \quad (7.1)$$

where the second product is over unordered pairs of distinct rows, and the sum is over all matchings. We assume that $0 < \Xi(E) < \infty$ for ν -almost every E . The matching law is the probability measure defined, for E in this full-measure set, by

$$\mu(dE, \{\tau\}, d\omega) = \nu(dE) \Xi(E)^{-1} M_\tau^E(d\omega)$$

on the state space X of triples (E, τ, ω) with $E \in E$, τ a matching, and $\omega \in P^{I(E)}$. For a matching τ , a group a , and a cell v , the count of accessible rows of group a , the bad set, and the vulnerable set of group a are

$$m_v^{(a)}(\tau) = \sum_{i \in I_v^{(a)}} A_{i\tau(i)}^E, \\ B_\tau^{(a)} = \{v \in \Lambda_{\text{act}} : m_v^{(a)}(\tau) < m_*\}, \quad V_\tau^{(a)} = \{v \in \Lambda_{\text{act}} : m_v^{(a)}(\tau) < m_* + 2\}.$$

We write a state as $x = (E, \sigma, \omega)$ and call σ the current matching. The state space X is a measurable subset of the disjoint union, over finite sets I_0, J_0 of the fixed countable set and bijections $\tau: I_0 \rightarrow J_0$, of the spaces $E \times P^{I_0}$, and it is standard Borel. We often fix E and drop it from the notation. For ν -almost every E , we write μ_E for the conditional law of (σ, ω) given E , that is, $\mu_E(\{\tau\}, d\omega) = \Xi(E)^{-1} M_\tau^E(d\omega)$.

Under μ_E , the matching has law $Z_\sigma(E)/\Xi(E)$, which is in general far from uniform. For example, two rows with masses $K_{11}(1) = K_{22}(1) = 1$ and $K_{12}(1) = K_{21}(1) = \varepsilon$, and no checks, give the identity matching probability $1/(1 + \varepsilon^2)$. Given σ , their joint law is the product of the normalized measures $K_{i\sigma(i)}$, conditioned on all pair checks. The measure K_{ij} contains all individual restrictions on the path of row i when it is paired with slot j . The pair checks are the only interactions between rows, and they do not depend on

the matching. We say that ω is individually permitted for a matching τ if $\omega_i \in \mathcal{P}_{i\tau(i)}$ for every i . Under μ , the paths are individually permitted for the current matching almost surely, and we remove the other states from X ; this does not change μ . The accessibility indicator $A_{i\tau(i)}$ depends only on the row i and its slot $\tau(i)$. A change of the matching changes the count $m_v^{(a)}$ only in cells that contain rows of group a whose slots change. Cells outside Λ_{act} are neither bad nor vulnerable, and (H5) below requires routes to stay in Λ_{act} .

In Section 9, a matching system describes one slab of the hard-sphere gas of duration $\delta = 2t$. The rows are the retained initial endpoints, the slots are the retained terminal endpoints, and the exposed data consist of the endpoint configurations, the eligibility indicators, the ordered lists of retained rows, the paths that are not retained, and the set of retained slots; the groups are successive blocks of the lists. The measure K_{ij}^E is the unnormalized Brownian bridge from the initial endpoint of row i to slot j , restricted by individual events and by avoidance of the paths that are not retained, and the pair checks are mutual avoidance. In Proposition 9.5, we prove that the conditional law of the slab is a matching law.

For a set $S \subset I$ of rows, a matching τ , and paths $\omega_{S^c} = (\omega_k)_{k \notin S}$, we write

$$M_\tau^S(d\omega_S; \omega_{S^c}) = \prod_{i \in S} K_{i\tau(i)}(d\omega_i) \prod_{\{i,k\} \cap S \neq \emptyset} h_{ik}(\omega_i, \omega_k), \quad Z_S(\tau; \omega_{S^c}) = M_\tau^S(1; \omega_{S^c}), \quad (7.2)$$

where the product is over unordered pairs of distinct rows that meet S . Let

$$N_\tau^{S^c}(d\omega_{S^c}) = \prod_{i \notin S} K_{i\tau(i)}(d\omega_i) \prod_{\{i,k\} \subset S^c} h_{ik}(\omega_i, \omega_k).$$

This measure on the outside paths includes no check joining S to S^c . Then

$$M_\tau(d\omega) = N_\tau^{S^c}(d\omega_{S^c}) M_\tau^S(d\omega_S; \omega_{S^c}), \quad (7.3)$$

and $N_\tau^{S^c}$ depends on τ only through its restriction to S^c . In particular, under μ_E , the conditional law of ω_S given σ and ω_{S^c} is $M_\sigma^S(\cdot; \omega_{S^c}) / Z_S(\sigma; \omega_{S^c})$.

7.2 Hypotheses

We say that a pair $(i, l) \in I \times J$ is a crossed pairing if there exist $k \in I$ and $j \in J$ such that $\text{anch}(i)$ and $\text{anch}(k)$ are equal or face neighbors and $A_{ij} = A_{kl} = 1$. Taking $k = i$ and $l = j$, we see that every accessible pairing is crossed.

Definition 7.2. Let $R_{\text{int}} \geq 0$ be an integer, let $0 < z_- \leq z_+ < \infty$, $\eta_{\text{glob}} \in [0, 1/2]$, $\eta_1 \geq 0$, and $p, p_\times \in [0, 1]$. A matching system satisfies (H1)–(H5) if (H1)–(H3) and (H3₁) hold for ν -almost every E , and (H4), (H4⁺), and (H5) hold.

- (H1) (Locality.) $h_{ik}^E \equiv 1$ whenever $d_\infty(\text{anch}(i), \text{anch}(k)) > R_{\text{int}}$.
- (H2) (Mass bounds for crossed pairings.) $z_- \leq K_{il}^E(1) \leq z_+$ for every crossed pairing (i, l) .
- (H3) (Insertion.) For every matching τ and row i such that $(i, \tau(i))$ is a crossed pairing, every set $I' \subset I \setminus \{i\}$, and all paths $\omega_k \in \mathcal{P}_{k\tau(k)}^E$, $k \in I'$,

$$\int \prod_{k \in I'} h_{ik}^E(\omega_i, \omega_k) K_{i\tau(i)}^E(d\omega_i) \geq (1 - \eta_{\text{glob}}) K_{i\tau(i)}^E(1).$$

- (H3₁) (One opponent.) For every matching τ and row i such that $(i, \tau(i))$ is a crossed pairing, every row $k \neq i$, and every path $\omega_k \in \mathcal{P}_{k\tau(k)}^E$,

$$K_{i\tau(i)}^E(\{\omega : h_{ik}^E(\omega, \omega_k) = 0\}) \leq \eta_1 K_{i\tau(i)}^E(1).$$

- (H4) (Rare vulnerable cells.) $\mathbb{P}_\mu(\mathcal{F} \subset V_\sigma^{(a)}) \leq p^{|\mathcal{F}|}$ for every group a and every finite set $\mathcal{F} \subset \mathbb{L}$.
- (H4⁺) (Rare vulnerable cells of two groups.) $\mathbb{P}_\mu(\mathcal{F} \subset V_\sigma^{(a)} \cup V_\sigma^{(b)}) \leq p_\times^{|\mathcal{F}|}$ for all groups $a \neq b$ and every finite set $\mathcal{F} \subset \mathbb{L}$.

- (H5) (Routes.) There exist a standard Borel space $\mathfrak{G}$ with a probability measure $\mathfrak{p}$, whose elements are called skeletons, and a routing map $\mathcal{R}$ that assigns to every set $B \subset \Lambda_{\text{act}}$ and every $\Gamma \in \mathfrak{G}$ either $\dagger$ or a route $\mathcal{R}(B, \Gamma)$, measurably in Γ . Every skeleton Γ determines, measurably in Γ , a finite sequence of cells, which we also denote by Γ . Fix integers $R \geq \max\{R_{\text{int}}, 1\}$ and $R_c \geq R + 2$. For $V \subset \Lambda_{\text{act}}$ and $\Gamma, \Gamma' \in \mathfrak{G}$, let $A_*(V; \Gamma, \Gamma')$ be the total number of cells in those components H of V , for the adjacency $0 < d_\infty(x, y) \leq R_c$, that satisfy $d_\infty(H, \Gamma) \leq R_c$ and $d_\infty(H, \Gamma') \leq R_c$. For $B_0 \subset V$, let $\mathcal{L}(V, B_0; \Gamma, \Gamma')$ be the set of pairs $(\mathcal{R}(B^1, \Gamma), \mathcal{R}(B^2, \Gamma'))$, where $B^1, B^2 \subset V$ satisfy

$$B^1 \triangle B_0 \subset \mathcal{N}_1(\mathcal{R}(B^2, \Gamma')), \quad B^2 \triangle B_0 \subset \mathcal{N}_1(\mathcal{R}(B^1, \Gamma)).$$

There exist constants $K_0, \kappa > 0, p_0 > 0, C_{\text{acc}}$, and C_{ch} , and, for each $c \geq 0$, numbers $p_{\text{env}}(c) > 0$ and $C_{\text{fin}}(c) < \infty$, none of which depends on K , such that the following hold for $K \geq K_0$.

- (a) Every route $\mathcal{R}(B, \Gamma)$ has all its cells in $\Lambda_{\text{act}} \setminus B$.
- (b) If B is a random subset of Λ_{act} that is independent of Γ , where Γ has law $\mathfrak{p}$, and $\mathbb{P}(\mathcal{F} \subset B) \leq p^{|\mathcal{F}|}$ for every finite $\mathcal{F} \subset \mathbb{L}$, where $p \leq p_0$, then $\mathbb{P}(\mathcal{R}(B, \Gamma) = \dagger) \leq C_{\text{acc}} p$.
- (c) Let $\Gamma, \Gamma' \in \mathfrak{G}$, let $V \subset \Lambda_{\text{act}}$ be finite, let $B_0 \subset V$, and write $A_* = A_*(V; \Gamma, \Gamma')$ and $\mathcal{L} = \mathcal{L}(V, B_0; \Gamma, \Gamma')$. Then:
 - (i) $\mathcal{L}$ has at most 4^{A_*} elements;

- (ii) if $A_* = 0$, then for every pair of routes (γ, γ') and all $B^1, B^2 \subset V$ with $B^1 \triangle B_0 \subset \mathcal{N}_1(\gamma')$ and $B^2 \triangle B_0 \subset \mathcal{N}_1(\gamma)$,

$$\mathbf{1}\{\mathcal{R}(B^1, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B^2, \Gamma') = \gamma'\} = \mathbf{1}\{\mathcal{R}(B_0, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B_0, \Gamma') = \gamma'\};$$

- (iii⁺) for all $B^1, B^2 \subset V$, the routes $\gamma = \mathcal{R}(B^1, \Gamma)$ and $\gamma' = \mathcal{R}(B^2, \Gamma')$ satisfy $J_R(\gamma, \gamma') \leq J_R(\Gamma, \Gamma') + C_{\text{ch}} A_*$; in particular, every pair of routes in $\mathcal{L}$ satisfies this bound.

- (d) Let $c \geq 0$, $p \leq p_{\text{env}}(c)$, and $0 \leq \varepsilon' \leq \kappa/2$. Let Γ, Γ' be independent with law $\mathbf{p}$, and let V be a random subset of Λ_{act} that is independent of (Γ, Γ') and satisfies $\mathbb{P}(\mathcal{F} \subset V) \leq p^{|\mathcal{F}|}$ for every finite $\mathcal{F} \subset \mathbb{L}$. Then

$$\mathbb{E} \exp(\varepsilon' J_R(\Gamma, \Gamma') + c A_*(V; \Gamma, \Gamma')) \leq 1 + \frac{C_{\text{fin}}(c)}{K}.$$

The properties in (H5) are proved in Section 6 for the skeletons and routing maps used there: the winding skeletons of Definition 6.4 with the winding routing map of Definition 6.14 on the torus, and the area skeletons of Definition 6.5 with the area routing map of Definition 6.14 in a container. Properties (a) and (b) are part of Proposition 6.15. Property (c) is Lemma 6.16, applied with perturbation radius one, and part (iii⁺) is its part (iii), which is stated for all $B^1, B^2 \subset V$; the condition $R_c \geq R + 2$ is the condition on the component range there. Property (d) combines Lemma 6.18 with the encounter bounds of Proposition 6.6, as in Corollary 6.19, whose hypotheses $2\varepsilon' \leq \kappa$, $(2R_c + 1)^6 p e^c \leq e^{-2}$, and $2\zeta(p, c) \leq \xi$ hold for $\varepsilon' \leq \kappa/2$ and $p \leq p_{\text{env}}(c)$ if $p_{\text{env}}(c)$ is chosen small enough; in a container, that corollary gives the bound C/K_{in} for the side K_{in} of the inner cube. In this application the scale K in (H5) and Theorem 7.14 is read as K_{in} (Section 11): the proof uses this scale only in the threshold $K \geq K_0$ and in (H5)(d), so the conclusion holds with K_{in} in place of K and with the same comparison constants. In (c), the sets B^1, B^2 are arbitrary subsets of V ; they need not be bad sets of actual matchings. Part (ii) compares the indicators of a prescribed pair of routes in both directions, which is needed to identify all four terms of the signed expansion below.

We record some comments on the hypotheses. Let $C_{\text{int}} = (2R_{\text{int}} + 1)^3$. By (H1) and $|I_v| \leq mM$, every row has nontrivial checks with at most $C_{\text{int}}mM$ other rows. Hence (H3₁) and a union bound imply (H3) with $\eta_{\text{glob}} = C_{\text{int}}mM\eta_1$, provided that this number is at most $1/2$; in Section 9, this is how (H3) is verified. We set $\eta_{\text{loc}} = 4C_{\text{int}}\eta_1$; this local error, used in Lemma 7.8, does not involve the number mM of rows per cell. Hypothesis (H2) concerns row-slot pairings that need not occur together in one matching: the rows i and k may coincide, the slots j and l may coincide, and the two accessibility conditions may hold in different matchings. This form is needed to bound masses in the four matchings of Lemma 7.4. In (H3), the other rows need only be individually permitted; they need not be accessible, and they need not pass their checks with each other. Since I' is arbitrary,

the bound persists when rows are deleted. In (H3₁), the opponent k may belong to any group. Hypotheses (H4) and (H4⁺) are statements about the joint law μ , including the exposed data; they are not assumed conditionally on E . Since $B_\sigma^{(a)} \subset V_\sigma^{(a)}$, they also hold for the bad sets.

The next lemma is the form in which (H3) is used. Fix E , a matching τ , and a set $O \subset I$ of rows with paths $\omega_O = (\omega_k)_{k \in O}$ that are individually permitted for τ . For $i \notin O$, we absorb the checks with O into the path measure of i and write

$$\widehat{K}_i^\tau(d\omega_i) = K_{i\tau(i)}(d\omega_i) \prod_{k \in O} h_{ik}(\omega_i, \omega_k), \quad \widehat{z}_i^\tau = \widehat{K}_i^\tau(1).$$

For disjoint sets $W, X \subset I \setminus O$ and paths ω_X that are individually permitted for τ , let

$$Z_W(\tau; \omega_X) = \int \prod_{i \in W} \widehat{K}_i^\tau(d\omega_i) \prod_{\{i,k\} \subset W} h_{ik}(\omega_i, \omega_k) \prod_{i \in W, k \in X} h_{ik}(\omega_i, \omega_k),$$

with $Z_\emptyset(\tau; \omega_X) = 1$. In $Z_W(\tau; \omega_X)$, the rows of W are integrated, the rows of $O \cup X$ are fixed opponents, and all other rows are deleted together with their checks. It depends on τ only through its restriction to W . With $O = I \setminus S$ and $X = \emptyset$, we recover $Z_S(\tau; \omega_{S^c})$ from (7.2).

Lemma 7.3. *Assume (H3) for E , and let τ , O , ω_O , X , and ω_X be as above. Let $W_0 \subset W \subset I \setminus (O \cup X)$.*

1. *We have*

$$Z_W(\tau; \omega_X) \leq \left(\prod_{i \in W \setminus W_0} \widehat{z}_i^\tau \right) Z_{W_0}(\tau; \omega_X). \quad (7.4)$$

If moreover $(i, \tau(i))$ is a crossed pairing for every $i \in W \setminus W_0$, then $(1 - \eta_{\text{glob}})K_{i\tau(i)}(1) \leq \widehat{z}_i^\tau \leq K_{i\tau(i)}(1)$ for these i , and

$$Z_W(\tau; \omega_X) \geq (1 - \eta_{\text{glob}})^{|W \setminus W_0|} \left(\prod_{i \in W \setminus W_0} \widehat{z}_i^\tau \right) Z_{W_0}(\tau; \omega_X). \quad (7.5)$$

2. *Assume moreover (H1) and (H3₁) for E , and that $(i, \tau(i))$ is a crossed pairing for every $i \in W \setminus W_0$. Let $N \geq 0$ be an integer such that every $i \in W \setminus W_0$ has at most N rows $k \in (W \cup X) \setminus \{i\}$ with $d_\infty(\text{anch}(i), \text{anch}(k)) \leq R_{\text{int}}$, and suppose that $2N\eta_1 \leq 1$. Then*

$$Z_W(\tau; \omega_X) \geq (1 - 2N\eta_1)^{|W \setminus W_0|} \left(\prod_{i \in W \setminus W_0} \widehat{z}_i^\tau \right) Z_{W_0}(\tau; \omega_X).$$

Proof. The upper bound (7.4) follows by bounding the checks incident to the rows of $W \setminus W_0$ by one. For the lower bound, the inequality $\widehat{z}_i^\tau \leq K_{i\tau(i)}(1)$ holds because the checks are at most one, and (H3) with $I' = O$ gives $\widehat{z}_i^\tau \geq (1 - \eta_{\text{glob}})K_{i\tau(i)}(1)$. We prove

(7.5) by induction on $|W \setminus W_0|$, the case $W = W_0$ being trivial. Let $i \in W \setminus W_0$ and $W_1 = W \setminus \{i\}$. Integrating ω_i last, we have

$$Z_W(\tau; \omega_X) = \int \Lambda_{W_1}(\mathrm{d}\omega_{W_1}) \int \widehat{K}_i^\tau(\mathrm{d}\omega_i) \prod_{k \in W_1 \cup X} h_{ik}(\omega_i, \omega_k),$$

where the measure

$$\Lambda_{W_1}(\mathrm{d}\omega_{W_1}) = \prod_{k \in W_1} \widehat{K}_k^\tau(\mathrm{d}\omega_k) \prod_{\{k, l\} \subset W_1} h_{kl}(\omega_k, \omega_l) \prod_{k \in W_1, l \in X} h_{kl}(\omega_k, \omega_l)$$

has total mass $Z_{W_1}(\tau; \omega_X)$. The measure Λ_{W_1} is carried by paths that are individually permitted for τ . For such ω_{W_1} , by (H3) with $I' = O \cup W_1 \cup X$, we have

$$\begin{aligned} \int \widehat{K}_i^\tau(\mathrm{d}\omega_i) \prod_{k \in W_1 \cup X} h_{ik}(\omega_i, \omega_k) &= \int K_{i\tau(i)}(\mathrm{d}\omega_i) \prod_{k \in O \cup W_1 \cup X} h_{ik}(\omega_i, \omega_k) \\ &\geq (1 - \eta_{\text{glob}}) K_{i\tau(i)}(1) \geq (1 - \eta_{\text{glob}}) \widehat{z}_i^\tau. \end{aligned}$$

We conclude that $Z_W(\tau; \omega_X) \geq (1 - \eta_{\text{glob}}) \widehat{z}_i^\tau Z_{W_1}(\tau; \omega_X)$, and the induction hypothesis applied to W_1 completes the proof of part (1).

For part (2), we use the same induction and the same representation of $Z_W(\tau; \omega_X)$. Fix paths ω_{W_1} that are individually permitted for τ . Since the checks take values in $\{0, 1\}$, we have $1 - \prod_k h_{ik} \leq \sum_k (1 - h_{ik})$, and since $\widehat{K}_i^\tau \leq K_{i\tau(i)}$,

$$\int \widehat{K}_i^\tau(\mathrm{d}\omega_i) \prod_{k \in W_1 \cup X} h_{ik}(\omega_i, \omega_k) \geq \widehat{z}_i^\tau - \sum_{k \in W_1 \cup X} K_{i\tau(i)}(\{\omega : h_{ik}(\omega, \omega_k) = 0\}).$$

By (H1), only the rows k with $d_\infty(\text{anch}(i), \text{anch}(k)) \leq R_{\text{int}}$ contribute, and there are at most N of them. For each of them, $k \neq i$ and $\omega_k \in \mathcal{P}_{k\tau(k)}$, so (H3₁) bounds its term by $\eta_1 K_{i\tau(i)}(1)$. By part (1), $K_{i\tau(i)}(1) \leq \widehat{z}_i^\tau / (1 - \eta_{\text{glob}}) \leq 2\widehat{z}_i^\tau$. Hence the left side is at least $(1 - 2N\eta_1) \widehat{z}_i^\tau$, and $Z_W(\tau; \omega_X) \geq (1 - 2N\eta_1) \widehat{z}_i^\tau Z_{W_1}(\tau; \omega_X)$. Since $W_1 \cup X \subset W \cup X$, the assumption on N holds for W_1 , and the induction hypothesis applied to W_1 completes the proof. The rows of O enter only through $\widehat{z}_i^\tau$, and they are not counted in N . $\square$

Used in: Lemma 7.6 and Lemma 7.8.

For the lower bound in part (1), we used the fact that the absorbed measure $\widehat{K}_i^\tau$ still passes the remaining checks with probability at least $1 - \eta_{\text{glob}}$ relative to its own mass: the numerator passes the outside and the remaining checks together with probability at least $1 - \eta_{\text{glob}}$ relative to $K_{i\tau(i)}(1)$, and the mass $\widehat{z}_i^\tau$ is at most $K_{i\tau(i)}(1)$.

7.3 Surgeries

Let $\gamma = (v_0, \dots, v_{r-1}, v_r = v_0)$ be a route other than $\dagger$. A labeling of γ is a family $(i_v)_{v \in \gamma}$ of rows with $i_v \in I_v$ for every $v \in \gamma$. The permutation of the labeled route is the

permutation π of I with $\pi(i_{v_k}) = i_{v_{k+1}}$ for $0 \leq k \leq r-1$, where $v_r = v_0$, and $\pi(i) = i$ for all other rows. Its support $S = \text{supp } \pi = \{i_v : v \in \gamma\}$ is the set of selected rows. It contains exactly one row in each cell of γ and no other row, and π is a cyclic permutation of S since $r \geq 3$. The labeled route is determined by π , and we identify the two. We write $\gamma(\pi)$ for the route of π , and Π_E for the finite set of permutations of labeled routes whose cells lie in Λ_{act} . For a group a , we write $\Pi_E^{(a)}$ for the set of $\pi \in \Pi_E$ whose selected rows belong to $I^{(a)}$, that is, with $i_v \in I_v^{(a)}$ for every $v \in \gamma(\pi)$. These sets are disjoint. Permutations act on the right: $(\sigma\pi)(i) = \sigma(\pi(i))$. In the matching $\sigma\pi$, the selected row of v_k is paired with $\sigma(i_{v_{k+1}})$, the slot of the selected row of the next cell along γ , and all other rows keep their slots.

Lemma 7.4. *Assume (H2) for E . Let σ be a matching, and let $\pi, \pi' \in \Pi_E$ have selected rows S, S' . Set $U = S \cup S'$ and*

$$\sigma_1 = \sigma\pi^{-1}, \quad \sigma_2 = \sigma\pi'^{-1}, \quad \sigma_{12} = \sigma\pi^{-1}\pi'^{-1}.$$

1. *If $A_{i\sigma(i)} = 1$ for every $i \in S$, then $(i, \sigma(i))$ and $(i, \sigma\pi(i))$ are crossed pairings for every $i \in S$.*
2. *If $A_{i\sigma_1(i)} = 1$ for every $i \in S$ and $A_{i\sigma_2(i)} = 1$ for every $i \in S'$, then $(i, \tau(i))$ is a crossed pairing for every $i \in U$ and every $\tau \in \{\sigma, \sigma_1, \sigma_2, \sigma_{12}\}$.*

In both cases, $z_- \leq K_{i\tau(i)}(1) \leq z_+$ for the indicated pairings, by (H2).

Proof. For part (1), let $i \in S$. The pairing $(i, \sigma(i))$ is accessible, hence crossed. Since $\sigma\pi(i) = \sigma(\pi(i))$, where $\pi(i)$ is the selected row of the next cell of the route, which is a face neighbor of $\text{anch}(i)$, and since $A_{i\sigma(i)} = A_{\pi(i)\sigma(\pi(i))} = 1$, the pairing $(i, \sigma\pi(i))$ is crossed, with $k = \pi(i)$ and $j = \sigma(i)$ in the definition.

For part (2), we consider the four matchings in turn. For $i \in S$, we have $\sigma(i) = \sigma_1(\pi(i))$ and $A_{i\sigma_1(i)} = A_{\pi(i)\sigma_1(\pi(i))} = 1$, and the anchors of i and $\pi(i)$ are face neighbors; this implies that $(i, \sigma(i))$ is crossed. For $i \in S' \setminus S$, the same argument with σ_2 and π' applies. This treats $\tau = \sigma$. For $\tau = \sigma_1$ and $i \in S$, the pairing $(i, \sigma_1(i))$ is accessible; for $i \in S' \setminus S$, we have $\sigma_1(i) = \sigma(i)$, which was treated. The case $\tau = \sigma_2$ is symmetric. For $\tau = \sigma_{12} = \sigma_1\pi'^{-1}$ and $i \in S \setminus S'$, we have $\sigma_{12}(i) = \sigma_1(i)$. For $i \in S'$, let $k = \pi'^{-1}(i)$, the selected row of the cell preceding $\text{anch}(i)$ on the route of π' , so that $\sigma_{12}(i) = \sigma_1(k)$. If $k \notin S$, then $\sigma_1(k) = \sigma(k) = \sigma_2(i)$, which was treated. If $k \in S$, then $A_{i\sigma_2(i)} = A_{k\sigma_1(k)} = 1$, and $\text{anch}(i)$ and $\text{anch}(k)$ are face neighbors, which implies that $(i, \sigma_1(k))$ is crossed. This includes the case $k \in S \cap S'$ and the case $i \in S \cap S'$. $\square$

Used in: Lemma 7.6 and Lemma 7.8.

Definition 7.5. Let a matching system satisfy (H1)–(H5). Fix a Borel set $E_0 \subset E$ with $\nu(E_0) = 1$ on which $0 < \Xi(E) < \infty$ and (H1)–(H3) and (H3₁) hold, with their full

universal path quantifiers. Such a set exists: after partitioning by the countably many finite index sets and matchings, failures of these hypotheses are projections of Borel sets of finite tuples of paths, hence analytic. They are null in the completion of ν and are contained in a Borel ν -null set; take its complement, also intersected with $\{0 < \Xi < \infty\}$. For a group a , $\pi \in \Pi_E^{(a)}$ with route γ and labeling $(i_v)_{v \in \gamma}$, a matching τ , and a skeleton $\Gamma \in \mathfrak{G}$, the selection rate is

$$R_\pi(\tau \mid \Gamma) = \mathbf{1}\{\mathcal{R}(B_\tau^{(a)}, \Gamma) = \gamma\} r_\pi(\tau), \quad r_\pi(\tau) = \prod_{v \in \gamma} \frac{A_{i_v \tau(i_v)}}{m_v^{(a)}(\tau)},$$

where a factor with $A_{i_v \tau(i_v)} = 0$ is zero; if $A_{i_v \tau(i_v)} = 1$, then $m_v^{(a)}(\tau) \geq 1$. For $E \in E_0$, we set $q_{\pi, \Gamma}(x, \cdot) = 0$ whenever $R_\pi(\sigma \mid \Gamma) = 0$, before forming any quotient. Every zero-rate contribution below is understood in this sense. When the rate is positive, the surgery along π maps a state $x = (E, \sigma, \omega)$ to the state $(E, \sigma\pi, \omega')$, where $\omega'_i = \omega_i$ for $i \notin S = \text{supp } \pi$, and ω'_S is a sample of

$$P_\pi(d\omega'_S \mid x) = \frac{M_{\sigma\pi}^S(d\omega'_S; \omega_{S^c})}{Z_S(\sigma\pi; \omega_{S^c})}.$$

The denominator is positive by Lemma 7.6(1). The surgery kernel of group a with skeleton Γ is the subprobability kernel

$$q_\Gamma^{(a)}(x, \cdot) = \sum_{\pi \in \Pi_E^{(a)}} q_{\pi, \Gamma}(x, \cdot), \quad q_{\pi, \Gamma}(x, \cdot) = R_\pi(\sigma \mid \Gamma) \delta_E \otimes \delta_{\sigma\pi} \otimes \delta_{\omega_{S^c}} \otimes P_\pi(\cdot \mid x),$$

where the formula for $q_{\pi, \Gamma}$ is used only when $R_\pi(\sigma \mid \Gamma) > 0$, and $Q_\Gamma^{(a)}$ is the completion of $q_\Gamma^{(a)}$. On the exceptional exposed fibers $E \notin E_0$, we set $q_{\pi, \Gamma}(x, \cdot) = q_\Gamma^{(a)}(x, \cdot) = 0$ and $Q_\Gamma^{(a)}(x, \cdot) = \delta_x$. The surgery kernel of group a is $Q^{(a)} = \int Q_\Gamma^{(a)} \mathfrak{p}(d\Gamma)$, and the surgery kernel is

$$Q = \frac{1}{m} \sum_{a=1}^m Q^{(a)}.$$

For $E \in E_0$, the surgery kernel acts as follows. Choose a group a uniformly at random and a skeleton Γ with law $\mathfrak{p}$, independently of each other and of the state, and let $\gamma = \mathcal{R}(B_\sigma^{(a)}, \Gamma)$. If $\gamma = \dagger$, do nothing. Otherwise, choose independently in each cell v of γ a row uniformly among the $m_v^{(a)}(\sigma)$ rows $i \in I_v^{(a)}$ with $A_{i\sigma(i)} = 1$, let π be the permutation of the resulting labeled route, and perform the surgery along π . Indeed, by (H5)(a), every cell of γ lies in $\Lambda_{\text{act}} \setminus B_\sigma^{(a)}$. Then $m_v^{(a)}(\sigma) \geq m_*$ for $v \in \gamma$, and the sum of $r_\pi(\sigma)$ over all $\pi \in \Pi_E^{(a)}$ with $\gamma(\pi) = \gamma$ is $\prod_{v \in \gamma} \sum_{i \in I_v^{(a)}} A_{i\sigma(i)} / m_v^{(a)}(\sigma) = 1$. In particular, the outgoing mass of $q_\Gamma^{(a)}$ is $\mathbf{1}\{\mathcal{R}(B_\sigma^{(a)}, \Gamma) \neq \dagger\}$. Since completion commutes with averaging, $Q^{(a)}$ is

the completion of $q^{(a)} = \int q_{\Gamma}^{(a)} p(d\Gamma)$, whose outgoing mass is the acceptance probability $p(\{\Gamma : \mathcal{R}(B_{\sigma}^{(a)}, \Gamma) \neq \dagger\})$ of group a , and Q is the completion of $q = m^{-1} \sum_a q^{(a)}$. The rates depend on (E, σ, Γ) only, and not on the paths. The surgery preserves E , and the new paths of the selected rows are sampled from their conditional law under μ_E given the matching $\sigma\pi$ and all other paths, including those of the other groups. By Lemma 7.6(1) below, the normalizer $Z_S(\sigma\pi; \omega_{S^c})$ is positive whenever $R_{\pi}(\sigma \mid \Gamma) > 0$.

This construction adapts [111, Sections 9.2 and 9.3]. There, one eligible slot is chosen uniformly in each cell of an open path of cells, the starting points of the chosen paths are shifted cyclically along the chosen tuple, and the chosen paths are resampled from their conditional interacting law; the change opens one path of a closed configuration. Here the route is closed, and the surgery maps a closed slab with one endpoint matching to a closed slab with another.

Lemma 7.6. Fix $E \in E_0$. Fix $\Gamma \in \mathfrak{G}$, a group a , and $\pi \in \Pi_E^{(a)}$ with selected rows S .

1. If $R_{\pi}(\sigma \mid \Gamma) > 0$ and ω is individually permitted for σ , then

$$Z_S(\sigma\pi; \omega_{S^c}) \geq \prod_{i \in S} (1 - \eta_{\text{glob}})^2 K_{i\sigma\pi(i)}(1) \geq ((1 - \eta_{\text{glob}})^2 z_-)^{|S|} > 0.$$

2. The measure $\mu_E q_{\pi, \Gamma}$ is absolutely continuous with respect to μ_E , with density

$$c_{\pi}(\sigma, \omega \mid \Gamma) = R_{\pi}(\sigma\pi^{-1} \mid \Gamma) \frac{Z_S(\sigma\pi^{-1}; \omega_{S^c})}{Z_S(\sigma; \omega_{S^c})}, \quad (7.6)$$

where the right side is zero if $R_{\pi}(\sigma\pi^{-1} \mid \Gamma) = 0$, and $Z_S(\sigma; \omega_{S^c}) > 0$ for μ_E -almost every (σ, ω) .

3. The measure $\mu_E Q_{\Gamma}^{(a)}$ is absolutely continuous with respect to μ_E , with density

$$L_{a, \Gamma} = 1 + \sum_{\pi \in \Pi_E^{(a)}} (c_{\pi}(\cdot \mid \Gamma) - R_{\pi}(\cdot \mid \Gamma)). \quad (7.7)$$

Proof. For part (1), if $R_{\pi}(\sigma \mid \Gamma) > 0$, then $A_{i\sigma(i)} = 1$ for every $i \in S$, and by Lemma 7.4(1), the pairing $(i, \sigma\pi(i))$ is crossed for every $i \in S$. The paths ω_{S^c} are individually permitted for $\sigma\pi$, since $\sigma\pi$ and σ agree off S . We apply Lemma 7.3(1) with $\tau = \sigma\pi$, $O = I \setminus S$, $X = \emptyset$, $W = S$, and $W_0 = \emptyset$. Since $Z_S(\sigma\pi; \omega_{S^c}) = Z_S(\sigma\pi; \emptyset)$ in the notation of that lemma, this implies the first inequality, and (H2) implies the second.

For part (2), let $f \geq 0$ be measurable. By (7.3) for the matching σ , and since $P_{\pi}(\cdot \mid x)$ does not depend on ω_S , we may integrate out ω_S to obtain

$$\begin{aligned} \int f d(\mu_E q_{\pi, \Gamma}) &= \frac{1}{\Xi} \sum_{\sigma} R_{\pi}(\sigma \mid \Gamma) \int N_{\sigma}^{S^c}(d\omega_{S^c}) \frac{Z_S(\sigma; \omega_{S^c})}{Z_S(\sigma\pi; \omega_{S^c})} \\ &\quad \times \int f(\sigma\pi, \omega_{S^c}, \omega'_S) M_{\sigma\pi}^S(d\omega'_S; \omega_{S^c}). \end{aligned}$$

By part (1), the divisor is positive for $N_\sigma^{S^c}$ -almost every ω_{S^c} whenever $R_\pi(\sigma \mid \Gamma) > 0$. We substitute $\tau = \sigma\pi$. Since σ and τ agree off S , we have $N_\sigma^{S^c} = N_\tau^{S^c}$, and by (7.3) for τ , we have

$$\begin{aligned} \int f \, d(\mu_E q_{\pi, \Gamma}) &= \frac{1}{\Xi} \sum_{\tau} \int M_{\tau}(\mathrm{d}\omega) R_{\pi}(\tau\pi^{-1} \mid \Gamma) \frac{Z_S(\tau\pi^{-1}; \omega_{S^c})}{Z_S(\tau; \omega_{S^c})} f(\tau, \omega) \\ &= \int f c_{\pi}(\cdot \mid \Gamma) \, d\mu_E. \end{aligned}$$

By part (1), applied to the matching $\tau\pi^{-1}$, the divisor $Z_S(\tau; \omega_{S^c})$ is positive whenever $R_{\pi}(\tau\pi^{-1} \mid \Gamma) > 0$, for $N_\tau^{S^c}$ -almost every ω_{S^c} . Moreover, the set $\{Z_S(\sigma; \omega_{S^c}) = 0\}$ is a μ_E -null set, since by (7.3) its measure is $\Xi^{-1} \sum_{\sigma} \int N_\sigma^{S^c}(\mathrm{d}\omega_{S^c}) \mathbf{1}\{Z_S(\sigma; \omega_{S^c}) = 0\} Z_S(\sigma; \omega_{S^c}) = 0$. This proves part (2).

Part (3) follows from part (2) and Lemma 5.2(1), since the outgoing mass of $q_{\Gamma}^{(a)}$ is $\sum_{\pi \in \Pi_E^{(a)}} R_{\pi}(\cdot \mid \Gamma)$ and its incoming density is $\sum_{\pi \in \Pi_E^{(a)}} c_{\pi}(\cdot \mid \Gamma)$. $\square$

Used in: Lemma 7.7, Lemma 7.8, Lemma 7.13, and Theorem 7.14.

Formula (7.6) has the structure of the density of the opened slab in [111, Equation (26)] and of the exact likelihood [112, Equation (9.10)]: the normalizer of the source matching appears in the numerator and that of the target matching in the denominator. Its derivation uses only the product form (7.1) of the matching law, with no assumption on the distribution of the matching. Since the rates do not depend on the paths, three of the four inner products in the expansion of $\langle L_{a, \Gamma} - 1, L_{b, \Gamma'} - 1 \rangle$ are given by exact identities.

Lemma 7.7. *Fix E as in Lemma 7.6. For groups a and b , $\Gamma, \Gamma' \in \mathfrak{G}$, $\pi \in \Pi_E^{(a)}$, and $\pi' \in \Pi_E^{(b)}$,*

$$\begin{aligned} \langle c_{\pi}(\cdot \mid \Gamma), R_{\pi'}(\cdot \mid \Gamma') \rangle_{\mu_E} &= \mathbb{E}_{\mu_E} [R_{\pi}(\sigma \mid \Gamma) R_{\pi'}(\sigma\pi \mid \Gamma')], \\ \langle R_{\pi}(\cdot \mid \Gamma), c_{\pi'}(\cdot \mid \Gamma') \rangle_{\mu_E} &= \mathbb{E}_{\mu_E} [R_{\pi}(\sigma\pi' \mid \Gamma) R_{\pi'}(\sigma \mid \Gamma')], \\ \langle R_{\pi}(\cdot \mid \Gamma), R_{\pi'}(\cdot \mid \Gamma') \rangle_{\mu_E} &= \mathbb{E}_{\mu_E} [R_{\pi}(\sigma \mid \Gamma) R_{\pi'}(\sigma \mid \Gamma')]. \end{aligned}$$

Proof. By Lemma 7.6(2), $\langle c_{\pi}(\cdot \mid \Gamma), f \rangle_{\mu_E} = \int q_{\pi, \Gamma} f \, d\mu_E$ for measurable $f \geq 0$. Since the function $f(\sigma, \omega) = R_{\pi'}(\sigma \mid \Gamma')$ does not depend on ω , we have $q_{\pi, \Gamma} f(\sigma, \omega) = R_{\pi}(\sigma \mid \Gamma) R_{\pi'}(\sigma\pi \mid \Gamma')$. This proves the first identity. The second is the first with the roles of (π, Γ) and (π', Γ') exchanged, and the third is the definition of the inner product. $\square$

7.4 Comparison of four matchings

It remains to bound the inner product of two incoming densities. In this section, we fix E as in Lemma 7.6 such that (H1) holds, a matching σ , and $\pi, \pi' \in \Pi_E$ with routes γ, γ' and selected rows S, S' . We set $U = S \cup S'$ and define $\sigma_1, \sigma_2, \sigma_{12}$ as in Lemma 7.4.

Matching	Assignment	Role
σ	σ	Current matching
σ_1	$\sigma\pi^{-1}$	Pre-surgery matching for π
σ_2	$\sigma\pi'^{-1}$	Pre-surgery matching for π'
σ_{12}	$\sigma\pi^{-1}\pi'^{-1}$	Composed matching

At a state with matching σ , the product $c_\pi(\cdot \mid \Gamma)c_{\pi'}(\cdot \mid \Gamma')$ contains the rates $R_\pi(\sigma_1 \mid \Gamma)$ and $R_{\pi'}(\sigma_2 \mid \Gamma')$, which vanish unless

$$A_{i\sigma_1(i)} = 1 \quad (i \in S), \quad A_{i\sigma_2(i)} = 1 \quad (i \in S'). \quad (7.8)$$

We fix paths ω_{U^c} that are individually permitted for σ and for the other three matchings, which agree with σ outside U . We use the notation of Lemma 7.3 with $O = U^c$, so that $\hat{K}_i^\tau$ and $\hat{z}_i^\tau$, for $i \in U$, contain the checks with the rows outside U . In this notation, $Z_S(\tau; \omega_{S^c}) = Z_S(\tau; \omega_{S' \setminus S})$ and $Z_{S'}(\tau; \omega_{S'^c}) = Z_{S'}(\tau; \omega_{S \setminus S'})$, the measure $M_\sigma^U(d\omega_U; \omega_{U^c})$ of (7.2) equals $\prod_{i \in U} \hat{K}_i^\sigma(d\omega_i) \prod_{\{i,k\} \subset U} h_{ik}(\omega_i, \omega_k)$, and $Z_U(\tau; \omega_{U^c}) = Z_U(\tau; \emptyset)$.

Let $R \geq \max\{R_{\text{int}}, 1\}$. We call a row $i \in S$ near if $d_\infty(\text{anch}(i), \gamma') \leq R$, and a row $i \in S'$ near if $d_\infty(\text{anch}(i), \gamma) \leq R$; other rows of S and S' are far. Let C_S, F_S be the near and far rows of S , let $C_{S'}, F_{S'}$ be those of S' , and let $C_U = C_S \cup C_{S'} = U \setminus (F_S \cup F_{S'})$. These sets have the following properties.

- (N1) Every row of $S \cap S'$ is anchored at a cell of $\gamma \cap \gamma'$, so it lies in $C_S \cap C_{S'}$. In particular, $F_S \subset S \setminus S'$, $F_{S'} \subset S' \setminus S$, $F_S \cap F_{S'} = \emptyset$, $C_U \cap S = C_S$, and $C_U \cap S' = C_{S'}$.
- (N2) Each near row of S is the selected row of a cell $v \in \gamma$ with $d_\infty(v, \gamma') \leq R$, and there are at most $J_R(\gamma, \gamma')$ such cells; the same holds for S' . This implies that $|C_S| + |C_{S'}| \leq 2J_R(\gamma, \gamma')$ and $|C_U| \leq 2J_R(\gamma, \gamma')$.
- (N3) A far row of S is anchored at distance more than $R \geq R_{\text{int}}$ from every row of S' . By (H1), it has no nontrivial check with any row of S' . The same holds with S and S' exchanged.
- (N4) $\sigma_{12} = \sigma_1$ on $S \setminus S'$, and $\sigma_{12} = \sigma_2$ on $F_{S'}$.

To prove (N4), we note that $\sigma_{12}(i) = \sigma_1(\pi'^{-1}(i))$, which equals $\sigma_1(i)$ if $i \notin S'$. If $i \in F_{S'}$, then $k = \pi'^{-1}(i)$ is the selected row of the cell preceding $\text{anch}(i)$ on γ' , which is at distance one from $\text{anch}(i)$. Since $d_\infty(\text{anch}(i), \gamma) > R \geq 1$, we have $k \notin S$ and $\sigma_{12}(i) = \sigma_1(k) = \sigma(k) = \sigma_2(i)$.

Lemma 7.8. Assume (H1)–(H3) and (H3₁) for E and $\eta_{\text{loc}} \leq 1/2$, and let $R \geq \max\{R_{\text{int}}, 1\}$. In the setting above, assume (7.8). Then all divisors in the following integral are positive, and

$$\int \frac{Z_S(\sigma_1; \omega_{S^c})}{Z_S(\sigma; \omega_{S^c})} \frac{Z_{S'}(\sigma_2; \omega_{S'^c})}{Z_{S'}(\sigma; \omega_{S'^c})} M_\sigma^U(d\omega_U; \omega_{U^c}) \leq G^{|S \cap S'|} (1 - \eta_{\text{loc}})^{-4J_R(\gamma, \gamma')} Z_U(\sigma_{12}; \omega_{U^c}), \quad (7.9)$$

where $G = (2z_+/z_-)^3$. Further, if $\pi \in \Pi_E^{(a)}$ and $\pi' \in \Pi_E^{(b)}$ for groups a and b , then for all $\Gamma, \Gamma' \in \mathfrak{G}$,

$$\langle c_\pi(\cdot | \Gamma), c_{\pi'}(\cdot | \Gamma') \rangle_{\mu_E} \leq G^{|S \cap S'|} (1 - \eta_{\text{loc}})^{-4J_R(\gamma, \gamma')} \mathbb{E}_{\mu_E} [R_\pi(\sigma\pi' | \Gamma) R_{\pi'}(\sigma\pi'\pi\pi'^{-1} | \Gamma')]. \quad (7.10)$$

Proof. We proceed in five steps.

Step 1: masses. By (7.8) and Lemma 7.4(2), every pairing $(i, \tau(i))$ with $i \in U$ and $\tau \in \{\sigma, \sigma_1, \sigma_2, \sigma_{12}\}$ is crossed. By Lemma 7.3(1) and (H2), we have

$$(1 - \eta_{\text{glob}})z_- \leq \hat{z}_i^\tau \leq z_+, \quad \text{so that} \quad \frac{\hat{z}_i^\tau}{\hat{z}_i^{\tau'}} \leq \frac{2z_+}{z_-} = G^{1/3} \quad (7.11)$$

for all such i and τ, τ' , where we used $\eta_{\text{glob}} \leq 1/2$. Recall that $\hat{z}_i^\tau$ depends on τ only through $\tau(i)$. Since S and S' contain exactly one row in each cell of γ and of γ' , respectively, U contains at most two rows in each cell. A cell has at most C_{int} cells within d_∞ -distance R_{int} , including itself. Hence every row of U has at most $N = 2C_{\text{int}} - 1$ other rows of U within d_∞ -distance R_{int} , and $2N\eta_1 \leq \eta_{\text{loc}} \leq 1/2$.

Step 2: normalizers of S and S' . Let $\tau \in \{\sigma, \sigma_1\}$. The paths $\omega_{S' \setminus S}$ under M_σ^U are individually permitted for σ and for σ_1 , which agrees with σ on $S' \setminus S$. By Lemma 7.3 with $W = S$, $W_0 = F_S$, and $X = S' \setminus S$, we have

$$(1 - \eta_{\text{loc}})^{|C_S|} \left(\prod_{i \in C_S} \hat{z}_i^\tau \right) Z_{F_S}(\tau; \omega_{S' \setminus S}) \leq Z_S(\tau; \omega_{S' \setminus S}) \leq \left(\prod_{i \in C_S} \hat{z}_i^\tau \right) Z_{F_S}(\tau; \omega_{S' \setminus S}). \quad (7.12)$$

The upper bound is part (1) of that lemma. The lower bound is part (2) with the number N of Step 1, which applies since $W \cup X = U$ and the pairings are crossed by Step 1, together with $1 - 2N\eta_1 \geq 1 - \eta_{\text{loc}}$. By (N3), the checks between F_S and $S' \setminus S$ are identically one, and we have $Z_{F_S}(\tau; \omega_{S' \setminus S}) = Z_{F_S}(\tau; \emptyset)$, which we abbreviate as $Z_{F_S}(\tau)$. It does not depend on $\omega_{S' \setminus S}$, and it depends on τ only through its restriction to F_S . By Lemma 7.3(1) with $W = F_S$ and $W_0 = \emptyset$, it is positive. In the same way, for $\tau \in \{\sigma, \sigma_2\}$, (7.12) holds with S, S' exchanged, and $Z_{F_{S'}}(\tau) > 0$. In particular, the divisors in (7.9) are positive.

Step 3: upper bound. We insert the upper bounds from (7.12) for the numerators in (7.9) and the lower bounds for the denominators. These bounds hold pointwise in ω_U , and the remaining integral of M_σ^U is $Z_U(\sigma; \emptyset)$. Then the left side of (7.9) is at most

$$(1 - \eta_{\text{loc}})^{-|C_S| - |C_{S'}|} \prod_{i \in C_S} \frac{\hat{z}_i^{\sigma_1}}{\hat{z}_i^\sigma} \prod_{i \in C_{S'}} \frac{\hat{z}_i^{\sigma_2}}{\hat{z}_i^\sigma} \cdot \frac{Z_{F_S}(\sigma_1) Z_{F_{S'}}(\sigma_2)}{Z_{F_S}(\sigma) Z_{F_{S'}}(\sigma)} \cdot Z_U(\sigma; \emptyset).$$

By (7.4) with $W = U$ and $W_0 = F_S \cup F_{S'}$, and since by (N3) there are no checks between F_S and $F_{S'}$, we have

$$Z_U(\sigma; \emptyset) \leq \left(\prod_{i \in C_U} \hat{z}_i^\sigma \right) Z_{F_S \cup F_{S'}}(\sigma; \emptyset) = \left(\prod_{i \in C_U} \hat{z}_i^\sigma \right) Z_{F_S}(\sigma) Z_{F_{S'}}(\sigma).$$

Canceling the far normalizers of σ , we find that the left side of (7.9) is at most

$$(1 - \eta_{\text{loc}})^{-|C_S| - |C_{S'}|} \left(\prod_{i \in C_U} \hat{z}_i^\sigma \right) \prod_{i \in C_S} \frac{\hat{z}_i^{\sigma_1}}{\hat{z}_i^\sigma} \prod_{i \in C_{S'}} \frac{\hat{z}_i^{\sigma_2}}{\hat{z}_i^\sigma} \cdot Z_{F_S}(\sigma_1) Z_{F_{S'}}(\sigma_2). \quad (7.13)$$

Step 4: the composed matching. By Lemma 7.3(2) with $\tau = \sigma_{12}$, $W = U$, $W_0 = F_S \cup F_{S'}$, $X = \emptyset$, and the number N of Step 1, which applies by Step 1, we have

$$\begin{aligned} Z_U(\sigma_{12}; \emptyset) &\geq (1 - \eta_{\text{loc}})^{|C_U|} \left(\prod_{i \in C_U} \hat{z}_i^{\sigma_{12}} \right) Z_{F_S}(\sigma_{12}) Z_{F_{S'}}(\sigma_{12}) \\ &= (1 - \eta_{\text{loc}})^{|C_U|} \left(\prod_{i \in C_U} \hat{z}_i^{\sigma_{12}} \right) Z_{F_S}(\sigma_1) Z_{F_{S'}}(\sigma_2). \end{aligned}$$

In the first step, we used (N3) to factorize the far normalizer, and in the second step, we used (N4) and $F_S \subset S \setminus S'$. Combining this with (7.13), and using (N2), we conclude that the left side of (7.9) is at most

$$(1 - \eta_{\text{loc}})^{-|C_S| - |C_{S'}| - |C_U|} \mathcal{Q} Z_U(\sigma_{12}; \emptyset) \leq (1 - \eta_{\text{loc}})^{-4J_R(\gamma, \gamma')} \mathcal{Q} Z_U(\sigma_{12}; \emptyset),$$

where

$$\mathcal{Q} = \prod_{i \in C_U} \mathcal{Q}_i, \quad \mathcal{Q}_i = \frac{\hat{z}_i^\sigma}{\hat{z}_i^{\sigma_{12}}} \left(\frac{\hat{z}_i^{\sigma_1}}{\hat{z}_i^\sigma} \right)^{\mathbf{1}_{\{i \in S\}}} \left(\frac{\hat{z}_i^{\sigma_2}}{\hat{z}_i^\sigma} \right)^{\mathbf{1}_{\{i \in S'\}}}.$$

Here we used $C_U \cap S = C_S$ and $C_U \cap S' = C_{S'}$ from (N1).

Step 5: the scalar ratio. We show that $\mathcal{Q} \leq G^{|S \cap S'|}$, which proves (7.9). Let $i \in C_U$. If $i \in S \setminus S'$, then $\mathcal{Q}_i = \hat{z}_i^{\sigma_1} / \hat{z}_i^{\sigma_{12}} = 1$ by (N4). If $i \in S' \setminus S$ and $k = \pi'^{-1}(i) \notin S$, then $\sigma_{12}(i) = \sigma_1(k) = \sigma(k) = \sigma_2(i)$, and we have $\mathcal{Q}_i = \hat{z}_i^{\sigma_2} / \hat{z}_i^{\sigma_{12}} = 1$. If $i \in S' \setminus S$ and $k = \pi'^{-1}(i) \in S$, then $k \in S \cap S'$ and $i \in \pi'(S \cap S')$, and $\mathcal{Q}_i \leq 2z_+ / z_-$ by (7.11). If $i \in S \cap S'$, then $\mathcal{Q}_i = \hat{z}_i^{\sigma_1} \hat{z}_i^{\sigma_2} / (\hat{z}_i^\sigma \hat{z}_i^{\sigma_{12}}) \leq (2z_+ / z_-)^2$. The number of rows in the third case is at most $|\pi'(S \cap S')| = |S \cap S'|$. Since $2z_+ / z_- \geq 1$, we conclude that $\mathcal{Q} \leq (2z_+ / z_-)^{3|S \cap S'|} = G^{|S \cap S'|}$.

Consequence. Let $\pi \in \Pi_E^{(a)}$ and $\pi' \in \Pi_E^{(b)}$, and write $F = (1 - \eta_{\text{loc}})^{-4J_R(\gamma, \gamma')}$. By (7.6),

$$\begin{aligned} \langle c_\pi(\cdot \mid \Gamma), c_{\pi'}(\cdot \mid \Gamma') \rangle_{\mu_E} &= \frac{1}{\Xi} \sum_{\sigma} R_\pi(\sigma_1 \mid \Gamma) R_{\pi'}(\sigma_2 \mid \Gamma') \int N_\sigma^{U^c}(\mathrm{d}\omega_{U^c}) \\ &\quad \times \int \frac{Z_S(\sigma_1; \omega_{S^c})}{Z_S(\sigma; \omega_{S^c})} \frac{Z_{S'}(\sigma_2; \omega_{S'^c})}{Z_{S'}(\sigma; \omega_{S'^c})} M_\sigma^U(\mathrm{d}\omega_U; \omega_{U^c}), \end{aligned}$$

where we used (7.3) with U in place of S and that the rates do not depend on ω . If the product of the two rates is positive, then (7.8) holds, and (7.9) applies for $N_\sigma^{U^c}$ -almost every ω_{U^c} . Since σ_{12} agrees with σ outside U , we have $N_\sigma^{U^c} = N_{\sigma_{12}}^{U^c}$, and by (7.3),

$$\int N_{\sigma_{12}}^{U^c}(\mathrm{d}\omega_{U^c}) Z_U(\sigma_{12}; \omega_{U^c}) = Z_{\sigma_{12}}(E).$$

We obtain

$$\langle c_\pi(\cdot \mid \Gamma), c_{\pi'}(\cdot \mid \Gamma') \rangle_{\mu_E} \leq G^{|S \cap S'|} F \frac{1}{\Xi} \sum_{\sigma} Z_{\sigma_{12}}(E) R_\pi(\sigma_1 \mid \Gamma) R_{\pi'}(\sigma_2 \mid \Gamma').$$

We have obtained the normalizing mass of the composed matching. We now sum using that matching as the base variable. The conjugated permutation in the second rate follows from the order of composition. The map $\sigma \mapsto \tau = \sigma \pi^{-1} \pi'^{-1}$ is a bijection of the set of matchings, with inverse $\sigma = \tau \pi' \pi$. In terms of τ , we have $\sigma_1 = \tau \pi'$ and $\sigma_2 = \tau \pi' \pi \pi'^{-1}$. Reindexing the sum by τ and using $Z_\tau(E)/\Xi(E) = \mu_E(\{\tau\} \times P^I)$ proves (7.10). $\square$

The second rate in (7.10) is evaluated at $\sigma \pi' \pi \pi'^{-1}$, which differs from $\sigma \pi$ when the two labeled routes share a row: two cyclic permutations with a common row need not commute.

