

A quantitative refinement of Lu's elementary prime number theorem

Balog's identity and a polylogarithmic loss in the square-root exponent

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Abstract

We refine Lu's elementary method for the prime number theorem to obtain, for every fixed $K > 400$,

$$\psi(X) - X = O_K \left(X \exp \left\{ -\frac{\sqrt{\log X}}{(\log \log X)^K} \right\} \right) \quad (X \rightarrow \infty),$$

where ψ is Chebyshev's function. The main ingredient is a comparison estimate that remains uniform as the degree grows and the short-interval parameter shrinks. Combining this estimate with exponential weights in Balog's identity, a single finite energy step and positivity-based interpolation yields the stated bound, retaining the full degree tail and all arithmetic correction terms. The resulting remainder strengthens Lu's fixed-parameter estimates and Goz 's quantitative refinement within the real-variable framework; stronger remainder estimates are known by other methods.

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1 Introduction

All logarithms are natural. Sums with real cutoffs include their integer upper endpoints, and $0^0 = 1$ in logarithmic polynomials. Let μ be the Möbius function, Λ the von Mangoldt function, $\psi(x) = \sum_{n \leq x} \Lambda(n)$, and γ Euler's constant. We also write $\pi(x) = \#\{p \leq x : p \text{ prime}\}$, $\vartheta(x) = \sum_{p \leq x} \log p$, $\tau(n) = \sum_{d|n} 1$, and $\varphi(n) = \#\{1 \leq a \leq n : (a, n) = 1\}$.

Theorem 1.1. *For every fixed real $K > 400$ there are constants $C_K, X_K > 0$ such that*

$$|\psi(X) - X| \leq C_K X \exp\{-\Phi_K(\log X)\} \quad (X \geq X_K), \quad \Phi_K(L) = \frac{\sqrt{L}}{(\log L)^K} \quad (L > 1).$$

In particular, $K = 512$ is admissible. Neither this value nor the threshold is asserted to be optimal.

1.1 Contribution and relation to earlier work

The elementary remainder method was developed by Bombieri [1] and Wirsing [5], who obtained arbitrary fixed logarithmic savings, and by Diamond and Steinig [2] and Lavrik and Sobirov [3], who obtained stronger exponential remainders. These results provide the historical setting for Lu's work. The fixed-power preliminary estimate used below belongs to this earlier theory; the growing-degree estimates are treated separately in the present proof.

Write $L = \log X$. Lu [4] proves, by elementary means, a remainder with decay profile $L^{1/2-\varepsilon}$ for every fixed $\varepsilon > 0$. We use the same arithmetic objects: the centered remainder R , the generalized von Mangoldt functions Λ_i , their smooth approximations m_i , and Balog's identity. The point of the present argument is quantitative uniformity as both the degree and the short-interval parameter vary. Its final parameter is

$$\lambda = K \frac{\log \log \log X}{\log \log X}.$$

This choice is justified by the estimates below; it cannot be inferred by substituting a varying parameter into Lu's fixed-parameter theorem. For fixed K and $\varepsilon > 0$,

$$\frac{\sqrt{L}/(\log L)^K}{L^{1/2-\varepsilon}} = \frac{L^\varepsilon}{(\log L)^K} \longrightarrow \infty.$$

Thus the stated profile is asymptotically stronger than that family of fixed-parameter estimates.

There is recent work on precisely this quantitative question. Gozé's 2024 thesis [8, Introduction, p. 4] proves that there are absolute constants $c, c' > 0$ such that

$$|R_1(X)| \leq X \exp\{-(\log X)^{1/2-\varepsilon(X)}\}, \quad \varepsilon(X) = \frac{c'}{\log \log \log X}, \quad \log \log \log X \geq c,$$

where $R_1(X) = \psi(X) - \lfloor X \rfloor + 2\gamma$. Its difference from $\psi(X) - X$ is bounded, so this gives the same asymptotic error profile for ψ . The result was also presented by Balazard and Martin [9, p. 3]. In terms of L , the profile is

$$\Phi_G(L) = L^{1/2-c'/\log \log L} = \sqrt{L} \exp\left(-\frac{c' \log L}{\log \log L}\right).$$