When $J_R(\gamma, \gamma') = 0$, the comparison is an identity, and the ordinary rates $r_\pi, r_{\pi'}$ are unchanged by the other surgery; for two moves of the same group, the route indicators still require the stability argument.

Lemma 7.9. *Assume (H1)–(H3) for E , and let $R \geq \max\{R_{\text{int}}, 1\}$. Let $\pi \in \Pi_E^{(a)}$ and $\pi' \in \Pi_E^{(b)}$ for groups a and b , with routes γ, γ' and selected rows S, S' , and suppose that $J_R(\gamma, \gamma') = 0$.*

1. *The sets S and S' are disjoint, the permutations π and π' commute, so that $\pi' \pi \pi'^{-1} = \pi$, and $h_{ik} \equiv 1$ for all $i \in S$ and $k \in S'$.*
2. *For every matching σ , we have $r_\pi(\sigma \pi') = r_\pi(\sigma)$ and $r_{\pi'}(\sigma \pi) = r_{\pi'}(\sigma)$.*
3. *In the setting of Lemma 7.8, the inequality (7.9) holds with equality. Consequently, for all $\Gamma, \Gamma' \in \mathfrak{G}$,*

$$\langle c_\pi(\cdot \mid \Gamma), c_{\pi'}(\cdot \mid \Gamma') \rangle_{\mu_E} = \mathbb{E}_{\mu_E} [R_\pi(\sigma \pi' \mid \Gamma) R_{\pi'}(\sigma \pi \mid \Gamma')]. \quad (7.14)$$

Proof. For part (1), every cell of γ is at d_∞ -distance more than R from every cell of γ' , since $J_R(\gamma, \gamma') = 0$. In particular, γ and γ' have no common cell. Each selected row is anchored at a cell of its route, so $S \cap S' = \emptyset$, and permutations with disjoint supports commute. Since $R \geq R_{\text{int}}$, (H1) implies that $h_{ik} \equiv 1$ for $i \in S$ and $k \in S'$.

For part (2), the matching $\sigma \pi'$ differs from σ only at the rows of S' , which are anchored at cells of γ' . No cell of γ contains such a row, and no selected row of π lies in S' . Since accessibility is row-local, we have $m_v^{(a)}(\sigma \pi') = m_v^{(a)}(\sigma)$ and $A_{i_v \sigma \pi'(i_v)} = A_{i_v \sigma(i_v)}$ for every $v \in \gamma$, and $r_\pi(\sigma \pi') = r_\pi(\sigma)$. The second identity follows in the same way.

For part (3), we use the notation of Lemma 7.3 with $O = U^c$. By part (1), there are no checks between S and S' . Then $Z_S(\tau; \omega_{S^c}) = Z_S(\tau; \emptyset)$ and $Z_{S'}(\tau; \omega_{S^c}) = Z_{S'}(\tau; \emptyset)$ do not depend on ω_U , and $M_\sigma^U(d\omega_U; \omega_{U^c})$ is the product of the measures $\prod_{i \in S} \hat{K}_i^\sigma(d\omega_i) \prod_{\{i,k\} \subset S} h_{ik}(\omega_i, \omega_k)$ and $\prod_{i \in S'} \hat{K}_i^\sigma(d\omega_i) \prod_{\{i,k\} \subset S'} h_{ik}(\omega_i, \omega_k)$, whose masses are $Z_S(\sigma; \emptyset)$ and $Z_{S'}(\sigma; \emptyset)$. These masses are positive by Step 2 of the proof of

Lemma 7.8. Integrating, we find that the left side of (7.9) equals $Z_S(\sigma_1; \emptyset) Z_{S'}(\sigma_2; \emptyset)$. For $i \in S$, we have $\pi'^{-1}(i) = i$ and $\sigma_{12}(i) = \sigma_1(i)$; for $i \in S'$, the row $\pi'^{-1}(i)$ lies in S' and not in S , and $\sigma_{12}(i) = \sigma_1(\pi'^{-1}(i)) = \sigma(\pi'^{-1}(i)) = \sigma_2(i)$. Since there are no checks between S and S' , we obtain $Z_U(\sigma_{12}; \emptyset) = Z_S(\sigma_1; \emptyset) Z_{S'}(\sigma_2; \emptyset)$. The right side of (7.9) is $Z_U(\sigma_{12}; \omega_{U^c}) = Z_U(\sigma_{12}; \emptyset)$, because $S \cap S' = \emptyset$ and $J_R(\gamma, \gamma') = 0$, which proves the equality in (7.9). The derivation of (7.10) from (7.9) in the proof of Lemma 7.8 uses no other inequality. Then (7.10) also holds with equality, with right side $\mathbb{E}_{\mu_E}[R_\pi(\sigma\pi' \mid \Gamma) R_{\pi'}(\sigma\pi'\pi\pi'^{-1} \mid \Gamma')]$. By part (1), $\pi'\pi\pi'^{-1} = \pi$, which gives (7.14). $\square$

Part (3) is the cancellation of conditional normalizing constants of [115, Lemma 8.3], reproduced as [112, Lemma 9.3], in its simplest form: when no check joins the two changes, the normalizers factor and cancel exactly. Together with part (2), it shows that a pair of labeled routes whose routes do not meet contributes exactly zero to the signed expansion, once the route indicators in the four terms agree; this is used in Step 4 of the proof of Theorem 7.14 and in Lemma 7.13.

Remark 7.10. The proof of Lemma 7.8 follows the proof of [115, Lemma 8.3], reproduced as [112, Lemma 9.3]. Rows far from the other route are separated from the near rows by deletion and insertion, the normalizers of the far rows cancel, and the cost of the insertions is confined to the near rows. The comparison of four states with a defect supported on the shared rows and their neighbors on a route adapts [111, Section 10.1], where the selected slots are partitioned into the shared set h and the set $d = h \cup \text{pred}(h)$; our set $(S \cap S') \cup \pi'(S \cap S')$ plays the role of d .

The fixed-reference-law comparison in [112, Lemma 9.3] cannot be applied directly when endpoints are reassigned. With the matching fixed, a path ending at a different slot is singular with respect to its former bridge law. If slots are instead included in the path coordinates, enforcing their distinctness does not satisfy the near-one insertion bound. We compare the four matching normalizers directly in Lemma 7.8. The crossed mass hypothesis (H2) controls their individual path masses, and the composed normalizer permits the reindexing back to the matching law.

7.5 Label averaging

By Lemmas 7.7 and 7.8, the four inner products for a pair of labeled routes are bounded by expectations of explicit functions of the matching. We now sum these functions over the labelings of a fixed pair of routes by rows of one group a . The next lemma describes how the bad sets of group a of the matchings that appear differ from $B_\sigma^{(a)}$.

Lemma 7.11. *Let σ be a matching, let a be a group, let $\pi, \pi' \in \Pi_E^{(a)}$ have routes γ, γ' and selected rows S, S' , and let $\tilde{\pi} = \pi'\pi\pi'^{-1}$. For $\tau \in \{\sigma\pi, \sigma\pi', \sigma\tilde{\pi}\}$, every cell contains at most two rows*

i with $\tau(i) \neq \sigma(i)$, and these rows belong to group a ; in particular, $|m_v^{(a)}(\tau) - m_v^{(a)}(\sigma)| \leq 2$ for every cell v , and $B_\tau^{(a)} \subset V_\sigma^{(a)}$. Further,

$$B_{\sigma\pi}^{(a)} \triangle B_\sigma^{(a)} \subset \gamma, \quad B_{\sigma\pi'}^{(a)} \triangle B_\sigma^{(a)} \subset \gamma', \quad B_{\sigma\tilde{\pi}}^{(a)} \triangle B_\sigma^{(a)} \subset \mathcal{N}_1(\gamma). \quad (7.15)$$

Proof. The matching $\sigma\pi$ changes the slots of exactly the rows of S , one in each cell of γ , and $\sigma\pi'$ those of S' . The permutation $\tilde{\pi}$ moves a row i if and only if $\pi'^{-1}(i) \in S$; its support is $\pi'(S) = (S \setminus S') \cup \pi'(S \cap S')$. The set $S \setminus S'$ has at most one row in each cell, and these cells lie on γ . The set $\pi'(S \cap S') \subset S'$ has at most one row in each cell, and the row $\pi'(k)$, for $k \in S \cap S'$, is anchored at the cell following $\text{anch}(k) \in \gamma$ on γ' , which is a face neighbor of a cell of γ . The rows of S and S' belong to group a . Since $A_{i\tau(i)}$ depends only on i and $\tau(i)$, we have $m_v^{(a)}(\tau) - m_v^{(a)}(\sigma) = \sum_{i \in I_v^{(a)}} (A_{i\tau(i)} - A_{i\sigma(i)})$, and only rows with $\tau(i) \neq \sigma(i)$ contribute. This proves the bound on the counts and (7.15). If $v \in B_\tau^{(a)}$, then $v \in \Lambda_{\text{act}}$ and $m_v^{(a)}(\sigma) \leq m_v^{(a)}(\tau) + 2 < m_* + 2$, so $v \in V_\sigma^{(a)}$. $\square$

Lemma 7.11 is the reason for the threshold $m_* + 2$ in the definition of $V_\sigma^{(a)}$: all bad sets that occur in the four terms are contained in the single set $V_\sigma^{(a)}$ of the base matching, which is the input of the route lists of (H5)(c). The enlarged set plays the role of the common enlarged set of unusable cells in [111, Lemma 10.2].

For a group a , $\pi, \pi' \in \Pi_E^{(a)}$ with routes γ, γ' , skeletons Γ, Γ' , and a matching σ , we define the four indicator products

$$\begin{aligned} \mathbf{1}^{\text{ii}} &= \mathbf{1}\{\mathcal{R}(B_{\sigma\pi'}^{(a)}, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B_{\sigma\tilde{\pi}}^{(a)}, \Gamma') = \gamma'\}, \\ \mathbf{1}^{\text{io}} &= \mathbf{1}\{\mathcal{R}(B_\sigma^{(a)}, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B_{\sigma\pi}^{(a)}, \Gamma') = \gamma'\}, \\ \mathbf{1}^{\text{oi}} &= \mathbf{1}\{\mathcal{R}(B_{\sigma\pi'}^{(a)}, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B_\sigma^{(a)}, \Gamma') = \gamma'\}, \\ \mathbf{1}^{\text{oo}} &= \mathbf{1}\{\mathcal{R}(B_\sigma^{(a)}, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B_\sigma^{(a)}, \Gamma') = \gamma'\}, \end{aligned}$$

with $\tilde{\pi} = \pi'\pi\pi'^{-1}$, and the four functions

$$\begin{aligned} T^{\text{ii}} &= G^{|S \cap S'|} (1 - \eta_{\text{loc}})^{-4J_R(\gamma, \gamma')} \mathbf{1}^{\text{ii}} r_\pi(\sigma\pi') r_{\pi'}(\sigma\tilde{\pi}), & T^{\text{io}} &= \mathbf{1}^{\text{io}} r_\pi(\sigma) r_{\pi'}(\sigma\pi), \\ T^{\text{oi}} &= \mathbf{1}^{\text{oi}} r_\pi(\sigma\pi') r_{\pi'}(\sigma), & T^{\text{oo}} &= \mathbf{1}^{\text{oo}} r_\pi(\sigma) r_{\pi'}(\sigma). \end{aligned} \quad (7.16)$$

The superscripts indicate whether the density of π , respectively π' , is incoming or outgoing. By Lemmas 7.7 and 7.8, the first inner product $\langle c_\pi, c_{\pi'} \rangle$ is at most $\mathbb{E}_{\mu_E} T^{\text{ii}}$; the other three, $\langle c_\pi, R_{\pi'} \rangle$, $\langle R_\pi, c_{\pi'} \rangle$, and $\langle R_\pi, R_{\pi'} \rangle$, equal $\mathbb{E}_{\mu_E} T^{\text{io}}$, $\mathbb{E}_{\mu_E} T^{\text{oi}}$, and $\mathbb{E}_{\mu_E} T^{\text{oo}}$, respectively. Here the two factors carry skeletons Γ and Γ' . By Lemma 7.9, the first bound is an equality when $J_R(\gamma, \gamma') = 0$. The indicator $\mathbf{1}^{\text{oo}}$ does not depend on the labelings.

Lemma 7.12. Assume (H5)(a) and $\eta_{\text{loc}} \leq 1/2$, and let $R \geq 1$, $G \geq 1$, and $\lambda \geq 1$ with $M \leq \lambda m_*$. Set

$$\varepsilon = \frac{C_1(G, \lambda)}{m_*} + C_2 \eta_{\text{loc}}, \quad C_1(G, \lambda) = 3 + 8G\lambda^2, \quad C_2 = 8. \quad (7.17)$$

Fix E , a group a , a matching σ , skeletons Γ, Γ' , and a pair of routes (γ, γ') in Λ_{act} , and write $J = J_R(\gamma, \gamma')$. Then

$$\sum_{\pi, \pi'} (T^{\text{ii}} + T^{\text{oo}}) \leq 2e^{\varepsilon J} \max \left\{ \mathbf{1}^{\text{oo}}, \max_{\pi, \pi'} \mathbf{1}^{\text{ii}} \right\}, \quad (7.18)$$

where the sum and the maximum are over all labelings of γ and γ' by rows of group a , that is, over all $\pi, \pi' \in \Pi_E^{(a)}$ with $\gamma(\pi) = \gamma$ and $\gamma(\pi') = \gamma'$. If either route has no such labeling, the sum and the maximum over labelings are defined to be zero.

Proof. If either route has no such labeling, the bound follows from the stated convention. Otherwise, we write $m_v = m_v^{(a)}(\sigma)$, and, for labelings $(i_v)_{v \in \gamma}$ and $(i'_w)_{w \in \gamma'}$, we let $\mathcal{S} = \{v \in \gamma \cap \gamma' : i_v = i'_v\}$ be the set of shared cells. Since a row is anchored at a single cell, $\mathcal{S} \cap \mathcal{S}' = \{i_v : v \in \mathcal{S}\}$ and $|\mathcal{S} \cap \mathcal{S}'| = |\mathcal{S}|$. For $w \in \gamma'$, let w^- be the cell preceding w on γ' . We use the following facts.

- (F1) If $\mathcal{R}(B_\tau^{(a)}, \Gamma) = \gamma$, then $m_v^{(a)}(\tau) \geq m_*$ for all $v \in \gamma$, by (H5)(a); the same holds for Γ' and γ' .
- (F2) By the proof of Lemma 7.11, $m_v^{(a)}(\sigma\pi')$ differs from m_v only for $v \in \gamma'$, by at most one, and $m_w^{(a)}(\sigma\tilde{\pi})$ differs from m_w only for $w \in \mathcal{N}_1(\gamma)$, by at most two.
- (F3) There are at most J cells of γ in γ' , and at most J cells of γ' in $\mathcal{N}_1(\gamma)$, since $R \geq 1$.
- (F4) The accessibility test of $v \in \gamma$ in $r_\pi(\sigma\pi')$ is $A_{i_v\sigma\pi'(i_v)}$, which equals $A_{i_v\sigma(i_v)}$ unless $i_v \in \mathcal{S}'$, that is, unless $v \in \mathcal{S}$. The test of $w \in \gamma'$ in $r_{\pi'}(\sigma\tilde{\pi})$ equals $A_{i'_w\sigma(i'_w)}$ unless $\pi'^{-1}(i'_w) = i'_{w^-} \in \mathcal{S}$, that is, unless $w^- \in \mathcal{S}$.

We also use an elementary inequality: if n, n' are positive integers with $n' \geq m_*$ and $n \leq n' + k$ for some $k \in \{1, 2\}$, then $n/n' \leq 1 + k/n' \leq e^{k/m_*}$. Finally, $F = (1 - \eta_{\text{loc}})^{-4J} \leq e^{8\eta_{\text{loc}}J}$, since $-\log(1 - \eta) \leq 2\eta$ for $\eta \in [0, 1/2]$.

By (F1) with $\tau = \sigma$, if $\mathbf{1}^{\text{oo}} = 1$, then $m_v \geq m_*$ for all $v \in \gamma \cup \gamma'$, and $\sum_{\pi, \pi'} T^{\text{oo}} = \prod_{v \in \gamma} \sum_{i \in I_v^{(a)}} A_{i\sigma(i)} / m_v \cdot \prod_{w \in \gamma'} \sum_{i \in I_w^{(a)}} A_{i\sigma(i)} / m_w = 1$; otherwise this sum vanishes. If $\mathbf{1}^{\text{ii}} = 0$ for all labelings, then $\sum_{\pi, \pi'} T^{\text{ii}} = 0$. Suppose that $\mathbf{1}^{\text{ii}} = 1$ for some labeling. By (F1), for this labeling, we have $m_v^{(a)}(\sigma\pi') \geq m_*$ for $v \in \gamma$ and $m_w^{(a)}(\sigma\tilde{\pi}) \geq m_*$ for $w \in \gamma'$, and by (F2), $m_v \geq m_* - 2 \geq m_*/2$ for every $v \in \gamma \cup \gamma'$. This lower bound on the counts of σ does not depend on the labelings.

Let $\bar{A}_v = A_{i_v\sigma(i_v)}$ for $v \notin \mathcal{S}$ and $\bar{A}_v = 1$ for $v \in \mathcal{S}$, and let $\bar{A}'_w = A_{i'_w\sigma(i'_w)}$ unless $w \in \mathcal{S}$ or $w^- \in \mathcal{S}$, in which case $\bar{A}'_w = 1$. For every labeling with $\mathbf{1}^{\text{ii}} = 1$, we use (F2)–(F4) and

the elementary inequality, with $k = 1$ for the counts of $\sigma\pi'$ on γ and $k = 2$ for those of $\sigma\tilde{\pi}$ on γ' , to obtain

$$T^{\text{ii}} \leq G^{|\mathcal{S}|} F e^{3J/m_*} \prod_{v \in \gamma} \frac{\bar{A}_v}{m_v} \prod_{w \in \gamma'} \frac{\bar{A}'_w}{m_w},$$

and this bound also holds for labelings with $\mathbf{1}^{\text{ii}} = 0$, for which $T^{\text{ii}} = 0$. We sum over all labelings with a fixed shared set $\mathcal{S} \subset \gamma \cap \gamma'$, dropping the condition $i_v \neq i'_v$ for $v \in (\gamma \cap \gamma') \setminus \mathcal{S}$, which only increases the sum. The sum then factorizes over cells. Since $m_v \geq m_*/2$ and $|I_v^{(a)}| \leq M$, a cell $v \in \mathcal{S}$ carries one common row and contributes at most $\sum_{i \in I_v^{(a)}} m_v^{-2} \leq 4M/m_*^2 \leq 4\lambda/m_*$. A cell $w \in \gamma' \setminus \mathcal{S}$ with $w^- \in \mathcal{S}$ contributes at most $\sum_{i \in I_w^{(a)}} m_w^{-1} \leq 2M/m_* \leq 2\lambda$, and there are at most $|\mathcal{S}|$ such cells. Every other cell of γ , and every other cell of γ' , contributes $\sum_{i \in I_v^{(a)}} A_{i\sigma(i)}/m_v = 1$. Including the factor $G^{|\mathcal{S}|}$, the labelings with shared set $\mathcal{S}$ contribute at most $F e^{3J/m_*} (8G\lambda^2/m_*)^{|\mathcal{S}|}$. Summing over $\mathcal{S} \subset \gamma \cap \gamma'$ and using (F3) and $F \leq e^{8\eta_{\text{loc}}J}$ yields

$$\sum_{\pi, \pi'} T^{\text{ii}} \leq F e^{3J/m_*} \left(1 + \frac{8G\lambda^2}{m_*}\right)^J \leq e^{(8\eta_{\text{loc}} + (3+8G\lambda^2)/m_*)J} = e^{\varepsilon J}.$$

Together with the bound for T^{oo} , this proves (7.18). $\square$

The bound of Lemma 7.12 is a statement about the sum over labelings after the comparison of normalizers and the reindexing of Lemma 7.8; it is not a pointwise bound on likelihoods. No signed estimate over labelings is needed: on $\{A_* = 0, J_R(\gamma, \gamma') = 0\}$, (H5)(c)(ii) identifies the four route-indicator products and Lemma 7.9 makes the signed sum vanish. On $\{A_* = 0, J_R(\gamma, \gamma') \geq 1\}$ and on $\{A_* \geq 1\}$, only the positive terms are kept. The coefficient ε must nevertheless be small, since it multiplies the encounter count in the exponential moments of (H5)(d). The large factor G is affordable only because a shared row has probability of order $1/m_*$ at each common cell, which is the mechanism of [112, Equation (9.16)] and [111, Section 10.5]. Here the selection probabilities are $1/m_v^{(a)}$ with counts that depend on the matching, and two kinds of corrections appear that are not present in those sources: the accessibility tests of shared rows and of the rows that follow them on γ' are relaxed, and the denominators of the compared matchings are compared with those of σ , through the baseline counts $m_v^{(a)} \geq m_* - 2$. The route-acceptance indicators for the pre-surgery matchings are retained until the denominators have been compared, so that no comparison passes through a vanishing count.

7.6 Moves of different groups

Lemma 7.13. *Let $K \geq K_0$, so that (H5)(a)–(d) hold, and let R be the integer of (H5). Fix E as in Lemma 7.6 such that (H1) and (H3₁) also hold, suppose that $\eta_{\text{loc}} \leq 1/2$, and write*

$F(J) = (1 - \eta_{\text{loc}})^{-4J}$ for integers $J \geq 0$. Let $a \neq b$ be groups, and let $\Gamma, \Gamma' \in \mathfrak{G}$. For $\pi \in \Pi_E^{(a)}$ and $\pi' \in \Pi_E^{(b)}$ with routes γ, γ' and selected rows S, S' , we write c_π and R_π for $c_\pi(\cdot \mid \Gamma)$ and $R_\pi(\cdot \mid \Gamma)$, and $c_{\pi'}$ and $R_{\pi'}$ for $c_{\pi'}(\cdot \mid \Gamma')$ and $R_{\pi'}(\cdot \mid \Gamma')$.

1. The sets S and S' are disjoint, and $\pi\pi' = \pi'\pi$. For every matching τ and every cell v , we have $m_v^{(a)}(\tau\pi') = m_v^{(a)}(\tau)$. Hence $B_{\tau\pi'}^{(a)} = B_\tau^{(a)}$, $V_{\tau\pi'}^{(a)} = V_\tau^{(a)}$, and $R_\pi(\tau\pi' \mid \Gamma) = R_\pi(\tau \mid \Gamma)$. The same holds with (a, π, Γ) and (b, π', Γ') exchanged.
2. With $J = J_R(\gamma, \gamma')$,

$$\langle c_\pi - R_\pi, c_{\pi'} - R_{\pi'} \rangle_{\mu_E} \leq [F(J) - 1] \mathbb{E}_{\mu_E} [R_\pi(\sigma \mid \Gamma) R_{\pi'}(\sigma \mid \Gamma')], \quad (7.19)$$

and the left side is zero if $J = 0$.

3. Consider the functions of the matching σ given by $\gamma_0 = \mathcal{R}(B_\sigma^{(a)}, \Gamma)$, $\gamma'_0 = \mathcal{R}(B_\sigma^{(b)}, \Gamma')$, and

$$X_{ab} = J_R(\Gamma, \Gamma') + C_{\text{ch}} A_*(B_\sigma^{(a)} \cup B_\sigma^{(b)}; \Gamma, \Gamma').$$

Then $J_R(\gamma_0, \gamma'_0) \leq X_{ab}$, and

$$\sum_{\pi \in \Pi_E^{(a)}} \sum_{\pi' \in \Pi_E^{(b)}} \langle c_\pi - R_\pi, c_{\pi'} - R_{\pi'} \rangle_{\mu_E} \leq \mathbb{E}_{\mu_E} [F(J_R(\gamma_0, \gamma'_0)) - 1] \leq \mathbb{E}_{\mu_E} [e^{8\eta_{\text{loc}} X_{ab}} - 1].$$

4. Let $\theta_\times = \min\{\kappa/2, 1/\max(1, C_{\text{ch}})\}$ and $c_\times = \theta_\times C_{\text{ch}}$. Let Γ, Γ' be independent with law $\mathbf{p}$, let V be a random subset of Λ_{act} that is independent of (Γ, Γ') and satisfies $\mathbb{P}(\mathcal{F} \subset V) \leq p_\times^{|\mathcal{F}|}$ for every finite set $\mathcal{F} \subset \mathbb{L}$, where $p_\times \leq p_{\text{env}}(c_\times)$, and let $X = J_R(\Gamma, \Gamma') + C_{\text{ch}} A_*(V; \Gamma, \Gamma')$. Then, for $0 \leq \theta \leq \theta_\times$,

$$\mathbb{E} e^{\theta X} - 1 \leq \frac{\theta}{\theta_\times} \frac{C_{\text{fin}}(c_\times)}{K}.$$

Proof. For part (1), we have $S \subset I^{(a)}$ and $S' \subset I^{(b)}$, so $S \cap S' = \emptyset$, and cyclic permutations with disjoint supports commute. The matching $\tau\pi'$ agrees with τ outside S' , and in particular on $I^{(a)}$. Since $A_{i\tau(i)}$ depends only on i and $\tau(i)$, we obtain $m_v^{(a)}(\tau\pi') = m_v^{(a)}(\tau)$ for every cell v , including the cells of $\gamma \cap \gamma'$. Then $B_{\tau\pi'}^{(a)} = B_\tau^{(a)}$ and $V_{\tau\pi'}^{(a)} = V_\tau^{(a)}$, the route indicator in $R_\pi(\tau\pi' \mid \Gamma)$ is that of τ , and so is every factor $A_{i_v\tau\pi'(i_v)}/m_v^{(a)}(\tau\pi')$ of $r_\pi(\tau\pi')$, since $i_v \in S \subset I^{(a)}$. The assertion with the two groups exchanged follows in the same way.

For part (2), Lemma 7.7 and part (1) give

$$\langle c_\pi, R_{\pi'} \rangle_{\mu_E} = \langle R_\pi, c_{\pi'} \rangle_{\mu_E} = \langle R_\pi, R_{\pi'} \rangle_{\mu_E} = \mathbb{E}_{\mu_E} [R_\pi(\sigma \mid \Gamma) R_{\pi'}(\sigma \mid \Gamma')].$$

Since $S \cap S' = \emptyset$ and $\pi'\pi\pi'^{-1} = \pi$, (7.10) reads $\langle c_\pi, c_{\pi'} \rangle_{\mu_E} \leq F(J) \mathbb{E}_{\mu_E} [R_\pi(\sigma\pi' \mid \Gamma) R_{\pi'}(\sigma\pi \mid \Gamma')]$, and by part (1), the last expectation equals $\mathbb{E}_{\mu_E} [R_\pi(\sigma \mid \Gamma) R_{\pi'}(\sigma \mid \Gamma')]$.

All four inner products are finite, since the rates are at most one and $c_\pi, c_{\pi'} \in L^2(\mu_E)$ by (7.10). Expanding the left side of (7.19) proves (7.19). If $J = 0$, then by Lemma 7.9(3) and part (1), $\langle c_\pi, c_{\pi'} \rangle_{\mu_E}$ equals the same expectation, and the left side of (7.19) is zero.

For part (3), we apply (H5)(c)(iii⁺) with $V = B_\sigma^{(a)} \cup B_\sigma^{(b)}$, $B^1 = B_\sigma^{(a)}$, and $B^2 = B_\sigma^{(b)}$, which gives $J_R(\gamma_0, \gamma'_0) \leq X_{ab}$; recall that J_R vanishes if one of its arguments is $\dagger$. By (7.19), the double sum is at most

$$\mathbb{E}_{\mu_E} \sum_{\pi, \pi'} [F(J_R(\gamma(\pi), \gamma(\pi')))) - 1] R_\pi(\sigma \mid \Gamma) R_{\pi'}(\sigma \mid \Gamma').$$

The rate $R_\pi(\sigma \mid \Gamma)$ vanishes unless $\gamma(\pi) = \gamma_0 \neq \dagger$, in which case it equals $r_\pi(\sigma)$, and the same holds for π' and γ'_0 . If $\gamma_0 \neq \dagger$, then by (H5)(a), as noted after Definition 7.5, the sum of $r_\pi(\sigma)$ over $\pi \in \Pi_E^{(a)}$ with $\gamma(\pi) = \gamma_0$ is one, and the same holds for γ'_0 . This proves the first inequality, since $F(0) = 1$. The second follows from $F(J) \leq e^{8\eta_{\text{loc}}J}$, which holds because $-\log(1 - \eta_{\text{loc}}) \leq 2\eta_{\text{loc}}$, and from $J_R(\gamma_0, \gamma'_0) \leq X_{ab}$.

For part (4), the function $f(\theta) = \mathbb{E}e^{\theta X} - 1$ is convex on $[0, \theta_\times]$ and $f(0) = 0$, so that $f(\theta) \leq (\theta/\theta_\times)f(\theta_\times)$. Since $\theta_\times X = \theta_\times J_R(\Gamma, \Gamma') + c_\times A_*(V; \Gamma, \Gamma')$, (H5)(d) with $\varepsilon' = \theta_\times \leq \kappa/2$, $c = c_\times$, and the base $p_\times \leq p_{\text{env}}(c_\times)$ in place of p gives $f(\theta_\times) \leq C_{\text{fin}}(c_\times)/K$. $\square$

Used in: Theorem 7.14.

Part (1) is an identity between functions of the matching. It does not assert that the conditional law of the matching, or of the paths of group a , is unchanged by a move of group b , and in general it is not, since the pair checks join rows of different groups. The comparison of four matchings gives the integrated upper bound (7.19); its left side is zero when $J = 0$.

7.7 The surgery theorem

Theorem 7.14. *Let a matching system satisfy (H1)–(H5), with the constants of Definition 7.2; these hypotheses include (H3₁), (H4) with base p , (H4⁺) with base $p_\times$, and (H5)(c)(iii⁺). Let $\lambda \geq 1$ with $M \leq \lambda m_*$, let $G = (2z_+/z_-)^3$, and let ε be given by (7.17). Let $0 < \varepsilon_{\text{max}} \leq \kappa/4$ and $c_{\text{rec}} \geq \log 4 + C_{\text{ch}}\varepsilon_{\text{max}}$, let $\theta_\times$ and $c_\times$ be as in Lemma 7.13, and suppose that $p \leq \min\{p_0, p_{\text{env}}(2c_{\text{rec}})\}$, $p_\times \leq p_{\text{env}}(c_\times)$, $\varepsilon \leq \varepsilon_{\text{max}}$, and $8\eta_{\text{loc}} \leq \theta_\times$. Let $\varepsilon_0 = \min\{\varepsilon_{\text{max}}, c_{\text{rec}}\}$. Let Q be the surgery kernel of Definition 7.5, and $q^{(a)} = \int q_\Gamma^{(a)} \mathbf{p}(d\Gamma)$ for every group a . Then the following hold for $K \geq K_0$.*

1. *The measure μQ is absolutely continuous with respect to μ , its density is in $L^2(\mu)$, and*

$$\chi^2(\mu Q \parallel \mu) \leq \frac{C_*}{mK} + \left(1 - \frac{1}{m}\right) \frac{C_\times \eta_{\text{loc}}}{K}, \quad (7.20)$$

where $C_* = 2C_{\text{fin}}(2c_{\text{rec}})/(1 - e^{-2\varepsilon_0})$ and $C_\times = 8C_{\text{fin}}(c_\times)/\theta_\times$.

2. For every group a , the acceptance probability of the surgeries of group a satisfies

$$\int q^{(a)}(x, X) \mu(dx) = \mathbb{P}(\mathcal{R}(B_\sigma^{(a)}, \Gamma) \neq \dagger) \geq 1 - C_{\text{acc}} p,$$

and the same bound holds for the acceptance probability $\int q(x, X) \mu(dx)$ of Q , where $q = m^{-1} \sum_a q^{(a)}$.

3. The kernel Q preserves E , and $\chi^2(\mu Q \| \mu) = \int \chi^2(\mu_E Q \| \mu_E) \nu(dE)$.

The constant C_* depends only on $\varepsilon_{\max}$, c_{rec} , and the routes of (H5), and $C_\times$ depends only on $\theta_\times$, $C_{\text{fin}}(c_\times)$, and the routes. Neither depends on K , m , M , m_* , η_{glob} , η_1 , $z_\pm$, p , $p_\times$, or the exposed data, and the threshold K_0 is that of (H5), which does not involve the groups. The estimate (7.20) concerns the unconditional law μ ; we do not assert a bound of order $1/K$ on each fiber μ_E , nor a lower bound on the acceptance probability for each E . Combining the theorem with Lemma 5.2, we obtain $(\mathbb{E}_{\mu Q} O - \mathbb{E}_\mu O)^2 \leq \chi^2(\mu Q \| \mu) \text{Var}_\mu(O)$ for every $O \in L^2(\mu)$, with the bound (7.20), and by (5.2), the gain $\mathbb{E}_{\mu Q} O - \mathbb{E}_\mu O$ is the mean change of O over accepted surgeries,

$$\begin{aligned} \mathbb{E}_{\mu Q} O - \mathbb{E}_\mu O &= \frac{1}{m} \sum_{a=1}^m \int \mathfrak{p}(d\Gamma) \int \mu(dx) \sum_{\pi \in \Pi_E^{(a)}} R_\pi(\sigma | \Gamma) \\ &\quad \times \int (O(E, \sigma\pi, \omega_{S^c}, \omega'_S) - O(x)) P_\pi(d\omega'_S | x). \end{aligned}$$

The gains of the winding and of the stochastic area are computed in Sections 10 and 11; the theorem itself supplies no gain.

Remark 7.15. The factor K^{-1} requires cancellation before any upper bound by positive terms. For two moves of the same group, when the routes have no encounters and their indicators agree, Lemma 7.9 makes the signed contribution vanish, and the proof discards the negative terms only on the remaining event, controlled by the skeleton encounter count and the vulnerable components approached by both skeletons. For two moves of different groups, the negative terms are kept on every event, and Lemma 7.13 bounds the signed contribution by the encounter loss of the two routes, which gives the factor η_{loc} in (7.20). A merely bounded second moment would give no positive thermodynamic lower bound; see Section 2.5. The antecedents for conditional centering are [113, Lemmas 6.6–6.8] and [114, Lemma 5.2].

Proof of Theorem 7.14. We fix $K \geq K_0$. Since $\theta_\times \leq 1$, the assumption $8\eta_{\text{loc}} \leq \theta_\times$ implies $\eta_{\text{loc}} \leq 1/8$, so that Lemmas 7.8, 7.12, and 7.13 apply. Steps 1–7 concern the surgeries of one group a , and Step 8 combines the groups.

Step 1: the likelihood on a fiber. Fix a group a and $E \in E_0$. By Lemma 7.6(3), for every Γ , the measure $\mu_E Q_\Gamma^{(a)}$ has the density $L_{a,\Gamma}$ of (7.7) with respect to μ_E . The set $\Pi_E^{(a)}$ is finite, the rates take values in $[0, 1]$, and by Lemma 7.8 with $\pi' = \pi$ and $\Gamma' = \Gamma$,

$$\|c_\pi(\cdot \mid \Gamma)\|_{L^2(\mu_E)}^2 \leq G^{|\gamma(\pi)|} (1 - \eta_{\text{loc}})^{-4J_R(\gamma(\pi), \gamma(\pi))}.$$

In particular, $\sup_\Gamma \|L_{a,\Gamma}\|_{L^2(\mu_E)} < \infty$. The density $L_{a,\Gamma}(\sigma, \omega)$ is jointly measurable in (Γ, σ, ω) , since $\mathcal{R}(B, \cdot)$ is measurable for each of the finitely many sets $B \subset \Lambda_{\text{act}}$. By Lemma 5.3, the measure $\mu_E Q^{(a)}$ is absolutely continuous with respect to μ_E , with density $\int L_{a,\Gamma} \mathfrak{p}(d\Gamma)$, and

$$\begin{aligned} \chi^2(\mu_E Q^{(a)} \parallel \mu_E) &= \iint \sum_{\pi, \pi' \in \Pi_E^{(a)}} \left(\langle c_\pi, c_{\pi'} \rangle_{\mu_E} - \langle c_\pi, R_{\pi'} \rangle_{\mu_E} \right. \\ &\quad \left. - \langle R_\pi, c_{\pi'} \rangle_{\mu_E} + \langle R_\pi, R_{\pi'} \rangle_{\mu_E} \right) \mathfrak{p}(d\Gamma) \mathfrak{p}(d\Gamma'), \end{aligned}$$

where c_π and R_π carry the skeleton Γ , and $c_{\pi'}$ and $R_{\pi'}$ carry Γ' . All sums are finite and all terms are finite.

Step 2: the four terms. By Lemmas 7.7 and 7.8, with the notation (7.16), the integrand above is at most $\mathbb{E}_{\mu_E} \sum_{\pi, \pi'} (T^{\text{ii}} - T^{\text{io}} - T^{\text{oi}} + T^{\text{oo}})$. We group the pairs (π, π') according to their routes (γ, γ') , and obtain

$$\chi^2(\mu_E Q^{(a)} \parallel \mu_E) \leq \mathbb{E}_{\mu_E \otimes \mathfrak{p} \otimes \mathfrak{p}} \sum_{(\gamma, \gamma')} \Sigma(\gamma, \gamma'), \quad \Sigma(\gamma, \gamma') = \sum_{\pi, \pi'} (T^{\text{ii}} - T^{\text{io}} - T^{\text{oi}} + T^{\text{oo}}), \quad (7.21)$$

where the outer sum is over ordered pairs of routes in Λ_{act} , and the inner sum is over their labelings by rows of group a . The function $\Sigma(\gamma, \gamma')$ depends on $(\sigma, \Gamma, \Gamma')$.

Step 3: the route indicators. Since $V_\sigma^{(a)} \subset \Lambda_{\text{act}}$ is finite, (H5)(c) applies with $V = V_\sigma^{(a)}$ and $B_0 = B_\sigma^{(a)}$. Fix σ , Γ , and Γ' , and write $A_* = A_*(V_\sigma^{(a)}; \Gamma, \Gamma')$, $J_0 = J_R(\Gamma, \Gamma')$, and $\mathcal{L} = \mathcal{L}(V_\sigma^{(a)}, B_\sigma^{(a)}; \Gamma, \Gamma')$. By Lemma 7.11, for every pair of labelings of (γ, γ') by rows of group a , each of the four indicator products has the form $\mathbf{1}\{\mathcal{R}(B^1, \Gamma) = \gamma\} \mathbf{1}\{\mathcal{R}(B^2, \Gamma') = \gamma'\}$ with $B^1, B^2 \subset V_\sigma^{(a)}$ and

$$B^1 \triangle B_\sigma^{(a)} \subset \mathcal{N}_1(\gamma'), \quad B^2 \triangle B_\sigma^{(a)} \subset \mathcal{N}_1(\gamma),$$

according to the following table.

	$\mathbf{1}^{\text{ii}}$	$\mathbf{1}^{\text{io}}$	$\mathbf{1}^{\text{oi}}$	$\mathbf{1}^{\text{oo}}$
B^1	$B_{\sigma\pi'}^{(a)}$	$B_\sigma^{(a)}$	$B_{\sigma\pi'}^{(a)}$	$B_\sigma^{(a)}$
B^2	$B_{\sigma\pi}^{(a)}$	$B_{\sigma\pi}^{(a)}$	$B_\sigma^{(a)}$	$B_\sigma^{(a)}$

Two consequences follow. First, if one of the four indicator products equals one, then $\gamma = \mathcal{R}(B^1, \Gamma)$ and $\gamma' = \mathcal{R}(B^2, \Gamma')$ for sets B^1, B^2 as in the definition of $\mathcal{L}$, which implies that $(\gamma, \gamma') \in \mathcal{L}$. Second, if $A_* = 0$, then (H5)(c)(ii) implies that all four indicator products equal $\mathbf{1}^{\text{oo}}$, for every pair of labelings.

Step 4: the event $A_* = 0$. Suppose that $A_* = 0$. By the second consequence in Step 3, all four indicator products equal $\mathbf{1}^{\text{oo}}$, for every pair of labelings. The indicator $\mathbf{1}^{\text{oo}}$ vanishes unless (γ, γ') is the pair $(\gamma_0, \gamma'_0) = (\mathcal{R}(B_\sigma^{(a)}, \Gamma), \mathcal{R}(B_\sigma^{(a)}, \Gamma'))$ and both γ_0 and γ'_0 are routes other than $\dagger$, so that $\Sigma(\gamma, \gamma') = 0$ for all other pairs. Suppose that γ_0 and γ'_0 are routes. Taking $B^1 = B^2 = B_\sigma^{(a)}$, we see that this pair lies in $\mathcal{L}$, and (H5)(c)(iii⁺) with $A_* = 0$ implies that $J_R(\gamma_0, \gamma'_0) \leq J_0$. We distinguish two cases.

If $J_R(\gamma_0, \gamma'_0) = 0$, then by Lemma 7.9(1), for every pair of labelings of (γ_0, γ'_0) , we have $S \cap S' = \emptyset$ and $\tilde{\pi} = \pi$, so the factor $G^{|S \cap S'|} (1 - \eta_{\text{loc}})^{-4J_R(\gamma_0, \gamma'_0)}$ in T^{ii} equals one. Since all indicators equal one, by Lemma 7.9(2), we have

$$T^{\text{ii}} - T^{\text{io}} - T^{\text{oi}} + T^{\text{oo}} = (r_\pi(\sigma\pi') - r_\pi(\sigma))(r_{\pi'}(\sigma\pi) - r_{\pi'}(\sigma)) = 0,$$

and $\Sigma(\gamma_0, \gamma'_0) = 0$. By Lemma 7.9(3), the bound of Step 2 is an equality for pairs of labeled routes whose routes do not meet; on the event considered here, these pairs contribute exactly zero, and not only at most zero. If $J_R(\gamma_0, \gamma'_0) \geq 1$, then $J_0 \geq 1$. We discard the nonpositive terms $-T^{\text{io}}$ and $-T^{\text{oi}}$, and by Lemma 7.12, we have $\Sigma(\gamma_0, \gamma'_0) \leq 2e^{\varepsilon J_R(\gamma_0, \gamma'_0)} \leq 2e^{\varepsilon_{\max} J_0}$. In both cases, we obtain

$$\sum_{(\gamma, \gamma')} \Sigma(\gamma, \gamma') \leq 2e^{\varepsilon_{\max} J_0 + c_{\text{rec}} A_*} \mathbf{1}\{J_0 + A_* \geq 1\}.$$

Step 5: the event $A_* \geq 1$. Suppose that $A_* \geq 1$. Since $T^{\text{io}}, T^{\text{oi}} \geq 0$, we have $\Sigma(\gamma, \gamma') \leq \sum_{\pi, \pi'} (T^{\text{ii}} + T^{\text{oo}})$. By Lemma 7.12 and the first consequence in Step 3, this is at most $2e^{\varepsilon J_R(\gamma, \gamma')} \mathbf{1}\{(\gamma, \gamma') \in \mathcal{L}\}$. By (H5)(c)(i) and (iii⁺), we obtain

$$\sum_{(\gamma, \gamma')} \Sigma(\gamma, \gamma') \leq 2 \cdot 4^{A_*} e^{\varepsilon(J_0 + C_{\text{ch}} A_*)} \leq 2e^{\varepsilon_{\max} J_0 + c_{\text{rec}} A_*} \mathbf{1}\{J_0 + A_* \geq 1\},$$

where we used $\varepsilon \leq \varepsilon_{\max}$ and $\log 4 + C_{\text{ch}} \varepsilon_{\max} \leq c_{\text{rec}}$. This is the same bound as in Step 4.

Step 6: the exponential bound. Let $X = \varepsilon_{\max} J_0 + c_{\text{rec}} A_*$. Since J_0 and A_* are nonnegative integers, we have $X \geq \varepsilon_0$ on the event $\{J_0 + A_* \geq 1\}$. For $x \geq \varepsilon_0$, we have $e^{2x} - 1 = e^{2x}(1 - e^{-2x}) \geq e^x(1 - e^{-2\varepsilon_0})$. Then

$$2e^X \mathbf{1}\{J_0 + A_* \geq 1\} \leq \frac{2}{1 - e^{-2\varepsilon_0}} (e^{2X} - 1),$$

where on the complementary event both sides vanish. By Steps 2, 4, and 5, for ν -almost every E ,

$$\chi^2(\mu_E Q^{(a)} \parallel \mu_E) \leq \frac{2}{1 - e^{-2\varepsilon_0}} \mathbb{E}_{\mu_E \otimes \mathbb{p} \otimes \mathbb{p}} \left[e^{2\varepsilon_{\max} J_R(\Gamma, \Gamma') + 2c_{\text{rec}} A_*(V_\sigma^{(a)}; \Gamma, \Gamma')} - 1 \right].$$

Step 7: one group. The kernel $Q^{(a)}$ preserves E , and for ν -almost every E , the measure $\mu_E Q^{(a)}$ has the density $L_a(E, \cdot) = \int L_{a,\Gamma} \mathfrak{p}(\mathrm{d}\Gamma)$ with respect to μ_E , and $L_{a,\Gamma}$ is jointly measurable in E, Γ , and the state, by the explicit formulas of Lemma 7.6 and Lemma 5.3, extended by $L_{a,\Gamma} = 1$ on $E \notin E_0$. Then $\mu Q^{(a)} = \int \mu_E Q^{(a)} \nu(\mathrm{d}E)$ is absolutely continuous with respect to μ , with density L_a , and $\int (L_a - 1)^2 \mathrm{d}\mu = \int \chi^2(\mu_E Q^{(a)} \parallel \mu_E) \nu(\mathrm{d}E)$. Under $\mu \otimes \mathfrak{p} \otimes \mathfrak{p}$, the vulnerable set $V_\sigma^{(a)}$ is a random subset of Λ_{act} that is independent of (Γ, Γ') and satisfies the prescribed-set bound (H4). We integrate the last display over E and apply (H5)(d) with $\varepsilon' = 2\varepsilon_{\max}$ and $c = 2c_{\text{rec}}$, which is allowed because $2\varepsilon_{\max} \leq \kappa/2$ and $p \leq p_{\text{env}}(2c_{\text{rec}})$. This gives

$$\chi^2(\mu Q^{(a)} \parallel \mu) \leq \frac{2}{1 - e^{-2\varepsilon_0}} \left(\mathbb{E} \left[e^{2\varepsilon_{\max} J_R(\Gamma, \Gamma') + 2c_{\text{rec}} A_*(V_\sigma^{(a)}; \Gamma, \Gamma')} \right] - 1 \right) \leq \frac{C_*}{K},$$

where the expectation is over independent (E, σ, ω) with law μ and Γ, Γ' with law $\mathfrak{p}$. In particular, $L_a \in L^2(\mu)$.

The outgoing mass of $q^{(a)}$ at x is $\mathfrak{p}(\{\Gamma : \mathcal{R}(B_\sigma^{(a)}, \Gamma) \neq \dagger\})$, as noted after Definition 7.5. Under $\mu \otimes \mathfrak{p}$, the bad set $B_\sigma^{(a)}$ is a random subset of Λ_{act} , independent of Γ , and $\mathbb{P}(\mathcal{F} \subset B_\sigma^{(a)}) \leq \mathbb{P}(\mathcal{F} \subset V_\sigma^{(a)}) \leq p^{|\mathcal{F}|}$ for every finite $\mathcal{F}$ by (H4). Since $p \leq p_0$, (H5)(b) implies that $\mathbb{P}(\mathcal{R}(B_\sigma^{(a)}, \Gamma) = \dagger) \leq C_{\text{acc}} p$. The acceptance probability of Q is the average of these probabilities over a , which proves part (2).

Step 8: the average over groups. By Definition 7.5 and Step 7, the measure μQ is absolutely continuous with respect to μ , with density $L = m^{-1} \sum_a L_a \in L^2(\mu)$. Since $\int L_a \mathrm{d}\mu = 1$ for every a ,

$$\chi^2(\mu Q \parallel \mu) = \int (L - 1)^2 \mathrm{d}\mu = \frac{1}{m^2} \sum_{a,b=1}^m \langle L_a - 1, L_b - 1 \rangle_\mu.$$

The kernel Q preserves E , and for ν -almost every E , the measure $\mu_E Q$ has the density $L(E, \cdot)$ with respect to μ_E , so that $\int (L - 1)^2 \mathrm{d}\mu = \int \chi^2(\mu_E Q \parallel \mu_E) \nu(\mathrm{d}E)$. This proves part (3). By Step 7, each of the m diagonal terms satisfies $\langle L_a - 1, L_a - 1 \rangle_\mu = \chi^2(\mu Q^{(a)} \parallel \mu) \leq C_*/K$.

Let $a \neq b$. Since $L_{a,\Gamma} \geq 0$, we have $|L_{a,\Gamma} - 1| \leq L_{a,\Gamma} + 1$, and Tonelli's theorem and the Cauchy-Schwarz inequality give

$$\begin{aligned} & \int \nu(\mathrm{d}E) \iint |\langle L_{a,\Gamma} - 1, L_{b,\Gamma'} - 1 \rangle_{\mu_E}| \mathfrak{p}(\mathrm{d}\Gamma) \mathfrak{p}(\mathrm{d}\Gamma') \\ & \leq \int \nu(\mathrm{d}E) \iint \langle L_{a,\Gamma} + 1, L_{b,\Gamma'} + 1 \rangle_{\mu_E} \mathfrak{p}(\mathrm{d}\Gamma) \mathfrak{p}(\mathrm{d}\Gamma') = \int (L_a + 1)(L_b + 1) \mathrm{d}\mu < \infty. \end{aligned}$$

The middle expression also bounds the integral of $|L_{a,\Gamma} - 1| |L_{b,\Gamma'} - 1|$ with respect to $\mu_E \otimes \mathfrak{p} \otimes \mathfrak{p}$ and $\nu(\mathrm{d}E)$. By Fubini's theorem, we obtain

$$\langle L_a - 1, L_b - 1 \rangle_\mu = \int \nu(\mathrm{d}E) \iint \langle L_{a,\Gamma} - 1, L_{b,\Gamma'} - 1 \rangle_{\mu_E} \mathfrak{p}(\mathrm{d}\Gamma) \mathfrak{p}(\mathrm{d}\Gamma').$$

For ν -almost every E and all Γ, Γ' , by (7.7), the inner product on the right side is the finite sum of the terms $\langle c_\pi - R_\pi, c_{\pi'} - R_{\pi'} \rangle_{\mu_E}$ over $\pi \in \Pi_E^{(a)}$ and $\pi' \in \Pi_E^{(b)}$, where c_π, R_π carry Γ and $c_{\pi'}, R_{\pi'}$ carry Γ' . By Lemma 7.13(3), this sum is at most $\mathbb{E}_{\mu_E}[e^{8\eta_{\text{loc}} X_{ab}} - 1]$, where $X_{ab} = J_R(\Gamma, \Gamma') + C_{\text{ch}} A_*(B_\sigma^{(a)} \cup B_\sigma^{(b)}; \Gamma, \Gamma')$. Under $\mu \otimes \mathfrak{p} \otimes \mathfrak{p}$, the set $B_\sigma^{(a)} \cup B_\sigma^{(b)} \subset V_\sigma^{(a)} \cup V_\sigma^{(b)}$ is a random subset of Λ_{act} that is independent of (Γ, Γ') , and by (H4⁺), it satisfies the prescribed-set bound with base $p_\times \leq p_{\text{env}}(c_\times)$. By Lemma 7.13(4) with $\theta = 8\eta_{\text{loc}} \leq \theta_\times$,

$$\langle L_a - 1, L_b - 1 \rangle_\mu \leq \mathbb{E}_{\mu \otimes \mathfrak{p} \otimes \mathfrak{p}}[e^{8\eta_{\text{loc}} X_{ab}} - 1] \leq \frac{8\eta_{\text{loc}}}{\theta_\times} \frac{C_{\text{fin}}(c_\times)}{K} = \frac{C_\times \eta_{\text{loc}}}{K}.$$

There are m diagonal and $m(m-1)$ off-diagonal terms, which gives (7.20) and completes the proof of part (1). $\square$

Used in: Propositions 10.1 and 11.9, together with the slab verification Corollary 9.12 and the gain estimates.

The common choice of constants is given in Section 10.3; Section 11.5 lists the additional container requirements.

PART III

The hard-sphere gas

Thermal inputs

In this section, we state the results on the hard-sphere Gibbs measure that are used in Sections 9 to 11. These are the scaled setting and the thermal and bridge estimates needed to analyze one short slab of imaginary time. On the torus they are quoted from OpenAI’s preprint [112], except Proposition 8.5, which extends a statement of [112] to higher temperatures and is proved in Appendix B; their analog in a container, Proposition 8.10, is proved in Appendix A. Section 8.1 fixes the scaled variables, and Section 8.2 identifies the thermal slab of [112] with one slab of the Bose path measure (Lemma 8.2). Sections 8.3 to 8.5 state the joint bound for failure of the local conditions, the estimate for a single replacement path, and the supply of useful labels. Section 8.6 treats the container.

Imported results. The periodic thermal results imported from the literature are listed below; the two lattice lemmas of [112, Lemmas 8.3 and 8.4] are proved in Section 6. The classical tools include the facts recalled in Section 3, the Markov property and Gaussian bridge computations, Doob’s maximal inequality, and Hoeffding’s sampling comparison [71, Theorem 4]. For the extension to higher temperatures and the transfer to a container, we also use the cited inputs enumerated in Tables A.1 and B.1, strong subadditivity of quantum entropy [98], and monotonicity of relative entropy [106], together with the other operator inequalities cited in Appendix A. Their adaptations are verified in Appendices A and B. Definition 8.3, Lemma 8.7, Remark 8.8, and Lemma 8.9 reproduce the cited statements of [112], with the notation changes in Table 8.1 and the explicitly stated omissions. Proposition 8.5 extends [112, Proposition 5.4] and is proved in Appendix B, following [112]. The one-slab identification and the container transfer are proved here and in Appendix A, respectively.

Much of [112, Sections 3–7] is reproduced there, with attribution, from the earlier OpenAI preprints [111, 115]: the energy and spatial estimates adapt [115, Section 3] and [111, Sections 3–4]; regular paths are those of [115, Definition 4.3]; the trajectory estimate behind Proposition 8.5, the positive-temperature counterpart of [115, Proposition 4.7], adapts the trace comparison of [111, Section 6, Lemma 6.2]; Lemma 8.7 and Remark 8.8 reproduce [115, Proposition 5.3 and Remark 5.4]; and Lemma 8.9 corresponds to [115, Lemma 6.1]. The main theorems of these three preprints concern Bose–Einstein condensation and are not used.

Table 8.1. Notation of [112] and of this paper.

Object	In [112]	Here
symmetric killed heat kernel; partition function	$B_s^{\text{sym}}; Z_N$	K_s^s (Section 3.2); $\text{tr } e^{-\beta H}$
purified slab law	$\mathcal{R}$	$\mathbb{P}^{\text{pur}}$ (8.3)
close-count fraction; separation constant	$\theta; b$	$\theta_0; b_0$
close count at a cell v	$C_v(X)$	$N_v^{\text{cl}}(X)$
scaled oscillation on a mesh interval	$A_k(\omega)$	$\text{osc}_k(\omega)$
regularity exponent	$\gamma = 1/12$	written out
trial laws	$Q_{y,z}^u, Q_{y,z}^B, Q_{y,z}^0, Q_i^u$	$P_{y,z}^u, P_{y,z}^B, P_{y,z}^0, P_i^u$

In a quoted statement, $c, C > 0$ denote constants independent of K, D , and T in the ranges under consideration; they may depend on parameters fixed earlier, including the window for α [112, Section 2].