For every fixed K and $c' > 0$,

$$\log \frac{\Phi_K(L)}{\Phi_G(L)} = \frac{c' \log L}{\log \log L} - K \log \log L \longrightarrow \infty.$$

This compares asymptotic profiles; it does not compare explicit thresholds. Gozé [8, §20.3, p. 132] also identifies the bound for the coefficients B_i in his Proposition 17.6 as a source of the restriction on the shrinking parameter. A detailed comparison with that estimate is needed to assess the originality of the uniform comparison used here. A March 2026 seminar announcement by Balazard [10] also describes ongoing joint work with Gozé and Martin on simplifying the proofs of Lu and Gozé. It does not specify a further remainder. Consequently, neither the first use of a shrinking parameter nor priority for simplifying this method is claimed here.

A different comparison is essential. Koukoulopoulos [7, §1, equation (1.1) and Theorem 1.1] proves the stronger Korobov–Vinogradov profile

$$\psi(X) - X = O\left(X \exp\left\{-c_0 \frac{L^{3/5}}{(\log L)^{1/5}}\right\}\right), \quad c_0 > 0,$$

using elementary estimates and Fourier inversion. That paper calls the method largely elementary and distinguishes it from proofs using no complex variables. Its profile dominates the present one since

$$\frac{c_0 L^{3/5}/(\log L)^{1/5}}{\Phi_K(L)} = c_0 L^{1/10} (\log L)^{K-1/5} \longrightarrow \infty.$$

Accordingly, this paper makes no claim to a new best numerical PNT remainder, or to a new best remainder among all methods described as elementary. Its scope is the real-variable Balog–Lu argument and the uniform estimates needed for the displayed profile. This distinction concerns the proof method, not an improvement over Koukoulopoulos's theorem. A focused literature review completed on 19 October 2026 did not establish historical priority for this restricted contribution.

1.2 Organization and proof dependencies

The proof follows Lu's progression from short-interval estimates to smoothing and a contraction for the prime remainder. Balog's identity relates that remainder to the generalized von Mangoldt coefficients and the centered divisor average. Its complete exponential generating form supplies the final forcing identity, with the factorial source summed explicitly.

We cite Lu's precise forms of the classical fixed-power preliminary estimates, the elementary interval sieve and the finite Wirsing inequality. Their hypotheses and normalizations are recorded

where used. The finite summation formulas in Section 3 prove the required degree-uniform Möbius bounds, and the prime-power, floor and unit corrections remain explicit. These inputs come from earlier elementary work and do not depend on Lu's final remainder theorem or on the present theorem. The energy parameter N_0 below is one sufficiently large absolute integer, fixed before the degree or interval margin varies.

Positivity transfers the terminal seed to a common domain, and the order-sensitive finite comparison in Lemma 4.2 controls all degrees needed by the exponential series. A single fixed smoothing cutoff and one finite energy step give the coefficient $3/4$ at degree one. The higher cutoff contribution costs at most $1/64$; the remaining normalized degree errors vanish, leaving room for the sufficient $7/8$ slope. The common shrinking-parameter schedule (61) and the full exponential tail are essential: a fixed-parameter estimate is not substituted at a varying parameter.

The last step centers at $1/\zeta(1+\sigma)$ and retains the complete exponential-average correction. Positive block mass controls the chord error with both integer endpoints. The mean retains its self term, and the finite envelope closes with coefficient $28/29$. No further energy iteration is used. All constants described as absolute are independent of the growing degree and the shrinking interval margin.

The choices made here do not establish the endpoint $\Phi(L) = \sqrt{L}$; $K = 512$ is a convenient, unoptimized example. The term elementary refers to finite divisor identities, an elementary sieve and real-variable arguments, including the cited classical preliminary estimate. No complex zero-free region or Fourier inversion is used. The comparisons in Section 1.1 delimit the contribution, whose historical priority remains unresolved.

2 The path of the proof and Balog's identity

Dirichlet convolution is $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$. Write $\mathbf{1}(n) = 1$ and $e_0(n) = \mathbf{1}_{n=1}$; hence $\mu * \mathbf{1} = e_0$. Prime factorization gives, independently,

$$\mathbf{1} * \Lambda = \log, \quad \sum_{d|n} \Lambda(d) = \log n. \quad (1)$$

For $j \geq 1$ define

$$\Lambda_j = \mu * (\log^j), \quad m_j(t) = \sum_{d \leq t} \frac{\mu(d)}{d} \log^j(t/d), \quad m(t) = m_0(t) = \sum_{d \leq t} \frac{\mu(d)}{d}.$$

Here $\log^j$ is an ordinary arithmetic function, not a convolution power. Put, for all $t > 0$,

$$\omega(n) = \Lambda(n) - 1, \quad R(t) = \sum_{n \leq t} \omega(n) + 2\gamma.$$

Thus $R(t) = 2\gamma$ below 1, $\omega(1) = -1$, and $R(t) = \psi(t) - [t] + 2\gamma$ above 1. For $j \geq 2$ set

$$R_j(t) = \sum_{n \leq t} (\Lambda_j(n) - j \log^{j-1} n), \quad R_1 = R, \quad Z_1 = R - 2\gamma, \quad Z_j = R_j \quad (j \geq 2).$$

All Z_j vanish below 1. Finally set

$$u_j(n) = \Lambda_j(n) - m_j(n), \quad U_j(t) = \sum_{n \leq t} u_j(n).$$

Lemma 2.1 (Positivity and the centered divisor average). *For $j \geq 1$,*

$$\Lambda_{j+1}(n) = (\log n)\Lambda_j(n) + (\Lambda * \Lambda_j)(n) \geq 0, \quad \Lambda_j(1) = 0.$$

Moreover, for $t \geq 1$,

$$\tilde{R}(t) := \sum_{n \leq t} R(t/n) = \sum_{n \leq t} \log n - \sum_{n \leq t} \tau(n) + 2\gamma \lfloor t \rfloor = O(\sqrt{t}). \quad (2)$$

Consequently, for $1 \leq y < x$,

$$\sum_{n \leq x} [R(x/n) - R(y/n)] = \tilde{R}(x) - \tilde{R}(y) - 2\gamma(\lfloor x \rfloor - \lfloor y \rfloor). \quad (3)$$

Proof. The operator $\mathcal{D}f(n) = (\log n)f(n)$ is a derivation for convolution. Differentiating $\mu * \mathbf{1} = e_0$ gives $\mathcal{D}\mu = -\mu * \Lambda$. Applying this to $\mu * \log^j$ proves the recurrence; induction from $\Lambda_1 = \Lambda$ proves positivity. Expanding the finite divisor average gives (2). To bound it, let $N = \lfloor \sqrt{t} \rfloor$. The elementary hyperbola formula is

$$\sum_{n \leq t} \tau(n) = 2 \sum_{n \leq N} \lfloor t/n \rfloor - N^2 = t \log t + (2\gamma - 1)t + O(\sqrt{t}),$$

using $\sum_{n \leq N} 1/n = \log N + \gamma + O(1/N)$. Integral comparison gives $\sum_{n \leq t} \log n = t \log t - t + O(\log(2t))$. All linear terms cancel, including $2\gamma t$. The terms with $y < n \leq x$ in the second sum of (3) have $R(y/n) = 2\gamma$, proving its floor correction. $\square$