8.1 The scaled setting

Let $K \geq 100$ be an integer, let $\Lambda_K = (\mathbb{R}/K\mathbb{Z})^3$ with the torus distance d , and for $N \geq 1$ set

$$V = K^3, \quad D = N/V, \quad r = \alpha/D, \quad \alpha \in [\alpha_0/2, 2\alpha_0], \quad 0 < T \leq \tau_0/\log D, \quad \beta = T^{-1}, \quad (8.1)$$

where $\tau_0 = \kappa_T \alpha_0$ is fixed in Appendix B; D is taken large enough that $\tau_0/\log D \leq 1/4$. With $0 < T \leq D^{-1}$ in place of the temperature condition, this is [112, (2.1)], the parameter range of [112, Theorem 2.1]; that theorem, a lower bound on the occupation of the constant orbital, is not used. The number $\alpha_0 \geq 2$ is fixed sufficiently large, and subsequently D is taken sufficiently large. Here D, r , and T are the density, the exclusion distance, and the temperature in scaled units. They are related to the physical variables of Section 3 by the dilation $x = s\tilde{x}$, $s = L/K$, under which $r = a/s$ and the scaled temperature is $s^2 T_{\text{phys}}$. The density relation is

$$D = \rho_L s^3, \quad \rho_L = \begin{cases} N/L^3 & \text{on the torus,} \\ N/(L^3|\Omega|) & \text{in a container,} \end{cases} \quad \rho_L \longrightarrow \rho.$$

Here ρ is the limiting density of the main theorems; see Sections 10 and 11.

The configuration domain, the Hamiltonian H , the kernels K_s and K_s^s , the bridge measures $\mathbb{W}_{X,Y}^s$, and the events $\mathcal{E}_s$ are those of Sections 3.1 and 3.2, with L replaced by K and the exclusion distance by r ; the identities of Section 3 hold for every exclusion

distance. A torus path has a continuous Euclidean lift, unique after the lift of its initial point is fixed; lifted increments and diameters do not depend on that choice. The unit cells are $B_v = v + [-1/2, 1/2]^3$, periodically interpreted, with $v \in \mathbb{L}_K = (\mathbb{Z}/K\mathbb{Z})^3$, and $n_v(X)$ is the number of positions of X in B_v . We write d_∞ for the periodic maximum distance on $\mathbb{L}_K$; distances to cells mean distances to their centers. For a configuration X , let $d_i(X) = \min_{k \neq i} d(x_i, x_k)$.

The spatial constants of [112, Section 4] are fixed in the following order: α_0 sufficiently large and then $\ell > 0$ sufficiently small, as in [112, Proposition 4.5]; then $\theta_0 > 0$ with $3\theta_0 < \ell/2$; then $b_0 \in (0, 1)$ sufficiently small, as in [112, Proposition 4.6]. Set

$$s_0 = b_0 D^{-1/3}, \quad N_v^{\text{cl}}(X) = \#\{i : d(x_i, v) \leq 5, d_i(X) \leq s_0\}, \quad (8.2)$$

and increase D so that $r < s_0 < 1/2$. These constants and the numerical constant C_{reg} of Definition 8.3 are fixed once, before all parameters introduced below, so that the requirements of Proposition 8.10 and the temperature-free requirements of Proposition B.2, with the constants of Proposition B.1, are also met; these sources allow α_0 to be increased and ℓ, θ_0, b_0 to be decreased.

Remark 8.1. The volume threshold K_1 of Proposition 8.5 depends on neither D , α , nor T (Lemma B.5); the volume-uniformity assertions for the occupation bound in [112, Theorem 2.1 and Section 9.6] are not used. In a container the volume threshold is K_Ω of Proposition 8.10, which depends on neither D nor T .

Used in: Sections 9 to 11 and Appendices A and B.

8.2 The thermal slab

Fix $0 < t \leq 1$, set $\delta = 2t$, and assume $\beta > \delta$; since $T \leq \tau_0 / \log D \leq 1/4$, this holds. The time t is fixed before the lower threshold on D . Let ν_N be the measure of N independent torus Brownian motions with generator Δ on $[0, \delta]$, with Lebesgue initial measure, and write $X(s) = (\omega_1(s), \dots, \omega_N(s))$. For $W' \in \Lambda_K^N$, the purified slab law of [112, (5.1)] is

$$\mathbb{P}^{\text{pur}}(d\omega, dW') = \frac{K_{\beta/2-t}^s(W', X(0)) K_{\beta/2-t}^s(X(\delta), W')}{\text{tr } e^{-\beta H}} \mathbf{1}_{\mathcal{E}_\delta}(\omega) \nu_N(d\omega) dW', \quad (8.3)$$

where $\mathbf{1}_{\mathcal{E}_\delta}(\omega) = 1$ if and only if $d(\omega_i(s), \omega_k(s)) > r$ for all $s \in [0, \delta]$ and $i < k$.

The slab variables of this paper are those of Section 3.6. Let $\mathbb{X}$ be the one-particle space (Λ_K , or $\mathbb{R}^3$ for a container) and $\mathcal{X}_N$ the configuration domain. For $0 < \delta < \beta$, the *one-slab law* is the probability measure on $\mathcal{X}_N \times \mathcal{X}_N \times S_N \times C([0, \delta]; \mathbb{X}^N)$ given by

$$\mathbb{P}(dY, dZ, \{\sigma\}, d\omega) = \frac{K_{\beta-\delta}^s(Z, Y)}{N! \text{tr } e^{-\beta H}} \mathbf{1}_{\mathcal{E}_\delta}(\omega) \mathbb{W}_{Y, \sigma Z}^\delta(d\omega) dY dZ, \quad (\sigma Z)_i = Z_{\sigma(i)}. \quad (8.4)$$

Under $\mathbb{P}$, the path ω_i runs from Y_i to $Z_{\sigma(i)}$. The labels of Y are those of the paths, the labels of Z name the terminal positions, which we call *slots*, and σ assigns to each path its slot. Summing over σ and using Lemma 3.6, we conclude that (8.4) has mass one. By Proposition 3.17(4), $\mathbb{P}$ is the marginal of each full slab in the marked representation $\bar{\mu}$, with the cut labels assigned independently as described after that proposition, and with its time interval translated to $[0, \delta]$.

Lemma 8.2. *On the torus, (8.3) defines a probability measure, and the law of the labeled path family $\omega = (\omega_1, \dots, \omega_N)$ is the same under $\mathbb{P}$ and under $\mathbb{P}^{\text{pur}}$, namely*

$$(\text{tr } e^{-\beta H})^{-1} K_{\beta-\delta}^s(\omega(\delta), \omega(0)) \mathbf{1}_{\mathcal{E}_\delta}(\omega) \nu_N(d\omega). \quad (8.5)$$

In particular, $\mathbb{P}(\omega \in G) = \mathbb{P}^{\text{pur}}(\omega \in G)$ for every measurable set G of path families. The same holds in a container, with wall killing included in $\mathcal{E}_\delta$ and ν_N the measure of independent Euclidean Brownian motions in (8.3).

Proof. By the symmetry of K^s and the semigroup identity of Lemma 3.6, integrating W' in the nonnegative measure (8.3) replaces the two outer kernels by $K_{\beta-\delta}^s(X(\delta), X(0))$, which gives (8.5) as its path marginal. For $\mathbb{P}$, let f be a bounded measurable function of ω . In the term of (8.4) indexed by σ , we change variables $Z' = \sigma Z$; by Lemma 3.6(1), $K_{\beta-\delta}^s(Z, Y) = K_{\beta-\delta}^s(Z', Y)$, so all $N!$ terms are equal and

$$\mathbb{E}f(\omega) = \frac{1}{\text{tr } e^{-\beta H}} \int dY \int dZ' K_{\beta-\delta}^s(Z', Y) \int f(\omega) \mathbf{1}_{\mathcal{E}_\delta}(\omega) W_{Y, Z'}^\delta(d\omega).$$

The kernel equals $K_{\beta-\delta}^s(\omega(\delta), \omega(0))$ on the support of $W_{Y, Z'}^\delta$, and $\nu_N = \int dY \int dZ' W_{Y, Z'}^\delta$ by the Markov property. This shows that $\mathbb{E}f(\omega)$ is the integral of f against (8.5). With $f = 1$, since $\mathbb{P}$ has mass one, so does $\mathbb{P}^{\text{pur}}$. The container case is identical. $\square$

Every event of the path family has the same probability under $\mathbb{P}^{\text{pur}}$, under $\mathbb{P}$, and under the law of each full slab of the Bose path measure. This is the only way in which Proposition 8.5 enters: as a statement about one slab.

Used in: Remark 8.6, for the joint bound (8.9).

8.3 Regular paths and local conditions

Following [112, (5.4)], we set

$$h_0 = D^{-1}, \quad m_t = \lceil t/r^2 \rceil, \quad \tau = t/m_t, \quad (8.6)$$

and take D large enough that $h_0 < t$ and $r^2/2 \leq \tau \leq r^2$. The intervals $[(k-1)\tau, k\tau]$, $1 \leq k \leq 2m_t$, divide each half of $[0, 2t]$ into m_t mesh intervals. For a path ω with lift $\tilde{\omega}$, let $\text{osc}_k(\omega) = r^{-1} \text{diam } \tilde{\omega}([(k-1)\tau, k\tau])$. The following is [112, Definition 5.2].

Definition 8.3. Fix a sufficiently large numerical constant C_{reg} . A path on $[0, 2t]$ is *regular* if

$$\max_k \text{osc}_k(\omega) \leq D^{1/12}, \quad \frac{1}{2m_t} \sum_{k=1}^{2m_t} (1 + \text{osc}_k(\omega))^3 \leq C_{\text{reg}}, \quad (8.7)$$

and its lifted diameter in each endpoint window $[0, h_0]$ and $[2t - h_0, 2t]$ is at most $s_0/8$.

Used in: Definition 8.4, Lemmas 8.7 and 8.9, and the half-path events of Section 9.2.

A path *visits* a set if it meets the set at some time. The next definition reproduces the local conditions stated before [112, Proposition 5.4], with H_1 as in Proposition 8.5. We retain the source's term “certificate” in Definition 8.4 and Proposition 8.5.

Definition 8.4. Let $v \in \mathbb{L}_K$, and write $Y = X(0)$ and $Z^\circ = X(2t)$. A “certificate at v ” consists of the following conditions on the full path sample:

1. $n_v(Y) \geq \ell D$ and $N_v^{\text{cl}}(Y), N_v^{\text{cl}}(Z^\circ) < \theta_0 D$;
2. at most $H_1 D$ paths visit the ball of radius 12 about v , every visiting path is regular, and fewer than $\theta_0 D$ visiting paths have lifted diameter greater than one.

These conditions define an event of the full path sample, invariant under relabeling; we do not condition on this event.

Proposition 8.5. *There are a numerical constant $\kappa_T \in (0, 1/16]$ and an integer K_1 , depending only on the constants fixed in Section 8.1, such that the following holds with $\tau_0 = \kappa_T \alpha_0$. For each prescribed $p_1 > 0$ there are H_1 and $\bar{t} \in (0, 1]$, and for each $t \in (0, \bar{t}]$ there is D_0 , such that, for every set A of distinct cell centers,*

$$\mathbb{P}^{\text{pur}}\{\text{no site of } A \text{ has a certificate}\} \leq p_1^{|A|} \quad (8.8)$$

whenever $K \geq K_1$, $D \geq D_0$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, and $0 < T \leq \tau_0 / \log D$.

For $0 < T \leq D^{-1}$, which lies in this window once D is large, and for K sufficiently large, this is [112, Proposition 5.4]. We prove the extension to the window, and the form of the volume threshold, in Appendix B, following the proof of [112, Proposition 5.4] and checking each use of the temperature.

Remark 8.6. The spatial constants α_0 , ℓ , θ_0 , b_0 , and C_{reg} are fixed before p_1 . After H_1 is fixed, Proposition 8.5 permits a further reduction of t before increasing D_0 . We use this permission for Lemma 8.9, whose requirement on t depends on H_1 . The quantities depending on t , including $\delta = 2t$, the mass bounds of Lemma 9.8, and the collision constant of Lemma 9.9, are fixed after this reduction. The group size M is chosen next; the number m of groups, which depends on D , is fixed in (10.7), and at most mM rows are retained per cell. The density threshold is increased last; the full parameter choice is

given in Section 10.3. The bound (8.8) is uniform in the window of (8.1) and imposes no relation between β and K .

The probability in (8.8) is an unconditional probability under $\mathbb{P}^{\text{pur}}$, that is, by Lemma 8.2, under the law of one slab of the canonical Gibbs path measure. It is not conditional on exposed data, on cut configurations, on a route, or on whether the local conditions hold at other cells, and no independence between distinct cells is assumed or asserted. For $A = \emptyset$ the bound is trivial.

By Lemma 8.2, (8.8) is equivalent to the following bound for the one-slab law $\mathbb{P}$ of (8.4):

$$\mathbb{P}\{\text{no cell of } A \text{ satisfies the conditions of Definition 8.4}\} \leq p_1^{|A|} \quad (8.9)$$

for every finite set A of distinct cells. We call (8.9) the joint bound with base p_1 . In a container it is required for sets of interior cells, and it is Proposition 8.10(i).

Used in: the application after Proposition 10.1 and the proof of Theorem 1.1; the joint bound (8.9) is a hypothesis of Propositions 9.11, 10.1, and 11.9 and Corollary 9.12; the order of choices is used in Sections 10 and 11.

8.4 Bridge estimates

Fix $0 < t \leq 1$. As in [112, (6.2)], we take D large enough that

$$r^2/2 \leq \tau \leq h_0/2, \quad h_0 < t/4, \quad rD^{1/12} < 1, \quad C\tau/t \leq r, \quad r < 3s_0/4, \quad (8.10)$$

for any fixed geometric constant C used in the proofs. Fix a cell B_v , and suppose that the outer endpoints y, z of a path have local lifts with $y \in B_v$ and $|z - v| \leq 3$. A *target cell* is B_v or one of its six face-neighbor cells. For u in a target cell B , let $P_{y,z}^u$ be the law obtained by joining two independent Euclidean Brownian bridges of generator Δ , from y to u and from u to z , each of duration t , projected to the torus. Let $P_{y,z}^B = \int_B P_{y,z}^u du$, a probability measure since $|B| = 1$, and let $P_{y,z}^0$ be the Euclidean Brownian bridge from y to z of duration $2t$, projected to the torus. We write $\mathcal{C}_v$ for the event that the path is regular and its local lift stays in the closed ball of radius 10 about v , and, for a deterministic torus path c ,

$$\mathcal{H}(c) = \{\omega : d(\omega(s), c(s)) \leq r \text{ for some } s \in [0, 2t]\}.$$

The next statement is [112, Lemma 6.3].

Lemma 8.7. *Let $y \in B_v$, $|z - v| \leq 3$, and let c be a deterministic regular path satisfying $d(y, c(0)) > s_0$ and $d(z, c(2t)) > s_0$. For either $P = P_{y,z}^B$, with any target cell B , or $P = P_{y,z}^0$, one has*

$$P(\mathcal{C}_v \cap \mathcal{H}(c)) \leq Ctr + C \frac{rD^{1/4}}{\sqrt{h_0}} \leq C\alpha(tD^{-1} + D^{-1/4}). \quad (8.11)$$

Thus this probability tends to zero with D for each fixed t .

Adapted in: the proof of Lemma 9.9, for the supports and endpoints of Section 9; the statement itself is not applied.

The next statement is [112, Remark 6.4], without its last sentence, which refers to later uses in [112].

Remark 8.8. The midpoint in $P_{y,z}^B$ is averaged over a unit cell. An estimate uniform in a prescribed midpoint is false: if $u = c(t)$, the trial collides at time t . Restricting a uniform midpoint to a measurable set of volume at least $a_0 > 0$ multiplies an upper bound for a nonnegative event by at most a_0^{-1} . Conditioning a trial on an event of probability at least a_0 has the same cost.

In Section 9 the midpoint is never prescribed: every collision bound integrates the full bridge law, and the only normalization is a division by a lower bound on a total mass.

Used in: the integrated bridge estimates of Sections 9.4 and 9.5; no estimate at a prescribed midpoint is asserted.

8.5 Useful labels under the local conditions

Label i refers to the path ω_i of the sample. Following [112, Section 7.1], a label i is *eligible at v in the sense of [112]* if

- (i) $\omega_i(0) \in B_v$ and $d(\omega_i(2t), v) \leq 3$;
- (ii) $d(\omega_i(0), \omega_k(0)) > s_0$ and $d(\omega_i(2t), \omega_k(2t)) > s_0$ for every $k \neq i$;
- (iii) its trajectory is regular and remains in the ball of radius 10 about v .

For such a label and u in a target cell, write $P_i^u = P_{\omega_i(0), \omega_i(2t)}^u$, with all lifts near v . For a collection F of fixed paths with labels other than i , let $C_i(F)$ be the event that the trial is regular, stays within distance 10 of v , and avoids every path in F at distance r , that is, stays at distance greater than r at every time. The next statement reproduces [112, Lemma 7.1], retaining its terminology.

Lemma 8.9. *At a cell with a certificate there are at least $\ell D/2$ eligible labels. After t has been chosen sufficiently small and D sufficiently large, at least nine tenths of these labels satisfy the following stronger property: for every target cell B_w ,*

$$|\{u \in B_w : P_i^u(C_i(F_i^{\text{all}})) \geq 0.9\}| \geq 0.99. \quad (8.12)$$

Here F_i^{all} consists of all other trajectories in the sampled configuration, and $|\cdot|$ denotes spatial volume.

We call an eligible label with property (8.12) *useful*. The requirement on t depends on H_1 , through [112, Corollary 6.5], which is why the reduction of t permitted in Proposition 8.5 is needed. Since every path that starts in B_v visits the ball of radius 12 about

v , Definition 8.4 implies that at most $H_1 D$ paths start in B_v at a cell satisfying the local conditions. By Lemma 8.9, we have at least $9\ell D/20$ useful labels there, a fraction at least $9\ell/(20H_1)$ of the paths starting in B_v .

Used in: Proposition 9.11.

8.6 The container

Let $\Omega \subset \mathbb{R}^3$ be a fixed nonempty bounded open set, and for an integer $K \geq 1$ let $U = K\Omega$, $V = |U|$, $D = N/V$, $r = \alpha/D$, with $\alpha \in [\alpha_0/2, 2\alpha_0]$ and $0 < T \leq \tau_0/\log D$. We consider N hard spheres of diameter r in U , with the Dirichlet conditions on the walls and on the collision set of Section 3.1, and the one-slab law $\mathbb{P}$ of (8.4) for U . Distances and diameters are Euclidean, and no lifts are needed. The cells are $B_v = v + [-1/2, 1/2]^3$ with $v \in \mathbb{Z}^3$. For $R_0 > 12$, a cell B_v is *interior* if $v \in U$ and $\text{dist}(v, \mathbb{R}^3 \setminus U) \geq R_0$. Local conditions, regular paths, eligible and useful labels, the trial laws P_i^μ , and the events $C_i(F)$ are defined as in Sections 8.3 and 8.5, with Euclidean distances; the trial laws consist of two independent unrestricted Euclidean bridges.

Proposition 8.10. *The constants α_0 , ℓ , θ_0 , b_0 , and C_{reg} of Section 8.1 can be chosen so that the following holds. Fix $R_0 > 12$. For every $p_1 > 0$, one can choose H_1 large, then $t \in (0, 1]$ sufficiently small, and then D_0 sufficiently large, and there exists $K_\Omega < \infty$, depending only on Ω and R_0 , such that, whenever $K \geq K_\Omega$, $D \geq D_0$, $N = DV \in \mathbb{N}$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, and $0 < T \leq \tau_0/\log D$:*

(i) *for every finite set A of distinct interior cells,*

$$\mathbb{P}(\text{every cell of } A \text{ fails the local conditions}) \leq p_1^{|A|}; \quad (8.13)$$

(ii) *at every interior cell v satisfying the local conditions, at most $H_1 D$ paths start in B_v , at least $\ell D/2$ labels are eligible at v , and at least nine tenths of these eligible labels are useful.*

The constants H_1 , t , and D_0 do not depend on Ω , K , or T . After H_1 is fixed, t may be decreased before D_0 is increased.

For an interior cell, the closed ball of radius 10 about v lies in U . The event $C_i(F)$ then implies survival at the walls, and the useful property is the same as on the torus. As in Remark 8.6, (8.13) is an unconditional bound under the law of one slab of the canonical Gibbs path measure in U , and the quantifiers are used in the same order. The proof in Appendix A follows the proofs of [112, Proposition 5.4 and Lemma 7.1], with the uses of the temperature checked as in Appendix B; the changes for the domain concern the insertion orbital, the lower bound on the free energy, the shortage bound, and the marked paths of the trajectory estimate, which are allowed to leave the container. The bridge

estimates of [112, Section 6] are Euclidean and local and apply without change at interior cells.

Used in: Section 10.3 and Section 11; part (i) is the joint bound (8.9) in a container.

The retained matching in one slab

In this section, we show that one slab of the hard-sphere path measure carries a matching system with m groups in the sense of Definition 7.1 that satisfies hypotheses (H1)–(H4), with (H3₁) and (H4⁺), of Section 7. At each active cell with at least mM eligible paths, we retain mM of them in a uniformly random order, and divide them into m groups of M ; otherwise we retain none. The rows label these retained eligible paths, each retained list has mM entries or is empty, and the slots are their terminal positions. The exposed data consist of everything except the assignment of the retained rows to these slots and the retained paths. The results are collected in Corollary 9.12, which is combined with Theorem 7.14 in Sections 10 and 11; hypothesis (H5) concerns the routes and is verified there. We rely on the one-slab law of Section 3.6 and on Section 8. Sections 9.1 and 9.2 fix the setting, eligibility, and the retained rows. Section 9.3 proves that the conditional law of the retained matching and paths is a matching system (Proposition 9.5) and checks (H1) (Lemma 9.4). Sections 9.4 to 9.6 prove (H2), (H3) with (H3₁), and (H4) with (H4⁺) (Lemmas 9.8 and 9.9 and Proposition 9.11).

The construction adapts the selection and exposure argument of [112, Section 7], which is that of [115, Section 6], in four ways. Eligibility is imposed separately on the two halves of a path, so that halves of permitted paths meeting at a common midpoint form a permitted path. The ordered list at a cell is divided into successive groups, as in [112, Section 7.2], but all groups remain retained in one matching fiber, whereas [112] exposes the paths outside a single selected group. The pairing of retained rows with their slots is not exposed. The conditional law is a weighted sum over matchings rather than the product law with fixed endpoints of [112, Lemma 7.2]. Finally, accessibility depends on the row and its slot. Our eligible list can be larger than the source list, so we estimate its useful fraction in Section 9.6.

9.1 Setting

Throughout this section, Λ is either the scaled torus Λ_K of Section 8.1 or the scaled container $U = K\Omega$ of Section 8.6, and the parameters satisfy (8.1), with the temperature-window condition in (8.1) replaced by $\beta > \delta$, and with $V = |U|$ in a container. We write d for the torus distance or the Euclidean distance, $\mathbb{L}$ for the cell lattice $\mathbb{L}_K$ or $\mathbb{Z}^3$, and $\mathcal{N}(v) = \{v, v \pm e_1, v \pm e_2, v \pm e_3\}$ for a cell v ; the cells B_w , $w \in \mathcal{N}(v)$, are the seven target

cells of v .

Active cells. On the torus, every cell is active: $\Lambda_{\text{act}} = \mathbb{L}_K$. In a container, set $R_{\text{act}} = 64$ and

$$\Lambda_{\text{act}} = \{v \in \mathbb{Z}^3 : v \in U, \text{dist}(v, \mathbb{R}^3 \setminus U) \geq R_{\text{act}}\}. \quad (9.1)$$

These are the interior cells of Section 8.6 for $R_0 = R_{\text{act}}$. In both settings, Λ_{act} is finite: the torus has K^3 cells, and the active cells in a container belong to the bounded set $U = K\Omega$. Rows are retained only at active cells, and the bad and vulnerable sets of Section 9.4 are subsets of Λ_{act} . The depth 64 ensures that permitted paths of rows at active cells stay in U (Lemma 9.4). That accepted routes lie in Λ_{act} is part of (H5) and is proved in Section 11.

The slab. Fix $t \in (0, 1]$ and set $\delta = 2t$. We write (Y, Z, σ, ω) for a sample of the one-slab law $\mathbb{P}$ of (8.4); the path ω_i runs from Y_i to $Z_{\sigma(i)}$. By Proposition 3.17(4), $\mathbb{P}$ is the law of every full slab of the Bose path measure, and by Lemma 8.2 the results of Section 8 apply to events of the path family under $\mathbb{P}$.

Standing assumptions. Unless stated otherwise, we assume

$$D > 2, \quad (8.10) \text{ holds with } C = 60, \quad \text{and } K \geq 2000 \text{ on the torus.} \quad (9.2)$$

The first two conditions hold for all D larger than a threshold depending only on t , α_0 , and b_0 . Under them, $r < 1$.

Local lifts. On the torus, for $K \geq 2000$ the projection $\mathbb{R}^3 \rightarrow \Lambda_K$ is an isometry from every Euclidean ball of radius 400 onto its image, since two points of such a ball are at distance at most $800 < K/2$. All events below concern paths and points within distance 400 of a cell center v ; we fix a lift $\tilde{v}$ and identify the two balls of radius 400 about v and $\tilde{v}$. Restricted to paths that stay in the ball, a torus bridge measure $\mathbb{W}_{x,y}^s$ of Section 3.2 is then the Euclidean bridge measure between the lifted endpoints, since in the sum over lifts defining $\mathbb{W}_{x,y}^s$ every other lift of y is at distance at least $K - 400$ from $\tilde{v}$. Let $\mathbb{W}_{\tilde{x},\tilde{y}}^{s,E}$ denote the unnormalized Euclidean bridge, of mass $p_s(\tilde{y} - \tilde{x})$, and write

$$\widehat{\mathbb{W}}_{\tilde{x},\tilde{y}}^{s,E} = \mathbb{W}_{\tilde{x},\tilde{y}}^{s,E} / p_s(\tilde{y} - \tilde{x}).$$

On the event that a path stays in the chosen chart, the torus bridge restriction is the projection of the corresponding Euclidean bridge restriction. In defining p_i^- and p_j^+ we use the normalized Euclidean bridges with these local lifts. In a container no lifts are needed, and $\tilde{x} = x$.

9.2 Eligible rows and ordered lists

We recall the mesh $h_0 = D^{-1}$, $m_t = \lceil t/r^2 \rceil$, $\tau = t/m_t$ of (8.6) and the oscillations $\text{osc}_k(\omega)$, $1 \leq k \leq 2m_t$. The intervals with $k \leq m_t$ lie in $[0, t]$, and since $h_0 < t/4$, the window $[0, h_0]$ lies in $[0, t]$ and the window $[2t - h_0, 2t]$ lies in $[t, 2t]$.

Definition 9.1. For $y \in \Lambda$, let $\mathcal{C}_y^-$ be the set of continuous paths ω on $[0, 2t]$ with $\omega(0) = y$ whose lift satisfies

- (i) $|\tilde{\omega}(s) - \tilde{\omega}(0)| \leq 20$ for $s \in [0, t]$;
- (ii) $\text{osc}_k(\omega) \leq D^{1/12}$ for $1 \leq k \leq m_t$, and $m_t^{-1} \sum_{k=1}^{m_t} (1 + \text{osc}_k(\omega))^3 \leq 2C_{\text{reg}}$;
- (iii) $\text{diam } \tilde{\omega}([0, h_0]) \leq s_0/8$.

For $z \in \Lambda$, let $\mathcal{C}_z^+$ be the set of paths ω with $\omega(2t) = z$ whose lift satisfies $|\tilde{\omega}(s) - \tilde{\omega}(2t)| \leq 20$ for $s \in [t, 2t]$, the two conditions in (ii) for $m_t < k \leq 2m_t$ (with the average over these m_t indices), and $\text{diam } \tilde{\omega}([2t - h_0, 2t]) \leq s_0/8$.

Definition 9.2. Let (Y, Z, σ, ω) be a sample. A row $i \in \{1, \dots, N\}$ is *eligible* if

- (E1) $d(Y_i, Y_k) > s_0$ for every $k \neq i$, and $d(Z_{\sigma(i)}, Z_l) > s_0$ for every $l \neq \sigma(i)$;
- (E2) $\omega_i \in \mathcal{C}_{Y_i}^- \cap \mathcal{C}_{Z_{\sigma(i)}}^+$.

Note that condition (E1) depends only on Y , on Z , and on the slot $\sigma(i)$; condition (E2) consists of a test on the first half of ω_i that refers to its initial point and a test on the second half that refers to its terminal point.

Lemma 9.3. Assume (9.2).

- (a) Let ω be a path on $[0, 2t]$ with $\omega(0) \in B_v$. If ω is regular and its lift stays within distance 10 of $\tilde{v}$, then $\omega \in \mathcal{C}_{\omega(0)}^- \cap \mathcal{C}_{\omega(2t)}^+$. In particular, every label that is eligible at v in the sense of [112] (Section 8.5) is an eligible row.
- (b) If $\omega \in \mathcal{C}_y^- \cap \mathcal{C}_z^+$, then $|\tilde{\omega}(2t) - \tilde{\omega}(0)| \leq 40$ and $|\tilde{\omega}(s) - \tilde{\omega}(0)| \leq 60$ for all $s \in [0, 2t]$. Moreover, $\text{osc}_k(\omega) \leq D^{1/12}$ for all $1 \leq k \leq 2m_t$, and the diameters of $\tilde{\omega}$ on the two windows of length h_0 are at most $s_0/8$.
- (c) For all y, z , the indicator of $\mathcal{C}_y^-$ is a function of the restriction of the path to $[0, t]$, and the indicator of $\mathcal{C}_z^+$ is a function of its restriction to $[t, 2t]$.

Proof. For (a), every point of the lift is within $10 + \sqrt{3}/2 < 20$ of $\tilde{\omega}(0)$, since $|\tilde{\omega}(0) - \tilde{v}| \leq \sqrt{3}/2$, and within 20 of $\tilde{\omega}(2t)$, since both lie within 10 of $\tilde{v}$. The maximum test in (ii) is part of (8.7). Since the summands in (8.7) are nonnegative, each half sum is at most the full sum, and we have

$$\frac{1}{m_t} \sum_{k \in \text{half}} (1 + \text{osc}_k(\omega))^3 \leq \frac{2}{2m_t} \sum_{k=1}^{2m_t} (1 + \text{osc}_k(\omega))^3 \leq 2C_{\text{reg}}.$$

The window conditions are those of Definition 8.3. A label eligible at v in the sense of [112] satisfies these hypotheses by its condition (iii), and condition (ii) there is (E1), since $Z_{\sigma(i)} = \omega_i(2t)$.

For (b), the point $\tilde{\omega}(t)$ is within 20 of both $\tilde{\omega}(0)$ and $\tilde{\omega}(2t)$, which gives the first bound. For $s \leq t$ we have $|\tilde{\omega}(s) - \tilde{\omega}(0)| \leq 20$, and for $s \geq t$ we have $|\tilde{\omega}(s) - \tilde{\omega}(0)| \leq |\tilde{\omega}(s) - \tilde{\omega}(2t)| + 40 \leq 60$. The remaining assertions are part of Definition 9.1. Part (c) holds because the mesh intervals and the window used in each half event lie in the corresponding half of $[0, 2t]$. $\square$

Ordered lists. For $v \in \Lambda_{\text{act}}$, let $\mathcal{I}_v^{\text{el}}$ be the set of eligible rows i with $Y_i \in B_v$, and let $n_v = |\mathcal{I}_v^{\text{el}}|$. Fix integers $m, M \geq 1$. Conditionally on the sample (Y, Z, σ, ω) , choose independently for every $v \in \Lambda_{\text{act}}$ a list L_v as follows. If $n_v \geq mM$, then L_v is a uniformly random ordered list of mM distinct elements of $\mathcal{I}_v^{\text{el}}$, each of the $(n_v)_{mM} = n_v! / (n_v - mM)!$ such lists having probability $1 / (n_v)_{mM}$. If $n_v < mM$, then L_v is empty. Let I_v be the set of entries of L_v for $v \in \Lambda_{\text{act}}$, and, for $1 \leq a \leq m$, let $I_v^{(a)}$ be the set of its entries in the positions $(a-1)M + 1, \dots, aM$. Set $I_v = I_v^{(a)} = \emptyset$ for $v \notin \Lambda_{\text{act}}$, and let

$$I = \bigsqcup_{v \in \mathbb{L}} I_v, \quad I^{(a)} = \bigsqcup_{v \in \mathbb{L}} I_v^{(a)}, \quad \text{anch}(i) = v \text{ for } i \in I_v, \quad J = \sigma(I).$$

The elements of I are the *retained rows*, $I^{(1)}, \dots, I^{(m)}$ are the *groups*, $\text{anch}(i)$ is the anchor of row i , and J is the set of *retained slots*. Each list has mM or no entries; in particular $|I_v^{(a)}| \leq M$ and $|I_v| \leq mM$. We keep the notation $\mathbb{P}$ and $\mathbb{E}$ for the joint law of the sample and the lists. Since the lists of a slab are chosen using that slab's sample only, the marked variables of distinct full slabs remain conditionally independent given the cut configurations, as Lemma 5.5 requires. In [112, Section 7.2] each cell carries $\lfloor \ell D / (4M) \rfloor$ groups of M labels; here the number m of groups is chosen in (10.7).

9.3 Exposed data and the conditional law

The *exposed data* E consist of the cut configurations Y and Z with their labels, the eligibility indicators $\varepsilon = (\varepsilon_k)_{k=1}^N$ of all rows, the lists $L = (L_v)_v$, and the slots $\sigma(k)$ and paths ω_k of all rows $k \notin I$. The set I , the groups, the anchors, and $J = \{1, \dots, N\} \setminus \sigma(I^c)$ are functions of E . The exposed data do not contain the restriction σ_I of σ to I , that is, the assignment of retained rows to retained slots, nor the retained paths $\omega_I = (\omega_i)_{i \in I}$. The sample and the lists are a measurable function of (E, σ_I, ω_I) , and conversely. Whether the local conditions hold at a cell is not included in E .

We write $F = (\omega_k)_{k \notin I}$ for the frozen paths, and let Av_F be the set of paths ω on $[0, 2t]$ with $d(\omega(s), \omega_k(s)) > r$ for all $s \in [0, 2t]$ and $k \notin I$ that, in a container, stay in U . For

$i \in I$ and $j \in J$, define the finite measure

$$K_{ij}^E(d\omega) = \mathbf{1}_{\mathcal{C}_{Y_i}^- \cap \mathcal{C}_{Z_j}^+ \cap \text{Av}_F}(\omega) \mathbb{W}_{Y_i, Z_j}^{2t}(d\omega), \quad (9.3)$$

and let $\mathcal{P}_{ij}^E$ be the set of paths from Y_i to Z_j in $\mathcal{C}_{Y_i}^- \cap \mathcal{C}_{Z_j}^+ \cap \text{Av}_F$. The measure K_{ij}^E is carried by $\mathcal{P}_{ij}^E$, and we call the elements of $\mathcal{P}_{ij}^E$ the *permitted paths* of the pairing (i, j) . Let $R_{\text{loc}} = 61$, $R_{\text{int}} = 123$, and $C_{\text{int}} = (2R_{\text{int}} + 1)^3$. For distinct $i, k \in I$, set

$$h_{ik}(\omega, \omega') = \mathbf{1}\{d(\omega(s), \omega'(s)) > r \text{ for all } s \in [0, 2t]\} \quad \text{if } d_\infty(\text{anch}(i), \text{anch}(k)) \leq R_{\text{int}},$$

and $h_{ik} \equiv 1$ otherwise. For a bijection $\sigma' : I \rightarrow J$, we define

$$\mathcal{M}_{\sigma'}^E(d\omega_I) = \prod_{i \in I} K_{i\sigma'(i)}^E(d\omega_i) \prod_{i < k} h_{ik}(\omega_i, \omega_k), \quad \Xi(E) = \sum_{\sigma' \in \text{Bij}(I, J)} \mathcal{M}_{\sigma'}^E(1). \quad (9.4)$$

Lemma 9.4. *Assume (9.2). For every value of E , every $i \in I$ and $j \in J$, every permitted path $\omega \in \mathcal{P}_{ij}^E$ stays within distance R_{loc} of the center of the cell $\text{anch}(i)$. If $i, k \in I$ satisfy $d_\infty(\text{anch}(i), \text{anch}(k)) > R_{\text{int}}$, then $d(\omega(s), \omega'(s)) > r$ for all s , all $\omega \in \mathcal{P}_{ij}^E$, and all $\omega' \in \mathcal{P}_{kl}^E$. For each $i \in I$, at most $C_{\text{int}}mM$ rows $k \in I$ satisfy $d_\infty(\text{anch}(i), \text{anch}(k)) \leq R_{\text{int}}$. In a container, every path in $\mathcal{C}_{Y_i}^- \cap \mathcal{C}_{Z_j}^+$ stays in U . These assertions do not depend on the matching.*

Proof. Let $v = \text{anch}(i)$. By Lemma 9.3(b), a permitted path stays within 60 of Y_i . Since $|Y_i - \tilde{v}| \leq \sqrt{3}/2$, the first assertion follows. If $d_\infty(v, v') > R_{\text{int}}$, then some coordinate of $v - v'$ has absolute value at least 124. Then $d(v, v') \geq 124$, and two paths within 61 of v and of v' are at distance at least $2 > r$ at all times. There are at most $(2R_{\text{int}} + 1)^3$ cells w with $d_\infty(v, w) \leq R_{\text{int}}$, and each carries at most mM retained rows. In a container, $\text{dist}(v, \mathbb{R}^3 \setminus U) \geq 64 > 61$ for $v \in \Lambda_{\text{act}}$. $\square$

By Lemma 9.4, replacing the avoidance indicator by one for rows with distant anchors does not change $\mathcal{M}_{\sigma'}^E$: on permitted paths both are equal to one.

Used in: Proposition 9.5 and Lemma 9.9, and Corollary 9.12, where it gives (H1).

Proposition 9.5. *Assume (9.2). For almost every value of E under $\mathbb{P}$, we have $0 < \Xi(E) < \infty$, and the conditional law of (σ_I, ω_I) given E is*

$$\mu_E(\{\sigma'\} \times d\omega_I) = \Xi(E)^{-1} \mathcal{M}_{\sigma'}^E(d\omega_I), \quad \sigma' \in \text{Bij}(I, J). \quad (9.5)$$

In particular, the joint law of (E, σ_I, ω_I) under $\mathbb{P}$ is a matching system in the sense of Definition 7.1, with exposed data E , rows $I = \bigsqcup_v I_v$ partitioned into the groups $I^{(1)}, \dots, I^{(m)}$ with $|I_v^{(a)}| \leq M$, anchors $\text{anch}(i)$, slots J , kernels K_{ij}^E , permitted sets $\mathcal{P}_{ij}^E$, pair checks h_{ik} , and active region Λ_{act} .

Proof. We first show that the exposed quantities that enter the density are the same for every matching of the retained rows. Fix a value of E such that $\varepsilon_i = 1$ for all $i \in I$; other values have probability zero, since the lists contain only eligible rows. Let $\sigma': I \rightarrow J$ be a bijection, let $\omega'_i \in \mathcal{P}_{i\sigma'(i)}^E$ for $i \in I$, and let $\mathbf{S}'$ be the sample with the cut configurations Y, Z , the matching equal to σ' on I and to the exposed slots on I^c , the paths ω'_I , and the frozen paths F . Then the following hold.

- (1) Every retained row i is eligible in $\mathbf{S}'$. Indeed, Y_i is separated from the other initial positions because i was eligible in the original sample and this condition depends on Y only. The slot $\sigma'(i) \in J$ is the slot of some retained row in the original sample; since that row was eligible, $Z_{\sigma'(i)}$ is separated from the other terminal positions. Condition (E2) holds because $\omega'_i \in \mathcal{P}_{i\sigma'(i)}^E$.
- (2) Every row $k \notin I$ has the same eligibility indicator in $\mathbf{S}'$ as in the original sample, because this indicator depends only on $Y, Z, \sigma(k)$, and ω_k , which are exposed.
- (3) By (1) and (2), the eligible sets $\mathcal{I}_v^{\text{el}}$ and the counts n_v of $\mathbf{S}'$ are those of the original sample, and the lists L have the same conditional probability $\prod_v s_v(L_v)$, where $s_v(L_v) = \mathbf{1}\{n_v \geq mM\} / (n_v)_{mM}$ if $L_v \neq \emptyset$ and $s_v(\emptyset) = \mathbf{1}\{n_v < mM\}$. Here we used that the entries of L_v are eligible rows starting in B_v , by (1).
- (4) The paths of $\mathbf{S}'$ avoid the frozen paths and, in a container, stay in U , because $\omega'_i \in \text{Av}_F$.

We now compute the joint law. Let Φ be a bounded nonnegative measurable function of E , and let Ψ be a bounded nonnegative measurable function of (E, σ_I, ω_I) . Fix the lists L , which determine I , an injective map $\varphi: I^c \rightarrow \{1, \dots, N\}$ prescribing the slots of the frozen rows, and the indicators ε ; then $J = \{1, \dots, N\} \setminus \varphi(I^c)$, and the permutations σ with $\sigma|_{I^c} = \varphi$ are the maps $\sigma = \varphi \cup \sigma'$ with $\sigma' \in \text{Bij}(I, J)$. By (8.4) and the definition of the lists, the contribution of these values to $\mathbb{E}[\Phi(E)\Psi(E, \sigma_I, \omega_I)]$ is

$$c_N \sum_{\sigma' \in \text{Bij}(I, J)} \int dY dZ K_{\beta-\delta}^s(Z, Y) \int \prod_{k \notin I} \mathbb{W}_{Y_k, Z_{\varphi(k)}}^{2t} (d\omega_k) \prod_{i \in I} \mathbb{W}_{Y_i, Z_{\sigma'(i)}}^{2t} (d\omega_i) \\ \times \mathbf{1}_{\mathcal{E}_{2t}}(\omega) \mathbf{1}\{\varepsilon(\mathbf{S}) = \varepsilon\} \prod_v s_v(L_v) \Phi \Psi,$$

where $c_N = (N! \text{tr } e^{-\beta H})^{-1}$ and $\mathbf{S} = (Y, Z, \varphi \cup \sigma', \omega)$. We split the indicator of $\mathcal{E}_{2t}$ into the pair conditions between frozen paths, between a retained and a frozen path, and between two retained paths, together with the wall condition in a container. The indicator $\mathbf{1}\{\varepsilon(\mathbf{S}) = \varepsilon\}$ is the product of the conditions $\varepsilon_k(\mathbf{S}) = \varepsilon_k$ for $k \notin I$, which are functions of E , and the conditions that each $i \in I$ satisfies (E1) and (E2). By the argument for (1), the product over $i \in I$ of the conditions (E1) equals

$$\prod_{i \in I} \mathbf{1}\{d(Y_i, Y_k) > s_0 \ \forall k \neq i\} \prod_{j \in J} \mathbf{1}\{d(Z_j, Z_l) > s_0 \ \forall l \neq j\},$$

which does not depend on σ' . By (3), the sampling factors are functions of E . Collecting the factors that depend only on E into a function $G(E) \geq 0$, and using (9.3), the definition of h_{ik} , and Lemma 9.4 for the pairs of retained paths, we factor the integrand as

$$c_N K_{\beta-\delta}^s(Z, Y) G(E) \Phi(E) \prod_{k \notin I} \mathbb{W}_{Y_k, Z_{\varphi(k)}}^{2t}(\mathrm{d}\omega_k) \times \Psi(E, \sigma', \omega_I) \mathcal{M}_{\sigma'}^E(\mathrm{d}\omega_I).$$

Let ν^{ref} be the measure on exposed data that, for each choice of L , φ , and ε , equals $c_N K_{\beta-\delta}^s(Z, Y) G(E) \mathrm{d}Y \mathrm{d}Z \prod_{k \notin I} \mathbb{W}_{Y_k, Z_{\varphi(k)}}^{2t}(\mathrm{d}\omega_k)$. Summing over the finitely many choices gives

$$\mathbb{E}[\Phi(E) \Psi(E, \sigma_I, \omega_I)] = \int \Phi(E) \sum_{\sigma' \in \text{Bij}(I, J)} \int \Psi(E, \sigma', \omega_I) \mathcal{M}_{\sigma'}^E(\mathrm{d}\omega_I) \nu^{\text{ref}}(\mathrm{d}E). \quad (9.6)$$

With $\Psi = 1$, (9.6) implies that the law of E is $\Xi(E) \nu^{\text{ref}}(\mathrm{d}E)$. In particular, the set $\{\Xi = 0\}$ has probability $\int_{\{\Xi=0\}} \Xi \mathrm{d}\nu^{\text{ref}} = 0$. Since each $\mathcal{M}_{\sigma'}^E(1)$ is at most a product of the finite masses $\mathbb{W}_{Y_i, Z_j}^{2t}(1)$, we have $\Xi(E) < \infty$. Then (9.6) states that (9.5) is a version of the conditional law of (σ_I, ω_I) given E . The exposed data take values in a finite disjoint union, over the choices of L , φ , and ε , of products of Euclidean spaces and path spaces, which is standard Borel. Since bridge measures depend measurably on their endpoints and the events in (9.3) are Borel sets of continuous paths, the maps $E \mapsto K_{ij}^E(A)$, $\mathcal{M}_{\sigma'}^E$, and Ξ are measurable. Each retained row uses the path space $C([0, 2t]; \mathbb{X})$ for all of its pairings, and the checks h_{ik} do not depend on the matching. This is the structure required in Definition 7.1; accessibility and the threshold m_* are defined in Section 9.4. $\square$

Remark 9.6. The retained matching is not uniformly distributed given E : its conditional probabilities are $\mathcal{M}_{\sigma'}^E(1)/\Xi(E)$. For example, $K_{ij}^E = 0$ when the lifted distance from Y_i to Z_j exceeds 40, by Lemma 9.3(b), and every matching pairing such i and j has probability zero, while the matching of the sample has positive mass almost surely. If σ_I is exposed as well, the conditional law is the product of the normalized enlarged kernels $K_{i\sigma_I(i)}^E$, conditioned on mutual avoidance. This has the same disintegration form as [112, Lemma 7.2], whose individual cutoffs and endpoint class are different. We cannot expose σ_I , since the surgeries of Section 7 change it. Apart from the eligibility and allocation marks whose fiber invariance was proved above, E imposes no further condition on the retained paths. Neither these local conditions nor route acceptance is exposed. Steps (1)–(3) of the proof, the invariance of eligibility and of the sampling factors on the matching fiber, are the reason for imposing eligibility separately on the two halves of a path.

Used in: Corollary 9.12; the form (9.5) is the matching law of Definition 7.1 used in Section 7.

9.4 Accessibility and masses

Fix a value of E . For $i \in I_v$, $j \in J$, and $u \in \mathbb{R}^3$ with $|u - \tilde{v}| \leq 400$ (on the torus, a point of the lifted ball about $\tilde{v}$), let $p_i^-(u)$ be the probability that a Brownian bridge of duration t from Y_i to u , with law $\widehat{W}_{Y_i, u}^{t, E}$, satisfies conditions (i)–(iii) of Definition 9.1 for $\mathcal{C}_{Y_i}^-$, avoids every frozen path at distance r during $[0, t]$, and, in a container, stays in U . Let $p_j^+(u)$ be the probability that a Brownian bridge of duration t from u to Z_j , with law $\widehat{W}_{u, Z_j}^{t, E}$ and run on the time interval $[t, 2t]$, satisfies the conditions of Definition 9.1 for $\mathcal{C}_{Z_j}^+$, avoids every frozen path at distance r during $[t, 2t]$, and, in a container, stays in U . On the torus, Z_j is lifted to the lift nearest to u ; if that lift is at distance more than 20 from u , then $p_j^+(u) = 0$.

Definition 9.7. For $i \in I_v$ and $j \in J$, set $A_{ij} = 1$ if, for every $w \in \mathcal{N}(v)$,

$$|\{u \in B_w : p_i^-(u) p_j^+(u) \geq 0.9\}| \geq 0.99, \quad (9.7)$$

and $A_{ij} = 0$ otherwise. For a bijection $\sigma' : I \rightarrow J$, a group a , and a cell v , let $m_v^{(a)}(\sigma') = \sum_{i \in I_v^{(a)}} A_{i\sigma'(i)}$. For an integer $m_* \geq 4$, the *bad set* and the *vulnerable set* of the group a are

$$B_{\sigma'}^{(a)} = \{v \in \Lambda_{\text{act}} : m_v^{(a)}(\sigma') < m_*\}, \quad V_{\sigma'}^{(a)} = \{v \in \Lambda_{\text{act}} : m_v^{(a)}(\sigma') < m_* + 2\}.$$

The quantities $p^\pm$ and A_{ij} are functions of E , i , j , and u ; they read no retained path, of any group. The probabilities p_i^- and p_j^+ are jointly measurable in (E, u) : bridge measures depend measurably on their endpoints, and their defining tests are jointly Borel in the exposed data and the continuous path, as in the proof of Proposition 9.5. Fubini's theorem, first for the bridge probabilities and then for the spatial volumes in (9.7), shows that A_{ij} is measurable in E ; see also [112, Section 7.3]. Changing σ' at one row changes $m_v^{(a)}(\sigma')$ only for the group a of that row and at its anchor, and by at most one. Cells outside Λ_{act} never belong to $B_{\sigma'}^{(a)}$ or $V_{\sigma'}^{(a)}$; in a container these sets consist of interior cells at depth at least R_{act} .

Lemma 9.8. Assume (9.2), and set

$$g_* = (4\pi t)^{-3/2} e^{-100/t}, \quad z_- = 0.98 (0.9)^2 g_*^2, \quad z_+ = (8\pi t)^{-3/2}.$$

For every value of E , all $i, k \in I$ whose anchors are equal or face neighbors, and all $j, l \in J$ with $A_{ij} = A_{kl} = 1$, we have

$$z_- \leq K_{il}^E(1) \leq z_+. \quad (9.8)$$

The cases $i = k$ and $j = l$ are included, and the pairings (i, j) and (k, l) need not belong to a common matching. Moreover, $K_{ij}^E(1) \leq z_+$ for all $i \in I$ and $j \in J$.

Proof. Let $v = \text{anch}(i)$ and $v' = \text{anch}(k) \in \mathcal{N}(v)$. Then $B_{v'}$ is a target cell of v , and it is the home cell of k . Since $A_{ij} = 1$ and $p_j^+ \leq 1$, the set $S_1 = \{u \in B_{v'} : p_i^-(u) \geq 0.9\}$ has volume at least 0.99. Since $A_{kl} = 1$, applied with the target cell $B_{v'}$ of k , the set $S_2 = \{u \in B_{v'} : p_l^+(u) \geq 0.9\}$ has volume at least 0.99. As $|B_{v'}| = 1$, we have $|S_1 \cap S_2| \geq 0.98$.

By the concatenation identity (3.12), $W_{Y_i, Z_l}^{2t} = \int W_{Y_i, u}^t \otimes W_{u, Z_l}^t du$. By Lemma 9.3(c), the indicator in (9.3) is the product of a function of the first half of the path and a function of the second half: avoidance of F on $[0, 2t]$ is avoidance on $[0, t]$ and on $[t, 2t]$, and the same holds for the wall condition. On the confinement event we pass to the corresponding Euclidean bridge restrictions. Normalizing the two Euclidean half bridges, we obtain

$$K_{il}^E(1) = \int p_t(u - Y_i) p_t(Z_l - u) p_i^-(u) p_l^+(u) du, \quad (9.9)$$

where on the torus the integral is over the lifted ball about $\tilde{v}$, by the convention on local lifts. For $u \in S_1 \cap S_2 \subset B_{v'}$, we have $|u - Y_i| \leq \sqrt{6} < 3$, since $Y_i \in B_v$ and $v' \in \mathcal{N}(v)$, and $|Z_l - u| \leq 20$, since $p_l^+(u) > 0$ requires a path from u to Z_l that stays within 20 of Z_l . Since p_t is radially decreasing, both heat kernels in (9.9) are at least $p_t(20e_1) = g_*$ on $S_1 \cap S_2$. Restricting the integral to $S_1 \cap S_2$ yields $K_{il}^E(1) \geq 0.98 (0.9)^2 g_*^2$.

For the upper bound, the confinement of Lemma 9.3(b) selects a unique Euclidean lift on the torus. We first pass to that Euclidean bridge restriction and then drop all restrictions to obtain $K_{ij}^E(1) \leq p_{2t}(Z_j - Y_i) \leq p_{2t}(0) = z_+$, with the selected lift of Z_j on the torus. If no lift is permitted, the mass is zero. $\square$

The constants $z_{\pm}$ depend only on t . Lemma 9.8 bounds the mass of the crossed pairing (i, l) ; it does not assert that this pairing is accessible. The intersection of two success sets of volume 0.99 is used in [112, Section 7.4], after Proposition 7.4 there, for two accessible labels sharing a midpoint target set, following [115, Proposition 6.4]. Here it bounds the unnormalized mass of a pairing whose two halves come from different accessible pairings.

Used in: Corollary 9.12, where it gives (H2); the lower bound z_- is also used in Lemma 9.9 and in the conditional area moments of Section 11.

9.5 Insertion

Lemma 9.9. *Assume (9.2). There exists $C_t < \infty$, depending only on t and α_0 , with the following property. Let E be a value of the exposed data satisfying $d(Y_i, Y_k) > s_0$ for every $i \in I$ and $k \neq i$, and $d(Z_j, Z_l) > s_0$ for every $j \in J$ and $l \neq j$. Every value produced by the selection procedure satisfies these conditions, since retained rows are eligible and $J = \sigma(I)$. Let $i \in I$ and $j \in J$ satisfy $K_{ij}^E(1) \geq z_-$, let $k \in I \setminus \{i\}$ and $j' \in J \setminus \{j\}$, and let $\omega' \in \mathcal{P}_{kj'}^E$. Then*

$$K_{ij}^E(\{\omega : h_{ik}(\omega, \omega') = 0\}) \leq \eta_1 K_{ij}^E(1), \quad \eta_1 = C_t D^{-1/4}. \quad (9.10)$$

Further, for every bijection $\sigma': I \rightarrow J$ with $K_{i\sigma'(i)}^E(1) \geq z_-$, every $I' \subset I \setminus \{i\}$, and every choice of $\omega_k \in \mathcal{P}_{k\sigma'(k)}^E$ for $k \in I'$,

$$\int \prod_{k \in I'} h_{ik}(\omega, \omega_k) K_{i\sigma'(i)}^E(d\omega) \geq (1 - \eta_{\text{glob}}) K_{i\sigma'(i)}^E(1), \quad \eta_{\text{glob}} = C_{\text{int}} m M C_t D^{-1/4}. \quad (9.11)$$

The opponents ω_k in (9.10) and (9.11) are only required to be permitted for their own pairings. They need not be accessible, they need not belong to the group of i , they need not avoid one another, and the frozen paths remain imposed through the kernels. Deleting rows from I' or checks from the product only increases the left side. By Lemma 9.8, every crossed pairing $(i, \sigma'(i))$ of (H2) has mass at least z_- . Hence (9.11) is hypothesis (H3) with collision error η_{glob} whenever $\eta_{\text{glob}} \leq 1/2$, and (9.10) is hypothesis (H3₁) with η_1 , since distinct rows have distinct slots under a bijection.

Proof. If $d_\infty(\text{anch}(i), \text{anch}(k)) > R_{\text{int}}$, then $h_{ik} \equiv 1$ and there is nothing to prove. Otherwise all paths involved stay within distance 400 of $\tilde{v}$, $v = \text{anch}(i)$, by Lemma 9.4, and we pass to the Euclidean bridge restriction in this chart. We write $y = Y_i$ and $z = Z_j$ in these Euclidean coordinates. Since $K_{ij}^E \leq \mathbf{1}_{\mathcal{C}_y^- \cap \mathcal{C}_z^+} \mathbb{W}_{y,z}^{2t,E}$, the restrictions to Av_F only decrease the left side of (9.10). It suffices to prove

$$\mathbb{W}_{y,z}^{2t,E}(\mathcal{C}_y^- \cap \mathcal{C}_z^+ \cap \mathcal{H}(\omega')) \leq C \alpha_0 t^{-3/2} D^{-1/4} \quad (9.12)$$

with a numerical constant C , where $\mathcal{H}(\omega')$ is the collision event of Section 8.4. Then (9.10) follows with $C_t = C \alpha_0 t^{-3/2} z_-^{-1}$, after division by $K_{ij}^E(1) \geq z_-$.

We first collect the properties of the opponent. Since $\omega' \in \mathcal{C}_{Y_k}^- \cap \mathcal{C}_{Z_{j'}}^+$, Lemma 9.3(b) implies that $\text{osc}_m(\omega') \leq D^{1/12}$ for all m , and the diameters of ω' on the two windows of length h_0 are at most $s_0/8$. Its initial point is Y_k , and $d(y, Y_k) > s_0$ because the retained row i is eligible. Its terminal point is $Z_{j'}$, and $d(z, Z_{j'}) > s_0$ because j is the slot of an eligible retained row in the original sample. Note that $j' \neq j$ is used here: the trial and the opponent have distinct terminal slots.

The endpoint windows. Let $\omega \in \mathcal{C}_y^-$ and $s \in [0, h_0]$. Then

$$d(\omega(s), \omega'(s)) \geq d(y, Y_k) - d(\omega(s), y) - d(\omega'(s), Y_k) > s_0 - \frac{s_0}{8} - \frac{s_0}{8} = \frac{3s_0}{4} > r,$$

where in the last inequality we used (8.10). The same holds on $[2t - h_0, 2t]$. It follows that on $\mathcal{C}_y^- \cap \mathcal{C}_z^+ \cap \mathcal{H}(\omega')$ a collision occurs during a mesh interval $[(m-1)\tau, m\tau]$ that is not contained in $[0, h_0] \cup [2t - h_0, 2t]$.

One mesh interval in the first half. Let $m \leq m_t$ with $m\tau > h_0$, and set $s = (m-1)\tau$; then $s > h_0 - \tau \geq h_0/2$, since $\tau \leq h_0/2$. By the concatenation identity, under $\mathbb{W}_{y,z}^{2t,E}$ the position $X_s = \omega(s)$ has the unnormalized density $p_s(x-y)p_{2t-s}(z-x)$, and conditionally

on $X_s = x$ the path $(\omega(s + s'))_{0 \leq s' \leq 2t-s}$ is a Brownian bridge from x to z of duration $t' = 2t - s \geq t$. We write it as

$$\omega(s + s') = x + \frac{s'}{t'}(z - x) + b(s'), \quad 0 \leq s' \leq t',$$

where b is a Brownian bridge from 0 to 0 of duration t' , whose law does not depend on x . Let $Y = r^{-1} \sup_{0 \leq s' \leq \tau} |b(s')|$. Writing $b(s') = \mathfrak{B}(s') - (s'/t')\mathfrak{B}(t')$ for a Brownian motion $\mathfrak{B}$ of generator Δ , Doob's maximal inequality and Gaussian moments give

$$\mathbb{E} \sup_{s' \leq \tau} |\mathfrak{B}(s')|^3 \leq \left(\frac{3}{2}\right)^3 \mathbb{E} |\mathfrak{B}(\tau)|^3 \leq C\tau^{3/2}, \quad \mathbb{E} \left(\frac{\tau}{t'} |\mathfrak{B}(t')|\right)^3 \leq C\tau^3 t'^{-3/2} \leq C\tau^{3/2},$$

where we used $\tau \leq t'$ in the last step. Since $\tau \leq r^2$, we obtain $\mathbb{E}Y^3 \leq C_Y$ with a numerical constant C_Y .

On $\mathcal{C}_y^- \cap \mathcal{C}_z^+$, Lemma 9.3(b) implies that $|z - X_s| \leq 60$, and the drift term satisfies $s'|z - X_s|/t' \leq 60\tau/t \leq r$ for $s' \leq \tau$, by (8.10) with $C = 60$. If, moreover, $d(\omega(s + s'), \omega'(s + s')) \leq r$ for some $s' \in [0, \tau]$, then

$$\begin{aligned} |X_s - \omega'(s)| &\leq |X_s - \omega(s + s')| + |\omega(s + s') - \omega'(s + s')| + |\omega'(s + s') - \omega'(s)| \\ &\leq (r + rY) + r + r \operatorname{osc}_m(\omega') = r(2 + \operatorname{osc}_m(\omega') + Y). \end{aligned}$$

This is a necessary condition for a collision during the m th interval on $\mathcal{C}_y^- \cap \mathcal{C}_z^+$. We now discard the cutoffs. Since Y is independent of X_s , $p_s \leq (4\pi s)^{-3/2}$, and $p_{2t-s} \leq (4\pi t)^{-3/2}$, the unnormalized mass of this necessary condition is at most

$$\frac{4\pi}{3} \frac{r^3}{(4\pi s)^{3/2} (4\pi t)^{3/2}} \mathbb{E} (2 + \operatorname{osc}_m(\omega') + Y)^3 \leq C \frac{r^3 D^{1/4}}{t^{3/2} s^{3/2}}.$$

In the last inequality, we used $(x + y)^3 \leq 4x^3 + 4y^3$, $\operatorname{osc}_m(\omega') \leq D^{1/12}$, and $\mathbb{E}Y^3 \leq C_Y$.

Summing over the intervals. The starting times $s = (m - 1)\tau \geq h_0/2$ are multiples of τ . With $k_0 = \lceil h_0/(2\tau) \rceil \geq 1$, we have

$$\sum_{k \geq k_0} (k\tau)^{-3/2} \leq \tau^{-3/2} (k_0^{-3/2} + 2k_0^{-1/2}) \leq 3\tau^{-3/2} k_0^{-1/2} \leq \frac{3\sqrt{2}}{\tau\sqrt{h_0}} \leq \frac{6\sqrt{2}}{r^2\sqrt{h_0}},$$

where we used $\tau \geq r^2/2$. The collisions during the first half contribute at most $Ct^{-3/2}rD^{1/4}h_0^{-1/2} = Ct^{-3/2}\alpha D^{-1/4} \leq 2C\alpha_0 t^{-3/2}D^{-1/4}$, since $r = \alpha/D$ and $h_0 = D^{-1}$.

The second half. The time reversal $\omega \mapsto \omega(2t - \cdot)$ maps $\mathbb{W}_{y,z}^{2t,E}$ to $\mathbb{W}_{z,y}^{2t,E}$ (Section 3.2). It maps $\mathcal{C}_z^+$ to $\mathcal{C}_z^-$ and $\mathcal{C}_y^-$ to $\mathcal{C}_y^+$, because the two halves of Definition 9.1 are mirror images, and it maps the mesh intervals of $[t, 2t]$ to those of $[0, t]$. The reversed opponent has the same oscillation and window bounds, and its endpoints are separated from z and y by more than s_0 . Collisions during the second half become collisions during the first half

of the reversed paths, and by the preceding argument, we obtain the same bound. This proves (9.12).

For (9.11), we have $1 - \prod_{k \in I'} h_{ik} \leq \sum_{k \in I'} (1 - h_{ik})$. Only rows k with $d_\infty(\text{anch}(i), \text{anch}(k)) \leq R_{\text{int}}$ contribute, and there are fewer than $C_{\text{int}} m M$ of them by Lemma 9.4. For each of them $\sigma'(k) \neq \sigma'(i)$, since σ' is a bijection, and $\omega_k \in \mathcal{P}_{k\sigma'(k)}^E$. Applying (9.10) with $j = \sigma'(i)$ and $j' = \sigma'(k)$ to each term and summing yields (9.11). $\square$

The normalization of the integrated bridge estimates is as in Remark 8.8, with divisor $K_{ij}^E(1) \geq z_-$. We adapt the proof of Lemma 8.7 for the whole bridge $P_{y,z}^0$, with three changes: permitted paths are confined to balls of radius 20 about their two endpoints instead of a ball of radius 10 about the home cell, and the drift bound uses the distance 60; the opponent is any permitted path of another retained row with a different terminal slot; and the normalization is by the mass bound of Lemma 9.8 instead of an accessibility probability as in [112, Proposition 7.4]. Only the maximum test on the opponent's oscillations is used, which is why the bound is $C_t D^{-1/4}$ rather than $C(tr + rD^{1/4}h_0^{-1/2})$.

Used in: Corollary 9.12, where it gives (H3) and (H3₁); the bound $(1 - \eta_{\text{glob}})z_- \geq z_-/2$ on the one-row normalizing masses after imposing the fixed opponents is also used in Section 11.

9.6 Vulnerable cells

We use the following form of the multiplicative Chernoff bound for sampling without replacement.

Lemma 9.10. *Let a set of n elements contain at least fn marked elements, where $0 < f \leq 1$, and let X be the number of marked elements among $M \leq n$ distinct elements chosen uniformly at random. Then $\mathbb{P}(X < fM/2) \leq e^{-fM/8}$.*

Proof. Let $f' \geq f$ be the fraction of marked elements, and let X' have the binomial distribution with parameters M and f' . By Hoeffding's comparison [71, Theorem 4] (see also [11, Lemma 1.1]), we have $\mathbb{E}\phi(X) \leq \mathbb{E}\phi(X')$ for every continuous convex function ϕ . For $\lambda > 0$, applying this with $\phi(x) = e^{-\lambda x}$ and Markov's inequality, we obtain

$$\mathbb{P}(X < fM/2) \leq e^{\lambda fM/2} \mathbb{E}e^{-\lambda X'} = e^{\lambda fM/2} (1 - f'(1 - e^{-\lambda}))^M \leq e^{\lambda fM/2 - fM(1 - e^{-\lambda})}.$$

In the last inequality, we used $1 - x \leq e^{-x}$ and $f' \geq f$. For $\lambda = \log 2$, the exponent equals $-fM(1 - \log 2)/2 \leq -fM/8$. $\square$

Used in: Proposition 9.11.

Proposition 9.11. *Let $p_1 \in (0, 1)$, and let $H_1 > 0$, $t \in (0, 1]$, and D_0 be such that the useful-label conclusion holds for these H_1 , t , and $D \geq D_0$: on the torus by Lemma 8.9, and*

in a container by the conclusion derived from Proposition A.23 in the proof of Theorem A.1(ii). Set $q_u = 9\ell/(20H_1)$, let M be an integer with $q_u M \geq 16$ and $p_1 + 2e^{-q_u M/8} \leq 1$, and let $m_* = \lfloor q_u M/4 \rfloor$. Assume (9.2), $D \geq D_0$, $m \geq 1$, $mM \leq \ell D/2$, $\eta_{\text{glob}} \leq 1/2$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, $\beta > \delta$, in a container $K \geq K_\Omega$, and that $\mathbb{P}$ satisfies the joint bound (8.9) with base p_1 (for sets of interior cells in a container). Then $m_* \geq 4$, $m_* + 2 < q_u M/2$, and $M/m_* \leq 8/q_u$, and, for every group a and every finite set A of distinct cells,

$$\mathbb{P}(A \subset V_\sigma^{(a)}) \leq (p_1 + e^{-q_u M/8})^{|A|}. \quad (9.13)$$

For all distinct groups a, b and every finite set A of distinct cells,

$$\mathbb{P}(A \subset V_\sigma^{(a)} \cup V_\sigma^{(b)}) \leq (p_1 + 2e^{-q_u M/8})^{|A|}. \quad (9.14)$$

Here σ stands for the retained matching σ_I of the sample, and $\mathbb{P}$ is the joint law of the slab sample and the lists. In particular, (H4) holds for every group with $p = p_1 + e^{-q_u M/8}$, and (H4⁺) holds with $p_\times = p_1 + 2e^{-q_u M/8} \leq 2p$.

Proof. Since $q_u M \geq 16$, we have $m_* \geq 4$ and $m_* + 2 \leq q_u M/4 + q_u M/8 < q_u M/2$; also $m_* \geq q_u M/4 - 1 \geq q_u M/8$, which gives $M/m_* \leq 8/q_u$. If $A \not\subset \Lambda_{\text{act}}$, the probabilities in (9.13) and (9.14) vanish, and we may assume $A \subset \Lambda_{\text{act}}$.

For $v \in \Lambda_{\text{act}}$, let G_v be the event that the local conditions of Definition 8.4 hold at v , with Euclidean distances in a container. Call a row *useful* if, as a label of the sample, it is eligible at some cell v in the sense of [112] and satisfies (8.12). Fix $v \in \Lambda_{\text{act}}$ and suppose that G_v holds. We claim that the following hold.

- (i) $n_v \geq mM$, so that $|I_v^{(a)}| = M$ for every group a ;
- (ii) at least $q_u n_v$ elements of $\mathcal{I}_v^{\text{el}}$ are useful;
- (iii) every useful row $i \in I_v$, in any group, satisfies $A_{i\sigma(i)} = 1$.

By the useful-label conclusion, which concerns every path configuration satisfying the local conditions at v , at least $\ell D/2$ labels are eligible at v in the sense of [112], at least nine tenths of them are useful, and at most $H_1 D$ paths start in B_v ; on the torus, the last bound is the remark after Lemma 8.9. By Lemma 9.3(a), all of these eligible labels belong to $\mathcal{I}_v^{\text{el}}$, and we have $n_v \geq \ell D/2 \geq mM$, which proves (i). Every row in $\mathcal{I}_v^{\text{el}}$ starts in B_v , so $n_v \leq H_1 D$. The number of useful rows is at least $9\ell D/20 \geq q_u H_1 D \geq q_u n_v$, which proves (ii).

For (iii), let $i \in I_v$ be useful, let $w \in \mathcal{N}(v)$, and let $u \in B_w$ satisfy $P_i^u(C_i(F_i^{\text{all}})) \geq 0.9$. Under P_i^u , the trial is the concatenation of independent bridges, b^- from Y_i to u and b^+ from u to $\omega_i(2t) = Z_{\sigma(i)}$, each of duration t , with the same local lifts as in the definition of $p^\pm$. On $C_i(F_i^{\text{all}})$ the trial is regular and stays within distance 10 of v . By Lemma 9.3(a), the bridge b^- then satisfies the conditions defining $\mathcal{C}_{Y_i}^-$ and b^+ those defining $\mathcal{C}_{Z_{\sigma(i)}}^+$. The trial avoids every path of F_i^{all} , which contains the frozen family F because $i \in I$. In a container

it stays in U , since the ball of radius 10 about an active cell lies in U . We conclude that $C_i(F_i^{\text{all}})$ is contained in the intersection of the event defining $p_i^-(u)$ for b^- and the event defining $p_{\sigma(i)}^+(u)$ for b^+ , and by the independence of b^- and b^+ , we have

$$0.9 \leq P_i^u(C_i(F_i^{\text{all}})) \leq p_i^-(u) p_{\sigma(i)}^+(u).$$

By (8.12), the set of such u has volume at least 0.99 in each target cell. Then $A_{i\sigma(i)} = 1$, which proves (iii). The point is that removing the retained paths from the obstacles can only increase success probabilities.

For a group a , let $N_v^{u,(a)}$ be the number of useful rows in $I_v^{(a)}$, and let $H_v^{(a)} = \{N_v^{u,(a)} < q_u M/2\}$. By (iii), $m_v^{(a)}(\sigma) \geq N_v^{u,(a)}$ on G_v . Since $m_* + 2 < q_u M/2$, we obtain

$$\{v \in V_\sigma^{(a)}\} \subset G_v^c \cup (G_v \cap H_v^{(a)}).$$

Usefulness and G_v are functions of the slab sample $\mathbf{S} = (Y, Z, \sigma, \omega)$, and $H_v^{(a)}$ is a function of $\mathbf{S}$ and L_v . Conditionally on $\mathbf{S}$, on G_v the list L_v is a uniformly random ordered list of mM distinct elements of $\mathcal{I}_v^{\text{el}}$, by (i). Each fixed block of M positions of L_v is then a uniformly random ordered list of M distinct elements of $\mathcal{I}_v^{\text{el}}$, since every such list has the same number $(n_v - M)!/(n_v - mM)!$ of completions. Thus $I_v^{(a)}$ is a uniformly random subset of M elements of $\mathcal{I}_v^{\text{el}}$, and (ii) and Lemma 9.10 with $f = q_u$ imply that, for every group a ,

$$\mathbb{P}(H_v^{(a)} \mid \mathbf{S}) \leq e^{-q_u M/8} \quad \text{on } G_v. \quad (9.15)$$

The blocks of different groups at one cell need not be independent; for $a \neq b$ we use only the consequence $\mathbb{P}(H_v^{(a)} \cup H_v^{(b)} \mid \mathbf{S}) \leq 2e^{-q_u M/8}$ on G_v .

We prove (9.14); the proof of (9.13) is the same, with $H_v^{(a)}$ in place of $H_v^{(a)} \cup H_v^{(b)}$ and $e^{-q_u M/8}$ in place of $2e^{-q_u M/8}$. Let $H_v = H_v^{(a)} \cup H_v^{(b)}$. On the event $\{A \subset V_\sigma^{(a)} \cup V_\sigma^{(b)}\}$, let A_0 be the set of cells of A at which G_v fails. By the inclusion above, every $v \in A \setminus A_0$ satisfies G_v and H_v . Summing over the possible values of A_0 , we obtain

$$\mathbb{P}(A \subset V_\sigma^{(a)} \cup V_\sigma^{(b)}) \leq \sum_{A_0 \subset A} \mathbb{E} \left[\prod_{v \in A_0} \mathbf{1}_{G_v^c} \prod_{v \in A \setminus A_0} \mathbf{1}_{G_v} \mathbf{1}_{H_v} \right].$$

Conditionally on $\mathbf{S}$, the lists at distinct cells are independent, and $\mathbf{1}_{H_v}$ depends only on $\mathbf{S}$ and L_v . Conditioning on $\mathbf{S}$ and using (9.15), we bound the term indexed by A_0 by

$$(2e^{-q_u M/8})^{|A \setminus A_0|} \mathbb{P} \left(\bigcap_{v \in A_0} G_v^c \right) \leq (2e^{-q_u M/8})^{|A \setminus A_0|} p_1^{|A_0|}.$$

In the last inequality, we used that $\bigcap_{v \in A_0} G_v^c$ is an event of the slab sample, whose probability under the joint law of the sample and the lists is its probability under the one-slab law, and the joint bound (8.9), where A_0 consists of interior cells in a container. Summing over $A_0 \subset A$ yields $(p_1 + 2e^{-q_u M/8})^{|A|}$. $\square$

All probabilities in Proposition 9.11 are unconditional, under the law of one slab of the canonical Gibbs path measure (Proposition 3.17(4)) together with its lists. The proof conditions only on the full slab sample, never on the exposed data, the matching, or a route, and it uses no independence of the local-condition events at distinct cells: the joint bound (8.9) is applied to the set A_0 . Only the dominating events are factored across cells; the vulnerable sets themselves depend on all retained lists, through the frozen family. This is the argument of [112, proof of Proposition 7.3] (see also [115, Proposition 6.3]), with the useful fraction $9/10$ replaced by q_u , because our eligible list may be larger than the list of labels eligible in the sense of [112], and with the sampling estimate replaced by Lemma 9.10. In a container the vulnerable sets are empty outside the interior cells; declaring the cells near the walls bad would violate (9.13).

Used in: Corollary 9.12, where it gives (H4) and (H4⁺).

9.7 Summary

We collect the results of this section in the form in which they are used.

Corollary 9.12. *Let $p_1 \in (0, 1)$, and let H_1 , t , D_0 , and $q_u = 9\ell/(20H_1)$ be as in Proposition 9.11. Let $z_{\pm}$ be as in Lemma 9.8 and C_t as in Lemma 9.9. Let M be an integer with $q_u M \geq 16$ and $p_1 + 2e^{-q_u M/8} \leq 1$, and let $m_* = \lfloor q_u M/4 \rfloor$. There exists $D_1 \geq \max\{D_0, 2M/\ell\}$, depending only on D_0 , t , M , and the constants of Section 8.1, such that the following holds. Let $D \geq D_1$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, and $\beta > \delta$, let $m \geq 1$ be an integer with $mM \leq \ell D/2$ and $\eta_{\text{glob}} = C_{\text{int}} m M C_t D^{-1/4} \leq 1/2$, let $K \geq 2000$ on the torus and $K \geq K_{\Omega}$ in a container, and suppose that $\mathbb{P}$ satisfies the joint bound (8.9) with base p_1 (for sets of interior cells in a container). Then the law of (E, σ_1, ω_1) under $\mathbb{P}$ is a matching system in the sense of Definition 7.1, with m groups of at most M rows per cell, active region Λ_{act} , accessibility A_{ij} , and threshold $m_* \geq 4$, and it satisfies*

(H1) *with $R_{\text{loc}} = 61$, $R_{\text{int}} = 123$, and at most $C_{\text{int}} m M = 247^3 m M$ rows with a nontrivial check with a given row;*

(H2) *with $z_- = 0.98 (0.9)^2 g_*^2$ and $z_+ = (8\pi t)^{-3/2}$, where $g_* = (4\pi t)^{-3/2} e^{-100/t}$;*

(H3) *with collision error η_{glob} ;*

(H3₁) *with $\eta_1 = C_t D^{-1/4}$;*

(H4) *for every group, with $p = p_1 + e^{-q_u M/8}$;*

(H4⁺) *with $p_{\times} = p_1 + 2e^{-q_u M/8} \leq 2p$.*

Moreover $M/m_* \leq 8/q_u$. The constants R_{loc} , R_{int} , and C_{int} are numerical, and $z_{\pm}$, q_u , and C_t depend only on H_1 , t , and the constants of Section 8.1. None of them depends on M , m , D , T , K , or the exposed data.

Proof. Let $D_1 \geq \max\{D_0, 2M/\ell\}$ be large enough that (9.2) holds for $D \geq D_1$, apart from the condition on K . By Proposition 9.5, we have a matching system with the groups $I^{(1)}, \dots, I^{(m)}$. Hypothesis (H1) follows from Lemma 9.4, and Lemma 9.8 implies (H2). By Lemma 9.9, we have (H3), since $\eta_{\text{glob}} \leq 1/2$, and (H3₁): for a matching τ , a crossed pairing $(i, \tau(i))$, and a row $k \neq i$ of any group, $K_{i\tau(i)}^E(1) \geq z_-$ by Lemma 9.8, and $\tau(k) \neq \tau(i)$, so that (9.10) applies with $j = \tau(i)$, $j' = \tau(k)$, and every $\omega' \in \mathcal{P}_{k\tau(k)}^E$. Finally, Proposition 9.11 implies (H4), (H4⁺), and the bound on M/m_* . $\square$

The full order of parameter choices is given in Section 10.3, with the permission to reduce t stated in Remark 8.6. Hypothesis (H5), including the containment of accepted routes in Λ_{act} in a container, is verified in Sections 10 and 11.

Used in: Sections 10 and 11, together with Theorem 7.14.

Proof of the theorem on the superfluid fraction

We apply Theorem 7.14 to the slab matching system, compute the exact winding gain, and use Proposition 5.6 to obtain the scaled lower bound. After fixing the parameters, we return to physical units and prove Theorem 1.1; the final subsection proves Corollary 1.4.

10.1 The scaled setting

In Sections 10.1 to 10.4, we work in the scaled units of Section 8.1. Here $K \geq 100$ is an integer, $\Lambda_K = (\mathbb{R}/K\mathbb{Z})^3$, $\mathbb{L}_K = (\mathbb{Z}/K\mathbb{Z})^3$ is the cell lattice with cells $B_v = v + [-1/2, 1/2]^3$, and N, D, r, α , and T satisfy (8.1), with the temperature-window condition in (8.1) replaced by $\beta > \delta$. We apply the results of Section 3 to N particles on Λ_K with exclusion distance r ; the side L and the exclusion distance a of Section 3 are then K and r . We write μ for the Bose path measure, W for its winding vector, and $f_s(N, K, T)$ for the superfluid fraction (3.5). Since $N = DK^3$, the identity (3.32) reads

$$f_s(N, K, T) = \frac{TK^2}{2N} \text{Var}_\mu(W_1) = \frac{T}{2DK} \text{Var}_\mu(W_1). \quad (10.1)$$

Slabs and marks. Let $t \in (0, 1]$ and the group size M be fixed as in Section 10.3, let $m = m(D)$ be the number of groups in (10.7), and set $\delta = 2t$ and $n_\beta = \lfloor \beta/\delta \rfloor \geq 1$. We use the slab decomposition of Proposition 3.17 with slab length δ , and we mark each full slab by the ordered lists of Section 9.2, of mM rows divided into m groups. The lists of a slab are chosen using that slab and its two cuts only, and they define a marking kernel m as in Proposition 5.6. We write μ_δ^m for the resulting marked one-slab law. In the notation of Section 9, it is the joint law $\mathbb{P}$ of the slab sample (Y, Z, σ, ω) and the lists, where Y and Z are the two cuts of the slab and ω_i runs from Y_i to $Z_{\sigma(i)}$. By Proposition 9.5, the marked one-slab variable and the triple (E, σ_I, ω_I) of exposed data, retained matching, and retained paths are measurable functions of each other, and we identify the two descriptions. As in Section 9, we write σ for the retained matching σ_I when no confusion can arise.

For a unit vector u , the function φ_u^W of Section 5.2 is, in the present notation,

$$\varphi_u^W = \frac{1}{K} \sum_{k=1}^N u \cdot d(\omega_k), \quad (10.2)$$

where $d(\omega_k)$ is the lifted displacement of the path ω_k (Section 3.2). By Proposition 3.17(4) and (5), it is the contribution of any full slab to $u \cdot W$, and it is square integrable under μ_δ^m .

The surgery kernel. Let Γ be a winding skeleton of Definition 6.4, sampled independently of the slab, and let $\mathcal{R}$ be the torus routing map of Definition 6.14. We use them in the kernels of Definition 7.5. For each group a , averaging the subprobability kernel of the group over the random skeleton gives a subprobability kernel $q^{(a)}$, whose outgoing mass $\bar{q}^{(a)}$ is the probability, over the random skeleton, that the routing map accepts the bad set $B_\sigma^{(a)}$. The surgery kernel $Q = m^{-1} \sum_a Q^{(a)}$, where $Q^{(a)}$ is the completion of $q^{(a)}$ (Definition 5.1), is the completion of $q = m^{-1} \sum_a q^{(a)}$, and it preserves the exposed data E , including the two cuts. For an accepted labeled route of the group a , the selected rows i_k belong to $I^{(a)}$, $\pi(i_k) = i_{k+1}$, and the matching changes from σ to $\sigma\pi$. We compute the resulting winding change below.

The following proposition is the scaled form of Theorem 1.1. Its constants are fixed in Section 10.3.

Proposition 10.1. *Let $\alpha_0, p_1, \delta, D_{\min}$, and $C_\sharp$ be the constants fixed in Section 10.3, and let K_0 be the threshold defined there, which depends only on the constants fixed there. Let $K \geq K_0$ and $N \geq 1$ with $D = N/K^3 \geq D_{\min}$, let $\alpha \in [\alpha_0/2, 2\alpha_0]$ and $r = \alpha/D$, and let $\beta > \delta$. Suppose that the one-slab law of μ satisfies the joint bound (8.9) with base p_1 . Then*

$$\text{Var}_\mu(W_1) \geq \frac{n_\beta K D^{1/4}}{4C_\sharp}, \quad f_s(N, K, T) \geq \frac{T n_\beta}{8C_\sharp D^{3/4}} \geq \frac{1}{16C_\sharp \delta D^{3/4}}. \quad (10.3)$$

By Proposition 8.5, the joint bound holds whenever $K \geq K_1$, $D \geq D_{\min}$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, and $0 < T \leq \tau_0 / \log D$.

Used in: the proof of Theorem 1.1 in Section 10.5.

10.2 The winding gain

We use the homology vector $\text{hom}(\gamma)$ of Section 6.1. If $\tilde{v}_0, \dots, \tilde{v}_{|\gamma|}$ is a lift of the oriented closed nearest-neighbor walk $\gamma = (v_0, \dots, v_{|\gamma|-1})$, then

$$\tilde{v}_{|\gamma|} - \tilde{v}_0 = K \text{hom}(\gamma). \quad (10.4)$$

The torus routing map ensures $\text{hom}_1(\gamma) \geq 1$ (Proposition 6.15).

The next lemma shows that a surgery along γ changes the winding vector by exactly $\text{hom}(\gamma)$, whatever the resampled paths are. The operation of moving terminal endpoints along a route of cells, with exact conditional resampling of the moved paths, is that of [111, Section 9], where the routes are open. For a closed route, the changes of the endpoints telescope, and the change of the observable is the homology class of the route.

Lemma 10.2. Assume $K \geq 2000$. Let E be a value of the exposed data of one slab, let $\gamma = (v_0, \dots, v_{|\gamma|-1})$ be an oriented nearest-neighbor cycle of distinct cells of $\mathbb{L}_K$, and let $i_k \in I_{v_k}$ for $k \in \mathbb{Z}/|\gamma|\mathbb{Z}$. Let π be the permutation of I with $\pi(i_k) = i_{k+1}$ that fixes all other rows, and let $S = \{i_k : k \in \mathbb{Z}/|\gamma|\mathbb{Z}\}$. Let $\sigma: I \rightarrow J$ be a bijection, and let ω_I and ω'_I be families of paths such that $\omega_i \in \mathcal{P}_{i\sigma(i)}^E$ for every $i \in I$, $\omega'_i = \omega_i$ for $i \notin S$, and $\omega'_i \in \mathcal{P}_{i,\sigma\pi(i)}^E$ for $i \in S$. Then

$$\frac{1}{K} \sum_{i \in I} (d(\omega'_i) - d(\omega_i)) = \text{hom}(\gamma). \quad (10.5)$$

If $(E, \sigma\pi, \omega'_I)$ is obtained from (E, σ, ω_I) by a move of the kernel q along γ , then $\phi_{e_1}^W$ increases by $\text{hom}_1(\gamma)$. If this move is applied to a full slab of the marked slab measure of Proposition 5.6, then the winding vector of the concatenated paths increases by $\text{hom}(\gamma)$.

Proof. We first show that the lifted displacement of a permitted path is determined by its endpoints. Let $i \in I_v$, $j \in J$, and $\omega \in \mathcal{P}_{ij}^E$. By Lemma 9.4, ω stays within distance $R_{\text{loc}} = 61$ of v . Fix a representative $\tilde{v} \in \mathbb{Z}^3$ of v . Since $K \geq 2000$, the torus ball of radius 61 about v is the isometric image of the Euclidean ball $B(\tilde{v}, 61)$, as in the paragraph on local lifts in Section 9.1, and the lift of ω that starts in $B(\tilde{v}, 61)$ stays there. We conclude that

$$d(\omega) = \tilde{z} - \tilde{y}, \quad (10.6)$$

where $\tilde{y}$ and $\tilde{z}$ are the unique representatives of $\omega(0)$ and $\omega(2t)$ in $B(\tilde{v}, 61)$.

Let $\tilde{v}_0, \dots, \tilde{v}_{|\gamma|}$ be the lifts of the cells of γ defined before (10.4). For $0 \leq k < |\gamma|$, let $\tilde{Y}_k$ be the representative of Y_{i_k} in $B(\tilde{v}_k, 61)$, and let $\tilde{Z}_k$ be the representative of $Z_{\sigma(i_k)}$ in $B(\tilde{v}_k, 61)$; both exist because $\omega_{i_k} \in \mathcal{P}_{i_k\sigma(i_k)}^E$ runs from Y_{i_k} to $Z_{\sigma(i_k)}$ within distance 61 of v_k . Set $\tilde{Z}_{|\gamma|} = \tilde{Z}_0 + K\text{hom}(\gamma)$. By (10.4), $\tilde{Z}_{|\gamma|}$ is the representative of $Z_{\sigma(i_0)}$ in $B(\tilde{v}_{|\gamma|}, 61)$. By (10.6), we have

$$d(\omega_{i_k}) = \tilde{Z}_k - \tilde{Y}_k, \quad 0 \leq k < |\gamma|.$$

Next, fix $0 \leq k < |\gamma|$. The path ω'_{i_k} belongs to $\mathcal{P}_{i_k, \sigma\pi(i_k)}^E$, and $\sigma\pi(i_k) = \sigma(i_{k+1})$. It runs from Y_{i_k} to $Z_{\sigma(i_{k+1})}$ and stays within distance 61 of v_k . By (10.6), we have $d(\omega'_{i_k}) = \tilde{Z}' - \tilde{Y}_k$, where $\tilde{Z}'$ is the representative of $Z_{\sigma(i_{k+1})}$ in $B(\tilde{v}_k, 61)$. On the other hand, $\tilde{Z}_{k+1}$ is a representative of the same point, and it lies in $B(\tilde{v}_{k+1}, 61) \subset B(\tilde{v}_k, 62)$, because $|\tilde{v}_{k+1} - \tilde{v}_k| = 1$; for $k = |\gamma| - 1$ we use the definition of $\tilde{Z}_{|\gamma|}$. Two representatives of one point of Λ_K differ by an element of $K\mathbb{Z}^3$, and $\tilde{Z}'$ and $\tilde{Z}_{k+1}$ both lie in $B(\tilde{v}_k, 62)$, whose diameter is less than K . This implies that $\tilde{Z}' = \tilde{Z}_{k+1}$, and we have

$$d(\omega'_{i_k}) = \tilde{Z}_{k+1} - \tilde{Y}_k.$$

The paths of the rows $i \notin S$ are unchanged. Summing over the selected rows gives

$$\sum_{i \in I} (d(\omega'_i) - d(\omega_i)) = \sum_{k=0}^{|\gamma|-1} (\tilde{Z}_{k+1} - \tilde{Z}_k) = \tilde{Z}_{|\gamma|} - \tilde{Z}_0 = K\text{hom}(\gamma),$$

which is (10.5). A move of q along γ is of this form: the selected rows lie in the distinct cells of γ , the paths in the support of the exact conditional law of the matching system belong to the permitted sets, and the frozen paths of the slab are part of E and do not change. By (10.2), $\varphi_{e_1}^W$ increases by $\text{hom}_1(\gamma)$. Finally, the move changes neither the cuts nor the other slabs. By (3.42), the winding vector of the concatenated paths changes by the change of the contribution of the slab, which is $\text{hom}(\gamma)$. $\square$

Used in: the proof of Proposition 10.1.

10.3 Choice of the parameters

We now fix the constants used in the proofs of Theorem 1.1 and Theorem 1.2, in the following order. Each step uses only the constants fixed in earlier steps. At every step, the choice is made so that the requirements of both theorems are met; Section 11.5 lists the additional requirements that come from the container. In particular, the constants α_0 , ℓ , θ_0 , b_0 , κ_T , H_1 , t , M , and $D_{\min}$, and the number $m(D)$ of groups, are common to the two theorems.

- Step 1. *Spatial constants and the temperature coefficient.* Fix $\alpha_0 \geq 2$, then ℓ , then θ_0 such that $3\theta_0 < \ell/2$, then b_0 , and the numerical constant C_{reg} , as in Section 8.1, so that the requirements of [112, Sections 4 and 5], the temperature-free requirements of Proposition B.2 with the constants of Proposition B.1, and the requirements of Proposition 8.10 are met. Then fix the numerical constant κ_T and $\tau_0 = \kappa_T \alpha_0$ of Proposition 8.5, the insertion constant C_{inc} and the color-gap base m_0 of Appendix B, and the volume threshold K_1 of Proposition 8.5, which depends only on the constants of this step.
- Step 2. *Geometry.* The eligibility class of Section 9.2 has half-path radius 20. It fixes the radii $R_{\text{loc}} = 61$ and $R_{\text{int}} = 123$ of (H1) and $C_{\text{int}} = 247^3$ (Lemma 9.4). Fix the radius R of the encounter count J_R as required in Section 7, the shell radius 1 and the perturbation radius 1 of Lemma 6.16, and a component range $R_c \geq R + 2$. Fix the winding skeleton law of Definition 6.4 and the torus routing map of Definition 6.14, with its size threshold $\lfloor K/20 \rfloor$. None of these choices depends on a constant fixed later.
- Step 3. *Exponential coefficients.* By Proposition 6.6, there exist $\kappa, \xi > 0$ and $C_h, C_s < \infty$ such that two independent winding skeletons Γ, Γ' satisfy $\mathbb{E} e^{\kappa J_R(\Gamma, \Gamma')} \leq 1 + C_h/K$ and $\mathbb{E} e^{\xi S_{R_c}(\Gamma, \Gamma')} \leq 1 + C_s/K$ for all sufficiently large K . Let C_{acc} be the acceptance constant of Proposition 6.15, C_{ch} the constant in the encounter bound of Lemma 6.16, and $\zeta(p, c)$ the coefficient of Lemma 6.18. The polygon skeletons of Section 11.1 have constants of the same kind, and we use the smaller exponential coefficients and the larger multiplicative constants of the

two families. We fix $\varepsilon_{\max} \in (0, \kappa/4]$, and then the coefficient $c_{\text{rec}} > \log 4 + C_{\text{ch}}\varepsilon_{\max} + 1$ of A_* in the exponential of Theorem 7.14. We also fix the cross-group exponent and coefficient of Lemma 7.13,

$$\theta_{\times} = \min\{\kappa/2, 1/\max(1, C_{\text{ch}})\}, \quad c_{\times} = \theta_{\times} C_{\text{ch}},$$

and the cross-group constant $C_{\times} = 8C_{\text{fin}}(c_{\times})/\theta_{\times}$ of Theorem 7.14, where $C_{\text{fin}}(c_{\times})$ is the constant of (H5)(d), the larger of its values for the two skeleton families. These choices are made before the thermal constants, and the base for bad cells in Step 4 does not depend on t .

- Step 4. *Base for bad cells.* Let $p_{\max} \in (0, 1)$ be such that, for every $p \in (0, p_{\max}]$, we have $p \leq p_0$, with p_0 as in Proposition 6.15, $C_{\text{acc}}p \leq 1/2$, the expansion of Lemma 6.18 applies with exponent $2c_{\text{rec}}$, and $2\zeta(p, 2c_{\text{rec}}) \leq \zeta$. Require moreover that $(2R_c + 1)^6(2p_{\max})e^{c_{\times}} \leq e^{-2}$ and $2\zeta(2p_{\max}, c_{\times}) \leq \zeta$, so that Corollary 6.19 applies with the coefficient $c_{\times}$ of A_* for every base at most $2p_{\max}$; that is, we may take $p_{\text{env}}(c_{\times}) = 2p_{\max}$ in (H5)(d). Fix $p_1 \in (0, p_{\max}/2)$.
- Step 5. *Thermal constants.* We apply Proposition 8.5, Lemma 8.9, and Proposition 8.10 with base p_1 and $R_0 = 64$, and we fix a common H_1 , then a common $t \in (0, 1]$, as permitted by Remark 8.6, together with the density thresholds D_0 . The larger of the two values of H_1 is allowed: increasing H_1 weakens the certificate conditions in Definition 8.4, and then t may be decreased to meet the useful-label requirements for this common value. Set $\delta = 2t$, the useful fraction $q_u = 9\ell/(20H_1)$ of Proposition 9.11, the masses $z_{\pm}$ of Lemma 9.8, $G = (2z_+/z_-)^3$, and the constant C_t of Lemma 9.9. No later step changes t .
- Step 6. *Group size M .* Set $\lambda_0 = 8/q_u$, and let M be an integer such that $q_u M \geq 16$, $e^{-q_u M/8} < p_{\max}/2$, and $(3 + 8G\lambda_0^2)/m_* < \varepsilon_{\max}/2$, where $m_* = \lfloor q_u M/4 \rfloor$ and $3 + 8G\lambda_0^2 = C_1(G, \lambda_0)$ is the constant of (7.17). By Proposition 9.11, $m_* \geq 4$, $m_* + 2 < q_u M/2$, and $M/m_* \leq \lambda_0$. Set $p = p_1 + e^{-q_u M/8}$ and $p_{\times} = p_1 + 2e^{-q_u M/8}$, so that $p < p_{\max}$ and $p_{\times} < 2p_{\max} < 1$.
- Step 7. *Density threshold and number of groups.* Let $D_{\min}$ be at least the threshold D_1 of Corollary 9.12 for the constants of Steps 5 and 6, which includes the threshold D_0 of Step 5 (and with it the thresholds of Proposition 8.5 and Appendix B), $2M/\ell$, and the requirement $D > 2$ of (9.2), and at least $(2C_{\text{int}}MC_t)^4$. Let $D_{\min}$ moreover be so large that

$$32C_{\text{int}}C_t D_{\min}^{-1/4} \leq \min\{\varepsilon_{\max}/2, \theta_{\times}\}, \quad D_{\min}^{3/4} \geq \frac{1}{\ell C_{\text{int}}C_t}, \quad \frac{\tau_0}{\log D_{\min}} \leq \frac{1}{4}.$$

For $D \geq D_{\min}$, the number of groups is

$$m(D) = \left\lfloor \frac{D^{1/4}}{2C_{\text{int}}MC_t} \right\rfloor. \quad (10.7)$$

Since $D \geq (2C_{\text{int}}MC_t)^4$, the argument of the floor is at least one, and $\lfloor x \rfloor \geq x/2$ for $x \geq 1$ gives

$$\frac{D^{1/4}}{4C_{\text{int}}MC_t} \leq m(D) \leq \frac{D^{1/4}}{2C_{\text{int}}MC_t}.$$

Hence $m(D) \geq 1$, the global collision error $\eta_{\text{glob}} = C_{\text{int}}m(D)MC_tD^{-1/4}$ of Lemma 9.9 satisfies $\eta_{\text{glob}} \leq 1/2$, and $m(D)M \leq D^{1/4}/(2C_{\text{int}}C_t) \leq \ell D/2$. With $\eta_1 = C_tD^{-1/4}$ and $\eta_{\text{loc}} = 4C_{\text{int}}\eta_1$, we have $8\eta_{\text{loc}} \leq \min\{\varepsilon_{\text{max}}/2, \theta_{\times}\}$, and therefore $\varepsilon = (3 + 8G\lambda_0^2)/m_* + 8\eta_{\text{loc}} < \varepsilon_{\text{max}}$. Since G was defined using only $\eta_{\text{glob}} \leq 1/2$, increasing D_{min} changes neither G nor M .

Step 8. *Comparison constants.* With the constants of Steps 1–7, Theorem 7.14 provides the constant C_* in its bound on the chi-square divergence; for the polygon skeletons of Section 11.1 it provides a constant of the same kind, and we let C_* be the larger of the two. Set

$$C_{\sharp} = 4C_{\text{int}}C_t(MC_* + C_{\times}).$$

The constants C_* , $C_{\times}$, and $C_{\sharp}$ depend only on the constants of Steps 1–6, and not on K , on $D \geq D_{\text{min}}$, on m , on α , or on T . Finally, we increase D_{min} , if necessary, so that $8C_{\sharp}\delta D_{\text{min}}^{3/4} \geq 1$; since $C_{\sharp}$ does not depend on D_{min} , this changes no constant fixed before, and the requirements of Step 7 persist. The constants of the area estimate are fixed at this step as well (Section 11.5).

Step 9. *Physical parameters, then the volume.* For Theorem 1.1, fix a and $0 < \rho_1 \leq \rho_2$ with $\rho_2 a^3 \leq 2c_0$, where $c_0 = \alpha_0^3/(16D_{\text{min}}^2)$. Only after that do we let N and L , and with them K , tend to infinity.

Volume thresholds. Let K_0 be the smallest integer $K_0 \geq 2000$ such that, for all $K \geq K_0$, the excess bounds of Step 3 for the winding skeletons, the acceptance bound of Proposition 6.15 for the torus routing map, and the volume condition of Theorem 7.14 hold. These are lattice conditions, which depend only on the constants of Steps 1–8. No condition involves D , α , T , or the number of groups, and none relates β to K .

10.4 Proof of the scaled bound

Proof of Proposition 10.1. Let K , N , α , and β be as in the statement, and let $m = m(D)$ be the number of groups (10.7). By assumption, $\beta > \delta$.

We first verify the hypotheses of Theorem 7.14 for one slab. By Step 7, we have $m \geq 1$, $mM \leq \ell D/2$, and $\eta_{\text{glob}} \leq 1/2$, and $D \geq D_{\text{min}}$ is at least the threshold D_1 of Corollary 9.12; moreover $K \geq K_0 \geq 2000$. Since the one-slab law satisfies the joint bound (8.9) with base p_1 , Corollary 9.12 shows that the marked one-slab law μ_{δ}^m is the law of a matching system with m groups of at most M rows per cell that satisfies (H1) with the radii of Step 2, (H2) with the masses $z_{\pm}$ of Step 5, (H3) with the collision error $\eta_{\text{glob}} \leq 1/2$, (H3₁) with

$\eta_1 = C_t D^{-1/4}$, (H4) with the base $p < p_{\max}$ of Step 6, and (H4⁺) with the base $p_{\times} < 2p_{\max}$ of Step 6. By Propositions 6.6 and 6.15 and Lemma 6.16, the winding skeletons and the torus routing map satisfy (H5), including (H5)(c)(iii⁺), with the constants of Step 3; on the torus every cell is active. By Steps 3, 4, 6, and 7, the constants $\varepsilon_{\max}$, c_{rec} , $\theta_{\times}$, p , $p_{\times}$, $\eta_{\text{loc}} = 4C_{\text{int}}\eta_1$ with $8\eta_{\text{loc}} \leq \theta_{\times}$, and $\varepsilon = (3 + 8G\lambda_0^2)/m_* + 8\eta_{\text{loc}} < \varepsilon_{\max}$ satisfy the requirements of Theorem 7.14, with $\lambda = \lambda_0$. Then Theorem 7.14 implies that $\mu_{\delta}^{\text{m}}Q$ is absolutely continuous with respect to μ_{δ}^{m} , that its density belongs to $L^2(\mu_{\delta}^{\text{m}})$, that each group accepts with probability at least $1 - C_{\text{acc}}p$, and that

$$\chi^2(\mu_{\delta}^{\text{m}}Q \parallel \mu_{\delta}^{\text{m}}) \leq \frac{C_*}{mK} + \left(1 - \frac{1}{m}\right) \frac{C_{\times}\eta_{\text{loc}}}{K}.$$

By the lower bound on $m(D)$ in Step 7, we have $C_*/m \leq 4C_{\text{int}}MC_t C_* D^{-1/4}$, and $C_{\times}\eta_{\text{loc}} = 4C_{\text{int}}C_t C_{\times} D^{-1/4}$. Therefore

$$\chi^2(\mu_{\delta}^{\text{m}}Q \parallel \mu_{\delta}^{\text{m}}) \leq \frac{C_{\sharp}D^{-1/4}}{K}, \quad \int \bar{q}^{(a)} d\mu_{\delta}^{\text{m}} \geq 1 - C_{\text{acc}}p \geq \frac{1}{2} \quad \text{for every group } a. \quad (10.8)$$

In the last inequality, we used $p < p_{\max}$ and $C_{\text{acc}}p_{\max} \leq 1/2$.

Next, we bound the gain. Since the function $\varphi_{e_1}^W$ of (10.2) and the density of $\mu_{\delta}^{\text{m}}Q$ are square integrable, and Q is the completion of $q = m^{-1} \sum_a q^{(a)}$, Lemma 5.2(2) implies that

$$\mathbb{E}_{\mu_{\delta}^{\text{m}}Q} \varphi_{e_1}^W - \mathbb{E}_{\mu_{\delta}^{\text{m}}} \varphi_{e_1}^W = \frac{1}{m} \sum_a \iint (\varphi_{e_1}^W(y) - \varphi_{e_1}^W(x)) q^{(a)}(x, dy) \mu_{\delta}^{\text{m}}(dx),$$

with absolutely convergent integrals. Fix a state $x = (E, \sigma, \omega_I)$ with $E \in E_0$ such that $\omega_i \in \mathcal{P}_{i\sigma(i)}^E$ for every $i \in I$, which holds for μ_{δ}^{m} -almost every x by (9.5). Fix also a group a and a random skeleton Γ for which the routing map accepts $B_{\sigma}^{(a)}$, with output γ . Every move of $q^{(a)}$ with this skeleton is a move along γ as in Lemma 10.2, since its selected rows lie in $I_v^{(a)}$ for the distinct cells v of γ , and the resampled paths are drawn from a law carried by the permitted sets $\mathcal{P}_{i\sigma\pi(i)}^E$, $i \in S$. By Lemma 10.2, every such move increases $\varphi_{e_1}^W$ by $\text{hom}_1(\gamma)$, and $\text{hom}_1(\gamma) \geq 1$ by Proposition 6.15. The selection probabilities of the rows sum to one. Averaging over the random skeleton, we obtain

$$\int (\varphi_{e_1}^W(y) - \varphi_{e_1}^W(x)) q^{(a)}(x, dy) = \mathbb{E}_{\Gamma} [\mathbf{1}\{\mathcal{R}(B_{\sigma}^{(a)}, \Gamma) \text{ accepts}\} \text{hom}_1(\mathcal{R}(B_{\sigma}^{(a)}, \Gamma))] \geq \bar{q}^{(a)}(x).$$

Integrating over x , averaging over the groups, and using (10.8), we obtain

$$\mathbb{E}_{\mu_{\delta}^{\text{m}}Q} \varphi_{e_1}^W - \mathbb{E}_{\mu_{\delta}^{\text{m}}} \varphi_{e_1}^W \geq \frac{1}{m} \sum_{a=1}^m \int \bar{q}^{(a)} d\mu_{\delta}^{\text{m}} \geq \frac{1}{2}. \quad (10.9)$$

The kernel Q preserves the cuts, and $0 < \delta < \beta$. By Proposition 5.6, applied with $u = e_1$, $g = 1/2$, and $h = C_{\sharp}D^{-1/4}/K$, and by (10.8) and (10.9), we have

$$\text{Var}_{\mu}(W_1) \geq \frac{n_{\beta}(1/2)^2}{C_{\sharp}D^{-1/4}/K} = \frac{n_{\beta}KD^{1/4}}{4C_{\sharp}}.$$

Inserting this bound into (10.1) yields

$$f_s(N, K, T) \geq \frac{T}{2DK} \cdot \frac{n_\beta K D^{1/4}}{4C_\#} = \frac{T n_\beta}{8C_\# D^{3/4}}.$$

Finally, since $\beta/\delta \geq 1$ and $\lfloor x \rfloor \geq x/2$ for $x \geq 1$, we have $T n_\beta = \lfloor \beta/\delta \rfloor / \beta \geq 1/(2\delta)$, which gives the last inequality in (10.3). $\square$

The temperature uniformity follows from $T n_\beta \geq 1/(2\delta)$; see Proposition 5.6. Since $f_s \leq 1$ by Proposition 3.14, the constants satisfy $16C_\# \delta D_{\min}^{3/4} \geq 1$.

10.5 Return to physical units

We follow the change of units of [112, Section 10]. The superfluid fraction is invariant under dilations, in the following sense. For an exclusion distance $a > 0$, we write $f_s(N, L, T; a)$ for the superfluid fraction of Definition 3.2.

Lemma 10.3. *Let $N \geq 1$ and $L, a, s > 0$, and set $K = L/s$. The configuration domains of N particles on Λ_L with exclusion distance a and on Λ_K with exclusion distance a/s are either both empty or both nonempty. In the second case, $F_{N,L,T}(\theta) = s^{-2} F_{N,K,s^2 T}(\theta)$ for every $\theta \in \mathbb{R}$, where the free energies are computed with the exclusion distances a and a/s , respectively, and*

$$f_s(N, L, T; a) = f_s(N, K, s^2 T; a/s). \quad (10.10)$$

Proof. The map $X \mapsto X/s$ is a bijection from Λ_L^N onto Λ_K^N , and $d_L(x, y) = s d_K(x/s, y/s)$. It maps the configuration domain with exclusion distance a onto the one with exclusion distance a/s , which proves the first claim. Define $(U\psi)(X) = s^{3N/2} \psi(sX)$ for $X \in \Lambda_K^N$. Then U is unitary from $L^2(\Lambda_L^N)$ onto $L^2(\Lambda_K^N)$, it preserves symmetry, and it maps smooth functions with compact support in the first domain onto those in the second. It also maps $\mathcal{Q}_N$ onto $\mathcal{Q}_N$. For smooth ψ , note that $\nabla_i(U\psi)(X) = s^{3N/2} s(\nabla_i \psi)(sX)$ and $\theta e_1/K = s \theta e_1/L$. Then

$$\sum_{i=1}^N \int |(-i\nabla_i + \theta e_1/K) U\psi|^2 dX = s^2 \sum_{i=1}^N \int |(-i\nabla_i + \theta e_1/L) \psi|^2 dX,$$

where we changed variables $X \mapsto sX$ on the right, with Jacobian s^{3N} . By density, this identity holds on $\mathcal{Q}_N$. The forms (3.3) then satisfy $q_\theta^{(L,a)} = s^{-2} q_\theta^{(K,a/s)} \circ U$, which implies that $H_{N,L}(\theta) = s^{-2} U^{-1} H_{N,K}(\theta) U$, with the indicated exclusion distances. It follows that $\text{tr} e^{-H_{N,L}(\theta)/T} = \text{tr} e^{-H_{N,K}(\theta)/(s^2 T)}$, and taking logarithms, we obtain the relation between the free energies. Differentiating twice at $\theta = 0$ and using (3.5) yields

$$f_s(N, L, T; a) = \frac{L^2}{2N} s^{-2} \partial_\theta^2 F_{N,K,s^2 T}(0) = \frac{K^2}{2N} \partial_\theta^2 F_{N,K,s^2 T}(0) = f_s(N, K, s^2 T; a/s).$$

This completes the proof. $\square$

Used in: the proof of Theorem 1.1.

Equivalently, the winding vector is invariant under the dilation of paths $x(\tau) \mapsto s^{-1}x(s^2\tau)$, and $TL^2 = (s^2T)K^2$.

Proof of Theorem 1.1. Let α_0 , p_1 , τ_0 , $D_{\min}$, δ , $C_{\sharp}$, and K_0 be the constants fixed in Section 10.3, and let K_1 be the threshold of Proposition 8.5; they are absolute. We prove the theorem with the constants $c_0 = \alpha_0^3/(16D_{\min}^2)$ and $c_1 = 1/(16C_{\sharp}\delta\alpha_0^{9/8})$. For $a, \rho > 0$, put

$$s_{\rho} = \left(\frac{\alpha_0}{a\rho}\right)^{1/2}, \quad D_{\rho} = \rho s_{\rho}^3 = \frac{\alpha_0^{3/2}}{(\rho a^3)^{1/2}}.$$

By (1.11), we have $T_*(a, \rho) = \tau_0/(s_{\rho}^2 \log D_{\rho})$.

We first prove the finite-volume form stated in Remark 1.7. Let $a > 0$ and $0 < \rho_1 \leq \rho_2$ with $\rho_2 a^3 \leq 2c_0$. We claim that there exists L_0 such that $f_s(N, L, T) \geq c_1(\rho_L a^3)^{3/8}$, where $\rho_L = N/L^3$, for all $L \geq L_0$, all N with $\rho_L \in [\rho_1, \rho_2]$, and all $0 < T \leq T_*(a, \rho_L)$. Let L , N , and T be such. As in [112, Section 10], with $\lceil \cdot \rceil$ in place of $\lfloor \cdot \rfloor$, let

$$K = \left\lceil \frac{L}{s_{\rho_L}} \right\rceil, \quad s = \frac{L}{K}, \quad D = \frac{N}{K^3} = \rho_L s^3, \quad r = \frac{a}{s}, \quad \alpha = Dr = \rho_L a s^2, \quad T_K = s^2 T.$$

Since $L/s_{\rho_L} \leq K < L/s_{\rho_L} + 1$, we have $(1 - 1/K)s_{\rho_L} \leq s \leq s_{\rho_L}$. Let $L_0 = s_{\rho_1} \max\{K_0, K_1, 8\}$. Since $s_{\rho_L} \leq s_{\rho_1}$, we have $K \geq \max\{K_0, K_1, 8\}$ for $L \geq L_0$, and then the following hold.