Lemma 2.2 (Balog's identity and its complete source). *For every integer $k \geq 0$ and every real $X \geq 1$,*

$$R(X)(\log X)^k + \sum_{i=1}^k \binom{k}{i} \sum_{n \leq X} \Lambda_i(n) \log^{k-i}(X/n) R(X/n) = \mathcal{B}_k(X), \quad (4)$$

where the complete right side is

$$\mathcal{B}_k(X) = \sum_{d \leq X} \mu(d) \log^k(X/d) \tilde{R}(X/d). \quad (5)$$

If $|\tilde{R}(t)| \leq C_0 \sqrt{t}$ for $t \geq 1$, then, uniformly in k ,

$$|\mathcal{B}_k(X)| \leq C_0 X 2^{k+1} k!. \quad (6)$$

Proof. For $k \geq 1$, (4) is Balog's identity as stated in Lu [4, §6, Lemma 20, p. 1024]. For $k = 0$, finite Möbius inversion gives $R(X) = \sum_{d \leq X} \mu(d) \tilde{R}(X/d)$, since $\sum_{d|n} \mu(d) = e_0(n)$. Thus $\Lambda_0 = e_0$ accounts for the degree-zero term. All upper cutoffs are inclusive, and the surviving unit $n = d = 1$ contributes $R(X)$.

The bound on $\tilde{R}$ was proved in Lemma 2.1. Since $t^{-1/2} \log^k(X/t)$ is nonnegative and decreasing on $(0, X]$,

$$|\mathcal{B}_k(X)| \leq C_0 \sqrt{X} \sum_{d \leq X} d^{-1/2} \log^k(X/d) \leq C_0 \sqrt{X} \int_0^X t^{-1/2} \log^k(X/t) dt = C_0 X 2^{k+1} k!.$$

The integral over $(0, 1]$ includes the first lattice term. No degree-dependent constant is hidden in this estimate. $\square$

The task in (4) is to control the coefficients Λ_i on short intervals. Following Lu, we separate them as $\Lambda_i = m_i + u_i$. The smooth moments m_i provide the mean; the local sums of u_i control the interpolation error. A direct two-lattice estimate gives the high-degree input. The positive recurrence for Λ_i then transfers that input through the lower degrees, and a finite real-convolution energy step gives a strict degree-one bound. Finally, we sum (4) with weights $\sigma^k/k!$, keep its full source and infinite tail, and close a finite-envelope contraction. The following lemma supplies the first local estimate.

Lemma 2.3 (One local estimate, uniform in the degree). *Let $1 \leq y < x$, $\Delta = x - y \geq \max(1, \sqrt{x})$, and $h = \log(x/\Delta)$. For every integer $i \geq 1$,*

$$|U_i(x) - U_i(y)| \leq \Delta E_i(h), \quad (7)$$

$$E_i(h) = 4Q_i(h) + (3h + 4)h^i, \quad Q_i(h) = \int_0^\infty e^{-z}(h+z)^i dz = i! \sum_{r=0}^i \frac{h^r}{r!}.$$

If $h \geq \max(1, 2i)$, then $Q_i(h) \leq 2h^i$ and $E_i(h) \leq 15h^{i+1}$.

Proof. Extend $f_i(t) = \log^i t$ by zero for $t < 1$. Finite rearrangement gives the exact identity

$$U_i(x) - U_i(y) = \sum_{d \leq x} \mu(d) \left\{ \sum_{y/d < a \leq x/d} f_i(a) - \frac{1}{d} \sum_{y < n \leq x} f_i(n/d) \right\}. \quad (8)$$

For $d \leq \Delta$, the two lattices have the same continuous integral. Monotone rectangle comparison bounds the sum of their two endpoint errors by $4\log^i(x/d)$. Its sum is at most

$$4 \int_0^\Delta \log^i(x/t) dt = 4\Delta Q_i(h).$$

For $d > \Delta$, reverse the first lattice and use $a \leq x/\Delta$. The number of available d for each a is at most $\Delta/a + 1$, and $f_i(a) \leq h^i$. Since $x/\Delta \leq \Delta$, this costs at most $\Delta(h+2)h^i$. In the second lattice there are at most $\Delta + 1 \leq 2\Delta$ integers n , and $\sum_{\Delta < d \leq n} 1/d \leq h + 1$. Its cost is at most $2\Delta(h+1)h^i$. This proves (7), with both integer-count corrections retained. Finally

$$Q_i(h) = h^i \sum_{r=0}^i \frac{i(i-1)\cdots(i-r+1)}{h^r} \leq h^i \sum_{r \geq 0} (i/h)^r \leq 2h^i.$$