- $(1 - 1/K)^3 D_{\rho_L} \leq D \leq D_{\rho_L}$. Since $\rho_L a^3 \leq 2c_0$, we have $D_{\rho_L} \geq \sqrt{8} D_{\min}$, and $(7/8)^3 \sqrt{8} \geq 1$ gives $D \geq D_{\min}$.
- $(1 - 1/K)^2 \alpha_0 \leq \alpha \leq \alpha_0$, so that $\alpha \in [\alpha_0/2, \alpha_0]$.
- Since $D \geq D_{\min} > 1$, both factors in $T_K \log D$ are nonnegative, and

$$T_K \log D \leq s_{\rho_L}^2 T \log D_{\rho_L} = \tau_0 \frac{T}{T_*(a, \rho_L)} \leq \tau_0.$$

- $\beta_K = 1/T_K \geq \log D_{\min}/\tau_0 \geq 4 > \delta$, since $\tau_0/\log D_{\min} \leq 1/4$ and $\delta = 2t \leq 2$.

By Proposition 8.5, in the form (8.9), the one-slab law of N particles on Λ_K with exclusion distance r and temperature T_K satisfies the joint bound with base p_1 ; the threshold K_1 depends on neither D nor T_K (Remark 8.1). By Lemma 10.3 and Proposition 10.1, we have

$$f_s(N, L, T; a) = f_s(N, K, T_K; r) \geq \frac{D^{-3/4}}{16C_{\sharp}\delta} \geq \frac{D_{\rho_L}^{-3/4}}{16C_{\sharp}\delta} = \frac{(\rho_L a^3)^{3/8}}{16C_{\sharp}\delta\alpha_0^{9/8}} = c_1(\rho_L a^3)^{3/8},$$

which proves the claim.

Now let $a, \rho > 0$ satisfy $\rho a^3 \leq c_0$, let $0 < T < T_*(a, \rho)$, and consider a sequence of N and L tending to infinity with $\rho_L = N/L^3 \rightarrow \rho$. Since $T_*(a, \cdot)$ is continuous,

for all sufficiently large L in the sequence we have $\rho_L \in [\rho/2, 2\rho]$ and $T \leq T_*(a, \rho_L)$. We apply the claim with $\rho_1 = \rho/2$ and $\rho_2 = 2\rho$, for which $\rho_2 a^3 \leq 2c_0$, and obtain $f_s(N, L, T) \geq c_1(\rho_L a^3)^{3/8}$ for all sufficiently large L in the sequence. Since $\rho_L \rightarrow \rho$, we conclude that

$$\liminf_{\substack{N, L \rightarrow \infty \\ N/L^3 \rightarrow \rho}} f_s(N, L, T) \geq c_1(\rho a^3)^{3/8},$$

which is the claim. $\square$

10.6 Scaling of the winding vector

Proof of Corollary 1.4. Fix a sequence of N and L tending to infinity with $N/L^3 \rightarrow \rho$, and write $\rho_L = N/L^3$ and $\hat{W} = W/(\beta\rho L)^{1/2}$. Let $\lambda_\rho = \sup \rho_L/\rho$, the supremum over the sequence, which is finite.

For $t \in \mathbb{R}^3$, we apply Theorem 3.10 to $t(\beta\rho L)^{-1/2}$ and obtain

$$\mathbb{E}_\mu e^{t \cdot \hat{W}} \leq \exp\left(\frac{\beta N |t|^2}{L^2 \beta \rho L}\right) = \exp\left(\frac{N |t|^2}{\rho L^3}\right).$$

Let u be a unit vector and $p \geq 1$. By Corollary 3.11, with $\sigma^2 = 2\beta N/L^2$ as in (3.25), we have $\mathbb{E}_\mu |u \cdot W|^p \leq p(2\sigma^2)^{p/2} \Gamma(p/2)$. Since $2\sigma^2/(\beta\rho L) = 4\rho_L/\rho \leq 4\lambda_\rho$, we obtain $\mathbb{E}_\mu |u \cdot \hat{W}|^p \leq p(4\lambda_\rho)^{p/2} \Gamma(p/2)$. By convexity, we have $|\hat{W}|^p \leq (|\hat{W}_1| + |\hat{W}_2| + |\hat{W}_3|)^p \leq 3^{p-1} \sum_b |\hat{W}_b|^p$, and then

$$\mathbb{E}_\mu |\hat{W}|^p \leq 3^p p(4\lambda_\rho)^{p/2} \Gamma(p/2)$$

for every element of the sequence.

Let $\hat{W}_\infty$ be the limit in distribution of $\hat{W}$ along a subsequence. By the last bound with $p = 4$, the variables $|\hat{W}|^2$ are uniformly integrable along the sequence, and we have $\mathbb{E}_\mu \hat{W} \rightarrow \mathbb{E} \hat{W}_\infty$ and $\mathbb{E}_\mu \hat{W} \hat{W}^T \rightarrow \mathbb{E} \hat{W}_\infty \hat{W}_\infty^T$ along the subsequence. Then Theorem 3.10 implies that $\mathbb{E}_\mu \hat{W} = 0$, which implies that $\mathbb{E} \hat{W}_\infty = 0$. By Theorem 3.10, $\text{Cov}_\mu(\hat{W}) = v_L \text{Id}$ with $v_L = \text{Var}_\mu(W_1)/(\beta\rho L)$. By (3.24) with $u = e_1$, we have $\text{Var}_\mu(W_1) \leq 2\beta N/L^2$, and then $v_L \leq 2\rho_L/\rho \rightarrow 2$. By (3.32), $\text{Var}_\mu(W_1) = 2\beta N f_s(N, L, T)/L^2$, that is, $v_L = 2(\rho_L/\rho) f_s(N, L, T)$, and Theorem 1.1 implies that $\liminf v_L \geq 2c_1(\rho a^3)^{3/8}$. Since $\text{Cov}(\hat{W}_\infty)$ is the limit of $v_L \text{Id}$ along the subsequence, we conclude that

$$2c_1(\rho a^3)^{3/8} \text{Id} \leq \text{Cov}(\hat{W}_\infty) \leq 2 \text{Id},$$

which completes the proof. $\square$

The upper and moment bounds use only Theorem 3.10; the lower covariance bound also uses Theorem 1.1. See the comments after Corollary 3.12 for the scope of the distributional conclusion.

Proof of the theorem on rotational inertia

In this section, we prove Theorem 1.2. The argument follows that of Section 10, with three changes. First, the gas lives in a bounded container, rows are retained only at cells at a fixed distance from the walls, and the routes are polygons in a cube inscribed in the container. Section 11.1 describes this setting and shows that the routes never leave the cells at which the matching system is active, which completes the verification of hypothesis (H5) of Section 7. Second, the observable is the stochastic area, and its change under a surgery is random. In Sections 11.3 and 11.4, we bound its conditional mean from below by the area of the routed polygon, up to an error proportional to the length of the polygon (Proposition 11.7). Third, to treat an arbitrary rotation axis, we rotate the container (Section 11.2), and the volume thresholds do not depend on the rotation. Section 11.5 lists the requirements on the parameters that come from the container, and Section 11.6 proves Theorem 1.2. We use the thermal estimates in a container of Proposition 8.10, which are proved in Appendix A, the matching system of Section 9, and the identities of Sections 3.5 and 3.6, which hold in every bounded open container.

11.1 The container in scaled units

Throughout this section, $\Omega \subset \mathbb{R}^3$ is a fixed nonempty bounded open set. We make no assumption on its regularity, connectivity, or symmetry. We use the scaled units of Section 8.6. For an integer $K \geq 1$, the container is $U = K\Omega$, the cells are $B_v = v + [-1/2, 1/2]^3$ with $v \in \mathbb{Z}^3$, and

$$D = \frac{N}{|U|} = \frac{N}{K^3|\Omega|}, \quad r = \frac{\alpha}{D}, \quad \alpha \in [\alpha_0/2, 2\alpha_0], \quad T > 0, \quad \beta = T^{-1} > \delta. \quad (11.1)$$

Here $\delta = 2t$, with t fixed in Section 10.3, and D is the number of particles per unit volume in scaled coordinates. It is not N divided by the number of cells that meet U , which can be much larger than $|U|$ when Ω is irregular. With this convention, D is the scaled counterpart of the finite-volume density $\rho_L = N/(L^3|\Omega|)$, which converges to the density ρ in Theorem 1.2. We apply the results of Section 3 to N particles in the container U with exclusion distance r , and we write μ for the Bose path measure and $\mathcal{A}$ for the stochastic area vector (3.19). By Proposition 3.16(3) and (4), for every unit vector n and

every $x_0 \in \mathbb{R}^3$,

$$I_{\text{cl}}(n, Kx_0) - I(n, Kx_0) = T \text{Var}_\mu(\mathcal{A}_n), \quad 0 \leq I(n, Kx_0) \leq I_{\text{cl}}(n, Kx_0), \quad (11.2)$$

where the axis passes through Kx_0 with direction n ; the matrix form is $I_{\text{cl}}(Kx_0) - I(Kx_0) = T \text{Cov}_\mu(\mathcal{A})$. No symmetry of Ω is used in (11.2).

An inscribed cube. Since Ω is open and nonempty, we may fix $x_* \in \Omega$ and $R_{\text{in}} > 0$ such that the closed ball $\bar{B}(x_*, 2R_{\text{in}})$ is contained in Ω . For K with $R_{\text{in}}K \geq 20$, set

$$H = \left\lfloor \frac{2R_{\text{in}}K}{\sqrt{3}} \right\rfloor - 2,$$

let $v_\Omega \in \mathbb{Z}^3$ be a lattice point nearest to $Kx_* - (H-1)(1,1,1)/2$, and let Ω_K be the union of the cells B_v with $v \in v_\Omega + \{0, \dots, H-1\}^3$. The set $\Omega_K = v_\Omega + [-1/2, H-1/2]^3$ is a cube of side H made of H^3 cells.

Lemma 11.1. *If $R_{\text{in}}K \geq 20$, then $R_{\text{in}}K \leq H \leq 2R_{\text{in}}K$, the cube Ω_K is contained in the ball $B(Kx_*, R_{\text{in}}K)$, and every point of Ω_K has distance at least $R_{\text{in}}K$ from $\mathbb{R}^3 \setminus U$.*

Proof. The upper bound on H is immediate. Since $2/\sqrt{3} - 1 > 0.15$ and $R_{\text{in}}K \geq 20$, we have $H \geq 2R_{\text{in}}K/\sqrt{3} - 3 \geq R_{\text{in}}K + 0.15R_{\text{in}}K - 3 \geq R_{\text{in}}K$. The center $c = v_\Omega + (H-1)(1,1,1)/2$ of Ω_K satisfies $|c - Kx_*| \leq \sqrt{3}/2$ by the choice of v_Ω , and every point of Ω_K is within distance $\sqrt{3}H/2$ of c . Then every point of Ω_K is within distance $\sqrt{3}(H+1)/2 \leq R_{\text{in}}K - \sqrt{3}/2$ of Kx_* , where we used $H+1 \leq 2R_{\text{in}}K/\sqrt{3} - 1$. The closed ball $\bar{B}(Kx_*, 2R_{\text{in}}K) = K\bar{B}(x_*, 2R_{\text{in}})$ is contained in U , which gives the last claim. $\square$

Used in: Lemma 11.2 and the proof of Proposition 11.9.

Active cells and the matching system. We use the matching system of Section 9 for the gas in U . By (9.1), the active cells are

$$\Lambda_{\text{act}} = \{v \in \mathbb{Z}^3 : v \in U, \text{dist}(v, \mathbb{R}^3 \setminus U) \geq 64\},$$

a finite set, since U is bounded. We recall the conventions of Section 9 that matter in a container. Rows are retained only at active cells. All other slab paths, in particular all paths that start near the walls, are frozen and exposed, and the wall condition is imposed on them. The bad sets $B_\sigma^{(a)}$ and the vulnerable sets $V_\sigma^{(a)}$ of the groups are subsets of Λ_{act} ; a prescribed set of cells that is not contained in Λ_{act} is never contained in $V_\sigma^{(a)} \cup V_\sigma^{(b)}$ for groups a, b , and the inactive cells do not affect (H4) and (H4⁺). The exposed data name the terminal positions as slots, and they do not contain the assignment of the retained rows to the retained slots (Proposition 9.5); exposing the terminal position of each retained row would fix the matching that the surgery changes. Restricting the

retained rows to the deterministic set Λ_{act} does not affect the proof of Proposition 9.5, since the absence of rows at inactive cells is not a random event. By Lemma 9.4, every permitted path of a retained row stays in U , within distance 61 of the center of its anchor; the walls do not restrict retained rows in any pairing, and the kernels K_{ij}^E of (9.3) are restrictions of Euclidean bridge measures W_{Y_i, Z_j}^{2t} . By Corollary 9.12, for $K \geq K_\Omega$ and $D \geq D_{\min}$, and whenever the one-slab law of the gas in U satisfies the joint bound (8.9) with base p_1 for sets of interior cells, the marked one-slab law μ_δ^m is the law of a grouped matching system with active region Λ_{act} and the number of groups of (10.7), which satisfies (H1)–(H4), (H3₁), and (H4⁺), with the same constants as on the torus. Here K_Ω is the threshold of Proposition 8.10 with $R_0 = 64$, the interior cells are the active cells, and μ_δ^m is defined as in Section 10.1.

The cells outside Λ_{act} never belong to a bad set, but they carry no rows, and a surgery cannot use them. Declaring them bad instead would violate (H4). Lemma 11.2 below shows that accepted routes never reach them.

Routes. For points $y_0, \dots, y_{m-1} \in \mathbb{R}^3$, with indices modulo m , we write

$$\text{ar}(y_0, \dots, y_{m-1}) = \frac{1}{2} \sum_{k=0}^{m-1} y_k \times y_{k+1} \quad (11.3)$$

for the area vector of the closed polygon with vertices $y_0, \dots, y_{m-1}$. It is invariant under translations $y_k \mapsto y_k + c$, since $\sum_k c \times (y_{k+1} - y_k) = 0$. The number $e_3 \cdot \text{ar}(y_0, \dots, y_{m-1})$ depends only on the projections of the y_k onto the plane orthogonal to e_3 ; it is the signed area enclosed by the projected polygon, counted positively for counterclockwise orientation. For an oriented cycle of cells $\gamma = (v_0, \dots, v_{|\gamma|-1})$, we write $\text{ar}(\gamma) = \text{ar}(v_0, \dots, v_{|\gamma|-1})$, using the cell centers $v_k \in \mathbb{Z}^3$.

Let $n_H = \lfloor H/64 \rfloor$, the scale denoted n in Definition 6.5. Let Γ be a polygonal skeleton of that definition with $K_{\text{in}} = H$ and inner cell cube $Q_{\text{in}} = v_\Omega + \{0, \dots, H-1\}^3$, the cell-index set of Ω_K . Let $\mathcal{R}$ be the area routing map of Definition 6.14.

Lemma 11.2. *Assume that $R_{\text{in}}K \geq 64$ and $H \geq 2^{14}$. For every polygonal skeleton Γ and every field $B \subset \mathbb{Z}^3$, every cell of an accepted output of the area routing map lies in $\mathcal{N}_{\lfloor \sqrt{n_H} \rfloor + 1}(\Gamma) \subset Q_{\text{in}} \subset \Lambda_{\text{act}}$.*

Proof. By Lemma 11.1, every cell center in Q_{in} has distance at least $R_{\text{in}}K \geq 64$ from $\mathbb{R}^3 \setminus U$. In particular, $Q_{\text{in}} \subset \Lambda_{\text{act}}$. The conclusion follows from Proposition 6.15(b), applied with $K_{\text{in}} = H$. $\square$

Used in: the verification of (H5) below.

The surgery theorem in the container. By Proposition 6.6, applied to the polygon law in a cube of side H , two independent polygonal skeletons satisfy the excess bounds of Step

3 in Section 10.3 with K replaced by H , for all sufficiently large H . By Proposition 6.15, if $n_H \geq 256$, then for a random set of bad cells that is independent of the random skeleton and satisfies the bound of (H4), the area routing map rejects with probability at most $C_{\text{acc}}p$; and every accepted output of the routing map is an oriented nearest-neighbor cycle γ of distinct cells outside the bad set, with $|\gamma| \leq 124n_H$ and $e_3 \cdot \text{ar}(\gamma) \geq n_H^2/250$. By Lemma 6.16, (H5)(c) holds, and by Lemma 11.2 the routes lie in Λ_{act} , as (H5)(a) requires. We conclude that (H5) holds, with the excess bounds stated in terms of H .

Let q be the subprobability kernel of Definition 7.5, built from the polygonal skeletons and the area routing map and averaged over the random skeleton and over the groups, and let Q be its completion. It acts on one slab as described in Section 10.1, with $\mathbb{L}_K$ replaced by Λ_{act} . By Theorem 7.14, with the constants of Section 10.3 and the excess bounds in terms of H , and by the bounds on the number of groups and on η_{loc} used for (10.8), the measure $\mu_\delta^m Q$ is absolutely continuous with respect to μ_δ^m with a square-integrable density, and

$$\chi^2(\mu_\delta^m Q \parallel \mu_\delta^m) \leq \frac{C_\# D^{-1/4}}{H}, \quad \int \bar{q} d\mu_\delta^m \geq 1 - C_{\text{acc}}p \geq \frac{1}{2}, \quad (11.4)$$

whenever K exceeds the volume thresholds collected in Section 11.5 and the joint bound holds.

11.2 Rotating the container

The routes of Section 11.1 produce area in the direction e_3 . For another direction, we rotate the container. We write $\text{SO}(3)$ for the group of rotations of $\mathbb{R}^3$, and μ_U for the Bose path measure of N particles in a container U with exclusion distance r .

Lemma 11.3. *Let $\Theta \in \text{SO}(3)$. The Bose path measure $\mu_{\Theta U}$ is the image of μ_U under the map $(\pi, X(\cdot)) \mapsto (\pi, \Theta X(\cdot))$, where $\Theta X(\cdot) = (\Theta x_1(\cdot), \dots, \Theta x_N(\cdot))$, and the stochastic area satisfies $\mathcal{A}(\Theta X(\cdot)) = \Theta \mathcal{A}(X(\cdot))$. In particular, $\text{Cov}_{\mu_{\Theta U}}(\mathcal{A}) = \Theta \text{Cov}_{\mu_U}(\mathcal{A}) \Theta^\top$, and $\text{Var}_{\mu_U}(n \cdot \mathcal{A}) = \text{Var}_{\mu_{\Theta U}}((\Theta n) \cdot \mathcal{A})$ for every unit vector n .*

Proof. Since $|\Theta x - \Theta y| = |x - y|$, the map $X \mapsto \Theta X$ sends $\mathcal{X}_N(U)$ onto $\mathcal{X}_N(\Theta U)$, preserves Lebesgue measure, and commutes with permutations of the particles. The heat kernel p_t is rotation invariant, and the image of the bridge measure $\mathbb{W}_{x,y}^t$ under $\omega \mapsto \Theta \omega$ is $\mathbb{W}_{\Theta x, \Theta y}^t$; this is the rotation invariance of Brownian motion with generator Δ . A configuration path stays in $\mathcal{X}_N(U)$ if and only if its image stays in $\mathcal{X}_N(\Theta U)$. Inserting these facts into (3.16) and (3.17), we conclude that the two partition functions are equal and that $\mu_{\Theta U}$ is the image of μ_U .

For $\Theta \in \text{SO}(3)$, recall that $\Theta a \times \Theta b = \Theta(a \times b)$ for all $a, b \in \mathbb{R}^3$. Each coordinate of the Stratonovich integral in (3.19) is a bilinear expression in the coordinates of the

integrand and of the integrator, and this identity carries over to the integrals. We obtain $\mathcal{A}(\Theta X(\cdot)) = \Theta \mathcal{A}(X(\cdot))$, computed with the reference points Θx_0 and x_0 ; recall that $\mathcal{A}$ does not depend on the reference point. The remaining claims follow. $\square$

Used in: the proof of Proposition 11.9.

If $\overline{B}(x_*, 2R_{\text{in}}) \subset \Omega$, then $\Theta\Omega$ is a nonempty bounded open set with $\overline{B}(\Theta x_*, 2R_{\text{in}}) \subset \Theta\Omega$ and $|\Theta\Omega| = |\Omega|$. The construction of Section 11.1 applies to $\Theta\Omega$, with the point Θx_* , the same radius R_{in} , and the same values of H , N , and D . Only the cell lattice $\mathbb{Z}^3$ is not rotated. The volume thresholds do not depend on Θ : conditions (K1) and (K3) of Section 11.5 involve only R_{in} and H , and for the threshold of Proposition 8.10 we use $g_{\Theta\Omega} = g_\Omega \circ \Theta^{-1}$ in (A.5), so that $K_{\Theta\Omega} = K_\Omega$ by Remark A.2(2).

11.3 Stochastic area of a Brownian bridge

For a path ω on $[0, \delta]$ and $y \in \mathbb{R}^3$, we write

$$\mathcal{B}_y(\omega) = \frac{1}{2} \int_0^\delta e_3 \cdot ((\omega(s) - y) \times \circ d\omega(s))$$

for the stochastic area of ω about y in the direction e_3 , defined almost everywhere under the bridge measures, as in (3.19). If $\omega(0) = y$ and $x' \in \mathbb{R}^3$, then, since $\omega - x' = (\omega - y) + (y - x')$ and $\int_0^\delta \circ d\omega = \omega(\delta) - y$,

$$\frac{1}{2} \int_0^\delta e_3 \cdot ((\omega(s) - x') \times \circ d\omega(s)) = \frac{1}{2} e_3 \cdot ((y - x') \times (\omega(\delta) - y)) + \mathcal{B}_y(\omega). \quad (11.5)$$

For the stochastic area of planar Brownian motion and its conditional characteristic function given the endpoint, see Lévy [97, Sections 1.2–1.3] and Yor [149, Equation (2)]. We need only the second moment of the area of a Brownian bridge. The proof below obtains it from the decomposition of Brownian motion into a bridge and its independent endpoint and from Itô's isometry [79, Theorems 2.1(I) and 2.2]; see also [128, Chapter I, Section 3, and Theorem IV.2.2].

Lemma 11.4. *Let $\delta > 0$ and $y, d \in \mathbb{R}^3$, and let ω be a Brownian bridge with generator Δ from y to $y + d$ of duration δ , with law $\widehat{\mathbb{W}}_{y, y+d}^\delta = \mathbb{W}_{y, y+d}^\delta / p_\delta(d)$. Then*

$$\mathbb{E} \mathcal{B}_y(\omega) = 0, \quad \mathbb{E} \mathcal{B}_y(\omega)^2 = \frac{\delta^2}{3} + \frac{\delta |d_\perp|^2}{6}, \quad (11.6)$$

where $d_\perp = (d_1, d_2, 0)$. Further, for $R \geq 0$ and $|d| \leq R$,

$$\int |\mathcal{B}_y| d\mathbb{W}_{y, y+d}^\delta \leq (4\pi\delta)^{-3/2} \left(\frac{\delta^2}{3} + \frac{\delta R^2}{6} \right)^{1/2}. \quad (11.7)$$

Proof. We write $\omega(s) = y + sd/\delta + b(s)$, where b is a Brownian bridge from 0 to 0 of duration δ , with $\mathbb{E}b_a(s)b_c(s') = 2\mathbf{1}\{a = c\}(\min(s, s') - ss'/\delta)$ for the coordinates $a, c \in \{1, 2, 3\}$. Since the coordinates of b are independent, the Stratonovich integrals of one coordinate against another coincide with the Itô integrals. Since $d \times d = 0$, we have

$$(\omega(s) - y) \times \circ d\omega(s) = \frac{s}{\delta} d \times db(s) + \frac{1}{\delta} b(s) \times d \, ds + b(s) \times \circ db(s).$$

Integration by parts gives $\int_0^\delta s \, db(s) = -\int_0^\delta b(s) \, ds$, because $b(\delta) = 0$. With $J = \int_0^\delta b(s) \, ds$, we obtain

$$\mathcal{B}_y(\omega) = A(b) + \frac{1}{\delta} e_3 \cdot (J \times d), \quad A(b) = \frac{1}{2} \int_0^\delta e_3 \cdot (b(s) \times \circ db(s)). \quad (11.8)$$

The law of b is invariant under $b \mapsto -b$, which leaves $A(b)$ unchanged and changes the sign of J . This implies that $\mathbb{E}J = 0$ and that $A(b)$ and $e_3 \cdot (J \times d)$ are uncorrelated. The law of b is also invariant under the reflection of its first coordinate, which changes the sign of $A(b)$. Then $\mathbb{E}A(b) = 0$. This proves the first identity in (11.6).

The coordinates of J are independent centered Gaussian variables with variance

$$2 \int_0^\delta \int_0^\delta \left(\min(s, s') - \frac{ss'}{\delta} \right) ds \, ds' = 2 \left(\frac{\delta^3}{3} - \frac{\delta^3}{4} \right) = \frac{\delta^3}{6}.$$

The variance of $\delta^{-1} e_3 \cdot (J \times d) = \delta^{-1} (J_1 d_2 - J_2 d_1)$ is $\delta |d_\perp|^2 / 6$.

To compute the variance of $A(b)$, let $\mathfrak{B}$ be a Brownian motion with generator Δ started at 0, and let $\mathfrak{U} = \mathfrak{B}(\delta)$. The process $\mathfrak{B}(s) - s\mathfrak{U}/\delta$ is a Brownian bridge from 0 to 0 of duration δ , independent of $\mathfrak{U}$, and we may take $b(s) = \mathfrak{B}(s) - s\mathfrak{U}/\delta$. Applying (11.8) with $y = 0$ to the path $\mathfrak{B}(s) = s\mathfrak{U}/\delta + b(s)$, whose computation above is pathwise, yields $A(\mathfrak{B}) = A(b) + \delta^{-1} e_3 \cdot (J \times \mathfrak{U})$. By Itô's isometry, since $d\langle \mathfrak{B}_a \rangle_s = 2 \, ds$ and $\mathbb{E}\mathfrak{B}_a(s)^2 = 2s$,

$$\mathbb{E}A(\mathfrak{B})^2 = \frac{1}{4} \int_0^\delta \mathbb{E}(\mathfrak{B}_1(s)^2 + \mathfrak{B}_2(s)^2) 2 \, ds = \frac{1}{2} \int_0^\delta 4s \, ds = \delta^2.$$

Since $\mathfrak{U}$ is centered and independent of b , the variables $A(b)$ and $\delta^{-1} e_3 \cdot (J \times \mathfrak{U})$ are uncorrelated, and the variance of the latter is

$$\delta^{-2} (\mathbb{E}J_1^2 \mathbb{E}\mathfrak{U}_2^2 + \mathbb{E}J_2^2 \mathbb{E}\mathfrak{U}_1^2) = \delta^{-2} \cdot 2 \cdot \frac{\delta^3}{6} \cdot 2\delta = \frac{2\delta^2}{3}.$$

We obtain $\mathbb{E}A(b)^2 = \delta^2 - 2\delta^2/3 = \delta^2/3$. Together with (11.8) and the computation for J , this proves the second identity in (11.6).

Finally, $W_{y, y+d}^\delta$ has total mass $p_\delta(d) \leq p_\delta(0) = (4\pi\delta)^{-3/2}$. By the Cauchy–Schwarz inequality, we have $\int |\mathcal{B}_y| \, dW_{y, y+d}^\delta = p_\delta(d) \mathbb{E}|\mathcal{B}_y(\omega)| \leq p_\delta(0) (\mathbb{E}\mathcal{B}_y(\omega)^2)^{1/2}$. Since $|d_\perp| \leq |d| \leq R$, this proves (11.7). $\square$

Used in: Lemma 11.5.

The stochastic area of a confined Brownian path is not bounded pathwise: a path that stays in a small ball can sweep a large signed area by winding many times. For this reason, we control the area gain of a surgery through first moments under the exact conditional laws of the matching system. Set

$$M_{\mathcal{B}} = (4\pi\delta)^{-3/2} \left(\frac{\delta^2}{3} + \frac{1600\delta}{6} \right)^{1/2}, \quad C_{\mathcal{B}} = \frac{2M_{\mathcal{B}}}{z_-}, \quad (11.9)$$

where z_- is the constant of Lemma 9.8.

Lemma 11.5. *Consider the matching system of one slab in $\mathcal{U}$, and assume that its global collision error satisfies $\eta_{\text{glob}} \leq 1/2$. Let E be a value of the exposed data satisfying the endpoint separation conditions of Lemma 9.9, let $\tau: I \rightarrow J$ be a bijection, and let $i \in I$ satisfy $K_{i\tau(i)}^E(1) \geq z_-$. Let $\omega_k \in \mathcal{P}_{k\tau(k)}^E$ for $k \in I \setminus \{i\}$, and let ν be the probability measure proportional to $\prod_{k \neq i} h_{ik}(\omega, \omega_k) K_{i\tau(i)}^E(d\omega)$. Then*

$$\int |\mathcal{B}_{Y_i}(\omega)| \nu(d\omega) \leq C_{\mathcal{B}}. \quad (11.10)$$

Proof. By (9.11) with $I' = I \setminus \{i\}$, the normalizing constant of ν is at least $(1 - \eta_{\text{glob}}) K_{i\tau(i)}^E(1) \geq z_-/2 > 0$. By (9.3), $K_{i\tau(i)}^E$ is bounded by the bridge measure $\mathbb{W}_{Y_i, Z_{\tau(i)}}^{2t}$, and the checks h_{ik} are at most one. Since $K_{i\tau(i)}^E(1) > 0$, there exists a permitted path from Y_i to $Z_{\tau(i)}$, and Lemma 9.3(b) implies that $|Z_{\tau(i)} - Y_i| \leq 40$. By (11.7) with $R = 40$, we have

$$\int |\mathcal{B}_{Y_i}(\omega)| \prod_{k \neq i} h_{ik}(\omega, \omega_k) K_{i\tau(i)}^E(d\omega) \leq \int |\mathcal{B}_{Y_i}| d\mathbb{W}_{Y_i, Z_{\tau(i)}}^{2t} \leq M_{\mathcal{B}}.$$

Dividing by the normalizing constant gives (11.10). $\square$

Used in: the proof of Proposition 11.7.

No independence between the area and the avoidance constraints is used: the constraints are dropped in the numerator and controlled in the denominator. The bound requires the insertion estimate for every permitted configuration of the other rows, including the rows of the other groups, and not only on average; this is the form of (H3), with the global error η_{glob} , proved in Lemma 9.9.

11.4 The mean area gain of a surgery

The change of the stochastic area under a surgery consists of a deterministic part, which is computed from the endpoints of the selected paths, and the change of the stochastic areas of the selected paths about their initial points. The deterministic part is the area of the polygon of initial points, up to an error from the chords between neighboring endpoints.

Lemma 11.6. *Let $m \geq 2$, let $Y_0, \dots, Y_{m-1}, Z_0, \dots, Z_{m-1}, x' \in \mathbb{R}^3$, with indices modulo m , and let $d_k = Z_k - Y_k$. Then*

$$\frac{1}{2} \sum_{k=0}^{m-1} (Y_k - x') \times (Z_{k+1} - Z_k) = \text{ar}(Y_0, \dots, Y_{m-1}) + \frac{1}{2} \sum_{k=0}^{m-1} (Y_{k-1} - Y_k) \times d_k. \quad (11.11)$$

If $Y_k = V_k + e_k$ for $V_k, e_k \in \mathbb{R}^3$, then

$$\text{ar}(Y_0, \dots, Y_{m-1}) - \text{ar}(V_0, \dots, V_{m-1}) = \frac{1}{2} \sum_{k=0}^{m-1} (e_k \times (V_{k+1} - V_{k-1}) + e_k \times e_{k+1}). \quad (11.12)$$

Proof. Since $\sum_k (Z_{k+1} - Z_k) = 0$, the terms with x' cancel. Next, note that $Z_{k+1} - Z_k = Y_{k+1} - Y_k + d_{k+1} - d_k$ and $Y_k \times Y_k = 0$. Expanding gives

$$\sum_k Y_k \times (Z_{k+1} - Z_k) = \sum_k Y_k \times Y_{k+1} + \sum_k Y_k \times d_{k+1} - \sum_k Y_k \times d_k.$$

Reindexing the second sum on the right as $\sum_k Y_{k-1} \times d_k$ proves (11.11). For (11.12), we expand $Y_k \times Y_{k+1} = V_k \times V_{k+1} + V_k \times e_{k+1} + e_k \times V_{k+1} + e_k \times e_{k+1}$ and use $\sum_k V_k \times e_{k+1} = \sum_k V_{k-1} \times e_k = -\sum_k e_k \times V_{k-1}$. $\square$

Used in: the proof of Proposition 11.7.

In the rest of this section, we assume the setting of Section 11.1, with K exceeding the volume thresholds of Section 11.5 and the one-slab law satisfying the joint bound, so that (11.4) holds and Lemma 11.2 applies. Fix $x' \in \mathbb{R}^3$, and let $\varphi = \varphi_{e_3}^A$ be the function of Section 5.2 with the reference point x' ; in the notation of Section 9,

$$\varphi = \frac{1}{2} \sum_{k=1}^N \int_0^\delta e_3 \cdot ((\omega_k(s) - x') \times \circ d\omega_k(s)),$$

the sum running over all paths of the slab. By Proposition 3.17(5), φ is square integrable under μ_δ^m . Let

$$C_{\text{err}} = 57 + 2C_{\mathcal{B}}. \quad (11.13)$$

Proposition 11.7. *Let a be a group, and let $q^{(a)}$ be the subprobability kernel of group a in Definition 7.5, averaged over the random skeleton. The double integral below converges absolutely, and*

$$\iint (\varphi(y) - \varphi(x)) q^{(a)}(x, dy) \mu_\delta^m(dx) \geq \mathbb{E} \left[\mathbf{1}\{\mathcal{R}(B_\sigma^{(a)}, \Gamma) \text{ accepts}\} (e_3 \cdot \text{ar}(\gamma) - C_{\text{err}} |\gamma|) \right], \quad (11.14)$$

where the expectation on the right is over (E, σ, ω_1) with law μ_δ^m and an independent polygonal skeleton Γ , and $\gamma = \mathcal{R}(B_\sigma^{(a)}, \Gamma)$.

Proof. Since q is the average of the group kernels, $q^{(a)}$ is at most the number of groups times q . The absolute convergence then follows from Lemma 5.2(2), applied to q , since φ and the density of $\mu_\delta^m Q$ are square integrable.

All fiber computations below are on $E \in E_0$ satisfying the endpoint separation conditions of Lemma 9.9, a set of full measure for the exposed data. For fixed Γ , the kernel $q_\Gamma^{(a)}$ is a finite sum over labeled routes as in Definition 7.5, with one row of group a in each cell. For $\gamma = (v_0, \dots, v_{|\gamma|-1})$ and a labeling $i_k \in I_{v_k}^{(a)}$, the rate is

$$R_\pi(\sigma | \Gamma) = \mathbf{1}\{\mathcal{R}(B_\sigma^{(a)}, \Gamma) = \gamma\} \prod_k \frac{A_{i_k \sigma(i_k)}}{m_{v_k}^{(a)}(\sigma)},$$

where π is the permutation with $\pi(i_k) = i_{k+1}$ that fixes all other rows (Definition 7.5). The rate depends only on E , σ , and Γ . Let $S = \{i_k\}$, and let P^{tgt} denote the exact target conditional law of Definition 7.5, given E , $\sigma\pi$, and $\omega_{I \setminus S}$. For an accepted route, the rates of its labelings sum to one.

Fix Γ , a state $x = (E, \sigma, \omega_I)$ with $\omega_i \in \mathcal{P}_{i\sigma(i)}^E$ for all $i \in I$, and a labeled route with $R_\pi(\sigma | \Gamma) > 0$. We write $Y_k = Y_{i_k}$, $Z_k = Z_{\sigma(i_k)}$, and $d_k = Z_k - Y_k$. The move replaces the path ω_{i_k} , from Y_k to Z_k , by a path ω'_{i_k} from Y_k to $Z_{\sigma\pi(i_k)} = Z_{k+1}$, and leaves all other paths of the slab unchanged. By (11.5), applied to ω_{i_k} and to ω'_{i_k} , and by (11.11), the new state y satisfies

$$\varphi(y) - \varphi(x) = e_3 \cdot \text{ar}(\gamma) + \mathcal{E}_1 + \mathcal{E}_2 + \sum_k (\mathcal{B}_{Y_k}(\omega'_{i_k}) - \mathcal{B}_{Y_k}(\omega_{i_k})), \quad (11.15)$$

where

$$\mathcal{E}_1 = \frac{1}{2} \sum_k e_3 \cdot ((Y_{k-1} - Y_k) \times d_k), \quad \mathcal{E}_2 = e_3 \cdot (\text{ar}(Y_0, \dots, Y_{|\gamma|-1}) - \text{ar}(\gamma)).$$

We bound the two errors. We write $e_k = Y_k - v_k$. Since $Y_k \in B_{v_k}$ and neighboring cell centers are at distance one, we have $|e_k| \leq \sqrt{3}/2$ and $|Y_{k-1} - Y_k| \leq 1 + \sqrt{3}$. The path ω_{i_k} is permitted, and Lemma 9.3(b) implies that $|d_k| \leq 40$. We obtain $|\mathcal{E}_1| \leq 20(1 + \sqrt{3})|\gamma|$. By (11.12) with $V_k = v_k$, and by $|v_{k+1} - v_{k-1}| \leq 2$, we have $|\mathcal{E}_2| \leq (\sqrt{3} + 3/4)|\gamma|/2 \leq 5|\gamma|/4$. Since $20(1 + \sqrt{3}) + 5/4 \leq 57$, we conclude from (11.15) that

$$\varphi(y) - \varphi(x) \geq e_3 \cdot \text{ar}(\gamma) - 57|\gamma| - \sum_k (|\mathcal{B}_{Y_k}(\omega'_{i_k})| + |\mathcal{B}_{Y_k}(\omega_{i_k})|). \quad (11.16)$$

We next bound the conditional means of the stochastic areas, starting with the target paths. Under P^{tgt} , conditionally on $\omega'_{S \setminus \{i_k\}}$, the path ω'_{i_k} has the law ν of Lemma 11.5 with $\tau = \sigma\pi$, $i = i_k$, and the opponents ω'_l , $l \in S \setminus \{i_k\}$, and ω_l , $l \notin S$, which are permitted for their pairings under $\sigma\pi$. Its pairing $(i_k, \sigma(i_{k+1}))$ has mass at least z_- by Lemma 9.8,

applied to the rows i_k and i_{k+1} , whose anchors v_k and v_{k+1} are face neighbors, and to $A_{i_k\sigma(i_k)} = A_{i_{k+1}\sigma(i_{k+1})} = 1$. Then Lemma 11.5 implies that $\int |\mathcal{B}_{Y_k}(\omega'_{i_k})| P^{\text{tgt}}(d\omega'_S) \leq C_B$.

We now turn to the source paths. By (9.5), under μ_δ^m and conditionally on $(E, \sigma, \omega_{I \setminus \{i_k\}})$, the path ω_{i_k} has the law ν of Lemma 11.5 with $\tau = \sigma$ and $i = i_k$. If $R_\pi(\sigma | \Gamma) > 0$, then $A_{i_k\sigma(i_k)} = 1$, and by Lemma 9.8, applied with equal rows and equal slots, we have $K_{i_k\sigma(i_k)}^E(1) \geq z_-$. Since $R_\pi(\sigma | \Gamma)$ is a function of (E, σ) for fixed Γ , by Lemma 11.5, we have

$$\begin{aligned} \int R_\pi(\sigma | \Gamma) |\mathcal{B}_{Y_k}(\omega_{i_k})| d\mu_\delta^m &= \int R_\pi(\sigma | \Gamma) \mathbb{E}[|\mathcal{B}_{Y_k}(\omega_{i_k})| \mid E, \sigma, \omega_{I \setminus \{i_k\}}] d\mu_\delta^m \\ &\leq C_B \int R_\pi(\sigma | \Gamma) d\mu_\delta^m. \end{aligned}$$

This is the point where we use that the rates do not read the paths that are integrated.

We now multiply (11.16) by $R_\pi(\sigma | \Gamma)$, integrate over P^{tgt} and over μ_δ^m , sum over the labeled routes, and average over Γ . All terms are integrable: $\text{ar}(\gamma)$ and $|\gamma|$ are bounded at fixed K , because Λ_{act} is finite, and the stochastic areas were just bounded. Since the rates of the labelings of an accepted route sum to one, the two bounds above give (11.14) with $C_{\text{err}} = 57 + 2C_B$. $\square$

Used in: Corollary 11.8.

In Proposition 11.7, the endpoint terms of (11.5) reproduce the area of the polygon of initial points through the chord identity (11.11), and the reference point x' drops out, in agreement with the independence of $\mathcal{A}$ from the reference point. The stochastic areas of the selected paths enter only through their conditional first moments, which is where the exact resampling of the target paths and the independence of the rates from the path values are used.

Corollary 11.8. *Suppose, in addition, that $H \geq 128$ and $n_H \geq \max\{256, 62000 C_{\text{err}}\}$. Then*

$$\mathbb{E}_{\mu_\delta^m} Q\varphi - \mathbb{E}_{\mu_\delta^m} \varphi \geq \frac{n_H^2}{1000} \geq c_g H^2, \quad c_g = 2^{-14} \cdot 10^{-3}. \quad (11.17)$$

Proof. By Lemma 5.2(2), the left side of (11.17) equals $\iint (\varphi(y) - \varphi(x)) q(x, dy) \mu_\delta^m(dx)$, which is the average over the groups a of the left sides of (11.14). By Proposition 6.15, every accepted route satisfies $e_3 \cdot \text{ar}(\gamma) \geq n_H^2/250$ and $|\gamma| \leq 124n_H$. Then

$$e_3 \cdot \text{ar}(\gamma) - C_{\text{err}} |\gamma| \geq \frac{n_H^2}{250} - 124 C_{\text{err}} n_H \geq \frac{n_H^2}{500},$$

where in the last inequality we used $n_H \geq 62000 C_{\text{err}}$. By Proposition 11.7, the left side of (11.14) is at least $n_H^2/500$ times the acceptance probability $\int \bar{q}^{(a)} d\mu_\delta^m$ of group a . The average of these acceptance probabilities over the groups is $\int \bar{q} d\mu_\delta^m \geq 1/2$, by (11.4). Hence the gain is at least $n_H^2/1000$. Finally, $n_H \geq H/64 - 1 \geq H/128$ since $H \geq 128$, and $(H/128)^2/1000 = c_g H^2$. $\square$

Used in: the proof of Proposition 11.9.

11.5 Choice of the parameters

We use the constants of Section 10.3. At the steps listed there, the choices also meet the following requirements of the container.

- *Step 1.* The spatial constants satisfy the requirements of Proposition 8.10.
- *Step 2.* The active cells have depth 64, as in (9.1). The polygonal skeleton law of Definition 6.5 and the area routing map of Definition 6.14 are fixed. The point x_* and the radius R_{in} are fixed properties of Ω ; they enter only the constant $c_\Omega^\#$ below and the volume thresholds.
- *Steps 3 and 4.* The skeleton constants, including $\theta_\times$ and $C_\times$, serve both skeleton families, and the area routing map accepts with probability at least $1 - C_{\text{acc}}p \geq 1/2$.
- *Steps 5 and 7.* The thermal constants H_1 and t are chosen for Proposition 8.10 with base p_1 and $R_0 = 64$ as well, and D_{min} is at least the threshold of Corollary 9.12 in a container, which includes the threshold D_0 of Proposition 8.10.
- *Step 8.* The area constants are M_B and $C_B = 2M_B/z_-$ of (11.9), $C_{\text{err}} = 57 + 2C_B$ of (11.13), $c_g = 2^{-14} \cdot 10^{-3}$, and

$$c_\Omega^\# = \frac{c_g^2 R_{\text{in}}^5}{2C_\# \delta}. \quad (11.18)$$

The constant C_B uses only the bound $\eta_{\text{glob}} \leq 1/2$. It does not depend on $D \geq D_{\text{min}}$. These constants depend on t , $C_\#$, and R_{in} only. They enter the volume thresholds and the final constants, and none of Steps 1–7.

Volume thresholds. Let K_0 be the smallest integer such that, for every $\Theta \in \text{SO}(3)$ and every $K \geq K_0$, the following hold for the container $\Theta\Omega$:

- (K1) $R_{\text{in}}K \geq 64$, $H \geq 2^{14}$, and $n_H \geq \max\{256, 62000 C_{\text{err}}\}$;
- (K2) $K \geq K_\Omega$, the threshold of Proposition 8.10 with $R_0 = 64$;
- (K3) the excess bounds of Proposition 6.6 for the polygon law in a cube of side H , the acceptance bound of Proposition 6.15 for the area routing map, and the volume condition of Theorem 7.14.

Conditions (K1) hold for all large K , since $H \geq R_{\text{in}}K$ and $n_H = \lfloor H/64 \rfloor \geq H/64 - 1 \geq R_{\text{in}}K/64 - 1$ by Lemma 11.1. The threshold K_0 depends only on Ω , through R_{in} and K_Ω , and on the constants of Section 10.3; it does not depend on Θ , D , α , or T , and no condition relates β to K .

11.6 Proof of the theorem

We first prove the lower bound on the area variance in a fixed direction, in scaled units.

Proposition 11.9. *Let Ω , x_* , and R_{in} be as in Section 11.1, let $c_\Omega^\sharp$ be the constant (11.18), and let K_0 be the threshold of Section 11.5. Let $K \geq K_0$ and $N \geq 1$ with $D = N/(K^3|\Omega|) \geq D_{\text{min}}$, let $\alpha \in [\alpha_0/2, 2\alpha_0]$ and $r = \alpha/D$, and let $\beta > \delta$. Let n be a unit vector and $\Theta \in \text{SO}(3)$ with $\Theta n = e_3$, and suppose that the one-slab law of N particles in $K\Theta\Omega$ with exclusion distance r satisfies the joint bound (8.9) with base p_1 for sets of interior cells. Then the Bose path measure μ of N particles in $U = K\Omega$ satisfies*

$$\text{Var}_\mu(n \cdot \mathcal{A}) \geq c_\Omega^\sharp D^{1/4} \beta K^5. \quad (11.19)$$

Proof. By Lemma 11.3, $\text{Var}_{\mu_U}(n \cdot \mathcal{A}) = \text{Var}_{\mu_{\Theta U}}(e_3 \cdot \mathcal{A})$, and $\Theta U = K(\Theta\Omega)$. By the comment after Lemma 11.3, the results of Sections 11.1 and 11.4 apply to the container ΘU , whose one-slab law satisfies the joint bound by assumption, with the point Θx_* and the same R_{in} , H , N , and D .

By assumption $\beta > \delta$. The completed surgery kernel Q preserves the exposed data, and in particular the cuts, and by (11.4) it satisfies $\chi^2(\mu_\delta^m Q \parallel \mu_\delta^m) \leq C_\sharp D^{-1/4}/H$. By Corollary 11.8, its gain for $\varphi_{e_3}^A$ is at least $c_g H^2$. By Proposition 5.6, applied in the container ΘU with the unit vector e_3 , $g = c_g H^2$, and $h = C_\sharp D^{-1/4}/H$, we obtain

$$\text{Var}_{\mu_{\Theta U}}(e_3 \cdot \mathcal{A}) \geq \frac{n_\beta c_g^2 H^4}{C_\sharp D^{-1/4}/H} = \frac{n_\beta c_g^2 H^5 D^{1/4}}{C_\sharp}.$$

Since $\beta > \delta$, we have $n_\beta = \lfloor \beta/\delta \rfloor \geq \beta/(2\delta)$, and $H \geq R_{\text{in}}K$ by Lemma 11.1. The right side is at least $c_g^2 R_{\text{in}}^5 D^{1/4} \beta K^5 / (2C_\sharp \delta) = c_\Omega^\sharp D^{1/4} \beta K^5$. $\square$

Used in: the proof of Theorem 1.2.

By Proposition 8.10, applied to $\Theta\Omega$ with $K_{\Theta\Omega} = K_\Omega$ (Section 11.2), the joint bound holds for every Θ whenever $D \geq D_{\text{min}}$, $0 < T \leq \tau_0/\log D$, and $K \geq K_0$.

We follow the change of units of [112, Section 10], with $\lceil \cdot \rceil$ in place of $\lfloor \cdot \rfloor$, as in Section 10.5. For the container, we use the following analog of Lemma 10.3.

Lemma 11.10. *Let $N \geq 1$, $L, a, s > 0$, $K = L/s$, and $x_0 \in \mathbb{R}^3$. Consider N particles in $L\Omega$ with exclusion distance a at temperature T , and N particles in $K\Omega$ with exclusion distance a/s at temperature $s^2 T$. Their configuration domains are both empty or both nonempty. In the second case, the inertia tensors of Definition 3.4 satisfy*

$$I_{bc}(Lx_0) = s^2 I_{bc}(Kx_0), \quad I_{\text{cl},bc}(Lx_0) = s^2 I_{\text{cl},bc}(Kx_0),$$

where the left sides refer to the first gas and the right sides to the second, and $\text{Cov}(\mathcal{A})$ for the first gas is s^4 times $\text{Cov}(\mathcal{A})$ for the second. In particular, $1 - I(n, Lx_0)/I_{\text{cl}}(n, Lx_0) = 1 - I(n, Kx_0)/I_{\text{cl}}(n, Kx_0)$ for every unit vector n .

Proof. The first claim and the properties of the unitary map U are as in the proof of Lemma 10.3, with Λ_L and Λ_K replaced by $L\Omega$ and $K\Omega$; in particular $q^{(L\Omega, a)} =$

$s^{-2}q^{(K\Omega, a/s)} \circ U$, so that $H^{(L\Omega, a)} = s^{-2}U^{-1}H^{(K\Omega, a/s)}U$. Under the change of variables $x = s\tilde{x}$, we have $x - Lx_0 = s(\tilde{x} - Kx_0)$ and $\nabla_x = s^{-1}\nabla_{\tilde{x}}$, and the angular momentum (3.6) about Lx_0 is conjugated by U to the angular momentum about Kx_0 . By Lemma 3.3, the operators $H - \omega \cdot \mathcal{L}$ are defined by their forms on the respective domains, and the last two facts give

$$\mathrm{tr} e^{-(H^{(L\Omega, a)} - \omega \cdot \mathcal{L})/T} = \mathrm{tr} e^{-(H^{(K\Omega, a/s)} - s^2\omega \cdot \mathcal{L})/(s^2T)}.$$

The rotating-frame free energies (3.9) then satisfy $F^{\mathrm{rot}}(\omega) = s^{-2}F^{\mathrm{rot}}(s^2\omega)$, with the first gas on the left and the second on the right. Differentiating twice at $\omega = 0$ gives $I_{bc}(Lx_0) = s^2I_{bc}(Kx_0)$. For a bounded measurable symmetric function f , the operator UfU^{-1} is multiplication by $f(s \cdot)$, and the Gibbs expectation of f in the first gas equals that of $f(s \cdot)$ in the second. The matrix function R of Section 3.5 with reference point Lx_0 , evaluated at sX , is s^2 times the one with reference point Kx_0 , evaluated at X . This gives $I_{cl, bc}(Lx_0) = s^2I_{cl, bc}(Kx_0)$. By (3.36) in both gases, $TCov(\mathcal{A})$ for the first gas equals $s^2(I_{cl} - I)$ for the second, which is $s^2 \cdot s^2TCov(\mathcal{A})$ for the second. Dividing by T gives the claim. $\square$

Used in: the proof of Theorem 1.2.

Proof of Theorem 1.2. Let α_0 , $D_{\min}$, δ , τ_0 , and $C_{\sharp}$ be the constants of Section 10.3, and let $c_{\Omega}^{\sharp}$ be the constant (11.18). We prove the theorem with the constants $c_0 = \alpha_0^3/(16D_{\min}^2)$ and $c_1 = c_{\Omega}^{\sharp}/\alpha_0^{9/8}$. The first is absolute; the second depends on Ω only through R_{in} . We first prove the following finite-volume bounds. Let $a > 0$ and $0 < \rho_1 \leq \rho_2$ with $\rho_2 a^3 \leq 2c_0$. There exists L_0 such that, for all $L \geq L_0$, all N with $\rho_L = N/(L^3|\Omega|) \in [\rho_1, \rho_2]$, all $0 < T \leq T_*(a, \rho_L)$, every unit vector n , and every $x_0 \in \mathbb{R}^3$,

$$\frac{\lambda_{\min}(\mathrm{Cov}_{\mu}(\mathcal{A}))}{\beta \rho_L L^5} \geq c_1(\rho_L a^3)^{3/8}, \quad 1 - \frac{I(n, Lx_0)}{I_{cl}(n, Lx_0)} \geq \frac{2c_1}{|\Omega|R_{\Omega}(x_0)^2}(\rho_L a^3)^{3/8},$$

where μ is the Bose path measure of N particles in $L\Omega$ with exclusion distance a at temperature T .

Change of units. For $\rho > 0$, let $s_{\rho} = (\alpha_0/(a\rho))^{1/2}$ and $D_{\rho} = \rho s_{\rho}^3 = \alpha_0^{3/2}(\rho a^3)^{-1/2}$, as in the proof of Theorem 1.1. By (1.11), $T_*(a, \rho) = \tau_0/(s_{\rho}^2 \log D_{\rho})$, and since $\rho_2 a^3 \leq 2c_0$, we have $D_{\rho} \geq D_{\rho_2} \geq \sqrt{8}D_{\min}$ for $\rho \leq \rho_2$. For N , L , and T as above, let

$$K = \left\lceil \left(\frac{aN}{\alpha_0|\Omega|L} \right)^{1/2} \right\rceil = \left\lceil \frac{L}{s_{\rho_L}} \right\rceil, \quad s = \frac{L}{K}, \quad D = \frac{N}{K^3|\Omega|} = \rho_L s^3,$$

$$r = \frac{a}{s}, \quad \alpha = Dr = \rho_L a s^2, \quad T_K = s^2 T, \quad \beta_K = \frac{1}{T_K}.$$

Since $K - 1 < L/s_{\rho_L} \leq K$, we have $(1 - 1/K)s_{\rho_L} < s \leq s_{\rho_L}$. If $K \geq 8$, then $D \in [(7/8)^3 D_{\rho_L}, D_{\rho_L}]$, so that $D \geq (7/8)^3 \sqrt{8} D_{\min} \geq D_{\min}$, and $\alpha \in [(7/8)^2 \alpha_0, \alpha_0] \subset [\alpha_0/2, 2\alpha_0]$. Since $D \leq D_{\rho_L}$ and $\log D > 0$,

$$T_K \log D \leq s_{\rho_L}^2 T \log D_{\rho_L} = \tau_0 \frac{T}{T_*(a, \rho_L)} \leq \tau_0,$$

and $\beta_K \geq \log D / \tau_0 \geq \log D_{\min} / \tau_0 \geq 4 > \delta$, since $\tau_0 / \log D_{\min} \leq 1/4$ by Step 7 of Section 10.3 and $\delta = 2t \leq 2$. Let K_0 be the threshold of Section 11.5, and let $L_0 = \max\{K_0, 8\} s_{\rho_1}$. For $L \geq L_0$, we have $K \geq L/s_{\rho_L} \geq L/s_{\rho_1} \geq \max\{K_0, 8\}$, since $\rho_L \geq \rho_1$. We denote by Cov_{sc} and Var_{sc} the covariance and variance under the Bose path measure of N particles in $K\Omega$ with exclusion distance r at temperature T_K .

All directions in scaled units. Let $L \geq L_0$. By the comment after Proposition 11.9, the joint bound holds for every $\Theta \in \text{SO}(3)$, since $D \geq D_{\min}$, $0 < T_K \leq \tau_0 / \log D$, and $K \geq K_0$. Since the threshold K_0 does not depend on the direction, Proposition 11.9 gives $\text{Var}_{\text{sc}}(n \cdot \mathcal{A}) \geq c_{\Omega}^{\#} D^{1/4} \beta_K K^5$ for every unit vector n . Hence

$$\lambda_{\min}(\text{Cov}_{\text{sc}}(\mathcal{A})) = \inf_{|n|=1} \text{Var}_{\text{sc}}(n \cdot \mathcal{A}) \geq c_{\Omega}^{\#} D^{1/4} \beta_K K^5.$$

Return to physical units. By Lemma 11.10, $\text{Cov}_{\mu}(\mathcal{A}) = s^4 \text{Cov}_{\text{sc}}(\mathcal{A})$. Since $\beta = s^2 \beta_K$, $L = sK$, and $D \leq D_{\rho_L}$, we obtain

$$\lambda_{\min}(\text{Cov}_{\mu}(\mathcal{A})) \geq c_{\Omega}^{\#} D^{1/4} s^4 \beta_K K^5 = c_{\Omega}^{\#} D^{1/4} \frac{\beta L^5}{s^3} = c_{\Omega}^{\#} \frac{\rho_L}{D^{3/4}} \beta L^5 \geq \frac{c_{\Omega}^{\#}}{\alpha_0^{9/8}} (\rho_L a^3)^{3/8} \beta \rho_L L^5,$$

which is the first finite-volume bound.

The inertia fraction. Fix a unit vector n and $x_0 \in \mathbb{R}^3$, and consider the physical gas in $L\Omega$ with the axis through Lx_0 with direction n . The classical moment of inertia $I_{\text{cl}}(n, Lx_0)$ is the Gibbs expectation of the bounded symmetric function $f(X) = \sum_j |n \times (x_j - Lx_0)|^2 / 2$. By Lemma 3.6(3), this expectation is $Z^{-1} \int f(X) K_{\beta}^s(X, X) dX$, with $Z = \int K_{\beta}^s(X, X) dX > 0$. The function f vanishes only on the Lebesgue null set of configurations with all particles on the axis, which implies that $I_{\text{cl}}(n, Lx_0) > 0$. Moreover, $I_{\text{cl}}(n, Lx_0) \leq NL^2 R_{\Omega}(x_0)^2 / 2$, since $|x - Lx_0| \leq LR_{\Omega}(x_0)$ for $x \in L\Omega$. By (3.36),

$$1 - \frac{I(n, Lx_0)}{I_{\text{cl}}(n, Lx_0)} = \frac{T \text{Var}_{\mu}(\mathcal{A}_n)}{I_{\text{cl}}(n, Lx_0)} \geq \frac{2\lambda_{\min}(\text{Cov}_{\mu}(\mathcal{A}))}{\beta NL^2 R_{\Omega}(x_0)^2} = \frac{\lambda_{\min}(\text{Cov}_{\mu}(\mathcal{A}))}{\beta \rho_L L^5} \cdot \frac{2}{|\Omega| R_{\Omega}(x_0)^2},$$

and the first finite-volume bound gives the second.

The limits. Let $a, \rho > 0$ satisfy $\rho a^3 \leq c_0$, let $0 < T < T_*(a, \rho)$, and consider a sequence of N and L tending to infinity with $\rho_L \rightarrow \rho$. Since $T_*(a, \cdot)$ is continuous, we have $\rho_L \in [\rho/2, 2\rho]$ and $T \leq T_*(a, \rho_L)$ for all sufficiently large L in the sequence. We apply the finite-volume bounds with $\rho_1 = \rho/2$ and $\rho_2 = 2\rho$. The second gives (1.14), and the first, multiplied by $\rho_L / \rho \rightarrow 1$, gives (1.13). This completes the proof. $\square$

The argument proves the uniform form stated in Remark 1.7. Together with the upper bound from (11.2), the covariance lower bound gives the two-sided area estimates discussed after Theorem 1.2.

For a cube centered at the origin, symmetry reduces the proof to one direction.

Remark 11.11. Let $\Omega = (-1/2, 1/2)^3$ and $x_0 = 0$. Every signed permutation matrix Θ maps $L\Omega$ onto itself. The argument of Lemma 11.3 applies to every orthogonal matrix, with $\mathcal{A}(\Theta X(\cdot)) = \det(\Theta) \Theta \mathcal{A}(X(\cdot))$, because $\Theta a \times \Theta b = \det(\Theta) \Theta(a \times b)$. This implies that $\text{Cov}_\mu(\mathcal{A}) = \Theta \text{Cov}_\mu(\mathcal{A}) \Theta^\top$ for every signed permutation matrix Θ . As in the proof of Theorem 3.10, invariance under changes of sign of single coordinates forces the off-diagonal entries to vanish, and invariance under permutations of the coordinates makes the diagonal entries equal; thus $\text{Cov}_\mu(\mathcal{A}) = v \text{Id}$. In the same way, the unitary map $\psi \mapsto \psi(\Theta^{-1} \cdot)$ commutes with the Hamiltonian, and the Gibbs state is invariant under Θ , and the matrix $\langle \sum_j x_j x_j^\top \rangle$ is a multiple of the identity. By (3.11), $I_{\text{cl}}(0) = w \text{Id}$ for some $w > 0$, and by (3.36), $I(0) = (w - Tv) \text{Id}$. It follows that $I(n, 0)$, $I_{\text{cl}}(n, 0)$, and the fraction $1 - I(n, 0)/I_{\text{cl}}(n, 0)$ do not depend on n , at every finite volume, and $\lambda_{\min}(\text{Cov}_\mu(\mathcal{A})) = \text{Var}_\mu(\mathcal{A}_3)$. In this case the proof of Theorem 1.2 needs only the direction e_3 , and Lemma 11.3 is not used.

Remark 11.12. The constant c_1 depends on Ω through R_{in}^5 , and the fraction bound (1.14) also involves $|\Omega|$ and $R_\Omega(x_0)$. The thermal constants are independent of Ω ; the volume thresholds may depend on its shape. For the interpretation of the rotational response and its dependence on the walls, see Remark 1.9. Unlike for winding in Corollary 1.4, no higher-moment or distributional-limit bound for the normalized area is proved here.

Thermal estimates in a container

In this appendix, we prove the estimates on the hard-sphere Gibbs measure that Sections 9 and 11 require for a gas in a container with hard walls. The container is the dilation of a fixed nonempty bounded open set $\Omega \subset \mathbb{R}^3$. We assume no regularity of $\partial\Omega$, no connectivity, and nothing about the equilibrium density profile. The main result, Theorem A.1, is Proposition 8.10: the analog in the container of the joint bound for failure of the local conditions [112, Proposition 5.4] and of the supply of useful labels [112, Lemma 7.1], for cells at a fixed distance from the walls.

The proof follows the chain of estimates in [112, Sections 3–7], much of which is reproduced there from [111, 115]. The largest changes are a lower bound on the free energy depending only on the volume and an insertion orbital satisfying the wall condition: a smooth orbital supported in the container and equal, before normalization, to one on a set of volume at least half the container volume. We also adapt the close-neighbor estimate, the entropy bound, the shortage argument, and the marked-path trace comparison. Table A.1 states which proofs are reproduced, modified, or restricted to interior cells. Upper bounds on local particle numbers hold at every unit cell, including cells cut by the walls; lower bounds require cells whose closures lie in the container. Auxiliary marked particles may leave the physical container, as explained in Section A.12. The domain arguments in Section A.3 apply to arbitrary open sets.

A.1 Setting and statement

The scaled container. We use the scaled container and parameter ranges of Section 8.6, with

$$\begin{aligned} U = K\Omega, \quad V = |U|, \quad D = N/V, \quad r = \alpha/D, \quad \beta = T^{-1}, \\ \alpha \in [\alpha_0/2, 2\alpha_0], \quad 0 < T \leq \tau_0/\log D, \end{aligned} \tag{A.1}$$

where $\tau_0 = \kappa_T \alpha_0$ and the numerical constant $\kappa_T \in (0, 1/16]$ is fixed in Appendix B. We take D so large that $\tau_0/\log D \leq 1/4$; then $T \leq 1/4$ and $\beta \geq 4$. When comparing different particle numbers j , we hold U , r , T , and the reference density $D = N/V$ fixed; in particular, r is never replaced by $\alpha V/j$.

Throughout this appendix, $c, C > 0$ denote constants that may change from line to line. Unless stated otherwise, they depend only on α_0 and on the constants $\ell, \theta_0, b_0, C_{\text{reg}}$

fixed below, and never on Ω , K , N , D , α , or T in the ranges under consideration. A constant called numerical does not depend on α_0 either.

Hamiltonians. For an open set $G \subset \mathbb{R}^d$, we write $H_0^1(G)$ for the closure in $H^1(\mathbb{R}^d)$ of the smooth functions with compact support in G , and we identify its elements with their extensions by zero. We use only this definition, and never a description of $H_0^1(G)$ as the set of H^1 functions vanishing outside G , which fails for some irregular boundaries. For an open set $U' \subset \mathbb{R}^3$ and $j \geq 1$, let

$$\mathcal{G}_j(U') = \{X = (x_1, \dots, x_j) \in (U')^j : |x_i - x_k| > r \text{ for } i \neq k\}, \quad \mathcal{G}_j = \mathcal{G}_j(U).$$

Let H_j^{lab} be the operator on $L^2(\mathcal{G}_j)$ associated with the closed form $q_j[f] = \sum_{i=1}^j \int |\nabla_i f|^2$ on $H_0^1(\mathcal{G}_j)$, and let H_j be its restriction to the symmetric subspace. In the notation of Section 3.1, $\mathcal{G}_N$ is the configuration domain $\mathcal{X}_N(U)$ and $H = H_N$ is the container Hamiltonian, with the exclusion distance r . For a bounded measurable function $W \geq 0$ on $\mathbb{R}^3$, set

$$H_j^{(-W)} = H_j - \sum_{i=1}^j W(x_i), \quad Z_j^{(-W)} = \text{tr} e^{-\beta H_j^{(-W)}}, \quad F_j^{(-W)} = -T \log Z_j^{(-W)},$$

where the trace is over the symmetric subspace. We omit the superscript when $W = 0$, and set $Z_0 = 1$ and $F_0 = 0$; thus $Z_N = \text{tr} e^{-\beta H}$. A symmetric trace-one state $\sigma = \sum_k \lambda_k |f_k\rangle \langle f_k|$ on $L^2(\mathcal{G}_j)$ has *finite kinetic energy* if $f_k \in H_0^1(\mathcal{G}_j)$ whenever $\lambda_k > 0$ and $\text{tr} H_j \sigma = \sum_k \lambda_k q_j[f_k] < \infty$; energies are always form expectations. For such σ we write $S(\sigma) = -\text{tr} \sigma \log \sigma$ and

$$\mathcal{F}_j^{(-W)}(\sigma) = \text{tr} H_j \sigma - \text{tr} \left(\sum_{i=1}^j W(x_i) \right) \sigma - TS(\sigma).$$

The Gibbs variational principle [56, Lemma 3.1], applied to $\beta H_j^{(-W)}$ with form energies, gives $\mathcal{F}_j^{(-W)}(\sigma) \geq F_j^{(-W)}$, with equality for the Gibbs state $\Gamma_j^{(-W)} = e^{-\beta H_j^{(-W)}} / Z_j^{(-W)}$ when $Z_j^{(-W)} > 0$; this state has finite kinetic energy, because $\text{tr} e^{-s H_j^{(-W)}} < \infty$ for every $s > 0$ (Lemma A.3(5)). We write $\mathbb{E}_{j,-W}$ for expectation under the position law of $\Gamma_j^{(-W)}$, and

$$\mu(dX) = Z_N^{-1} K_\beta^s(X, X) dX$$

for the equilibrium position law of N particles (Lemma 3.6). For a bounded measurable function f on $\mathbb{R}^3$, let $n_f(X) = \sum_i f(x_i)$, and let $\mathbb{E}_\sigma$ denote expectation under the position law of a state σ .

Cells. The cells are the half-open unit cubes $B_v = v + [-1/2, 1/2)^3$, $v \in \mathbb{Z}^3$, and $n_v(X) = \#\{i : x_i \in B_v\}$. Since configurations lie in U , the count n_v refers to the points in $B_v \cap U$. This set may be empty or of arbitrarily small volume, and we never divide by its volume. For $A \subset \mathbb{Z}^3$ and $x \in \mathbb{R}^3$, let $d(x, A) = \min_{v \in A} |x - v|$, and let $|A|$ be the number of elements of A . Fix $R_0 > 12$. As in Section 8.6, a cell B_v is *interior* if $v \in U$ and $\text{dist}(v, \mathbb{R}^3 \setminus U) \geq R_0$. Then the closed ball $\bar{B}(v, 12)$ and the closure of B_v lie in U . Distinct interior cells are disjoint unit cubes in U . A set A of interior cells satisfies $|A| \leq V$.

Separation and the slab. Let $s_0 = b_0 D^{-1/3}$. For a configuration X of j points, let $d_i(X) = \min_{k \neq i} |x_i - x_k|$, with $d_i = +\infty$ if $j = 1$, and let $N_v^{\text{cl}}(X) = \#\{i : |x_i - v| \leq 5, d_i(X) \leq s_0\}$, as in (8.2). Fix $t \in (0, 1]$ and set $\delta = 2t < \beta$. Let ν_N be the measure of N independent Euclidean Brownian motions with generator Δ on $[0, \delta]$ with Lebesgue initial measure, write $X(s) = (\omega_1(s), \dots, \omega_N(s))$, $Y = X(0)$, and $Z^\circ = X(\delta)$, and let $\mathcal{E}_\delta$ be the event that $X(s) \in \mathcal{G}_N$ for all $s \in [0, \delta]$. On this event no two paths come within distance r and no path leaves U . The purified slab law is

$$\mathbb{P}^{\text{pur}}(d\omega, dW') = \frac{1}{Z_N} K_{\beta/2-t}^s(W', Y) K_{\beta/2-t}^s(Z^\circ, W') \mathbf{1}_{\mathcal{E}_\delta}(\omega) \nu_N(d\omega) dW', \quad (\text{A.2})$$

where K_s^s are the symmetrized killed kernels (3.13) of $\mathcal{G}_N$, which vanish outside $\mathcal{G}_N \times \mathcal{G}_N$. This is (8.3) for the container. By Lemma 8.2, every event of the path family ω has the same probability under $\mathbb{P}^{\text{pur}}$ as under the one-slab law $\mathbb{P}$ of (8.4) for U .

Regular paths, local conditions, and useful labels. We use Definitions 8.3 and 8.4 and Section 8.5 with Euclidean distances and diameters, including the definitions of eligible and useful labels and the unrestricted trial laws $P_{y,z}^u$. The mesh variables m_t and τ and the oscillations osc_k are those of Definition 8.3; set $h_0 = D^{-1}$. The mesh conditions are

$$\max_k \text{osc}_k(\omega) \leq D^{1/12}, \quad \frac{1}{2m_t} \sum_{k=1}^{2m_t} (1 + \text{osc}_k(\omega))^3 \leq C_{\text{reg}}, \quad (\text{A.3})$$

together with the two endpoint-window conditions in that definition. The local conditions quoted from [112, Definition 5.2 and Proposition 5.4] are those of Definition 8.4; their six failure tests are listed in Section A.14. We read the ball of radius 12 there as $B(v, 12)$; the proof also applies to $\bar{B}(v, 12)$.

For an eligible label i , usefulness means

$$|\{u \in B_w : P_i^u(C_i(F_i^{\text{all}})) \geq 0.9\}| \geq 0.99, \quad F_i^{\text{all}} = \{\omega_k : k \neq i\}, \quad (\text{A.4})$$

for each target cell B_w of v , with P_i^u and C_i as in Section 8.5. At an interior cell, $\bar{B}(v, 10) \subset U$. A trial in $C_i(F_i^{\text{all}})$ stays in U .

The shape threshold. By inner regularity of Lebesgue measure, Ω contains a compact set of volume greater than $|\Omega|/2$. We fix once and for all a smooth function g_Ω with compact support in Ω , with $0 \leq g_\Omega \leq 1$, and equal to one on such a compact set. Then

$$\int g_\Omega^2 \geq \frac{|\Omega|}{2}, \quad e_\Omega = \frac{\int |\nabla g_\Omega|^2}{\int g_\Omega^2}, \quad K_\Omega = \max\{1, |\Omega|^{-1/3}, e_\Omega^{1/2}\}. \quad (\text{A.5})$$

We prove Proposition 8.10 in the following form.

Theorem A.1. *With the parameters, ranges, and order of choices specified in Proposition 8.10, in particular for $0 < T \leq \tau_0 / \log D$, the following hold:*

(i) *for every finite set A of distinct interior cells,*

$$\mathbb{P}(\text{every cell of } A \text{ fails the local conditions}) \leq p_1^{|A|}; \quad (\text{A.6})$$

(ii) *at every interior cell v satisfying the local conditions, at most $H_1 D$ paths start in B_v , at least $\ell D/2$ labels are eligible at v , and at least nine tenths of these eligible labels are useful.*

The constants H_1 , t , and D_0 are independent of Ω , K , and T , and t may be decreased after H_1 is fixed and before D_0 is increased.

Used in: Proposition 8.10, and through it Section 11; the proof of part (ii) is used in Proposition 9.11.

The proof gives the following additional information.

- Remark A.2.** 1. There exist numerical constants $\alpha_0^* \geq 2$, $\ell^* > 0$, C_{reg}^* and, for each $\theta_0 > 0$, a numerical $b^*(\theta_0) \in (0, 1)$ such that the theorem holds whenever $\alpha_0 \geq \alpha_0^*$, $0 < \ell \leq \ell^*$, $0 < \theta_0 < \ell/6$, $0 < b_0 \leq b^*(\theta_0)$, and $C_{\text{reg}} \geq C_{\text{reg}}^*$. This is the form in which the theorem enters Section 8.1, where these constants are fixed once for the torus and the container.
2. One may take K_Ω as in (A.5). It depends only on Ω , not on R_0 , D , T , or p_1 . The constants H_1 , t , D_0 depend only on p_1 and the constants of part (1), and not on R_0 . No inradius, boundary area, or bound on the volume near ∂U enters. For a rotated container $\Theta\Omega$, $\Theta \in \text{SO}(3)$, we take $g_{\Theta\Omega} = g_\Omega \circ \Theta^{-1}$; then $K_{\Theta\Omega} = K_\Omega$. This is used in Section 11.
3. The threshold D_0 can be chosen so that, in addition, the scale restrictions (8.10) hold, $2s_0 < 1$, and $r < 1/32$.
4. The proof uses two kinds of local estimates, with different ranges of cells. Upper bounds on local particle numbers (Corollary A.13) hold for every cell B_v , $v \in \mathbb{Z}^3$, including cells that meet $\mathbb{R}^3 \setminus U$; they bound n_v , not $n_v/|B_v \cap U|$. Lower bounds (Proposition A.14) are proved only for cells whose closure lies in U . No lower bound holds for cells cut by the walls in general: if $B_v \cap U$ lies in a ball of radius

$r/2$, then $n_v \leq 1$. Applied to one interior cell, Proposition A.14 gives $\mathbb{E}_\mu n_v \geq \ell D(1 - e^{-cD})$; this lower bound on the mean occupation is a consequence of the free-energy comparisons, not an assumption on the density profile.

A.2 The torus estimates in the container

Table A.1 lists the statements of [112, Sections 3–7] used to prove [112, Proposition 5.4 and Lemma 7.1], together with their status in the container. A statement marked *same proof* follows from the cited argument after replacing periodic distances by Euclidean distances, using the container configuration domain $\mathcal{G}_j$, and omitting sums over periodic lifts. Auxiliary free paths, bridges, marked particles, and neighborhoods of cells lie in $\mathbb{R}^3$. A statement marked *modified proof* requires an additional argument, which is given below. The table also records restrictions to interior cells. Its statuses concern the change of domain. The standing temperature range of [112] is $0 < T \leq D^{-1}$. For the window in (A.1), we rely on the cited proofs for the unchanged domain-transfer steps, and each use of the temperature is one checked in this appendix or in Appendix B, where Table B.1 records the use of the temperature in each of these statements. The origins in [111, 115] are those given in [112]. The statements of [112, Sections 7.2–7.4, 8, and 9] are not used in this appendix.

Table A.1. Status in the container of the statements of [112] used in the proof of Theorem A.1.

Statement of [112]	Status in the container	Here
Lemma 3.1, cube energy (from [115, Lemma 3.1])	Same proof for the cube inequality. The cell inequality is applied to the zero extension on full cubes, summed over all cells, including cells cut by the walls.	Lemma A.5
Lemma 3.2, close-neighbor energy (from [115, Lemma 3.2])	Modified proof: translated Euclidean grid of side $8s$.	Lemma A.5
Lemma 3.3, entropy (from [111, Lemma 3.2])	Modified proof: open sets of finite volume in $\mathbb{R}^3$; the term V^{-1} disappears.	Lemma A.4
Lemma 3.4, normalized insertion (from [115, Lemma 3.3])	Same proof for orbitals with compact support in U .	Lemma A.8
Proposition 3.5, lower bound $F_N \geq c\alpha N$	Modified proof: new proof using only $ U $.	Lemma A.6, Corollary A.7

Statement of [112]	Status in the container	Here
Proposition 3.5, monotonicity of F_j/j (from [111, Proposition 3.3])	Same proof at fixed U, r, T, D .	Proposition A.9
Proposition 3.5, upper increments	Modified proof: orbital w_K and Fourier frame on a cube containing U .	Proposition A.9
Lemma 4.1, two-color localization (from [111, Lemma 4.1])	Same proof; first color on $U \cap O$, second color on U .	Lemma A.10
Lemma 4.2, diagonal tilt (from [111, Lemma 4.1])	Same proof.	Lemma A.11
Proposition 4.3, local penalty (from [115, Lemma 3.6])	Same proof with Euclidean distances, for every finite set of cells; temperature uses as in Proposition B.2.	Proposition A.12
Corollary 4.4, density in a small field	Same proof, for every finite set of cells; temperature uses as in Proposition B.2.	Corollary A.13
Proposition 4.5, low counts (method of [111, Section 4])	Restricted to cells with closure in U ; modified proof (filling inside U , $ A \leq V$); temperature uses as in Proposition B.2.	Proposition A.14
Proposition 4.6, close counts (method of [111, Section 4]; geometry from [115, Section 3.5])	Same proof with Euclidean distances, for every finite set of cells; temperature uses as in Proposition B.2.	Proposition A.15
Proposition 5.1, purification	Modified proof: kernels include wall killing; the occupation identity is not used.	Proposition A.16
Definition 5.2, regular path (from [115, Definition 4.3])	Same proof, with Euclidean diameters.	(A.3)
Lemma 5.3, free cutoffs (from [115, Lemma 4.4])	Same proof for Euclidean free laws.	Lemma A.17
Proposition 5.4, [BEC]'s "certificates"	Restricted to interior cells; extended to the window as in Proposition B.4, with the remainder r_D^{wall} of Section A.13.	Theorem A.1(i)
Lemma 5.5, marked kernel (adapting [111, Section 6])	Modified proof: Euclidean neighborhoods, no lifts; marked paths without wall killing.	Lemma A.18, Lemma A.19
Lemma 5.6, trace comparison (from [111, Lemma 6.2])	Same proof; hypotheses verified for the mixed color space.	Lemma A.21

Statement of [112]	Status in the container	Here
Proposition 5.7, trajectory moment (adapting [111, Proposition 6.1])	Modified proof: unmarked physical particles in U , all-cell density bound, no term V^{-1} .	Proposition A.22
Lemma 6.1, Gaussian density (from [115, Lemma 5.1])	Same proof.	Proposition A.23
Lemma 6.2, averaged collision (from [115, Proposition 5.2])	Same proof in $\mathbb{R}^3$; the paragraph on lifts is deleted.	Proposition A.23
Lemma 6.3, single replacement (from [115, Proposition 5.3])	Same proof in $\mathbb{R}^3$.	Proposition A.23
Remark 6.4, midpoints (from [115, Remark 5.4])	Same proof.	Proposition A.23
Corollary 6.5, successful labels (from [115, Corollary 5.5])	Same proof in $\mathbb{R}^3$.	Proposition A.23
Lemma 7.1, useful labels (from [115, Lemma 6.1])	Restricted to interior cells; same proof.	Theorem A.1(ii)

No auxiliary torus is used. In [112], several neighborhoods of sets of cells are allowed to be the whole torus; in $\mathbb{R}^3$ they are bounded, and this case does not occur. The only place where periodic functions enter is Parseval's identity on a cube containing U , in the proof of Proposition A.9.

A.3 Open-domain preliminaries

The following facts hold for arbitrary open sets. They are used without comment in the rest of the appendix.

Lemma A.3. *Let $G \subset \mathbb{R}^d$ be open.*

1. *If $f \in H_0^1(G)$, then $f = 0$ and $\nabla f = 0$ almost everywhere on $\mathbb{R}^d \setminus G$.*
2. *Let $m: \mathbb{R}^d \rightarrow \mathbb{C}$ be bounded and Lipschitz, and let $G' \subset G$ be open with $\text{supp } m \cap G \subset G'$. Then $f \mapsto mf$ is bounded from $H_0^1(G)$ to $H_0^1(G')$, and $\nabla(mf) = m\nabla f + f\nabla m$ almost everywhere. If $G_1 \subset \mathbb{R}^{d_1}$ and $G_2 \subset \mathbb{R}^{d_2}$ are open, $f_1 \in H_0^1(G_1)$, and $f_2 \in H_0^1(G_2)$, then $f_1 \otimes f_2 \in H_0^1(G_1 \times G_2)$.*
3. *Let $G_1 \subset \mathbb{R}^{d_1}$ and $G_2 \subset \mathbb{R}^{d_2}$ be open with $G \subset G_1 \times G_2$, and let ∇_1 be the gradient in the first d_1 variables. Let $\rho = \sum_k \lambda_k |f_k\rangle\langle f_k|$ be a positive trace-class operator on $L^2(G)$ with $\lambda_k \geq 0$, $f_k \in H_0^1(G)$, and $\sum_k \lambda_k \|\nabla_1 f_k\|^2 < \infty$. Then the partial trace ρ_1 of ρ over the*

last d_2 variables is a positive trace-class operator on $L^2(G_1)$, its eigenvectors with nonzero eigenvalues lie in $H_0^1(G_1)$, and its kinetic energy $\sum_l \mu_l \|\nabla h_l\|^2$, computed in an eigenbasis $\rho_1 = \sum_l \mu_l |h_l\rangle \langle h_l|$, equals $\sum_k \lambda_k \|\nabla_1 f_k\|^2$.

4. Let W be a bounded measurable real function on G , and let h be the operator of the closed form $\int |\nabla f|^2 - \int W|f|^2$ on $H_0^1(G)$. For $s > 0$, the operator e^{-sh} has the nonnegative kernel

$$\int \mathbf{1}\{\gamma(\tau) \in G \text{ for all } \tau \in [0, s]\} \exp\left(\int_0^s W(\gamma(\tau)) d\tau\right) \mathbb{W}_{x,y}^s(d\gamma),$$

where $\mathbb{W}_{x,y}^s$ is the Brownian bridge measure of Section 3.2, on $\mathbb{R}^d$, of total mass $(4\pi s)^{-d/2} e^{-|y-x|^2/(4s)}$.

5. If $d = 3$ and $|G| < \infty$, then $\text{tr } e^{s\Delta_G} \leq |G|(4\pi s)^{-3/2}$ for $s > 0$, where Δ_G is the Dirichlet Laplacian of G . If G is bounded, the operator h of part (4) has compact resolvent and $\text{tr } e^{-sh} < \infty$ for $s > 0$.

Proof. For part (1), approximants vanish outside G together with their gradients, and both properties pass to limits in L^2 . For part (2), let $f \in C_c^\infty(G)$. Then mf is Lipschitz, and its support lies in the compact set $\text{supp } f \cap \text{supp } m$, which is contained in $G \cap \text{supp } m \subset G'$. Mollification shows that Lipschitz functions with compact support in G' belong to $H_0^1(G')$. The product rule for a Lipschitz function and a Sobolev function gives $\|mf\|_{H^1} \leq C_m \|f\|_{H^1}$, and since $H_0^1(G')$ is closed, approximation proves the first assertion for every $f \in H_0^1(G)$. Products of approximants of f_1 and f_2 are smooth, have compact support in $G_1 \times G_2$, and converge to $f_1 \otimes f_2$ in H^1 .

Part (3) is the domain argument of [112, Section 3.2]. Let (e_m) be an orthonormal basis of $L^2(G_2)$, and for $f \in H_0^1(G)$ set $f_m(x) = \int f(x, y) \overline{e_m(y)} dy$. If $f \in C_c^\infty(G)$, then f_m is smooth with support in the projection of $\text{supp } f$, which is a compact subset of G_1 , and Parseval's identity in y gives $\sum_m \|f_m\|^2 = \|f\|^2$ and $\sum_m \|\nabla f_m\|^2 = \|\nabla_1 f\|^2$. Both identities extend to $f \in H_0^1(G)$ by approximation, and $f_m \in H_0^1(G_1)$ because $H_0^1(G_1)$ is closed. Then $\rho_1 = \sum_{k,m} \lambda_k |f_{k,m}\rangle \langle f_{k,m}|$ with $\sum_{k,m} \lambda_k \|f_{k,m}\|_{H^1}^2 < \infty$. If $\rho_1 h = \mu h$ with $\mu > 0$ and $\|h\| = 1$, then $h = \mu^{-1} \sum_{k,m} \lambda_k \langle f_{k,m}, h \rangle f_{k,m}$. By the Cauchy–Schwarz inequality, $\sum_{k,m} \lambda_k |\langle f_{k,m}, h \rangle| \|f_{k,m}\|_{H^1} < \infty$. This series converges in H^1 , and $h \in H_0^1(G_1)$. With A the Dirichlet Laplacian of G_1 with the sign reversed, both expressions for the kinetic energy equal $\text{tr}(A^{1/2} \rho_1 A^{1/2})$.

For part (4), the case $W = 0$ with G a configuration domain is Proposition 3.5(1). Its proof exhausts G by relatively compact open subsets with smooth boundaries, applies the Feynman–Kac formula with killing [32, Proposition 2.9] on each of them, and passes to the limit by strong convergence of the semigroups and monotone convergence of the killed path integrals. The only property of G that it uses is that a continuous path which stays in G on $[0, s]$ has compact image in G , and therefore stays in one of the exhausting subsets. The argument applies to every open set G and every bounded W ; it is the open-domain argument of [112, Section 2.1] and [115, Section 4]. Kernel identities

hold almost everywhere, and traces are evaluated after a split at a positive time, as in Lemma 3.6(3).

For part (5), let p_s^G be the kernel of $e^{s\Delta_G}$, and p_s the free heat kernel. By part (4), $0 \leq p_s^G(x, y) \leq p_s(y - x)$ almost everywhere. Since $e^{s\Delta_G} = (e^{s\Delta_G/2})^2$,

$$\mathrm{tr} e^{s\Delta_G} = \iint p_{s/2}^G(x, y)^2 dx dy \leq \int_G \int_{\mathbb{R}^3} p_{s/2}(y - x)^2 dy dx = |G| p_s(0) = |G| (4\pi s)^{-3/2}.$$

If G is bounded, extension by zero maps $H_0^1(G)$ isometrically into $H_0^1(Q)$ for an open cube $Q \supset G$, and the min-max principle compares the eigenvalues of $h + \|W\|_\infty$ with those of the Dirichlet Laplacian on Q , as in the proof of Lemma 3.1. $\square$

The configuration domains $\mathcal{G}_j$ are bounded open sets. By Lemma A.3(5), $H_j^{(-W)}$ has compact resolvent and $Z_j^{(-W)} < \infty$ for every j and every bounded measurable $W \geq 0$. That $Z_j^{(-W)} > 0$ for $j \leq 2N$ is proved in Proposition A.9.

Lemma A.4. *Let $O \subset \mathbb{R}^3$ be open with $|O| < \infty$, let $\varsigma \in (0, 1]$, and let $j \geq 0$. For every symmetric trace-one state σ on $L^2(O^j)$ of finite kinetic energy,*

$$\varsigma \mathrm{tr} \left(\sum_{i=1}^j -\Delta_{O,i} \right) \sigma - TS(\sigma) \geq -Tj - C_\varsigma T^{5/2} |O|, \quad (\text{A.7})$$

where $\Delta_{O,i}$ is the Dirichlet Laplacian of O in the i th coordinate. In particular $S(\sigma) < \infty$. The same bound holds for symmetric states of finite kinetic energy on $L^2(\mathcal{G}_j(O'))$ for every open set $O' \subset O$.

Proof. This is [112, Lemma 3.3], the ideal Bose entropy bound of [111, Lemma 3.2], with the periodic heat-kernel bound replaced by Lemma A.3(5). Let $h = -\varsigma \Delta_O$. For $0 < z < 1$, define the ideal grand partition function by $\Xi(z) = \sum_{j \geq 0} z^j \mathrm{tr}_{\mathrm{sym}} e^{-\beta \sum_{i=1}^j h_i}$. We use $z = e^{-1}$, for which

$$\log \Xi(e^{-1}) = \sum_{m \geq 1} \frac{e^{-m}}{m} \mathrm{tr} e^{m\beta\varsigma\Delta_O} \leq |O| (4\pi\beta\varsigma)^{-3/2} \sum_{m \geq 1} e^{-m} m^{-5/2} \leq C_\varsigma |O| T^{3/2}.$$

Its j -particle term is e^{-j} times the canonical partition function, and the Gibbs variational principle gives (A.7) for states of finite rank. The general case, and the finiteness of $S(\sigma)$, follow as in [112] from the nonnegativity of the relative entropy with respect to the ideal canonical Gibbs state. For the last statement, extension by zero maps $H_0^1(\mathcal{G}_j(O'))$ into $H_0^1(O^j)$ and preserves the kinetic form. $\square$

A.4 Local kinetic estimates on full cubes

For a set I of labels, we write $q_I[F] = \sum_{i \in I} \int |\nabla_i F|^2$, let $n_Q^I(X)$ be the number of $i \in I$ with $x_i \in Q$, and let $d_i^I(X) = \min_{k \in I \setminus \{i\}} |x_i - x_k|$, with $d_i^I = +\infty$ if $I = \{i\}$. For a function F of X , $\mathbb{E}_F$ denotes integration against $|F|^2$.

Lemma A.5. *Let $U' \subset \mathbb{R}^3$ be open, $F \in H_0^1(\mathcal{G}_j(U'))$, and $I \subset \{1, \dots, j\}$.*

1. *For every $l > 0$ and every partition of $\mathbb{R}^3$ into the cubes Q of a grid $z + l\mathbb{Z}^3 + [0, l)^3$, $z \in \mathbb{R}^3$,*

$$q_I[F] \geq c_{\text{cube}} r l^{-3} \mathbb{E}_F \sum_Q n_Q^I (n_Q^I - 1), \quad c_{\text{cube}} = \frac{1}{64\sqrt{3}}.$$

In particular $q_I[F] \geq c_{\text{cube}} r \mathbb{E}_F \sum_{v \in \mathbb{Z}^3} n_v^I (n_v^I - 1)$.

2. *If $0 < r < s < 1$, then*

$$q_I[F] \geq c_{\text{near}} r s^{-3} \mathbb{E}_F \# \{i \in I : d_i^I \leq 2s\}, \quad c_{\text{near}} = \frac{27}{262144\sqrt{3}}.$$

The sums run over all cubes, including those that meet $\mathbb{R}^3 \setminus U'$, and n_Q^I counts the points in $Q \cap U'$.

Proof. We extend F by zero; it vanishes almost everywhere where some pair has distance at most r (Lemma A.3(1)). The first assertion of [112, Lemma 3.1] states that, for a cube Q of side l and a function in $H^1(Q^n)$ that vanishes almost everywhere where some pair has distance at most r , with no condition on the faces of Q , the kinetic energy is at least $c_{\text{cube}} r l^{-3} n(n-1)$ times the squared norm. Its proof applies the ray inequality

$$\int_Q |f|^2 \leq \frac{\sqrt{3}l^3}{r} \int_Q |\nabla f|^2 \quad \text{for } f \in H^1(Q) \text{ vanishing on } \{x \in Q : |x - y| \leq r\}, y \in Q, \quad (\text{A.8})$$

on the full cube Q , followed by a subdivision of Q . Partition configuration space according to the cube that contains each coordinate in I . In a sector in which n_Q of these coordinates lie in Q , fix all other coordinates; for almost every choice, the restriction of F to the product of the cubes Q^{n_Q} lies in H^1 of that product and vanishes where two coordinates of I in the same cube are within distance r . The cube inequality in each cube with $n_Q \geq 2$, integrated over the fixed coordinates and summed over sectors, gives part (1). Restriction to sectors is Neumann bracketing: no derivative of a sector indicator appears. The set $Q \cap U'$ plays no role; it may be nonconvex, disconnected, or of small volume.

For part (2), let $\ell_s = 8s$ and, for $z \in [0, \ell_s)^3$, consider the grid $z + \ell_s \mathbb{Z}^3 + [0, \ell_s)^3$. In each sector of this grid and for each $i \in I$ that has another coordinate of I in its cube, choose such a partner by a fixed order of the labels, and apply (A.8) in the coordinate x_i with the partner and all other coordinates fixed. Each coordinate is used once. Then

$q_I[F] \geq (\sqrt{3}\ell_s^3)^{-1}r \mathbb{E}_F \#\{i \in I : i \text{ has a partner in its cube}\}$. Now average over z uniform in $[0, \ell_s)^3$. For fixed X and $i \in I$ with $d_i^l \leq 2s$, let k be a nearest partner. The two points lie in one cube with probability $\prod_{m=1}^3 (1 - |x_{i,m} - x_{k,m}|/\ell_s)_+ \geq (3/4)^3 = 27/64$. Since $\ell_s^3 = 512s^3$, this proves part (2) with the constant $27/(32768\sqrt{3}) \geq c_{\text{near}}$. This is the translated-grid argument of [112, Lemma 3.2], with a Euclidean grid in place of a periodic one. $\square$

A.5 A lower bound on the free energy

The proof of the bound $F_N \geq c\alpha N$ in [112, Proposition 3.5] uses the inequality $\sum_v n_v(n_v - 1) \geq N(D - 1)$, which on the torus follows from the Cauchy–Schwarz inequality over the V cells. In a container, the analogous inequality would involve the number of cells that meet U , which need not be comparable to V . For example, let $C_1 \subset [0, 1]$ be a closed nowhere dense set of measure close to one, and let $\Omega = (0, 1)^3 \setminus C_1^3$. Then Ω is open and dense in $(0, 1)^3$, its volume is small, and every cell in $K(0, 1)^3$ meets $K\Omega$. The next lemma uses only the volume of U . We state it for an arbitrary bounded open set.

Lemma A.6. *Let $U \subset \mathbb{R}^3$ be a bounded open set, $N \geq 1$, $D = N/|U|$, and $r = \alpha/D$ with $\alpha \geq 1$. For every $F \in H_0^1(\mathcal{G}_N)$,*

$$q_N[F] \geq \left(\frac{3c_{\text{cube}}}{16} \alpha N - \frac{c_{\text{cube}}}{2} \frac{\alpha^{5/2}}{D} N \right) \|F\|^2.$$

Proof. We may assume $\|F\| = 1$. Let $l = \alpha^{-1/2}$, and partition $\mathbb{R}^3$ into the cubes Q of the grid $l\mathbb{Z}^3 + [0, l)^3$. We call Q *dense* if $|Q \cap U| > l^3/2$ and *sparse* otherwise. The dense cubes are disjoint, and each contains a part of U of volume greater than $l^3/2$. Their number M_d satisfies $M_d \leq 2V/l^3$, where $V = |U|$. For a configuration X , let $m(X)$ be the number of its points in dense cubes.

We first bound the kinetic energy of the points in sparse cubes. Let Q be a cube of side l , and let $g \in H^1(Q)$ vanish almost everywhere on a measurable set $E \subset Q$ with $|E| \geq |Q|/2$. Let g_Q be the mean of g over Q . Then $|E||g_Q|^2 = \int_E |g - g_Q|^2 \leq \int_Q |g - g_Q|^2$, and by the Poincaré inequality for the Neumann Laplacian of Q , whose smallest nonzero eigenvalue is π^2/l^2 , we have

$$\int_Q |g|^2 = \int_Q |g - g_Q|^2 + |Q||g_Q|^2 \leq 3 \int_Q |g - g_Q|^2 \leq \frac{3l^2}{\pi^2} \int_Q |\nabla g|^2.$$

For almost every choice of the coordinates other than x_i , the function $x_i \mapsto F(X)$ lies in $H^1(\mathbb{R}^3)$, its gradient is $\nabla_i F$, and it vanishes almost everywhere outside U by Lemma A.3(1). In a sparse cube Q , the set $Q \setminus U$ has volume at least $l^3/2$. We apply the

last display in every sparse cube, and sum over the cubes and over i . Integrating over the other coordinates yields

$$q_N[F] \geq \frac{\pi^2}{3l^2} \mathbb{E}_F(N - m) = c_P \alpha \mathbb{E}_F(N - m), \quad c_P = \frac{\pi^2}{3}. \quad (\text{A.9})$$

If $M_d = 0$, then $m = 0$, and (A.9) implies the lemma. Suppose that $M_d > 0$. By Lemma A.5(1) with the grid of side l , keeping only the dense cubes, and by the Cauchy–Schwarz inequality $\sum_{Q \text{ dense}} n_Q^2 \geq m^2 / M_d$, we have

$$q_N[F] \geq c_{\text{cube}} r l^{-3} \mathbb{E}_F \sum_{Q \text{ dense}} n_Q (n_Q - 1) \geq c_{\text{cube}} r l^{-3} \mathbb{E}_F \left(\frac{l^3 m^2}{2V} - m \right). \quad (\text{A.10})$$

We average (A.9) and (A.10), so that each kinetic energy is used once, and use $m \leq N$. This gives

$$q_N[F] \geq \mathbb{E}_F \left[\frac{c_{\text{cube}} r}{4V} m^2 + \frac{c_P \alpha}{2} (N - m) \right] - \frac{c_{\text{cube}} r}{2l^3} N.$$

Since $r/V = \alpha/N$ and $c_P/2 \geq c_{\text{cube}}/4$, the expectation is at least $(c_{\text{cube}}/4)\alpha N \mathbb{E}_F[(m/N)^2 + 1 - m/N] \geq (3c_{\text{cube}}/16)\alpha N$, where we used $x^2 + 1 - x \geq 3/4$. The last term equals $(c_{\text{cube}}/2)\alpha^{5/2}N/D$, because $rl^{-3} = \alpha^{5/2}/D$. $\square$

Corollary A.7. *There exists D_1 , depending only on α_0 , such that the following holds for every bounded open set U , every N with $D = N/|U| \geq D_1$, every $\alpha \in [\alpha_0/2, 2\alpha_0]$, and every $0 < T \leq \tau_0/\log D$. Every symmetric trace-one state σ on $L^2(\mathcal{G}_N)$ of finite kinetic energy satisfies $\mathcal{F}_N(\sigma) \geq c_* \alpha N$, with $c_* = c_{\text{cube}}/16$. In particular $F_N \geq c_* \alpha N$.*

Proof. By Lemma A.6 and the spectral decomposition of σ , $\text{tr } H_N \sigma \geq (3c_{\text{cube}}/16)\alpha N - (c_{\text{cube}}/2)\alpha^{5/2}N/D$. We use half of the kinetic energy for this bound and the other half for Lemma A.4 with $O = U$ and $\zeta = 1/2$; the latter is applicable by its last statement. We obtain

$$\begin{aligned} \mathcal{F}_N(\sigma) &\geq \frac{3c_{\text{cube}}}{32} \alpha N - \frac{c_{\text{cube}}}{4} \frac{\alpha^{5/2}}{D} N - TN - CT^{5/2}V \\ &\geq N \left(\frac{3c_{\text{cube}}}{32} \alpha - \frac{c_{\text{cube}}}{4} \frac{\alpha^{5/2}}{D} - \frac{\tau_0}{\log D} - \frac{C\tau_0^{5/2}}{D(\log D)^{5/2}} \right), \end{aligned}$$

where in the second inequality we used $T \leq \tau_0/\log D$ and $V = N/D$. Since $\alpha \geq 1$, $\alpha \leq 2\alpha_0$, and $\tau_0 = \kappa_T \alpha_0$, the bracket is at least $c_{\text{cube}}\alpha/16$ once D is large depending on α_0 . If $Z_N > 0$, the Gibbs variational principle gives $F_N = \mathcal{F}_N(\Gamma_N) \geq c_* \alpha N$; if $Z_N = 0$, then $F_N = +\infty$. $\square$

A.6 Insertion, the Fourier frame, and particle-number comparisons

Lemma A.8. *Let w be a real smooth function with compact support in U and $\|w\|_2 = 1$. Let $n \geq 0$, let $F \in H_0^1(\mathcal{G}_n)$, and suppose that on the support of F at most k old positions lie within distance $2r$ of $\text{supp } w$. If $C\|w\|_\infty^2 k r^3 \leq 1/4$, there exists $\tilde{F} \in H_0^1(\mathcal{G}_{n+1})$ of the form $\tilde{F}(X, y) = F(X)\varphi_X(y)$, where φ_X is real, $|\varphi_X| \leq 2|w|$, φ_X is a symmetric function of X , and, for X in the support of F ,*

$$\begin{aligned} \int \varphi_X^2 dy &= 1, & \int \varphi_X \nabla_{x_i} \varphi_X dy &= 0, \\ \int \left(|\nabla_y \varphi_X|^2 + \sum_{i=1}^n |\nabla_{x_i} \varphi_X|^2 \right) dy &\leq C(\|\nabla w\|_2^2 + \|w\|_\infty^2 k r). \end{aligned} \tag{A.11}$$

In particular $\|\tilde{F}\| = \|F\|$, $\tilde{F}$ has the same old-coordinate position marginal as F , and

$$q_{n+1}[\tilde{F}] \leq q_n[F] + C(\|\nabla w\|_2^2 + \|w\|_\infty^2 k r) \|F\|^2.$$

Proof. This is [112, Lemma 3.4], the configurationwise normalized insertion of [115, Lemma 3.3], which refines the nearest-neighbor trial construction of [102, Section 3.2 in arXiv version 1]. Its proof applies with Euclidean distances, and we recall the construction because it is used below. Let $h_r: [0, \infty) \rightarrow [0, 1]$ be nondecreasing, equal to zero on $[0, 1.1r]$ and to one on $[2r, \infty)$, with Lipschitz constant at most C/r . For $X \in (\mathbb{R}^3)^n$ and $y \in \mathbb{R}^3$, let

$$f_X(y) = w(y) \min_{i \leq n} h_r(|y - x_i|), \quad m(X) = \int f_X^2 dy, \quad \varphi_X = \frac{f_X}{\max\{m(X)^{1/2}, 1/2\}},$$

with the empty minimum equal to one. On the support of F , the hypothesis implies that $1 - m(X) \leq \|w\|_\infty^2 k |B(0, 2r)| \leq 1/4$. The proof of (A.11) and of the energy bound in [112] uses only these definitions and Euclidean geometry, and applies after the stated substitutions.

It remains to verify that $\tilde{F} \in H_0^1(\mathcal{G}_{n+1})$; this is where the walls enter. The function $(X, y) \mapsto \varphi_X(y)$ is bounded and Lipschitz, and $\int |\nabla_X \varphi_X(y)|^2 dy$ is bounded uniformly in X . The map $F \mapsto F\varphi$ is bounded from $H^1(\mathbb{R}^{3n})$ to $H^1(\mathbb{R}^{3n+3})$. If $F' \in C_c^\infty(\mathcal{G}_n)$, then $F'\varphi$ is Lipschitz and supported in $\{(X, y) : X \in \text{supp } F', y \in \text{supp } w, |y - x_i| \geq 1.1r \text{ for all } i\}$. This set is compact and contained in $\mathcal{G}_{n+1}$: on $\text{supp } F'$ the mutual distances of the old points are bounded below by a number larger than r , the new point lies in the compact set $\text{supp } w \subset U$, and it has distance at least $1.1r > r$ from the old points. We have $F'\varphi \in H_0^1(\mathcal{G}_{n+1})$, and approximation of F proves the claim. The Dirichlet condition at the walls for the new coordinate comes from the compact support of w in U , and for the old coordinates from F . $\square$

We now fix the insertion orbital. With g_Ω and e_Ω as in (A.5), let

$$w_K(x) = \frac{K^{-3/2}g_\Omega(x/K)}{\|g_\Omega\|_2}. \quad (\text{A.12})$$

Then w_K is real and smooth with compact support in U , $\|w_K\|_2 = 1$, $\|w_K\|_\infty^2 \leq K^{-3}(|\Omega|/2)^{-1} = 2/V$, and $\|\nabla w_K\|_2^2 = e_\Omega K^{-2} \leq 1$ for $K \geq K_\Omega$. This is the only place where a threshold depending on the shape of Ω enters. An orbital supported in a single ball inside U would have a supremum controlled by the volume of that ball rather than by V ; and a constant orbital, as in [112], does not satisfy the Dirichlet condition at the walls.

Proposition A.9. *There exist a numerical constant $c_* > 0$ and a threshold D_1 , depending only on α_0 , such that the following holds whenever $K \geq K_\Omega$, $D \geq D_1$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, and $0 < T \leq \tau_0/\log D$.*

1. *For $0 \leq j \leq 2N$, the space $H_0^1(\mathcal{G}_j) \cap L_{\text{sym}}^2(\mathcal{G}_j)$ is nonzero. Hence $0 < Z_j^{(-W)} < \infty$ for every bounded measurable $W \geq 0$.*
2. $F_N \geq c_* \alpha N$.
3. $F_i/i \leq F_j/j$ for $1 \leq i \leq j \leq 2N$, and $F_{N+k} \geq F_N + c_* \alpha k$ for $0 \leq k \leq N$.
4. $F_j^{(-W)} - F_{j-1}^{(-W)} \leq C_{\text{inc}} \alpha$ for $1 \leq j \leq 2N$ and every bounded measurable $W \geq 0$ on $\mathbb{R}^3$, with the numerical constant C_{inc} of Appendix B.

All quantities are computed in the same container U , at the same exclusion distance $r = \alpha/D$ and temperature T , with $D = N/V$.

Used in: Propositions A.12, A.14, and A.22, including the energy-shift choice in Section A.12.

Proof. For $j \leq 2N$ and the orbital w_K , the smallness condition of Lemma A.8 holds with $k \leq 2N$, because $C\|w_K\|_\infty^2(2N)r^3 \leq 4CNr^3/V = 4C\alpha^3/D^2 \leq 1/4$ for D large. By (A.11), we have, uniformly in X ,

$$|\varphi_X(y)|^2 \leq \frac{8}{V}, \quad \int \left(|\nabla_y \varphi_X|^2 + \sum_i |\nabla_{x_i} \varphi_X|^2 \right) dy \leq C(1 + 4\alpha) \leq C'\alpha, \quad (\text{A.13})$$

since $\|\nabla w_K\|^2 \leq 1$ and $\|w_K\|_\infty^2(2N)r \leq 4Nr/V = 4\alpha$.

For part (1), we apply Lemma A.8 with $w = w_K$ successively, starting from the zero-particle vector 1. We obtain a nonzero labeled vector in $H_0^1(\mathcal{G}_j)$ for every $j \leq 2N$. Let Ψ be such a vector. The function $((j!)^{-1} \sum_\pi |\Psi \circ \pi|^2)^{1/2}$ is nonzero and symmetric. It belongs to $H_0^1(\mathcal{G}_j)$: for approximants $\Psi_m \in C_c^\infty(\mathcal{G}_j)$ of Ψ , the corresponding functions are Lipschitz with compact support in $\mathcal{G}_j$, they converge in L^2 , and their gradients are bounded in L^2 by the gradient inequality for the Euclidean norm of a finite vector of

functions; hence a subsequence converges weakly in H^1 , and $H_0^1(\mathcal{G}_j)$ is weakly closed. Part (2) is Corollary A.7.

For part (3), we follow [112, Proposition 3.5], which takes the argument from [111, Proposition 3.3]. Let $1 \leq i < j \leq 2N$, let $\sigma = \Gamma_j$ be the j -particle Gibbs state, and for $0 \leq m \leq j$ let σ_m be its trace-one m -particle marginal and $S_m = S(\sigma_m)$, with $S_0 = 0$. Since $\mathcal{G}_j \subset \mathcal{G}_m \times U^{j-m}$, Lemma A.3(3) shows that σ_m is a symmetric state of finite kinetic energy on $L^2(\mathcal{G}_m)$ with $\text{tr } H_m \sigma_m = (m/j) \text{tr } H_j \sigma$, by symmetry; its entropy is finite by Lemma A.4. Strong subadditivity of the entropy [98, Theorem 2(i) and Section 4], applied to σ_{m+1} with the labels split into $\{1\}$, $\{2, \dots, m\}$, and $\{m+1\}$, and the symmetry of σ give $S_{m+1} + S_{m-1} \leq 2S_m$ for $1 \leq m < j$; for $m = 1$ this is ordinary subadditivity. Then $m \mapsto S_m$ is concave with $S_0 = 0$, and S_m/m is nonincreasing. We obtain

$$F_i \leq \mathcal{F}_i(\sigma_i) = \frac{i}{j} \text{tr } H_j \sigma - TS_i \leq \frac{i}{j} (\text{tr } H_j \sigma - TS_j) = \frac{i}{j} F_j.$$

With $i = N$ and $j = N + k$, part (2) gives $F_{N+k} \geq (1 + k/N)F_N \geq F_N + c_* \alpha k$. This argument uses neither translation invariance nor a uniform one-body density. It does use that U , r , and T are the same for all particle numbers.

For part (4), we adapt the insertion-frame argument in the proof of [112, Proposition 3.5, especially Equation (3.12)], using w_K and Fourier modes on a cube containing U . To bound the partition function from below after adding a particle, we distinguish the new particle and insert it in many low-frequency states. A Fourier frame bound controls the overlap between these states, and positivity of the kernel then returns to the symmetric partition function.