No constant here depends on i . □

3 The preliminary estimates and their exact normalizations

The preliminary estimate needed in the subsequent proof is

$$r(x) := \sum_{n \leq x} \frac{1 - \Lambda(n)}{n} - 2\gamma = O((\log x)^{-3}) \quad (x \geq 2). \quad (9)$$

We take this classical fixed-power estimate from Lu [4, §6, p. 1026, equation (114) and the proof of Lemma 21], with the preliminary exponent fixed at 3 once and for all. Lu attributes that estimate to the earlier elementary results of Bombieri [1], Wirsing [5], and Diamond–Steinig [2]; its use does

not depend on Lu's final theorem or on the present remainder. Equation (114) has exactly the centering -2γ and inclusive cutoff displayed above.

For the fixed Möbius estimates we also cite Lu's Lemma 21, retaining the exact normalization identities in Lemma 3.2. The interval sieve and finite Wirsing inequality are cited separately below. All imported exponents and constants are fixed before the degree or interval parameter varies; the degree-uniform consequences are proved here.

We shall freely use finite summation by parts in the form

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t) dt, \quad A(t) = \sum_{n \leq t} a_n.$$

The atom at 1 is included because $A(1-) = 0$. For the elementary preliminary estimates only, write

$$H_j(x) = \sum_{n \leq x} \frac{(\log n)^j}{n} \quad (j = 0, 1).$$

The harmonic and factorial estimates are

$$H_0(x) = \log x + \gamma + O(x^{-1}), \quad \sum_{n \leq x} \log n = x \log x - x + O(\log(2x)).$$

The first follows by summing $0 \leq 1/n - \log(1 + 1/n) \leq 1/(2n^2)$ and bounding its tail; the second follows from the upper and lower integral sums for $\log t$.

The binomial coefficient $\binom{2N}{N} \leq 4^N$ gives $\vartheta(2N) - \vartheta(N) \leq 2N \log 2$. Dyadic summation, followed by $\psi(x) = \sum_{j \geq 1} \vartheta(x^{1/j})$, proves $\psi(x) \ll x$. Summing $\mathbf{1} * \Lambda = \log$ and retaining the floors then gives

$$P(x) := \sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1), \quad \sum_{n \leq x} \frac{\Lambda(n) - 1}{n} = O(1), \quad (10)$$

because the error in replacing $\lfloor x/n \rfloor$ by x/n is at most $\psi(x) \ll x$.

The exact centered divisor average and its below-one floor correction were proved in Lemma 2.1; in particular $\tilde{R}(x) = 0$ for $x < 1$, whereas $R(x) = 2\gamma$ there.

3.1 A complete interval sieve

Lemma 3.1. *Uniformly for real $A \geq 0$ and $H \geq 3$,*

$$\pi(A + H) - \pi(A) \leq \frac{2H}{\log H} \left(1 + O \left(\frac{\log \log(3H)}{\log H} \right) \right).$$

In particular, for each fixed $0 < \eta < 1$,

$$|R(x) - R(y)| \leq (x - y) \{1 + O_\eta((\log x)^{-\eta})\} \quad (11)$$

uniformly when $x/2 \leq y < x$ and $x - y \geq x \exp[-(\log x)^{1-\eta}]$.

Proof. The first assertion is Selberg's elementary interval-sieve estimate in Lu [4, §3, p. 985, equation (17)]; the error there is explicitly independent of both the starting point and the interval length. Replacing $\log \log H$ by $\log \log(3H)$ and increasing the absolute constant covers bounded $H \geq 3$. The case $A = 0$ follows by a right limit. We record the conversion to R because its prime-power and integer corrections are used later.