Fix $1 \leq j \leq 2N$ and W . Let $\mathcal{H}_{\text{dist}}$ be the space of functions in $L^2(\mathcal{G}_j)$ that are symmetric in the first $j-1$ coordinates, write (X, y) with $X \in (\mathbb{R}^3)^{j-1}$ and $y \in \mathbb{R}^3$, and let H_{dist} be the restriction of $H_j^{\text{lab}} - \sum_i W(x_i) - W(y)$ to $\mathcal{H}_{\text{dist}}$ and $Z_{\text{dist}} = \text{tr } e^{-\beta H_{\text{dist}}}$. Fix an integer $\hat{K} \geq 100$ and a cube Q_U of side $\hat{K}$ containing U , and let $\hat{V} = \hat{K}^3$. For $\mathbf{k} \in (2\pi/\hat{K})\mathbb{Z}^3$ and a symmetric $(j-1)$ -particle form vector F , let

$$v_{\mathbf{k}} F(X, y) = F(X) \varphi_X(y) e^{i\mathbf{k} \cdot y},$$

with φ_X the normalized orbital of Lemma A.8 for $w = w_K$. The proof of Lemma A.8, with the bounded Lipschitz factor $e^{i\mathbf{k} \cdot y}$ included, shows that $v_{\mathbf{k}} F$ is a form vector in $\mathcal{H}_{\text{dist}}$, and $v_{\mathbf{k}}$ is an isometry. The Fourier phase need not satisfy any boundary condition, since its product with φ_X does. Since φ_X is real, the kinetic energy added by the new coordinate is $\int (|\nabla_y \varphi_X|^2 + |\mathbf{k}|^2 \varphi_X^2) dy$, and the cross terms in the old coordinates vanish by (A.11). The old field terms are unchanged, and the new field term is nonpositive. By (A.13),

$$\langle v_{\mathbf{k}} F, H_{\text{dist}} v_{\mathbf{k}} F \rangle \leq \langle F, H_{j-1}^{(-W)} F \rangle + (C' \alpha + |\mathbf{k}|^2) \|F\|^2.$$

Next we prove a frame bound. For $u \in \mathcal{H}_{\text{dist}}$ and fixed X , extend $y \mapsto \varphi_X(y)u(X, y)$ by zero from U to Q_U . Parseval's identity in $L^2(Q_U)$, with the orthonormal basis $\widehat{V}^{-1/2}e^{ik \cdot y}$, and (A.13) give

$$\sum_{\mathbf{k}} \left| \int_U \varphi_X(y) e^{-ik \cdot y} u(X, y) dy \right|^2 = \widehat{V} \int_U \varphi_X(y)^2 |u(X, y)|^2 dy \leq \frac{8\widehat{V}}{V} \int_U |u(X, y)|^2 dy. \quad (\text{A.14})$$

The integral on the left is $(v_{\mathbf{k}}^* u)(X)$; it is symmetric in X , because φ_X and u are. Integrating in X and restricting the sum, we obtain $\sum_{|\mathbf{k}| \leq 1} v_{\mathbf{k}} v_{\mathbf{k}}^* \leq (8\widehat{V}/V)I$ on $\mathcal{H}_{\text{dist}}$. The number of $\mathbf{k}$ with $|\mathbf{k}| \leq 1$ is at least $c\widehat{V}$, since $\widehat{K} \geq 100$. Let (E_l, f_l) be an orthonormal eigenbasis of $H_{j-1}^{(-W)}$. Jensen's inequality for the spectral measure of the unit vector $v_{\mathbf{k}} f_l$ gives $\langle v_{\mathbf{k}} f_l, e^{-\beta H_{\text{dist}}} v_{\mathbf{k}} f_l \rangle \geq e^{-\beta(E_l + C'\alpha + 1)}$ for $|\mathbf{k}| \leq 1$. Summing over l and over $|\mathbf{k}| \leq 1$, we find

$$\frac{8\widehat{V}}{V} Z_{\text{dist}} \geq \sum_{|\mathbf{k}| \leq 1} \text{tr}(v_{\mathbf{k}}^* e^{-\beta H_{\text{dist}}} v_{\mathbf{k}}) \geq c\widehat{V} e^{-\beta(C'\alpha + 1)} Z_{j-1}^{(-W)}.$$

The factor $\widehat{V}$ cancels, and $Z_{\text{dist}} \geq cV e^{-\beta(C'\alpha + 1)} Z_{j-1}^{(-W)}$. By Lemma A.3(4), the labeled kernel of $e^{-\beta(H_j^{\text{lab}} - \Sigma W)}$ is nonnegative. In the formula for $Z_j^{(-W)}$ as the average over $\pi \in S_j$ of $\int K(X, \pi X) dX$ (Lemma 3.6(3)), we retain only the $(j-1)!$ permutations that fix the label j ; this gives $Z_j^{(-W)} \geq j^{-1} Z_{\text{dist}}$. We obtain

$$F_j^{(-W)} - F_{j-1}^{(-W)} \leq C'\alpha + 1 + T \log_+(Cj/V).$$

Here $\log_+ x = \max\{\log x, 0\}$. This bound holds for every $T > 0$, and the constants $C_{\text{ins}} \geq C'$ and $C_f \geq C$ of Appendix B are chosen to dominate its constants. Since $j/V \leq 2D$, the logarithm is at most $\log(2C_f D) \leq 2 \log D$ for $D \geq 2C_f$. Since $T \leq \tau_0 / \log D$ with $\tau_0 = \kappa_T \alpha_0$, $\kappa_T \leq 1/16$, and $\alpha \geq \alpha_0/2$, the last term is at most $2\kappa_T \alpha_0 \leq \alpha_0/8 \leq \alpha/4$. Since $\alpha \geq 1$, we obtain part (4) with $C_{\text{inc}} = C_{\text{ins}} + 5/4$, as in Appendix B. The constant C_{inc} is uniform in W , including its size, in T , and in the shape of Ω , once $K \geq K_\Omega$. $\square$

A.7 Localization and diagonal tilting

Lemma A.10. *Let $p, q: \mathbb{R}^3 \rightarrow \mathbb{R}$ be Lipschitz with $p^2 + q^2 = 1$, and let $O \subset \mathbb{R}^3$ be a bounded open set that contains the compact set $\text{supp } p$. Let $0 \leq J \leq N$, and let σ be a symmetric trace-one state on $L^2(\mathcal{G}_j)$ of finite kinetic energy. Introduce an auxiliary label with two values by the isometry $f \mapsto (pf, qf)$ on each coordinate, and call the resulting state σ_{loc} . If P_j projects onto the subspace with exactly j labels of the first type, replace σ_{loc} by $\sum_j P_j \sigma_{\text{loc}} P_j$; this removes terms between distinct particle-count subspaces. We call the two label types colors. Let (ω_j) be the distribution of j , and let σ_j^{in} and σ_j^{out} be the conditional marginal states of the two colors. Then*

σ_j^{in} is a symmetric state of finite kinetic energy on $L^2(\mathcal{G}_j(U \cap O))$, σ_j^{out} is one on $L^2(\mathcal{G}_{J-j})$ whose eigenvectors vanish unless every coordinate lies in $\text{supp } q$, and for every bounded measurable $W \geq 0$,

$$\mathcal{F}_j^{(-W)}(\sigma) + \mathbb{E}_\sigma n_{|\nabla p|^2 + |\nabla q|^2} \geq \sum_j \omega_j [\mathcal{F}_j^{(-W)}(\sigma_j^{\text{in}}) + \mathcal{F}_{J-j}^{(-W)}(\sigma_j^{\text{out}})] - T\mathcal{S}(\omega), \quad (\text{A.15})$$

where $\mathcal{F}_j^{(-W)}(\sigma_j^{\text{in}})$ uses the Dirichlet form of $\mathcal{G}_j(U \cap O)$ and $\mathcal{S}(\omega) = -\sum_j \omega_j \log \omega_j$. Moreover,

$$\sum_j j\omega_j = \mathbb{E}_\sigma n_{p^2}, \quad \mathcal{S}(\omega) \leq -\log(1 - e^{-1}) + \mathbb{E}_\sigma n_{p^2}. \quad (\text{A.16})$$

Conditional on the positions, the particles receive the first color independently with probabilities $p(x_i)^2$, and for every nonnegative measurable symmetric function g , $\sum_j \omega_j \mathbb{E}_{\sigma_j^{\text{in}}} g$ is the expectation of g evaluated at the first-color points.

Proof. This is [112, Lemma 4.1], the two-color localization of [111, Lemma 4.1]; the geometric step localizes the many-particle kinetic energy by a partition of unity, see [40, Section 3, Lemma 3.1]. Its proof applies after the stated substitutions once the following domain statements are verified. In the sector of color assignments with first-color label set S , the localized vector is $\prod_{i \in S} p(x_i) \prod_{i \notin S} q(x_i) F(X)$, where F is an eigenvector of σ . By Lemma A.3(2), applied with $G = \mathcal{G}_J$ and $G' = \{X \in \mathcal{G}_J : x_i \in O \text{ for } i \in S\}$, this vector lies in $H_0^1(G')$. After grouping the coordinates by color, G' is contained in $\mathcal{G}_{|S|}(U \cap O) \times \mathcal{G}_{J-|S|}$; dropping the exclusions between the colors only enlarges the domain. The wall condition for both colors comes from F , and the Dirichlet condition on ∂O for the first color comes from the compact support of p in O . This order matters when $\text{supp } p$ meets ∂U : the mask multiplies approximants of F , which already vanish near the walls. By Lemma A.3(3), the conditional marginals have finite kinetic energy, and their kinetic energies add up to the localized kinetic energy. Their entropies are finite by Lemma A.4. Every second-color factor contains $q(x_i)$, which gives the statement on the support of σ_j^{out} . $\square$

Lemma A.11. *Let G be a bounded real symmetric Lipschitz function of N positions, $g = e^G$, $M_G = \mathbb{E}_\mu e^{2G}$, and $\sigma_g = M_G^{-1} g \Gamma_N g$. The state σ_g has finite kinetic energy, and*

$$\mathcal{F}_N(\sigma_g) - F_N \leq \mathbb{E}_{\sigma_g} |\nabla G|^2 + T(2\mathbb{E}_{\sigma_g} G - \log M_G), \quad |\nabla G|^2 = \sum_i |\nabla_i G|^2.$$

Proof. This is [112, Lemma 4.2], the diagonal tilt of [111, Lemma 4.1], and its proof applies after the stated substitutions. The only domain property it uses is that multiplication by the bounded Lipschitz symmetric functions g and g^2 preserves $H_0^1(\mathcal{G}_N) \cap L_{\text{sym}}^2$, which is Lemma A.3(2) with $G' = G$; this allows a weak eigenfunction ψ of H_N to be tested against $g^2\psi$. Energies and entropies are finite by Lemma A.3(5) and Lemma A.4, and the relative-entropy step uses monotonicity under discarding a finite classical label [106, Lemmas 1

and 2]; see also [109, Theorem 1] for monotonicity under positive trace-preserving maps on separable Hilbert spaces. No regularity of ∂U and no preservation of operator domains is used. $\square$

A.8 The local penalty and the density in a small field

For a finite nonempty set $A \subset \mathbb{Z}^3$ and a configuration X of $J \leq N$ points, let

$$f_A(X) = \sum_{i=1}^J e^{-d(x_i, A)}, \quad k_A(X) = \#\{i : d(x_i, A) \leq 6, d_i(X) \leq 2s_0\}.$$

Here A is any finite set of cells. Its cells may meet $\mathbb{R}^3 \setminus U$, or lie outside U .

Proposition A.12. *There exist numerical constants $c, C > 0$, in particular independent of b_0 , such that the following holds for $K \geq K_\Omega$ and D sufficiently large depending on b_0 and α_0 . For every finite nonempty $A \subset \mathbb{Z}^3$, every $0 \leq J \leq N$, every measurable W with $0 \leq W \leq 1$, and every symmetric trace-one state σ on $L^2(\mathcal{G}_J)$ of finite kinetic energy,*

$$\mathcal{F}_J^{(-W)}(\sigma) - F_J^{(-W)} \geq \alpha [\mathbb{E}_\sigma(f_A + cb_0^{-3}k_A) - CD|A|]. \quad (\text{A.17})$$

Proof. We follow the proof of [112, Proposition 4.3], which combines the exponential localization of [115, Lemma 3.6] with the two-color localization. We give it in full, because the inside states now live in $U \cap O_h$, the outside states in U , and the cell energies run over all cells B_v , $v \in \mathbb{Z}^3$. We take D so large that $r < s_0 < 1/2$ and Proposition A.9 applies.

For $h \geq 0$, let $\phi_h(x) = \min\{\max\{d(x, A) - h, 0\}, 1\}$, $p_h = \cos(\pi\phi_h/2)$, and $q_h = \sin(\pi\phi_h/2)$. Then $p_h = 1$ where $d(x, A) \leq h$, $p_h = 0$ where $d(x, A) \geq h + 1$, and $|\nabla p_h|^2 + |\nabla q_h|^2 \leq (\pi^2/4)\mathbf{1}\{h < d(x, A) < h + 1\}$. Let $O_h = \{x \in \mathbb{R}^3 : d(x, A) < h + 2\}$. This is a bounded open set containing $\text{supp } p_h$, and $|O_h| \leq C(h + 4)^3|A|$. We apply Lemma A.10.

Outside, by Proposition A.9(4), we have $\mathcal{F}_{J-j}^{(-W)}(\sigma_j^{\text{out}}) \geq F_{J-j}^{(-W)} \geq F_J^{(-W)} - C_{\text{inc}}\alpha j$. Inside, we divide the kinetic energy of σ_j^{in} into three equal parts. The first part pays for the entropy: by Lemma A.4 with $O = O_h$, $O' = U \cap O_h$, and $\varsigma = 1/3$, it is at least $TS(\sigma_j^{\text{in}}) - Tj - CT^{5/2}(h + 4)^3|A|$. To the second and third parts we apply Lemma A.5(1) with the unit cells and Lemma A.5(2) with $s = s_0$, with $U' = U \cap O_h$ and all labels retained. The field costs at most j , since $W \leq 1$, and the color entropy costs at most $T(C + \mathbb{E}_\sigma n_{p_h^2})$ by (A.16). Using $\sum_j j\omega_j = \mathbb{E}_\sigma n_{p_h^2}$, $T \leq 1$, and $\alpha \geq 1$, we obtain

$$\begin{aligned} \mathcal{F}_J^{(-W)}(\sigma) - F_J^{(-W)} &\geq cr \mathbb{E} \sum_v n_v^{\text{in}} (n_v^{\text{in}} - 1) + crs_0^{-3} \mathbb{E} k^{\text{in}} \\ &\quad - C\alpha \mathbb{E}_\sigma [n_{p_h^2} + n_{\mathbf{1}\{h < d(\cdot, A) < h+1\}}] - C(h + 4)^3|A|, \end{aligned}$$

where n_v^{in} is the number of first-color points in B_v , k^{in} is the number of first-color points with a first-color partner within $2s_0$, and $\mathbb{E}$ is the expectation in the description of the colors in Lemma A.10.

We average this bound over h with the density e^{-h} on $[0, \infty)$. For every x ,

$$\int_0^\infty p_h(x)^2 e^{-h} dh \leq e e^{-d(x,A)}, \quad \int_0^\infty \mathbf{1}\{h < d(x,A) < h+1\} e^{-h} dh \leq e e^{-d(x,A)},$$

and the averaged losses are at most $C\alpha\mathbb{E}_\sigma f_A + C|A|$. Let $b_v = e^{-d(v,A)}$. If $h \geq d(v,A) + 1$, then $p_h = 1$ on B_v , because $|x - v| < 1$ for $x \in B_v$, and every point in B_v has the first color. This holds whether or not B_v is contained in U . Averaging, the cell term is at least $cr\mathbb{E}_\sigma \sum_v b_v n_v (n_v - 1)$. A point counted by k_A has a partner within $2s_0 < 1$, and both points have distance less than 7 from A ; for $h \geq 8$ both have the first color. Averaging, the close term is at least $crs_0^{-3}\mathbb{E}_\sigma k_A = c\alpha b_0^{-3}\mathbb{E}_\sigma k_A$. We have proved

$$\mathcal{F}_J^{(-W)}(\sigma) - F_J^{(-W)} \geq \mathbb{E}_\sigma \left[cr \sum_v b_v n_v (n_v - 1) - C\alpha f_A + c\alpha b_0^{-3} k_A \right] - C|A|.$$

For $x \in B_v$, we have $d(x,A) \geq d(v,A) - 1$. Then $f_A \leq e \sum_v b_v n_v$. For $n \geq 0$ and $D \geq 1$, completing the square gives $(c/D)n(n-1) - (C+1)en \geq -C'D$. Multiplying by αb_v and summing over $v \in \mathbb{Z}^3$, and using $r = \alpha/D$, we obtain $cr \sum_v b_v n_v (n_v - 1) - (C+1)\alpha f_A \geq -C'\alpha D \sum_v b_v$. Finally, $\sum_{v \in \mathbb{Z}^3} b_v \leq \sum_{w \in A} \sum_{v \in \mathbb{Z}^3} e^{-|v-w|} \leq C|A|$. Adding αf_A to both sides of the previous inequality proves (A.17). The completion of the square uses the integers n_v and the weights b_v only; it does not use the volume of $B_v \cap U$. $\square$

Corollary A.13. *There exists a constant C_{dens} such that the following holds under the hypotheses of Proposition A.12. For every $0 \leq J \leq N$, every measurable W with $0 \leq W \leq 1$, and every finite set $A \subset \mathbb{Z}^3$,*

$$\mathbb{E}_{J,-W} \sum_{v \in A} n_v \leq C_{\text{dens}} D |A|.$$

Used in: Proposition A.22.

Proof. This is [112, Corollary 4.4]. The left side of (A.17) vanishes for the Gibbs state $\Gamma_J^{(-W)}$. Dropping the term with k_A and using $\sum_{v \in A} n_v \leq e f_A$, we obtain the claim. The case $A = \emptyset$ is trivial. $\square$

The reference density in Corollary A.13 is $D = N/V$ for every $J \leq N$. The bound is needed in Proposition A.22, where the neighborhoods of the marked colors can meet the walls; see Remark A.2(4).

A.9 Shortages at interior cells

Proposition A.14. *There exist numerical constants α_0^* , $\ell^* > 0$, and $c_{\text{sh}} > 0$ such that the following holds for $\alpha_0 \geq \alpha_0^*$, $0 < \ell \leq \ell^*$, $K \geq K_\Omega$, and D sufficiently large depending on α_0 . For every finite set $A \subset \mathbb{Z}^3$ of cells whose closures lie in U , in particular for every finite set of interior cells,*

$$\mu\{n_v < \ell D \text{ for every } v \in A\} \leq e^{-c_{\text{sh}} D |A|}.$$

Used in: Section A.14.

Proof. We follow the proof of [112, Proposition 4.5]; the filling construction is now carried out inside U . Let A be nonempty. Since the cells of A are disjoint unit cubes in U , we have $|A| \leq V$. For $x \in \mathbb{R}^3$ let $|x|_\infty$ be the maximum norm. Define p in each cell $v \in A$ by $p(x) = \cos((\pi/2) \min\{\max\{12(|x - v|_\infty - 1/4), 0\}, 1\})$, so that $p = 1$ where $|x - v|_\infty \leq 1/4$ and $p = 0$ where $|x - v|_\infty \geq 1/3$, let $p = 0$ outside the cells of A , and let $q = (1 - p^2)^{1/2}$, the corresponding sine. Then $|\nabla p|^2 + |\nabla q|^2 \leq C$. Let O be the union of the open cubes $\{|x - v|_\infty < 1/2\}$, $v \in A$; it is a bounded open subset of U containing $\text{supp } p$. Let $\phi = \phi_0^2$, where ϕ_0 is smooth, $0 \leq \phi_0 \leq 1$, $\phi_0 = 1$ where $|x - v|_\infty \leq 1/3$ for some $v \in A$, and $\text{supp } \phi_0 \subset O$. Then $|\nabla \phi|^2 \leq C\phi$ and

$$p^2 + |\nabla p|^2 + |\nabla q|^2 \leq C\phi. \quad (\text{A.18})$$

The filling vector. Fix a real smooth function g_0 supported in $(-1/8, 1/8)^3$ with $\|g_0\|_2 = 1$, and let $w = |A|^{-1/2} \sum_{v \in A} g_0(\cdot - v)$. Then w is smooth with compact support in U , $\|w\|_2 = 1$, $\|w\|_\infty^2 \leq C/|A|$, and $\|\nabla w\|_2^2 \leq C$. Let $\varepsilon_f \in (0, 1)$ be fixed below, and let $k = \lfloor \varepsilon_f D |A| \rfloor$. We apply Lemma A.8 with this w successively k times, starting from the zero-particle vector 1. At every step, at most k old points are present, and the smallness condition holds because $C\|w\|_\infty^2 k r^3 \leq C\varepsilon_f D r^3 = C\varepsilon_f \alpha^3 / D^2 \leq 1/4$ for large D . Since $kr/|A| \leq \varepsilon_f \alpha$, the result is a normalized labeled vector $\Psi \in H_0^1(\mathcal{G}_k)$ with

$$q_k[\Psi] \leq Ck(1 + rk/|A|) \leq Ck(1 + \varepsilon_f \alpha), \quad (\text{A.19})$$

which vanishes unless every coordinate lies in $\text{supp } w$. Let $Y_k = ((k!)^{-1} \sum \pi |\Psi \circ \pi|^2)^{1/2}$. As in the proof of Proposition A.9(1), Y_k is a symmetric normalized vector in $H_0^1(\mathcal{G}_k)$ with $q_k[Y_k] \leq q_k[\Psi]$, and it vanishes unless every coordinate lies in $\text{supp } w$.

The filling map. The support of q lies where $|x - v|_\infty \geq 1/4$ for every $v \in A$, and $\text{supp } w$ lies where $|x - v|_\infty \leq 1/8$ for some $v \in A$; the two sets have distance at least $1/8$. Let σ be a symmetric state of finite kinetic energy on $L^2(\mathcal{G}_N)$, and let σ_j^{out} be the outside states of Lemma A.10 for these masks and $J = N$. For a symmetric $(N - j)$ -particle form vector f that vanishes unless every coordinate lies in $\text{supp } q$, let

$$\mathcal{U}_j f = \binom{N - j + k}{k}^{1/2} P_{\text{sym}}(f \otimes Y_k).$$

The terms in $P_{\text{sym}}(f \otimes Y_k)$ that correspond to different choices of the k coordinates in $\text{supp } w$ have disjoint supports. Then $\mathcal{U}_j$ is an isometry, and the kinetic energy of $\mathcal{U}_j f$ is $q_{N-j}[f]\|Y_k\|^2 + q_k[Y_k]\|f\|^2$. The vector $\mathcal{U}_j f$ is admissible. Indeed, $f \otimes Y_k \in H_0^1(\mathcal{G}_{N-j} \times \mathcal{G}_k)$ by Lemma A.3(2). Let θ_1, θ_2 be Lipschitz functions on $\mathbb{R}^3$ with $\theta_1 = 1$ on $\text{supp } q$, $\theta_1 = 0$ where $|x - v|_\infty \leq 3/16$ for some $v \in A$, $\theta_2 = 1$ on $\text{supp } w$, and $\theta_2 = 0$ where $|x - v|_\infty \geq 5/32$ for every $v \in A$. Then $f \otimes Y_k$ is unchanged by multiplication with $m = \theta_1^{\otimes(N-j)} \otimes \theta_2^{\otimes k}$, and on $\text{supp } m$ every old point has distance at least $1/32 > r$ from every new point. Then $\text{supp } m \cap (\mathcal{G}_{N-j} \times \mathcal{G}_k) \subset \mathcal{G}_{N-j+k}$, and by Lemma A.3(2), we have $f \otimes Y_k \in H_0^1(\mathcal{G}_{N-j+k})$. The walls are respected because both factors vanish outside U , and the new exclusions because of the separation of supports. The isometry preserves the nonzero eigenvalues of σ_j^{out} , hence its entropy. The Gibbs variational principle and (A.19) give

$$\mathcal{F}_{N-j}(\sigma_j^{\text{out}}) \geq F_{N-j+k} - Ck(1 + \varepsilon_f \alpha).$$

Since $k \leq \varepsilon_f D|A| \leq \varepsilon_f N$, all particle numbers below are at most $2N$. By Proposition A.9(3) at $N + k$ and then part (4) applied j times,

$$\mathcal{F}_{N-j}(\sigma_j^{\text{out}}) \geq F_N + c_* \alpha k - C_{\text{inc}} \alpha j - Ck(1 + \varepsilon_f \alpha). \quad (\text{A.20})$$

The constant C in (A.20) depends only on g_0 , which was fixed before α_0 . Let $\alpha_0^* = 8C/c_*$ and $\varepsilon_f = c_*/(4C)$. For $\alpha_0 \geq \alpha_0^*$ and $\alpha \geq \alpha_0/2$, we have $C(1 + \varepsilon_f \alpha) \leq c_* \alpha/2$. For $D \geq 2/\varepsilon_f$, we have $k \geq \varepsilon_f D|A|/2$. Then (A.20) implies that $\mathcal{F}_{N-j}(\sigma_j^{\text{out}}) \geq F_N + (c_* \varepsilon_f/4) \alpha D|A| - C_{\text{inc}} \alpha j$.

The inside states. The state σ_j^{in} lives on $\mathcal{G}_j(U \cap O)$, and $|O| = |A|$. By Lemma A.4 with the full kinetic energy, $\mathcal{F}_j(\sigma_j^{\text{in}}) \geq -Tj - CT^{5/2}|A|$. Inserting both bounds into (A.15), and using (A.16) and (A.18), we conclude that for every symmetric state σ of finite kinetic energy

$$\begin{aligned} \mathcal{F}_N(\sigma) - F_N &\geq \frac{c_* \varepsilon_f}{4} \alpha D|A| - C|A| - (C_{\text{inc}} \alpha + 2T) \mathbb{E}_\sigma n_{p^2} - \mathbb{E}_\sigma n_{|\nabla p|^2 + |\nabla q|^2} \\ &\geq c_1 \alpha D|A| - C_2 \alpha \mathbb{E}_\sigma n_\phi, \end{aligned}$$

with $c_1 = c_* \varepsilon_f/8$ and a numerical C_2 , once D is large enough to absorb $C|A|$. This is a free-energy inequality for arbitrary states; no density profile has been assumed.

The tilt. For $0 \leq \lambda \leq 1$, let $G = -\lambda n_\phi$, a bounded Lipschitz symmetric function, and $L(\lambda) = -\log \mathbb{E}_\mu e^{-2\lambda n_\phi}$. By Lemma A.11, with $|\nabla G|^2 \leq C\lambda^2 n_\phi$ from $|\nabla \phi|^2 \leq C\phi$ and with $2\mathbb{E}_{\sigma_g} G \leq 0$, we have $\mathcal{F}_N(\sigma_g) - F_N \leq C\mathbb{E}_{\sigma_g} n_\phi + TL(\lambda)$. Comparing with the previous display for $\sigma = \sigma_g$, and using that the position law of σ_g is $e^{2G} \mu/M_G$, we find

$$C_3 \alpha L'(\lambda) + TL(\lambda) \geq c_1 \alpha D|A|, \quad L'(\lambda) = 2\mathbb{E}_{\sigma_g} n_\phi,$$

with a numerical C_3 . The differentiation is justified because $0 \leq n_\phi \leq N$. Let $a_* = \min\{c_1/2, c_1/(2C_3)\}$. If $L(1) < a_* D|A|$, then $L(\lambda) < a_* D|A|$ for all $\lambda \in [0, 1]$, since

L is nondecreasing. Using $T \leq 1$ and $\alpha \geq 1$, we have $TL(\lambda) \leq c_1 \alpha D|A|/2$. Then $L' \geq c_1 D|A|/(2C_3)$, and integration from $L(0) = 0$ gives $L(1) \geq a_* D|A|$, a contradiction. We conclude that $L(1) \geq a_* D|A|$. On the event of the proposition, $n_\phi \leq \sum_{v \in A} n_v < \ell D|A|$, and by the exponential Markov inequality, we have the bound $e^{2\ell D|A|} \mathbb{E}_\mu e^{-2n_\phi} \leq e^{-(a_* - 2\ell)D|A|}$. With $\ell^* = a_*/4$ and $c_{\text{sh}} = a_*/2$, this proves the proposition. The constants $\varepsilon_f, c_1, C_3, a_*$ are numerical, and ℓ^* does not depend on α_0 . $\square$

Each cell in Proposition A.14 contains the fixed-size region used for the inserted particles, and the comparison shows that leaving all these regions nearly empty costs too much free energy. Nothing is asserted about cells cut by the walls; see Remark A.2(4).

A.10 Close counts

Proposition A.15. *For every fixed $\theta_0 > 0$ there exists a numerical $b^*(\theta_0) \in (0, 1)$ such that, for $0 < b_0 \leq b^*(\theta_0)$, $K \geq K_\Omega$, D sufficiently large depending on b_0 and α_0 , and every finite set $A \subset \mathbb{Z}^3$,*

$$\mu\{N_v^{\text{cl}} \geq \theta_0 D \text{ for every } v \in A\} \leq \exp(-c_{\text{cl}} \theta_0 b_0 D^{2/3} |A|),$$

where $c_{\text{cl}} > 0$ is numerical.

Used in: Section A.14.

Proof. The proof of [112, Proposition 4.6] applies after the stated substitutions with Euclidean distances, Proposition A.12 in place of [112, Proposition 4.3], and Lemma A.11 in place of [112, Lemma 4.2]. It tilts the Gibbs state by λm_A , $0 \leq \lambda \leq s_0$, where $m_A = \sum_i u_d(d(x_i, A)) u_n(d_i/s_0)$ for squared Lipschitz cutoffs u_d, u_n equal to one on $[0, 5]$ and $[0, 1]$, and zero on $[6, \infty)$ and $[2, \infty)$, respectively. Then $0 \leq m_A \leq k_A$ and $\sum_{v \in A} N_v^{\text{cl}} \leq C_0 m_A$, where C_0 bounds the number of lattice sites within distance 5 of a point. We note the two points where the domain could have entered. First, the bound $|\nabla m_A|^2 \leq C s_0^{-2} m_A$ rests on the fact that a point is the nearest neighbor of a numerically bounded number of other points: if x_k is the nearest neighbor of x_i and of x_j , then $|x_i - x_j|$ is the largest side of the triangle, so the angle at x_k is at least $\pi/3$. This is Euclidean geometry of the physical points, and restricting them to U changes none of it. Second, λm_A is a bounded Lipschitz symmetric function. Then Lemma A.11 applies. The temperature enters only through the term $2T\lambda \leq 1$ in the upper bound on the tilted free energy, since $T \leq 1$ and $\lambda \leq s_0 < 1/2$. The conditions on b_0 in the proof, namely that $c\alpha b_0^{-3}$ dominate the constant in the upper bound on the tilted free energy and that $Cb_0^3 \leq \theta_0/C_0$, where C_0 bounds the number of points of $\mathbb{Z}^3$ within distance 5 of a point, involve only numerical constants and are monotone in b_0 . This gives $b^*(\theta_0)$. $\square$

A.11 The slab and free cutoff bounds

Proposition A.16. *Let $0 < \delta < \beta$. Then (A.2) defines a probability measure. At every fixed time $s \in [0, \delta]$, the position marginal of $X(s)$ is μ . The joint law of $X(t)$ and W' has density $\Phi(X, W')^2$, where $\Phi = Z_N^{-1/2} K_{\beta/2}^s$ is nonnegative, symmetric separately in both arguments, and has squared integral one. The marginal of the path family ω is $Z_N^{-1} K_{\beta-\delta}^s(Z^\circ, Y) \mathbf{1}_{\mathcal{E}_\delta}(\omega) \nu_N(d\omega)$, the law (8.5) for the container.*

Proof. This is [112, Proposition 5.1] without its occupation identity, with the killed kernels of $\mathcal{G}_N$, which include the walls. The proof there uses only the Feynman–Kac representation of the labeled killed kernel, which is Proposition 3.5(1) for the container; the facts that P_{sym} commutes with the labeled semigroup and $P_{\text{sym}}^2 = P_{\text{sym}}$; and the semigroup and trace identities, which are Lemma 3.6. These hold for every open configuration domain, and the proof applies after the stated substitutions. The last statement is the first step of the proof of Lemma 8.2. $\square$

Used in: Section A.14.

Lemma A.17. *Fix $t \in (0, 1]$. For every $q > 0$, a Euclidean Brownian motion with generator Δ on $[0, 2t]$ fails to be regular with probability at most $C_{q,t} D^{-q}$, uniformly in its starting point and in $\alpha \in [\alpha_0/2, 2\alpha_0]$. The same holds for each of the tests restricted to one half of $[0, 2t]$, namely its endpoint window, the maximum over its mesh intervals, and the average over its mesh intervals compared with C_{reg} . The same assertions hold for a Euclidean Brownian bridge of duration $2t$, and for two independent Euclidean bridges of duration t joined at a prescribed midpoint, when the prescribed points lie in a fixed bounded region. The constant C_{reg} can be chosen as one numerical constant for all these laws, and any larger value is allowed. If the three prescribed points lie in $B(v, 3)$, the probability that the joined path leaves $\bar{B}(v, 10)$ is at most $Ce^{-c/t}$ for $0 < t \leq 1$.*

Proof. This is [112, Lemma 5.3], from [115, Lemma 4.4]. Its proof concerns free Brownian motion and free bridges only, and applies after the stated substitutions in $\mathbb{R}^3$, where no lifts are needed. We use these bounds only for free laws, never for a bridge conditioned to survive the walls. $\square$

Used in: Lemma A.18 and Proposition A.23.

A.12 Marked colors without wall killing

Fix a finite nonempty set A of cells. For a continuous path $\omega: [0, \delta] \rightarrow \mathbb{R}^3$, let $J_A(\omega)$ be the number of $v \in A$ such that ω visits $B(v, 12)$. Consider the three events

$$E_{\text{all}} = \{\text{all paths}\}, \quad E_{\text{diam}} = \{\text{diam } \omega([0, \delta]) > 1\}, \quad E_{\text{irr}} = \{\omega \text{ is not regular}\},$$

and for one of them, E , and $\lambda > 0$, the marked moment

$$\mathcal{M} = \mathbb{E}_{\text{ppur}} \exp\left(\lambda \sum_{i=1}^N J_A(\omega_i) \mathbf{1}_E(\omega_i)\right) = \mathbb{E}_{\text{ppur}} \prod_{i=1}^N [1 + (e^{\lambda J_A(\omega_i)} - 1) \mathbf{1}_E(\omega_i)]. \quad (\text{A.21})$$

As in [112, Section 5.3], on $[0, t]$ mark the first exit from the unit ball about $\omega(0)$, then the first exit from the unit ball about that exit point, and so on up to time t ; on $[t, \delta]$ make the same construction backward from $\omega(\delta)$. Let $n(\omega)$ be the total number of completed exits. Between consecutive marks the path stays in a unit ball, so on $\{n(\omega) = n\}$,

$$J_A(\omega) \leq C(n+1), \quad |\omega(\delta) - \omega(0)| \leq n+2, \quad (\text{A.22})$$

and if $J_A(\omega) > 0$, both endpoints lie within distance $n+15$ of A , and the closed unit neighborhood of the segment $[\omega(0), \omega(\delta)]$ lies in $\{x : d(x, A) < 32(n+1)\}$. Let

$$O_n = \{x \in \mathbb{R}^3 : d(x, A) < 32(n+1) + 2\},$$

and let $p_n : \mathbb{R}^3 \rightarrow [0, 1]$ be Lipschitz with $p_n = 1$ where $d(x, A) \leq 32(n+1)$, $p_n = 0$ where $d(x, A) \geq 32(n+1) + 1$, and $|\nabla p_n| \leq 1$. Then O_n is a bounded open set containing $\text{supp } p_n$, $|O_n| \leq C(n+1)^3 |A|$, and O_n is covered by at most $C(n+1)^3 |A|$ cells. The sets O_n are auxiliary: they need not be contained in U . Let

$$\mathcal{K}_n(y, z) = \int (e^{\lambda J_A(\omega)} - 1) \mathbf{1}_E(\omega) \mathbf{1}\{n(\omega) = n\} W_{y,z}^\delta(d\omega)$$

be the free Euclidean kernel of duration δ with the marking weight and the restriction $n(\omega) = n$, without walls or exclusion.

Lemma A.18. *For $y, z \in \mathbb{R}^3$ and $n \geq 0$,*

$$\mathcal{K}_n(y, z) \leq l_n p_n(y) p_n(z) [e^{3\delta \Delta_{O_n}}](y, z), \quad l_n = C\lambda(n+1) e^{C\lambda(n+1)} e^{-cn^2/t} \epsilon_E^{1/3},$$

where Δ_{O_n} is the Dirichlet Laplacian of O_n , and $\epsilon_E = 1, Ce^{-c/t}, C_t D^{-12}$ for $E = E_{\text{all}}, E_{\text{diam}}, E_{\text{irr}}$. The constants for E_{all} and E_{diam} are uniform in $0 < t \leq 1$.

Proof. The proof of [112, Lemma 5.5] applies after the stated substitutions in $\mathbb{R}^3$, with the sum over lifts of z deleted and Lemma A.17 in place of [112, Lemma 5.3]. It bounds the kernel without the marking factor by the geometric mean of three free bounds, which is a constant times $g_{3\delta}(z-y) e^{-cn^2/t} \epsilon_E^{1/3}$ with $g_s(u) = (4\pi s)^{-3/2} e^{-|u|^2/(4s)}$, and the marking factor by (A.22). The only lower bound in the proof is the following. If $\mathcal{K}_n(y, z) \neq 0$, then the closed unit neighborhood of the segment $[y, z]$ lies in O_n , and $p_n(y) = p_n(z) = 1$. A Brownian bridge of duration $3\delta \leq 6$ from y to z stays within distance one of this segment with probability bounded below by a numerical constant, by Brownian scaling. Then $[e^{3\delta \Delta_{O_n}}](y, z) \geq c g_{3\delta}(z-y)$ by Lemma A.3(4). This lower bound concerns the auxiliary set O_n , not U . $\square$

We now set up the colored operator of [112, Section 5.4]. Fix $n_* \geq 0$, and let $\mathcal{M}_{n_*}$ be the moment (A.21) with $\mathbf{1}_E(\omega_i)$ replaced by $\mathbf{1}_E(\omega_i)\mathbf{1}\{n(\omega_i) \leq n_*\}$. Let C_{inc} and $m_0 = \max\{2, 4(2C_{\text{inc}}\alpha_0 + 1)\}$ be the constants of Appendix B; C_{inc} is the constant of Proposition A.9(4), and m_0 depends only on α_0 . Let

$$\mathfrak{g}_n = \frac{m_0 + n}{\delta}, \quad \varkappa_n = l_n e^{m_0 + n}. \quad (\text{A.23})$$

Since $\delta \leq 2$ and $\alpha \leq 2\alpha_0$, we have $\mathfrak{g}_n/2 \geq m_0/4 \geq C_{\text{inc}}\alpha + 1$, and since $\beta > \delta$, we have $\beta\mathfrak{g}_n/2 > (m_0 + n)/2 \geq 1$. The direct sum below is an auxiliary comparison space. Its first summand retains a physical particle, while the additional summand indexed by n replaces a marked path by the heat-kernel bound of Lemma A.18. The factor $\varkappa_n$ and the energy shift $\mathfrak{g}_n$ satisfy $\varkappa_n e^{-\delta\mathfrak{g}_n} = l_n$. The one-particle space is

$$\mathfrak{h}_e = L^2(U) \oplus \bigoplus_{n=0}^{n_*} L^2(O_n).$$

The first summand represents unmarked physical particles, and the others represent auxiliary types indexed by the exit count n . For a color assignment $\mathbf{c} = (c_1, \dots, c_N) \in \{0, 1, \dots, n_*\}^N$, let $S(\mathbf{c}) = \{i : c_i \neq 0\}$ and

$$G_{\mathbf{c}} = \{X \in (\mathbb{R}^3)^N : (x_i)_{i \notin S(\mathbf{c})} \in \mathcal{G}_{N-|S(\mathbf{c})|}, x_i \in O_{c_i} \text{ for } i \in S(\mathbf{c})\}.$$

On $\mathcal{H}_e^{\text{lab}} = \bigoplus_{\mathbf{c}} L^2(G_{\mathbf{c}})$, let $\mathbb{H}^{\text{lab}}$ act on $L^2(G_{\mathbf{c}})$ as the operator of the closed form

$$\sum_{i \notin S(\mathbf{c})} \int |\nabla_i f|^2 + \sum_{i \in S(\mathbf{c})} \int (3|\nabla_i f|^2 + \mathfrak{g}_{c_i}|f|^2) \quad \text{on } H_0^1(G_{\mathbf{c}}).$$

Unmarked physical particles remain in U with their mutual hard-core exclusions. A particle marked by n instead moves in O_n with a Dirichlet condition on ∂O_n ; it is subject neither to the physical walls ∂U nor to pair exclusions. Simultaneous permutations of positions and colors commute with $\mathbb{H}^{\text{lab}}$. Let $\mathcal{H}_e$ be the invariant subspace and $\mathbb{H}$ the restriction of $\mathbb{H}^{\text{lab}}$ to it. For $f \in L^2(U)$, let $v_e f = (f, (\varkappa_n^{1/2} p_n f)_{n \leq n_*})$, where $p_n f$ is regarded as an element of $L^2(O_n)$; let $V_{\text{lab}} = v_e^{\otimes N}$ on $L^2(\mathcal{G}_N)$, so that its component in $L^2(G_{\mathbf{c}})$ is $\prod_{i \in S(\mathbf{c})} \varkappa_{c_i}^{1/2} p_{c_i}(x_i) F(X)$, and let V_e be its restriction to symmetric functions. Set $\mathcal{T}_{\text{lab}} = V_{\text{lab}}^* e^{-\delta \mathbb{H}^{\text{lab}}} V_{\text{lab}}$ and $\mathcal{T} = V_e^* e^{-\delta \mathbb{H}} V_e$.

Lemma A.19. 1. The map V_{lab} is bounded, maps $H_0^1(\mathcal{G}_N)$ into $\bigoplus_{\mathbf{c}} H_0^1(G_{\mathbf{c}})$, and intertwines permutations of the coordinates with simultaneous permutations of positions and colors.

2. We have

$$\mathcal{M}_{n_*} \leq \frac{\text{tr}_{\text{sym}}(e^{-(\beta-\delta)H_N} \mathcal{T})}{Z_N} \leq \left(\frac{\text{tr} \mathcal{T}^{\beta/\delta}}{Z_N} \right)^{\delta/\beta}. \quad (\text{A.24})$$

Proof. For part (1), $\|V_{\text{lab}}F\|^2 = \int \prod_i (1 + \sum_n \varkappa_n p_n(x_i)^2) |F|^2 \leq (1 + \sum_n \varkappa_n)^N \|F\|^2$. Let $F \in H_0^1(\mathcal{G}_N)$ and let $\mathbf{c}$ be a color assignment with $S = S(\mathbf{c})$. We apply Lemma A.3(2) with $G = \mathcal{G}_N$, the bounded Lipschitz function $m(X) = \prod_{i \in S} \varkappa_{c_i}^{1/2} p_{c_i}(x_i)$, and $G' = \{X \in \mathcal{G}_N : x_i \in O_{c_i} \text{ for } i \in S\}$. Since $\text{supp } p_n \subset O_n$, we have $\text{supp } m \cap \mathcal{G}_N \subset G'$, and $mF \in H_0^1(G')$. Since $G' \subset G_c$, extension by zero gives $mF \in H_0^1(G_c)$. The coordinates of the unmarked physical particles keep the walls and their mutual exclusions from F ; the marked coordinates acquire the Dirichlet condition on ∂O_{c_i} from the compact support of p_{c_i} . The image vanishes at further configurations, for example where a marked coordinate lies outside U ; this is allowed, since the target domain does not require it. Marked colors can lose the wall condition while the input of the map remains admissible. The intertwining property is immediate from the coordinatewise definition.

For part (2), expand the product in $\mathcal{M}_{n_*}$. The term indexed by a set S of marked labels is

$$\frac{1}{Z_N} \int K_{\beta-\delta}^s(Z^\circ, Y) \mathbf{1}_{\mathcal{E}_\delta}(\omega) \prod_{i \in S} (e^{\lambda J_A(\omega_i)} - 1) \mathbf{1}_E(\omega_i) \mathbf{1}\{n(\omega_i) \leq n_*\} \nu_N(d\omega),$$

after integrating out W' by Proposition A.16. For every $i \in S$, we drop the wall killing of ω_i and every exclusion between ω_i and another path: $\mathbf{1}_{\mathcal{E}_\delta}(\omega)$ is at most the indicator that $(\omega_i)_{i \notin S}$ stays in $\mathcal{G}_{N-|S|}$. The unmarked physical paths keep all their walls and mutual exclusions. All factors, including $K_{\beta-\delta}^s$, are nonnegative. The integral does not decrease. Integrating the paths with fixed endpoints, Proposition 3.5(1) for the domain $\mathcal{G}_{N-|S|}$ and the definition of $\mathcal{K}_n$ show that the term is at most

$$\frac{1}{Z_N} \int K_{\beta-\delta}^s(Z^\circ, Y) K_\delta^{(S^c)}(Y_{S^c}, Z_{S^c}^\circ) \prod_{i \in S} \sum_{n_i \leq n_*} \mathcal{K}_{n_i}(y_i, z_i^\circ) dY dZ^\circ,$$

where $K_\delta^{(S^c)}$ is the labeled killed kernel of $\mathcal{G}_{N-|S|}$. Let $V_{\text{lab},c}$ be the component of V_{lab} in $L^2(G_c)$. By Lemma A.18 and $l_n = \varkappa_n e^{-\delta g_n}$, the integrand is at most $K_{\beta-\delta}^s(Z^\circ, Y)$ times the sum, over the assignments $\mathbf{c}$ with $S(\mathbf{c}) = S$, of the kernel of $V_{\text{lab},c}^* e^{-\delta H^{\text{lab}}} V_{\text{lab},c}$, which equals

$$K_\delta^{(S^c)}(Y_{S^c}, Z_{S^c}^\circ) \prod_{i \in S} \varkappa_{c_i} p_{c_i}(y_i) p_{c_i}(z_i^\circ) e^{-\delta g_{c_i}} [e^{3\delta \Delta_{O_{c_i}}}] (y_i, z_i^\circ).$$

Every choice of marked labels and exit counts appears exactly once in the direct sum over $\mathbf{c}$, and summing over S bounds the tested slab kernel pointwise by the kernel of $\mathcal{T}_{\text{lab}}$. This is a comparison of nonnegative kernels; it does not compare normalized laws. The rest of the proof is that of [112, Section 5.4]: both $e^{-(\beta-\delta)H_N^{\text{lab}}}$ and $\mathcal{T}_{\text{lab}}$ commute with $P_{\text{sym}} = (N!)^{-1} \sum_\pi U_\pi$, and the kernel of $e^{-(\beta-\delta)H_N^{\text{lab}}} P_{\text{sym}}$ is $K_{\beta-\delta}^s$. Then

$$\int K_{\beta-\delta}^s(Z^\circ, Y) K_{\mathcal{T}_{\text{lab}}}(Y, Z^\circ) dY dZ^\circ = \text{tr}(e^{-(\beta-\delta)H_N^{\text{lab}}} P_{\text{sym}} \mathcal{T}_{\text{lab}}) = \text{tr}_{\text{sym}}(e^{-(\beta-\delta)H_N} \mathcal{T}).$$

The only permutation normalization is the one in P_{sym} ; no factorial enters the color expansion. This proves the first inequality of (A.24), and the second is Hölder's inequality for Schatten norms [134, Theorem 2.8] with exponents $\beta/(\beta - \delta)$ and β/δ . All traces are finite by Lemma A.3(5). $\square$

Remark A.20. The marked particles must be allowed to leave U , because there is no uniform lower bound on the probability that a short bridge near a wall survives. For example, a Brownian bridge with generator Δ and duration s , from a point at distance h from a plane back to the same point, stays on one side of the plane with probability $1 - e^{-h^2/s}$, which tends to zero as $h \rightarrow 0$. The lower bound in Lemma A.18 concerns a tube contained in the auxiliary set O_n , which is not contained in U .

A.13 A trace comparison and the path moment bound

Let $\Sigma_c(x) = \sum_{n \leq n_*} \varkappa_n p_n(x)^2$, $\Theta(x) = (1/2) \log(1 + \Sigma_c(x))$, and $\hat{\Theta}(X) = \sum_i \Theta(x_i)$, with the same multiplier in every color. The map $\tilde{V}_e = e^{-\hat{\Theta}} V_e$ is an isometry, because the squared pointwise norm of its one-coordinate color vector is one. With $\mathcal{Y} = e^{\hat{\Theta}} e^{-\delta \mathbb{H}} e^{\hat{\Theta}}$, we have $\mathcal{T} = \tilde{V}_e^* \mathcal{Y} \tilde{V}_e$. The min-max principle for compressions of positive compact operators gives $\text{tr } \mathcal{T}^q \leq \text{tr } \mathcal{Y}^q$ for every $q > 0$.

Lemma A.21. *Let $W_0 = 2\Theta/\delta + 3|\nabla\Theta|^2$ and $\tilde{\mathbb{H}} = \mathbb{H} - \sum_i W_0(x_i)$. For every real $q > 0$, $\text{tr } \mathcal{Y}^q \leq \text{tr } e^{-q\delta\tilde{\mathbb{H}}}$.*

Proof. This is [112, Lemma 5.6], the trace estimate for $e^{\hat{\Theta}} e^{-\delta \mathbb{H}} e^{\hat{\Theta}}$ with a Lipschitz multiplier of [111, Lemma 6.2]. Its proof uses the following properties of the colored operator, which we verify for the present domains. (i) There are finitely many color assignments, and in each, $\mathbb{H}^{\text{lab}}$ is given by a closed form on $H_0^1(G_c)$ with kinetic coefficients 1 or 3 and nonnegative constant energy shifts $\mathfrak{g}_n$. Since U and the sets O_n are bounded, every G_c is a bounded open set, and Lemma A.3(5) shows that $\mathbb{H}$ and $\tilde{\mathbb{H}}$ have compact resolvent and finite heat traces; W_0 is bounded. (ii) Multiplication by $e^{u\hat{\Theta}}$, $|u| \leq 1$, preserves each $H_0^1(G_c)$ and the simultaneous permutation symmetry, by Lemma A.3(2) with $G' = G$; this retains the walls and exclusions of the unmarked physical particles and the boundaries ∂O_n of the marked colors. (iii) The product rule holds almost everywhere. With these properties, the proof in [112] applies after the stated substitutions. It pairs form-domain vectors with operator-domain vectors only, so no preservation of operator domains by the multipliers is needed, and its final integration against $q^2 e^{qh} dh$ gives the bound for every real $q > 0$, including $q = \beta/\delta$. $\square$

Proposition A.22. *Let E be one of E_{all} , E_{diam} , E_{irr} , and let l_n , $\varkappa_n$, $\mathfrak{g}_n$ be as in Lemma A.18 and (A.23). There is a numerical C_0 such that, if*

$$\frac{C_0}{\delta} \sum_{n \geq 0} \varkappa_n \leq 1, \quad (\text{A.25})$$

then, under the hypotheses of Proposition A.12 and for every finite nonempty $A \subset \mathbb{Z}^3$,

$$\begin{aligned} \log \mathbb{E}_{\text{ppur}} \exp \left(\lambda \sum_{i=1}^N J_A(\omega_i) \mathbf{1}_E(\omega_i) \right) &\leq CD|A| \sum_{n \geq 0} (n+1)^3 \varkappa_n \\ &\quad + C\delta T^{5/2} |A| \sum_{n \geq 0} (n+1)^3 e^{-\beta \mathfrak{g}_n / 2}. \end{aligned}$$

The constants do not depend on n_* , K , Ω , D , T , or $0 < t \leq 1$.

Used in: Section A.14.

Proof. We follow [112, Proposition 5.7]; the unmarked physical particles remain in U , and the marked traces are Euclidean. By the Cauchy–Schwarz inequality and $|\nabla p_n| \leq \mathbf{1}_{O_n}$,

$$|\nabla \Theta|^2 \leq \frac{\Sigma_c}{(1 + \Sigma_c)^2} \sum_{n \leq n_*} \varkappa_n |\nabla p_n|^2 \leq \sum_{n \leq n_*} \varkappa_n \mathbf{1}_{O_n}, \quad 2\Theta \leq \Sigma_c \leq \sum_{n \leq n_*} \varkappa_n \mathbf{1}_{O_n}.$$

Since $\delta \leq 2$, this implies that $0 \leq W_0 \leq (C/\delta) \sum_{n \leq n_*} \varkappa_n \mathbf{1}_{O_n} \leq 1$, by (A.25) with $C_0 \geq C$.

The unmarked physical particles. For $0 \leq J \leq N$ and $0 \leq u \leq 1$, the Duhamel formula and cyclicity of the trace give

$$\partial_u \log Z_J^{(-uW_0)} = \beta \mathbb{E}_{J, -uW_0} \sum_i W_0(x_i);$$

the field is bounded and the traces are finite. We cover each O_n by at most $C(n+1)^3|A|$ cells, and apply Corollary A.13 with the field uW_0 , which lies in $[0, 1]$. Some of these cells may meet the walls, or lie outside U ; the corollary applies to all of them. Integrating in u gives

$$\log \frac{Z_J^{(-W_0)}}{Z_J} \leq \frac{\beta C}{\delta} D|A| \sum_{n \leq n_*} (n+1)^3 \varkappa_n. \quad (\text{A.26})$$

By Proposition A.9(4) with $W = 0$, $F_N - F_J \leq C_{\text{inc}} \alpha(N - J)$, that is,

$$Z_J / Z_N \leq e^{\beta C_{\text{inc}} \alpha(N - J)}. \quad (\text{A.27})$$

The decomposition into auxiliary types. The space $\mathcal{H}_e$ decomposes according to the number J of unmarked physical particles and the numbers k_n of particles of color n , with $J + \sum_n k_n = N$. Normalized symmetrization over the mutually orthogonal color

assignments identifies each such sector with one tensor product of $L_{\text{sym}}^2(\mathcal{G}_J)$ and the symmetric spaces $L_{\text{sym}}^2(O_n^{k_n})$; there is no multinomial multiplicity. On this tensor product, $\tilde{H}$ is the sum of $H_J^{(-W_0)}$ and, for each n , the sum, over the k_n coordinates of this type, of $-3\Delta_{O_n} + \mathfrak{g}_n - W_0$. The trace of $e^{-\beta\tilde{H}}$ over the sector is $Z_J^{(-W_0)}$ times the canonical traces of the marked colors. By (A.27), $Z_J \leq Z_N \prod_n e^{\beta C_{\text{inc}} \alpha k_n}$, and since $W_0 \leq 1$ and $\mathfrak{g}_n/2 \geq C_{\text{inc}} \alpha + 1$, each marked one-particle operator with $C_{\text{inc}} \alpha$ subtracted is at least $-3\Delta_{O_n} + \mathfrak{g}_n/2$ in the sense of forms. The min-max principle then bounds each marked canonical trace by that of $-3\Delta_{O_n} + \mathfrak{g}_n/2$. We drop the restriction $\sum_n k_n \leq N$; since J is determined by the k_n , this introduces no sum over J . Using (A.26), we obtain

$$\frac{\text{tr } e^{-\beta\tilde{H}}}{Z_N} \leq \exp\left(\frac{\beta C}{\delta} D|A| \sum_{n \leq n_*} (n+1)^3 \varkappa_n\right) \prod_{n \leq n_*} \Xi_n,$$

where Ξ_n is the ideal Bose grand partition function of $-3\Delta_{O_n} + \mathfrak{g}_n/2$ at inverse temperature β .

The marked colors. By Lemma A.3(5) and $\beta\mathfrak{g}_n/2 \geq 1$,

$$\begin{aligned} \log \Xi_n &= \sum_{m \geq 1} \frac{e^{-m\beta\mathfrak{g}_n/2}}{m} \text{tr } e^{3m\beta\Delta_{O_n}} \leq |O_n| (12\pi\beta)^{-3/2} \sum_{m \geq 1} m^{-5/2} e^{-m\beta\mathfrak{g}_n/2} \\ &\leq C(n+1)^3 |A| T^{3/2} e^{-\beta\mathfrak{g}_n/2}. \end{aligned}$$

No volume of U and no volume of a containing set appears here. Combining the last two displays with Lemma A.19(2), the compression inequality $\text{tr } \mathcal{T}^{\beta/\delta} \leq \text{tr } \mathcal{Y}^{\beta/\delta}$, and Lemma A.21 with $q = \beta/\delta$, and multiplying the logarithm by $\delta/\beta = \delta T$, we conclude that the bound of the proposition holds for $\mathcal{M}_{n_*}$.

As $n_* \rightarrow \infty$, the truncated marking weight of every path increases to the full weight, and so does the finite product over labels. The bounds are uniform in n_* , and their right sides are summable, as shown in (A.28) below. Monotone convergence removes the truncation. $\square$

We record the sizes of the two terms. Since $t \leq 1$, by the tail bound for the exit count in l_n , we have

$$\sum_{n \geq 0} (n+1)^3 \varkappa_n \leq C \lambda \epsilon_E^{1/3} e^{m_0} \sum_{n \geq 0} (n+1)^4 e^{C\lambda(n+1)+n-cn^2} = C_\lambda \lambda \epsilon_E^{1/3}, \quad (\text{A.28})$$

where C_λ is bounded on bounded intervals of λ and depends on α_0 through m_0 . The same bound holds without the factor $(n+1)^3$. For the second term, $\beta\mathfrak{g}_n/2 = (m_0 + n)x$ with $x = (2\delta T)^{-1} \geq (4T)^{-1} \geq 1$, since $\delta \leq 2$ and $T \leq 1/4$. Then $\sum_{n \geq 0} (n+1)^3 e^{-(m_0+n)x} \leq C e^{-m_0 x} \leq C e^{-m_0/(4T)}$. Since $T \mapsto T^{5/2} e^{-m_0/(4T)}$ is increasing and $T \leq \tau_0/\log D$, the second term in Proposition A.22 is at most $r_D^{\text{wall}} |A|$, uniformly in $0 < t \leq 1$, where

$$r_D^{\text{wall}} = C \frac{\tau_0^{5/2}}{(\log D)^{5/2}} D^{-m_0/(4\tau_0)} \longrightarrow 0 \quad (D \rightarrow \infty).$$

A.14 Completion of the joint failure estimate

Proof of Theorem A.1(i). We follow the six-test assembly in [112, Section 5.6, proof of Proposition 5.4], substituting the container estimates proved above. Let $\alpha_0 \geq \alpha_0^*$, $0 < \ell \leq \ell^*$, $0 < \theta_0 < \ell/6$, $0 < b_0 \leq b^*(\theta_0)$, and $C_{\text{reg}} \geq C_{\text{reg}}^*$, where α_0^* and ℓ^* are given by Proposition A.14, $b^*(\theta_0)$ by Proposition A.15, and C_{reg}^* is the numerical constant of Lemma A.17; this is the range of constants in Remark A.2(1). We may assume $p_1 < 1$. Let A be a finite nonempty set of interior cells; for $A = \emptyset$ there is nothing to prove. By Lemma 8.2, it suffices to bound the probability under $\mathbb{P}^{\text{pur}}$. A cell fails the local conditions if and only if one of six tests fails: $n_v(Y) < \ell D$; $N_v^{\text{cl}}(Y) \geq \theta_0 D$; $N_v^{\text{cl}}(Z^\circ) \geq \theta_0 D$; more than $H_1 D$ paths visit $B(v, 12)$; some path visiting $B(v, 12)$ is irregular; at least $\theta_0 D$ paths visiting $B(v, 12)$ have diameter greater than one. It suffices to show that, for a given $p' \in (0, 1)$, the parameters can be chosen so that for every test and every finite set A' of interior cells, the probability that the test fails at every cell of A' is at most $(p')^{|A'|}$. Indeed, given failures at every cell of A , assign to each cell a failing test. There are at most $6^{|A|}$ assignments, and in each, some test is assigned to a set A' of at least $|A|/6$ cells. The probability that every cell of A fails is at most $(6(p')^{1/6})^{|A|}$, and we choose p' with $6(p')^{1/6} \leq p_1$. No independence between cells or between tests is used.

Endpoint tests. By Proposition A.16, Y and Z° have law μ . By Proposition A.14, we bound the first test by $e^{-c_{\text{sh}} D |A'|}$, and by Proposition A.15, we bound the second and third by $\exp(-c_{\text{cl}} \theta_0 b_0 D^{2/3} |A'|)$. Both bases are at most p' for D large.

Visitors. We take $E = E_{\text{all}}$, for which $\epsilon_E = 1$, and $\lambda = c't$ with a constant $c' > 0$. By (A.28), $\sum_n \varkappa_n \leq C\lambda = Cc't$ with C independent of t . Then (A.25) holds for all $t \in (0, 1]$ once c' is small, and the first term in Proposition A.22 is at most $C_4 \lambda D |A'|$ with C_4 independent of t . If every cell of A' has more than $H_1 D$ visitors, then $\sum_i J_{A'}(\omega_i) > H_1 D |A'|$. By the exponential Markov inequality and Proposition A.22, we have the bound

$$\exp\{[-\lambda(H_1 - C_4)D + r_D^{\text{wall}}]|A'|\}.$$

We choose $H_1 = C_4 + 1$ now. For every subsequently fixed t , the base $\exp(-c'tD + r_D^{\text{wall}})$ is at most p' for D large. Reducing t does not affect the choice of H_1 .

Visitors of large diameter. We set $E = E_{\text{diam}}$ and $\lambda = 1$. Then $\epsilon_E^{1/3} \leq Ce^{-c/t}$ and $\sum_n (n+1)^3 \varkappa_n \leq Ce^{-c/t}$. Fix t so small that (A.25) holds and the first coefficient in Proposition A.22 is at most $\theta_0 D/2$ per cell. If every cell of A' is visited by at least $\theta_0 D$ paths of diameter greater than one, then $\sum_i J_{A'}(\omega_i) \mathbf{1}_E(\omega_i) \geq \theta_0 D |A'|$, and the probability is at most $\exp[-(\theta_0 D/2 - r_D^{\text{wall}})|A'|]$. This reduction of t is permitted, and any further reduction preserves the bound.

Irregular visitors. With t now fixed, take $E = E_{\text{irr}}$ and $\lambda = \log(2/p')$. Then $\epsilon_E^{1/3} = C_t D^{-4}$, and we have $\sum_n (n+1)^3 \varkappa_n \leq C_{\lambda,t} D^{-4}$, (A.25) holds for D large, and the first term in Proposition A.22 is at most $C_{\lambda,t} D^{-3} |A'|$. If every cell of A' is visited by an irregular

path, then $\sum_i J_{A'}(\omega_i) \mathbf{1}_E(\omega_i) \geq |A'|$, even if a single path visits many of these cells. The probability is at most $\exp[-(\lambda - C_{\lambda,t} D^{-3} - r_D^{\text{wall}}) |A'|] \leq (p')^{|A'|}$ for D large.

Order of choices. The constants $\alpha_0, \ell, \theta_0, b_0, C_{\text{reg}}$ are fixed first, then κ_T and $\tau_0 = \kappa_T \alpha_0$, then C_{inc} and m_0 of Appendix B, and c' . Given p_1 , we fix p' with $6(p')^{1/6} \leq p_1$, then H_1 , then t , which may be reduced further for the requirement of Section A.15, then λ for the irregular test, and finally D_0 , larger than all the thresholds above and than those of Propositions A.9, A.12, A.14, A.15, and A.23. We also require (8.10), $2s_0 < 1$, and $r < 1/32$, which proves Remark A.2(3). The shape of Ω enters only through $K \geq K_\Omega$, in Proposition A.9; R_0 enters only through the set of cells to which the theorem applies. No lower bound on T other than positivity is used. $\square$

A.15 Euclidean bridge estimates and useful labels

For $y, z \in \mathbb{R}^3$ and a cell B , let $P_{y,z}^B = \int_B P_{y,z}^u du$, and let $P_{y,z}^0$ be the Euclidean Brownian bridge with generator Δ from y to z of duration $2t$. For $v \in \mathbb{Z}^3$, let $\mathcal{C}_v$ be the event that a path on $[0, 2t]$ is regular and stays in $\bar{B}(v, 10)$, and for a deterministic continuous path $c: [0, 2t] \rightarrow \mathbb{R}^3$ let $\mathcal{H}(c) = \{\omega : |\omega(s) - c(s)| \leq r \text{ for some } s \in [0, 2t]\}$.

Proposition A.23. *Fix $t \in (0, 1]$, and let D be large enough that (8.10) holds. Then [112, Lemmas 6.1, 6.2, and 6.3, Remark 6.4, and Corollary 6.5] hold for the Euclidean laws $P_{y,z}^u, P_{y,z}^B$, and $P_{y,z}^0$ with the same constants at every $v \in \mathbb{Z}^3$. The obstacle paths are deterministic Euclidean paths. In particular:*

1. *If $y \in B_v$, $|z - v| \leq 3$, and c is a deterministic regular path with $|y - c(0)| > s_0$ and $|z - c(2t)| > s_0$, then, for $P = P_{y,z}^B$ with any target cell B of v , or $P = P_{y,z}^0$,*

$$P(\mathcal{C}_v \cap \mathcal{H}(c)) \leq Ctr + C \frac{rD^{1/4}}{\sqrt{h_0}} \leq C\alpha(tD^{-1} + D^{-1/4}).$$

The midpoint is averaged as in Remark 8.8; no uniform prescribed-midpoint estimate is used.

2. *Let $\mathfrak{F}$ be a collection of deterministic continuous paths on $[0, 2t]$, at most $H_1 D$ of which visit $B(v, 12)$, every visiting path being regular. Let a list of $m \geq \ell D/2$ distinct paths $c_i \in \mathfrak{F}$ have endpoints $y_i = c_i(0) \in B_v$ and $z_i = c_i(2t)$ with $|z_i - v| \leq 3$, such that $|y_i - c(0)| > s_0$ and $|z_i - c(2t)| > s_0$ for every $c \in \mathfrak{F} \setminus \{c_i\}$. For t sufficiently small depending on H_1 , and then D sufficiently large depending on t , at least $9m/10$ of the listed labels have the following property: for every target cell B of v , the set of $u \in B$ with $P_{y_i, z_i}^u(\mathcal{C}_v \cap \bigcap_{c \in \mathfrak{F} \setminus \{c_i\}} \mathcal{H}(c)^c) \geq 9/10$ has volume at least $99/100$.*

The choices are uniform in $\mathfrak{F}$, in v , and in $\alpha \in [\alpha_0/2, 2\alpha_0]$.

Used in: Theorem A.1(ii), and through its proof Proposition 9.11; part (1) is the Euclidean form of Lemma 8.7, whose proof is adapted in Lemma 9.9.

Proof. The statements of [112, Section 6], reproduced there from [115, Section 5], concern free Euclidean Brownian bridges and deterministic obstacle paths. The torus enters their proofs only through the projection of the trial paths to Λ_K , and through the paragraph on lifts in the proof of [112, Lemma 6.2]. In that paragraph, the lift of the obstacle near v is selected for each mesh interval. In $\mathbb{R}^3$ we delete the projection and that paragraph; every other line applies unchanged, with Lemma A.17 in place of [112, Lemma 5.3]. In the proof of [112, Corollary 6.5], a trial in $\bar{B}(v, 10)$ can collide only with an obstacle that visits $B(v, 12)$, since $r < 1$. For each such obstacle, we first average the collision probability over the m listed labels using [112, Lemma 6.2], obtaining $Ctr + CrD^{1/4}/(D\sqrt{h_0})$, omitting the obstacle's own label when present but retaining the divisor m . By the union bound over the at most H_1D such obstacles, we bound the averaged failure probability by $Ce^{-c/t} + \varepsilon_D(t) + CH_1\alpha(t + D^{-1/4})$, with $\varepsilon_D(t) \rightarrow 0$ as $D \rightarrow \infty$; this is the reason t is chosen small depending on H_1 . The container U does not appear in any of these statements. $\square$

Proof of Theorem A.1(ii). The proof of [112, Lemma 7.1] applies after the stated substitutions at interior cells; we recall it. Let v be an interior cell at which the sample satisfies the local conditions. Every path that starts in B_v visits $B(v, 12)$. At most H_1D paths start in B_v . By item 1 of Definition 8.4, at least ℓD paths start in B_v ; item 2 makes all of them regular. We discard the fewer than $\theta_0 D$ of them with diameter greater than one. The remaining paths stay within distance $\sqrt{3}/2 + 1 < 2$ of v ; in particular they stay in $\bar{B}(v, 10)$ and end within distance 3 of v . We discard also those whose initial point has another initial point within distance s_0 , and those whose terminal point has another terminal point within distance s_0 . These endpoints lie within distance 5 of v , so by item 1 of Definition 8.4 each of the two losses is less than $\theta_0 D$. The remaining labels are eligible, and their number is at least $(\ell - 3\theta_0)D \geq \ell D/2$.

We now apply Proposition A.23(2) to the entire list of labels eligible at v , not only to the labels just counted, with $\mathfrak{F} = \{\omega_k : 1 \leq k \leq N\}$. Every eligible label has its initial point in B_v , its terminal point within distance 3 of v , and both endpoints at distance greater than s_0 from the corresponding endpoints of all other paths. By item 2 of Definition 8.4, at most H_1D paths visit $B(v, 12)$, and all of them are regular. For an eligible label i , the event in Proposition A.23(2) is $C_i(F_i^{\text{all}})$. For t small depending on H_1 and D large depending on t , we conclude that at least nine tenths of the eligible labels are useful. Since $\bar{B}(v, 10) \subset U$, a successful trial stays in U . The useful property needs no factor for survival inside the container. The requirement on t is the further reduction of t after H_1 is fixed, which is permitted by part (i). $\square$

Thermal estimates in a larger temperature window

In this appendix, we prove Proposition 8.5: on the torus, the joint bound of [112, Proposition 5.4] for failure of the local conditions holds uniformly for $0 < T \leq \tau_0 / \log D$, in place of $0 < T \leq D^{-1}$, with a volume threshold that depends on neither D , α , nor T . The proof is that of [112, Sections 3–5], much of which is reproduced there from [111, 115], with the origins recorded in Table A.1; we check each use of the temperature. Propositions B.1 to B.4 extend [112, Proposition 3.5], [112, Proposition 4.3, Corollary 4.4, and Propositions 4.5 and 4.6], [112, Proposition 5.7], and [112, Proposition 5.4], respectively, and their proofs display every inequality in which the temperature enters differently. The supply of useful labels, Lemma 8.9, involves no temperature. The statements of [112, Sections 7.2–7.4, 8, and 9] and the occupation bound [112, Theorem 2.1] are not used.

B.1 The window and the order of choices

We use the scaled torus setting of Section 8.1: $K \geq 100$ is an integer, $V = K^3$, $D = N/V$, $r = \alpha/D$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, and $\beta = T^{-1}$, with the spatial constants α_0 , ℓ , θ_0 , b_0 , and C_{reg} fixed there. For $j \geq 0$ and a bounded measurable function $W \geq 0$ on Λ_K , we use the operators $H_j^{(-W)}$, the partition functions $Z_j^{(-W)} = \text{tr } e^{-\beta H_j^{(-W)}}$, the free energies $F_j^{(-W)} = -T \log Z_j^{(-W)}$, and the functionals $\mathcal{F}_j^{(-W)}(\sigma)$ of states σ of finite kinetic energy, with the entropy $S(\sigma)$, of [112, Section 2]. They are defined as in Section A.1, with Λ_K in place of U and the torus distance in place of the Euclidean distance. Thus $Z_N = \text{tr } e^{-\beta H}$, and $\mu(dX) = Z_N^{-1} K_\beta^s(X, X) dX$ is the equilibrium position law. We write $\mathbb{E}_{J, -W}$ for the expectation under the position law of the Gibbs state $e^{-\beta H_j^{(-W)}} / Z_j^{(-W)}$, $\mathbb{E}_\sigma$ for the expectation under the position law of a state σ , and $n_f(X) = \sum_i f(x_i)$. Throughout this appendix, $c, C > 0$ denote constants that may change from line to line; unless stated otherwise, they depend only on the spatial constants, and not on K , N , D , α , T , or t .

The window. Let $\alpha_+ = 2\alpha_0$. We fix

$$\kappa_T = \frac{1}{16}, \quad \tau_0 = \kappa_T \alpha_0, \quad 0 < T \leq \frac{\tau_0}{\log D}. \quad (\text{B.1})$$

This is the window of (8.1). Let $C_{\text{ins}} \geq 1$ and $C_f \geq 1$ be numerical constants such that, once D is large depending on α_0 , the bound

$$F_j^{(-W)} - F_{j-1}^{(-W)} \leq C_{\text{ins}}\alpha + 1 + T \log_+(C_f j/V), \quad 1 \leq j \leq 2N, \quad (\text{B.2})$$

holds for every $T > 0$ and every bounded measurable $W \geq 0$, both on the torus and in every container with $V = |U|$ and $K \geq K_\Omega$. Here $\log_+ x = \max\{\log x, 0\}$. On the torus, (B.2) is the bound obtained in the proof of [112, Proposition 3.5] before the temperature is restricted (see the proof of Proposition B.1); in a container, it is the corresponding bound in the proof of Proposition A.9. Every threshold on D below is at least $\max\{2C_f, e^{4\tau_0}\}$. Then $\log(2C_f D) \leq 2 \log D$, and in the window

$$T \leq \frac{\tau_0}{\log D} \leq \frac{1}{4}, \quad \beta \geq 4 > \delta \quad \text{for every } \delta = 2t \text{ with } t \in (0, 1], \quad (\text{B.3})$$

$$T \log(2C_f D) \leq 2T \log D \leq 2\tau_0 = \frac{\alpha_0}{8} \leq \frac{\alpha}{4}. \quad (\text{B.4})$$

We set

$$C_{\text{inc}} = C_{\text{ins}} + \frac{5}{4}, \quad m_0 = \max\{2, 4(C_{\text{inc}}\alpha + 1)\}. \quad (\text{B.5})$$

For $j \leq 2N$ we have $j/V \leq 2D$, and (B.4) and $\alpha \geq 1$ bound the right side of (B.2) by $C_{\text{inc}}\alpha$ in the window. The constant C_{inc} is numerical, and m_0 depends only on α_0 .

Order of choices. The constants are chosen in the following order, and each depends only on those chosen before it:

1. the spatial constants $\alpha_0, \ell, \theta_0, b_0$, and C_{reg} of Section 8.1;
2. $\kappa_T = 1/16$ and $\tau_0 = \kappa_T \alpha_0$;
3. C_{inc} , and then m_0 , by (B.5);
4. the prescribed base p_1 , and a base $p' \in (0, 1)$ for the six tests with $6(p')^{1/6} \leq p_1$ (for $p_1 \geq 1$ there is nothing to prove);
5. H_1 ;
6. $\bar{t} \in (0, 1]$, and then the time $t \in (0, \bar{t}]$, with $\delta = 2t$;
7. the fixed tilt $\lambda = \log(2/p')$ of the test for irregular visitors;
8. the threshold $D_0 = D_0(t)$;
9. the volume threshold $K_1 = 100$ of Lemma B.5.

In particular, κ_T, C_{inc} , and m_0 do not depend on p_1, H_1 , or t . The thermal threshold $D_0(t)$ is distinct from the density threshold D_{min} of Section 10.3, which is chosen later and also involves M and the constants of the surgery.

B.2 Statements whose proofs hold at every temperature

The setting of [112, Section 2] involves the temperature only through $\beta > 0$: the killed heat kernels are nonnegative, heat traces are finite at every positive time, and the Gibbs variational principle holds for every $T > 0$. In [112], the standing range is $0 < T \leq D^{-1}$ [112, (2.1)], and the statements of [112, Sections 3–5] without an explicit range are proved in it. Table B.1 lists the statements of [112, Sections 3–7] used to prove [112, Proposition 5.4 and Lemma 7.1]. For each statement whose proof holds for every $T > 0$, or for every $\beta > \delta$, the table records the reason, and we use the statement without change. The other statements are extended in Sections B.3 to B.6. The endpoint-window length $h_0 = D^{-1}$, the mesh scales of (8.6), and the scale restrictions (8.10) are spatial; they are kept, and the midpoint stays averaged, as in Remark 8.8.

Table B.1. Use of the temperature in the statements of [112] used to prove [112, Proposition 5.4 and Lemma 7.1].

Statement of [112]	Use of the temperature	Here
Lemma 3.1, cube energy	None: a kinetic inequality for hard-core functions.	Propositions B.1 and B.2
Lemma 3.2, close-neighbor energy	None.	Proposition B.2
Lemma 3.3, entropy	Proved for every $T > 0$. Its error $-Tj - CT O (V^{-1} + T^{3/2})$, including the term V^{-1} from the zero mode of the torus, is kept at every use.	Propositions B.1 and B.2
Lemma 3.4, normalized insertion	None: its smallness condition is geometric.	Propositions B.1 and B.2
Proposition 3.5, particle-number bounds	Stated for $0 < T \leq D^{-1}$. Extended.	Proposition B.1
Lemma 4.1, two-color localization	Proved for every $T > 0$. The color entropy $-TS(\omega)$, with $S(\omega) \leq C + \mathbb{E}_\sigma n_{p^2}$, is kept. Each sector of fixed color numbers is one tensor product, with no binomial factor.	Proposition B.2
Lemma 4.2, diagonal tilt	Proved for every $T > 0$. The term $T(2\mathbb{E}_{\sigma_g} G - \log M_G)$ is kept; the gradient term has no factor β .	Proposition B.2
Proposition 4.3 and Corollary 4.4, local penalty and density in a field	Standing range; the proof uses Proposition 3.5 and $T \leq 1$. Extended.	Proposition B.2 (1), (2)

Statement of [112]	Use of the temperature	Here
Propositions 4.5 and 4.6, low and close counts	Stated for $0 < T \leq D^{-1}$. Extended.	Proposition B.2 (3), (4)
Proposition 5.1, purification	Only $\beta > \delta$, so that the outer times $\beta/2 - t$ are positive.	Propositions B.3 and B.4
Definition 5.2, regular path	None: $h_0 = D^{-1}$ is an endpoint-window length on $[0, 2t]$.	Definition 8.3
Lemma 5.3, free cutoffs	None: free paths and bridges of duration $2t$, or two legs of duration t .	Proposition B.3
Proposition 5.4, certificates	Stated for $0 < T \leq D^{-1}$ and K sufficiently large. Extended.	Proposition B.4
Lemma 5.5, marked kernel	None: free kernels of duration δ .	Proposition B.3
Lemma 5.6, trace comparison	Proved for every real $q > 0$, and used at $q = \beta/\delta$; the Hölder step before it has the exponents $\beta/(\beta - \delta)$ and β/δ , and requires $\beta > \delta$.	Proposition B.3
Proposition 5.7, trajectory moment	Standing range; the proof uses $\beta \geq D$ in the choice of the color gaps and Proposition 3.5 for the removal of particles. Extended.	Proposition B.3
Lemmas 6.1–6.3, Remark 6.4, and Corollary 6.5, bridge estimates	None: deterministic estimates for free bridges of duration t and $2t$; the scale restrictions relate D to t , not to β .	Lemmas 8.7 and 8.9 and Remark 8.8
Lemma 7.1, useful labels	None: deterministic on the event of a certificate.	Lemma 8.9

B.3 Free energies and particle numbers

The next proposition extends [112, Proposition 3.5], which is stated uniformly in $K \geq 100$ and $0 < T \leq D^{-1}$. We follow its proof. We write $c_{\text{cube}} = 1/(64\sqrt{3})$ for the constant of [112, Lemma 3.1].

Proposition B.1. *There is D_1 , depending only on α_0 , such that the following hold for every integer $K \geq 100$, every N with $D = N/V \geq D_1$, every $\alpha \in [\alpha_0/2, 2\alpha_0]$, and every $0 < T \leq \tau_0/\log D$.*

1. *For $0 \leq j \leq 2N$, the symmetric hard-core form domain of j particles on Λ_K is nonzero. Hence $0 < Z_j^{(-W)} < \infty$ for every bounded measurable $W \geq 0$.*
2. $F_N \geq (c_{\text{cube}}/4)\alpha N$.
3. $F_i/i \leq F_j/j$ for $1 \leq i \leq j \leq 2N$, and $F_{N+k} \geq F_N + (c_{\text{cube}}/4)\alpha k$ for $0 \leq k \leq N$.

4. $F_j^{(-W)} - F_{j-1}^{(-W)} \leq C_{\text{inc}}\alpha$ for $1 \leq j \leq 2N$ and every bounded measurable $W \geq 0$, with the numerical constant C_{inc} of (B.5).

All quantities are computed on the same torus Λ_K , at the same exclusion distance $r = \alpha/D$ and temperature T , with $D = N/V$.

Used in: Propositions B.2 and B.3.

Proof. All particle-number comparisons keep the reference domain Λ_K , the exclusion distance r , the temperature T , and the reference density $D = N/V$ fixed; we establish the nonzero form domains through $2N$ in part (1), before using their partition-function ratios.

Part (1) is proved in [112] by inserting particles one at a time with [112, Lemma 3.4] and the orbital $V^{-1/2}$, starting from the vacuum. Its smallness condition is at most $C(2N)r^3/V = 2C\alpha^3/D^2$, and it does not involve T .

For part (2), we reserve half of the kinetic energy for [112, Lemma 3.3] with $O = \Lambda_K$, and apply [112, Lemma 3.1] with $\sum_v n_v(n_v - 1) \geq N(D - 1)$ to the other half. As in [112], this gives, for every $T > 0$ and every symmetric trace-one N -particle state σ of finite kinetic energy,

$$\frac{\mathcal{F}_N(\sigma)}{N} \geq \frac{c_{\text{cube}}}{2}\alpha(1 - D^{-1}) - T - \frac{CT}{N} - \frac{CT^{5/2}}{D}. \quad (\text{B.6})$$

The last two terms are the error $CTV(V^{-1} + T^{3/2})$ of [112, Lemma 3.3] divided by $N = DV$; the term CT/N comes from the zero mode of the torus. In [112], the negative terms are absorbed using $T \leq D^{-1}$. In the window, since $N \geq 100^3 D$ and $\alpha \geq 1$,

$$T + \frac{CT}{N} + \frac{CT^{5/2}}{D} \leq \frac{\tau_0}{\log D} + \frac{C\tau_0}{N \log D} + \frac{C\tau_0^{5/2}}{D(\log D)^{5/2}} \leq \frac{c_{\text{cube}}}{8}\alpha$$

once D is large depending on α_0 , uniformly in $K \geq 100$. For $D \geq 4$, (B.6) then gives $\mathcal{F}_N(\sigma) \geq (c_{\text{cube}}/4)\alpha N$, and the Gibbs variational principle gives part (2).

For part (3), [112], following [111, Proposition 3.3], shows that F_j/j is nondecreasing for $1 \leq j \leq 2N$. The proof uses strong subadditivity of the entropies of the marginals of the Gibbs state [98], the fact that taking a marginal preserves the hard-core form domain and the kinetic energy per particle, and the variational principle. Writing S_i for the entropy of the i -particle marginal of the j -particle Gibbs state, strong subadditivity gives $S_i/i \geq S_j/j$. Hence $-TS_i/i \leq -TS_j/j$ for every $T > 0$, which, together with preservation of kinetic energy per particle and the variational principle, proves the monotonicity in the window. With part (2), $F_{N+k} \geq (1 + k/N)F_N \geq F_N + (c_{\text{cube}}/4)\alpha k$.

For part (4), the insertion-frame argument of [112] distinguishes the last label, inserts it with the normalized orbital of [112, Lemma 3.4] for $w = V^{-1/2}$ and the plane waves $e^{ip \cdot y}$, $p \in (2\pi/K)\mathbb{Z}^3$, $|p| \leq 1$, bounds the frame operator of these insertions, applies Jensen's inequality in the spectral measure of each inserted vector, and returns to the symmetric

partition function by positivity of the labeled kernel. These steps hold for every $\beta > 0$ and every bounded measurable $W \geq 0$: the old field terms are unchanged, and the new one is nonpositive. For D large depending on α_0 , they give (B.2) with numerical constants. In [112], the last term of (B.2) is absorbed using $T \leq D^{-1}$. In the window, $j/V \leq 2D$ and (B.4) give $T \log_+(C_f j/V) \leq T \log(2C_f D) \leq \alpha/4$; hence, since $\alpha \geq 1$,

$$F_j^{(-W)} - F_{j-1}^{(-W)} \leq C_{\text{ins}}\alpha + 1 + \frac{\alpha}{4} \leq C_{\text{inc}}\alpha.$$

The constant does not depend on W , including its size. $\square$

B.4 Spatial estimates

The next proposition extends [112, Proposition 4.3 and Corollary 4.4], which are proved in the standing range $0 < T \leq D^{-1}$ of [112], and [112, Propositions 4.5 and 4.6], which are stated uniformly in $K \geq 100$ and $0 < T \leq D^{-1}$. We follow their proofs. For a nonempty set $A \subset \mathbb{L}_K$ and a configuration X of $J \leq N$ points, let

$$f_A(X) = \sum_{i=1}^J e^{-d(x_i, A)}, \quad k_A(X) = \#\{i : d(x_i, A) \leq 6, d_i(X) \leq 2s_0\},$$

as in [112, Section 4.2], with $s_0 = b_0 D^{-1/3}$. These proofs impose on α_0 , ℓ , θ_0 , and b_0 the requirements of [112, Propositions 4.5 and 4.6], with the constants of Proposition B.1. These requirements involve no temperature, and they are preserved when α_0 is increased and ℓ , θ_0 , and b_0 are decreased; we take the constants of Section 8.1 to meet them.

Proposition B.2. *There are constants $c, C > 0$, independent of b_0 , and a threshold $D_2 \geq D_1$ such that the following hold for every integer $K \geq 100$, every N with $D = N/V \geq D_2$, every $\alpha \in [\alpha_0/2, 2\alpha_0]$, and every $0 < T \leq \tau_0/\log D$.*

1. *For every nonempty $A \subset \mathbb{L}_K$, every $0 \leq J \leq N$, every measurable W with $0 \leq W \leq 1$, and every symmetric trace-one J -particle state σ of finite kinetic energy,*

$$\mathcal{F}_J^{(-W)}(\sigma) - F_J^{(-W)} \geq \alpha [\mathbb{E}_\sigma(f_A + cb_0^{-3}k_A) - CD|A|].$$

2. *For every $A \subset \mathbb{L}_K$, every $0 \leq J \leq N$, and every measurable W with $0 \leq W \leq 1$, we have $\mathbb{E}_{J, -W} \sum_{v \in A} n_v \leq CD|A|$.*
3. *For every $A \subset \mathbb{L}_K$, we have $\mu\{n_v < \ell D \text{ for every } v \in A\} \leq e^{-cD|A|}$.*
4. *For every $A \subset \mathbb{L}_K$, we have $\mu\{N_v^{\text{cl}} \geq \theta_0 D \text{ for every } v \in A\} \leq \exp(-c\theta_0 b_0 D^{2/3}|A|)$.*

Used in: Propositions B.3 and B.4, and in Appendix A for the uses of the temperature.

Proof. The proofs in [112] use [112, Lemmas 3.1–3.4, 4.1, and 4.2], which hold for every $T > 0$ (Table B.1), and [112, Proposition 3.5], for which we use Proposition B.1. The temperature enters only in the inequalities displayed below. We use $T \leq 1/4$ from (B.3), and $\alpha \geq 1$.

Part (1). For $h \geq 0$, let p_h , q_h , and the inside region O_h be as in the proof of [112, Proposition 4.3], so that $|O_h| \leq \min\{V, C(h+4)^3|A|\}$. We apply [112, Lemma 4.1] to σ , with the distribution ω of the number of particles of the first color, its entropy $\mathcal{S}(\omega)$, and the inside and outside states σ_j^{in} and σ_j^{out} . Outside, Proposition B.1(4), applied j times with the field W , gives

$$\mathcal{F}_{J-j}^{(-W)}(\sigma_j^{\text{out}}) \geq F_{J-j}^{(-W)} \geq F_J^{(-W)} - C_{\text{inc}}\alpha j.$$

Inside, one third of the kinetic energy and [112, Lemma 3.3] with $O = O_h$ pay for the entropy $TS(\sigma_j^{\text{in}})$, with the error $-Tj - CT|O_h|(V^{-1} + T^{3/2})$; the field costs at most j ; and the color entropy costs $TS(\omega) \leq T(C + \mathbb{E}_{\sigma} n_{p_h^2})$. Since $\sum_j j\omega_j = \mathbb{E}_{\sigma} n_{p_h^2}$, the losses that are linear in the inside number are at most

$$(C_{\text{inc}}\alpha + 2T + 1) \mathbb{E}_{\sigma} n_{p_h^2} \leq C\alpha \mathbb{E}_{\sigma} n_{p_h^2},$$

and the remaining entropy error is at most

$$CT[1 + |O_h|(V^{-1} + T^{3/2})] \leq C(h+4)^3|A|,$$

since $T \leq 1$, $|O_h| \leq V$, and $|A| \geq 1$. These are the two inequalities of the proof of [112, Proposition 4.3] that use the temperature: there, the first relies on [112, Proposition 3.5] in its range, and the second uses $T \leq 1$. The term V^{-1} is kept. The rest of that proof, namely the averaging over h with the density e^{-h} , the cell and close-neighbor energies of [112, Lemmas 3.1 and 3.2], and the completion of the square, does not involve the temperature, and it gives part (1) with constants independent of b_0 .

Part (2). The left side of part (1) vanishes for the Gibbs state $e^{-\beta H_J^{(-W)}} / Z_J^{(-W)}$, which exists by Proposition B.1(1). Dropping k_A and using $\sum_{v \in A} n_v \leq ef_A$, we obtain part (2), as in the proof of [112, Corollary 4.4].

Part (3). Let $A \neq \emptyset$, and let the masks p , q , the squared cutoff ϕ with $p^2 + |\nabla p|^2 + |\nabla q|^2 \leq C\phi$, the fraction η , the number $k = \lfloor \eta D|A| \rfloor$, the symmetric filling vector ψ_k , and the filling map $\mathcal{U}_j f = \binom{N-j+k}{k}^{1/2} P_{\text{sym}}(f \otimes \psi_k)$ be as in the proof of [112, Proposition 4.5]. The map $\mathcal{U}_j$ is an isometry, because the terms for different choices of the k added labels have disjoint supports; it preserves the nonzero spectrum, and hence the entropy, of the outside state. The binomial coefficient is its normalization, and no binomial term enters the free energy. Since $|A| \leq V$, we have $k \leq \eta N$, so all particle numbers below are at most $2N$. Proposition B.1(3) at $N+k$, and then Proposition B.1(4) applied j times, give

$$\mathcal{F}_{N-j}(\sigma_j^{\text{out}}) \geq F_{N-j+k} - Ck(1 + rk/|A|) \geq F_N + \frac{c_{\text{cube}}}{4}\alpha k - C_{\text{inc}}\alpha j - Ck(1 + rk/|A|).$$

With α_0 large and then η small, as in [112], and with D large, the last term is at most $(c_{\text{cube}}/8)\alpha k$, and $k \geq \eta D|A|/2$. Inside, [112, Lemma 3.3] with the full kinetic energy, on the union of the interiors of the cells of A , gives $\mathcal{F}_j(\sigma_j^{\text{in}}) \geq -Tj - CT|A|(V^{-1} + T^{3/2})$. We insert both bounds and the color entropy $T\mathcal{S}(\omega) \leq T(C + \mathbb{E}_\sigma n_{p^2})$ into [112, Lemma 4.1]. For every symmetric trace-one N -particle state σ of finite kinetic energy, we obtain

$$\begin{aligned} \mathcal{F}_N(\sigma) - F_N &\geq \frac{c_{\text{cube}}\eta}{16}\alpha D|A| - (C_{\text{inc}}\alpha + 2T)\mathbb{E}_\sigma n_{p^2} - \mathbb{E}_\sigma n_{|\nabla p|^2 + |\nabla q|^2} \\ &\quad - CT - CT|A|(V^{-1} + T^{3/2}) \\ &\geq c_p\alpha D|A| - C_p\alpha \mathbb{E}_\sigma n_\phi, \end{aligned}$$

with $c_p = c_{\text{cube}}\eta/32$, using $T \leq 1$, $V \geq 1$, $p^2 + |\nabla p|^2 + |\nabla q|^2 \leq C\phi$, and D large to absorb $C|A|$.

For $0 \leq \lambda \leq 1$, let $G = -\lambda n_\phi$ and $L(\lambda) = -\log \mathbb{E}_\mu e^{-2\lambda n_\phi}$, and let σ_g be the tilted state of [112, Lemma 4.2]. That lemma, with $|\nabla G|^2 \leq C\lambda^2 n_\phi$ and $2\mathbb{E}_{\sigma_g} G \leq 0$, gives $\mathcal{F}_N(\sigma_g) - F_N \leq C\mathbb{E}_{\sigma_g} n_\phi + TL(\lambda)$. With the previous display for $\sigma = \sigma_g$, we obtain

$$C'_p\alpha \mathbb{E}_{\sigma_g} n_\phi + TL(\lambda) \geq c_p\alpha D|A|, \quad L'(\lambda) = 2\mathbb{E}_{\sigma_g} n_\phi,$$

with $C'_p = C_p + C$. Let $a' = \min\{c_p/2, c_p/(2C'_p)\}$. Whenever $L(\lambda) \leq a'D|A|$, we have

$$TL(\lambda) \leq a'D|A| \leq \frac{c_p}{2}\alpha D|A|,$$

since $T \leq 1 \leq \alpha$. This is the only use of the temperature in the tilt, and the factor $D|A|$ appears on both sides. Then $L'(\lambda) \geq 2a'D|A|$ wherever $L(\lambda) \leq a'D|A|$, and the argument of [112] gives $L(1) \geq a'D|A|$ and the bound $\exp[-(a' - 2\ell)D|A|]$ for the event in part (3). The requirement $2\ell < a'$ involves no temperature.

Part (4). Let m_A be the Lipschitz enlargement of the close count in the proof of [112, Proposition 4.6], which satisfies $m_A \leq k_A$, $\sum_{v \in A} N_v^{\text{cl}} \leq C m_A$, and $|\nabla m_A|^2 \leq C s_0^{-2} m_A$. For $G = \lambda m_A$ with $0 \leq \lambda \leq s_0$, [112, Lemma 4.2] and $M_G \geq 1$ give

$$\mathcal{F}_N(\sigma_g) - F_N \leq (C\lambda^2 s_0^{-2} + 2T\lambda)\mathbb{E}_{\sigma_g} m_A \leq (C + 1)\mathbb{E}_{\sigma_g} m_A,$$

since $\lambda \leq s_0 < 1/2$ and $T \leq 1$. Part (1) with $W = 0$ and $J = N$ gives the lower bound $c\alpha b_0^{-3}\mathbb{E}_{\sigma_g} m_A - C\alpha D|A|$, with c independent of b_0 . The rest of the proof of [112, Proposition 4.6], namely the choice of b_0 , the integration of the logarithmic derivative of $\mathbb{E}_\mu e^{2\lambda m_A}$, and the exponential Markov inequality, does not involve the temperature. $\square$

B.5 The trajectory moment

Let $t \in (0, 1]$ and $\delta = 2t$. For a nonempty set $A \subset \mathbb{L}_K$, let $J_A(\omega)$ be the number of sites of A whose balls of radius 12 a path ω on $[0, \delta]$ visits. Let the path events E_{all} , E_{diam} , and E_{irr} , the exit count $n(\omega)$, the sets O_n with their cutoffs p_n , and the coefficients

$$l_n = C\lambda(n+1)e^{C\lambda(n+1)}e^{-cn^2/t}\epsilon_E^{1/3}, \quad \epsilon_E = 1, C e^{-c/t}, C_t D^{-12} \quad \text{for } E = E_{\text{all}}, E_{\text{diam}}, E_{\text{irr}},$$

of the marked kernel be as in [112, Section 5.3 and Lemma 5.5], for D large enough that this lemma applies; their Euclidean versions are written out in Section A.12. The sets O_n may be the whole torus. With m_0 from (B.5), the color gaps and coefficients of [112, Section 5.4] are

$$\mathfrak{g}_n = \frac{m_0 + n}{\delta}, \quad \varkappa_n = l_n e^{m_0 + n}.$$

The next proposition extends [112, Proposition 5.7], which is proved in the standing range $0 < T \leq D^{-1}$ of [112]. We follow its proof and the colored comparison of [112, Section 5.4] that precedes it.

Proposition B.3. *There are a numerical constant C_0 and a constant C , independent of K , D , α , T , t , and λ , such that the following holds for every integer $K \geq 100$, every N with $D = N/V \geq D_2$ also large enough, depending only on t and the spatial constants, for [112, Lemma 5.5] to apply, every $\alpha \in [\alpha_0/2, 2\alpha_0]$, every $0 < T \leq \tau_0/\log D$, every $\lambda > 0$, every $E \in \{E_{\text{all}}, E_{\text{diam}}, E_{\text{irr}}\}$, and every nonempty $A \subset \mathbb{L}_K$. If $(C_0/\delta) \sum_{n \geq 0} \varkappa_n \leq 1$, then*

$$\log \mathbb{E}_{\mathbb{P}^{\text{pur}}} \exp \left(\lambda \sum_{i=1}^N J_A(\omega_i) \mathbf{1}_E(\omega_i) \right) \leq CD|A| \sum_{n \geq 0} (n+1)^3 \varkappa_n + R_{D,K,\delta} |A|, \quad (\text{B.7})$$

where

$$R_{D,K,\delta} = C\delta T(V^{-1} + T^{3/2}) \sum_{n \geq 0} (n+1)^3 e^{-(m_0+n)/(2\delta T)}.$$

Moreover, uniformly in $t \in (0, 1]$ and $K \geq 100$,

$$\begin{aligned} R_{D,K,\delta} &\leq C \left[\frac{\tau_0}{V \log D} + \frac{\tau_0^{5/2}}{(\log D)^{5/2}} \right] D^{-m_0/(4\tau_0)} \leq r_D, \\ r_D &= C \left[\frac{\tau_0}{\log D} + \frac{\tau_0^{5/2}}{(\log D)^{5/2}} \right] D^{-m_0/(4\tau_0)}, \end{aligned} \quad (\text{B.8})$$

and $r_D \rightarrow 0$ as $D \rightarrow \infty$.

Used in: Proposition B.4.

Proof. The proof of [112, Proposition 5.7] bounds the moment, with the bins truncated at $n \leq n_*$, by the colored comparison of [112, Section 5.4]: the marked kernel of [112, Lemma 5.5], the color map, the pointwise domination of the tested slab kernel, the symmetric trace identity, the Schatten Hölder inequality with exponents $\beta/(\beta - \delta)$ and β/δ , the compression inequality, and [112, Lemma 5.6] at $q = \beta/\delta$. These steps hold for every $\beta > \delta$ (Table B.1), and $\beta > \delta$ by (B.3). As in [112], the smallness condition gives a field W_0 with $0 \leq W_0 \leq (C/\delta) \sum_{n \leq n_*} \varkappa_n \mathbf{1}_{O_n} \leq 1$, and the comparison reduces the claim to a bound on $\log(\text{tr } e^{-\beta \tilde{\mathbf{H}}} / Z_N)$, where $\tilde{\mathbf{H}}$ is the colored operator with W_0 subtracted. The temperature enters this bound at the following places.

The color gaps. In [112], m_0 is chosen so that $\mathfrak{g}_n/2 \geq C_{\text{inc}}\alpha + 1$ and $\beta\mathfrak{g}_n/2 \geq 1$, the latter using $\beta \geq D \geq 1$. With (B.5) and $\delta \leq 2$,

$$\frac{\mathfrak{g}_n}{2} = \frac{m_0 + n}{2\delta} \geq \frac{m_0}{4} \geq C_{\text{inc}}\alpha + 1 \geq C_{\text{inc}}\alpha + 1, \quad \frac{\beta\mathfrak{g}_n}{2} > \frac{m_0 + n}{2} \geq 1, \quad (\text{B.9})$$

where the last inequalities use only $\beta > \delta$ and $m_0 \geq 2$.

The original color. For $0 \leq J \leq N$ and $0 \leq u \leq 1$, we have $\partial_u \log Z_J^{(-uW_0)} = \beta \mathbb{E}_{J, -uW_0} \sum_i W_0(x_i)$. Proposition B.2(2), applied to covers of the sets O_n by at most $C(n+1)^3|A|$ cells, gives, after integration in u ,

$$\log \frac{Z_J^{(-W_0)}}{Z_J} \leq \frac{\beta C}{\delta} D|A| \sum_{n \leq n_*} (n+1)^3 \varkappa_n. \quad (\text{B.10})$$

Removal and the gaps, paid once. By Proposition B.1(4) with $W = 0$, applied $N - J$ times, $Z_J/Z_N \leq e^{\beta C_{\text{inc}}\alpha(N-J)}$. The enlarged symmetric space decomposes into sectors with J original-color particles and k_n particles of color n , where $J = N - \sum_n k_n$. Each sector is one normalized tensor product of symmetric species spaces, with no multinomial multiplicity, as shown in the proof of [112, Proposition 5.7]. We attach the factor $e^{\beta C_{\text{inc}}\alpha}$ to each marked particle. With $W_0 \leq 1$ and (B.9), the one-particle operator of color n then satisfies, in the sense of forms,

$$-3\Delta_{O_n} + \mathfrak{g}_n - W_0 - C_{\text{inc}}\alpha \geq -3\Delta_{O_n} + \frac{\mathfrak{g}_n}{2}. \quad (\text{B.11})$$

This single inequality pays for the removal of the marked particles and for the attractive field W_0 , and it leaves the positive gap $\mathfrak{g}_n/2$. The insertion entropy of the removal is contained in C_{inc} by (B.4), and no other factor is charged for it. The variational principle then bounds each marked canonical trace by that of $-3\Delta_{O_n} + \mathfrak{g}_n/2$. Since J is determined by the occupations k_n , dropping the restriction $\sum_n k_n \leq N$ introduces no sum over J .

The marked traces. For $s > 0$, the periodic Gaussian image sum and killing give $\text{tr} e^{3s\Delta_{O_n}} \leq C|O_n|(V^{-1} + s^{-3/2})$, at every $s > 0$. Let Ξ_n be the ideal Bose grand partition function of $-3\Delta_{O_n} + \mathfrak{g}_n/2$ at inverse temperature β . Since $\beta\mathfrak{g}_n/2 \geq 1$ by (B.9),

$$\log \Xi_n = \sum_{a \geq 1} \frac{e^{-a\beta\mathfrak{g}_n/2}}{a} \text{tr} e^{3a\beta\Delta_{O_n}} \leq C|O_n|(V^{-1} + T^{3/2})e^{-\beta\mathfrak{g}_n/2}.$$

The term V^{-1} , from the zero mode of the torus, is kept. Combining the last three steps with $|O_n| \leq C(n+1)^3|A|$, we obtain

$$\log \frac{\text{tr} e^{-\beta\tilde{\mathcal{H}}}}{Z_N} \leq \frac{\beta C}{\delta} D|A| \sum_{n \leq n_*} (n+1)^3 \varkappa_n + C|A|(V^{-1} + T^{3/2}) \sum_{n \leq n_*} (n+1)^3 e^{-\beta\mathfrak{g}_n/2}.$$

The factor δ/β . The comparison bounds the logarithm of the truncated moment by $\delta/\beta = \delta T$ times the last display. This factor cancels the factor β/δ of (B.10) exactly, and it turns the second term into $R_{D,K,\delta}|A|$, since $\beta g_n/2 = (m_0 + n)/(2\delta T)$. The bounds do not depend on n_* , and monotone convergence removes the truncation, as in [112]. This proves (B.7).

The remainder. By (B.3), $T \leq 1/4$, and $\delta \leq 2$; hence $x = 1/(2\delta T) \geq 1$ and

$$\sum_{n \geq 0} (n+1)^3 e^{-(m_0+n)x} \leq e^{-m_0x} \sum_{n \geq 0} (n+1)^3 e^{-n} \leq C e^{-m_0x}.$$

Since $\delta \leq 2$, this gives $R_{D,K,\delta} \leq CT(V^{-1} + T^{3/2})e^{-m_0/(4T)}$. The functions $T \mapsto Te^{-m_0/(4T)}$ and $T \mapsto T^{5/2}e^{-m_0/(4T)}$ are increasing on $(0, \infty)$, and $e^{-m_0/(4T)} = D^{-m_0/(4\tau_0)}$ at $T = \tau_0/\log D$. This gives the first inequality in (B.8) for every $0 < T \leq \tau_0/\log D$, and the second follows from $V \geq 1$. $\square$

B.6 The joint certificate bound

The next proposition extends [112, Proposition 5.4], which is stated for $D \geq D_0$, $0 < T \leq D^{-1}$, and K sufficiently large. We follow its proof in [112, Section 5.6], with the order of choices of Section B.1. Certificates are those of Definition 8.4.

Proposition B.4. *For each $p_1 > 0$ there are H_1 and $\bar{t} \in (0, 1]$, and for each $t \in (0, \bar{t}]$ there is D_0 , such that, for every set A of distinct cell centers,*

$$\mathbb{P}^{\text{pur}}\{\text{no site of } A \text{ has a certificate}\} \leq p_1^{|A|}$$

whenever $K \geq 100$, $D \geq D_0$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, and $0 < T \leq \tau_0/\log D$.

Used in: Lemma B.5 and the proof of Proposition 8.5; also Appendix A.

Proof. We may assume $p_1 < 1$ and $A \neq \emptyset$. Fix $p' \in (0, 1)$ with $6(p')^{1/6} \leq p_1$. A site v has no certificate if and only if one of six tests fails there: $n_v(X(0)) < \ell D$; $N_v^{\text{cl}}(X(0)) \geq \theta_0 D$; $N_v^{\text{cl}}(X(2t)) \geq \theta_0 D$; more than $H_1 D$ paths visit the ball of radius 12 about v ; some path visiting this ball is not regular; at least $\theta_0 D$ paths visiting it have lifted diameter greater than one. As in [112], it suffices to show that, for each test and each $A' \subset \mathbb{L}_K$, the probability that the test fails at every site of A' is at most $(p')^{|A'|}$. Indeed, given failures at every site of A , we assign a failing test to each site. There are at most $6^{|A|}$ assignments, in each some test is assigned to a set A' with $|A'| \geq |A|/6$, and the union bound gives $(6(p')^{1/6})^{|A|} \leq p_1^{|A|}$. No independence between sites or between tests is used. Below, r_D is the remainder of (B.8); it depends on neither t , K , nor T , and it tends to zero as $D \rightarrow \infty$. We also use the bound $\sum_n (n+1)^3 \varkappa_n \leq C_\lambda \lambda \epsilon_E^{1/3}$ from the Gaussian tail of the bins, and

the same bound without the factor $(n+1)^3$ [112, Section 5.6]; here C_λ is bounded for λ in bounded intervals, uniformly in $t \in (0, 1]$.

Endpoint tests. Since $\beta > \delta$, [112, Proposition 5.1] shows that $X(0)$ and $X(2t)$ have law μ under $\mathbb{P}^{\text{pur}}$. By Proposition B.2(3) and (4), the three endpoint tests fail at every site of A' with probabilities at most $e^{-cD|A'|}$ and $\exp(-c\theta_0 b_0 D^{2/3}|A'|)$, and both bases are at most p' for D large.

Visitors. We take $E = E_{\text{all}}$, for which $\epsilon_E = 1$, and $\lambda = c't$ with a constant $c' \in (0, 1]$. Then $\sum_n \varkappa_n \leq Cc't$ with C independent of t , so the smallness condition of Proposition B.3 holds for every $t \in (0, 1]$ once c' is small, and the first term in (B.7) is at most $C_4\lambda D|A'|$ with C_4 independent of t . If every site of A' has more than $H_1 D$ visitors, then $\sum_i J_{A'}(\omega_i) > H_1 D|A'|$, and the exponential Markov inequality and Proposition B.3 bound the probability by

$$\exp\{-\lambda(H_1 - C_4)D + r_D\}|A'|.$$

We choose $H_1 = C_4 + 1$. For each $t \in (0, 1]$, the base $\exp(-c'tD + r_D)$ is at most p' once D is large depending on t . Decreasing t does not affect H_1 .

Visitors of large diameter. We take $E = E_{\text{diam}}$ and $\lambda = 1$. Then $\epsilon_E^{1/3} \leq Ce^{-c/t}$, and both $\sum_n \varkappa_n$ and $\sum_n (n+1)^3 \varkappa_n$ are at most $Ce^{-c/t}$. We choose $\bar{t} \in (0, 1]$ such that, for every $t \in (0, \bar{t}]$, the smallness condition holds and the first term in (B.7) is at most $(\theta_0 D/2)|A'|$. If every site of A' is visited by at least $\theta_0 D$ paths of diameter greater than one, then $\sum_i J_{A'}(\omega_i) \mathbf{1}_E(\omega_i) \geq \theta_0 D|A'|$, and the probability is at most $\exp[-(\theta_0 D/2 - r_D)|A'|] \leq (p')^{|A'|}$ for D large. We now fix $t \in (0, \bar{t}]$.

Irregular visitors. We take $E = E_{\text{irr}}$ and $\lambda = \log(2/p')$. Then $\epsilon_E^{1/3} = C_t D^{-4}$, so that $\sum_n (n+1)^3 \varkappa_n \leq C_{p',t} D^{-4}$; the smallness condition holds for D large, and the first term in (B.7) is at most $C_{p',t} D^{-3}|A'|$. If every site of A' is visited by an irregular path, then $\sum_i J_{A'}(\omega_i) \mathbf{1}_E(\omega_i) \geq |A'|$, even if one path visits several of these sites. The probability is at most

$$\exp[-(\log(2/p') - C_{p',t} D^{-3} - r_D)|A'|] \leq (p')^{|A'|}$$

once $C_{p',t} D^{-3} + r_D \leq \log 2$. Here the remainder enters once per site, without a factor D , so $r_D \rightarrow 0$ is needed.

The threshold. Let $D_0 = D_0(t)$ be at least D_2 , at least the thresholds of the six tests above, and large enough that the scale conditions after (8.6) hold and [112, Lemma 5.5] applies. None of these thresholds depends on K or T . $\square$

B.7 The volume threshold

Lemma B.5. *The bounds of Propositions B.1 to B.4 hold for every integer $K \geq 100$ and every $0 < T \leq \tau_0 / \log D$, with thresholds on D and constants that depend on neither K nor T ; no relation between β and K is used. Consequently, the volume threshold of Proposition 8.5 can be taken to be $K_1 = 100$.*

Used in: the proof of Proposition 8.5, and Remark 8.1.

Proof. Propositions 3.5, 4.5, and 4.6 of [112] are stated uniformly in $K \geq 100$, the constants of [112, Proposition 5.7] do not depend on K , the bounds of [112, Lemma 5.3] are uniform in K , and, by the convention of [112, Section 2], the constants of [112, Proposition 4.3 and Corollary 4.4] do not depend on K . The marked kernel of [112, Lemma 5.5] allows O_n to be the whole torus and sums over the lifts of the endpoints, so it requires no injectivity of lifts. The proof of [112, Proposition 5.4] combines these statements with [112, Proposition 5.1], and it imposes no other condition on K . In the extended proofs, the volume enters only through the following explicit terms, each of which is bounded uniformly in $K \geq 100$:

1. the term CT/N in (B.6), which is at most $C\tau_0/(100^3 D \log D)$, since $N = DV \geq 100^3 D$;
2. the terms $CT|O_h|V^{-1}$ and $CT|A|V^{-1}$ from [112, Lemma 3.3] in the proof of Proposition B.2, which are at most CT and $CT|A|$, since $|O_h| \leq V$ and $V \geq 1$;
3. the term $\tau_0/(V \log D)$ in (B.8), which is at most $\tau_0/\log D$.

The temperature enters through (B.3), (B.4), and the thermal errors displayed in Sections B.3 to B.5. Each of these errors was bounded either by means of $T \leq 1$, or by the value at $T = \tau_0/\log D$ of an increasing function of T , namely a sum of terms T^a and $T^a e^{-m_0/(4T)}$, $a > 0$, with coefficients that do not depend on T . The bounds therefore hold for every $0 < T \leq \tau_0/\log D$, including the limit $T \downarrow 0$. No step compares β with K . Hence the thresholds D_1 , D_2 , and $D_0(t)$ depend only on the constants of Section B.1 and on p_1 and t , and the propositions hold for every $K \geq 100$. $\square$

Proof of Proposition 8.5. We take $\kappa_T = 1/16$, as in (B.1), and $K_1 = 100$, as in Lemma B.5. Given $p_1 > 0$, Proposition B.4 provides H_1 and $\bar{t} \in (0, 1]$, and, for each $t \in (0, \bar{t}]$, a threshold D_0 such that (8.8) holds whenever $K \geq K_1$, $D \geq D_0$, $\alpha \in [\alpha_0/2, 2\alpha_0]$, and $0 < T \leq \tau_0/\log D$. $\square$

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