Proper prime powers contribute at most $O(\sqrt{x} \log^2 x)$, so for $H = x - y \geq 3$,

$$\psi(x) - \psi(y) \leq \frac{2H \log x}{\log H} \left(1 + O\left(\frac{\log \log(3H)}{\log H}\right) \right) + O(\sqrt{x} \log^2 x).$$

This also proves $\psi(x) - \psi(y) \ll H$ whenever $H \geq x^{2/3}$. On the range in (11), $\log H = \log x + O((\log x)^{1-\eta})$ and the prime-power and floor errors are $O_\eta(H(\log x)^{-\eta})$; the same is true of the error caused by the sieve correction because $\log \log x / \log x = O_\eta((\log x)^{-\eta})$. Subtracting $\lfloor x \rfloor - \lfloor y \rfloor = H + O(1)$ from the nonnegative prime sum proves both sides of (11). $\square$

3.2 The two Möbius estimates, with the unit atoms retained

Lemma 3.2. *The classical fixed-exponent Möbius estimates give*

$$m(x) = O((\log x)^{-3}), \quad m_1(x) = 1 + O((\log x)^{-2}).$$

Consequently

$$K_m := \int_1^\infty |m(t)| \frac{dt}{t} < \infty, \quad |m_j(x)| \leq K_m j (\log x)^{j-1} \quad (j \geq 1, x > 1), \quad (12)$$

where the last constant is independent of j . Put $J = \int_1^\infty |m_1(t) - 1| dt/t < \infty$. For every $j \geq 2$,

$$|m_j(x) - j(\log x)^{j-1}| \leq Jj(j-1)(\log x)^{j-2} \quad (x > 1). \quad (13)$$

Proof. The first two estimates follow from the transfer recorded in Lu [4, §6, p. 1026, proof of Lemma 21], taking exponent 3 in his equation (114): it gives $m = O((\log x)^{-3})$ and $m_1 - 1 = O((\log x)^{-3} \log \log x)$, hence the weaker displayed bound. Thus K_m and J are finite, including the compact initial interval. We record the exact normalization identities of that transfer before deriving the degree-uniform bounds.

If F, G are right-continuous functions of locally bounded variation, zero below 1, and $XY = x$, their multiplicative hyperbola formula is

$$\int_{[1,x]} F(x/t) dG(t) = \int_{[1,X]} F(x/t) dG(t) + \int_{[1,Y]} \{G(x/t) - G(X)\} dF(t). \quad (14)$$

It follows by splitting the finite-variation product measure on $uv \leq x$ into $u \leq X$ and $u > X$. At $v = Y$ the subtracted difference is zero; all atoms, including the atoms at 1, are counted once.

Let $F(t) = \log t - P(t) - \gamma$ for $t \geq 1$ and zero below 1. Then $F = r + O(t^{-1}) = O((\log t)^{-3})$ and direct expansion, using $\mu * \Lambda = -\mu \log$, gives

$$\int_{[1,x]} F(x/t) dm(t) = (\log x - \gamma)m(x). \quad (15)$$

In particular $dF = dt/t - \sum \Lambda(n)\delta_n/n - \gamma\delta_1$; the atom $-\gamma\delta_1$ is not dropped.

Now let $q(t) = \log t + \gamma - H_0(t)$ for $t \geq 1$, zero below 1. We have $q(t) = O(t^{-1})$, $|dq|([1, Y]) \ll \log(2Y)$, and exactly

$$\int_{[1,x]} q(x/t) dm(t) = m_1(x) + \gamma m(x) - 1. \quad (16)$$

The constant -1 is the unit convolution $\mu * \mathbf{1}$. These identities retain the centering terms in the cited estimates; no atom is absorbed by a change of normalization.

Finite-sum integration gives, including the atom $m(t) = 1$ on $[1, 2)$,

$$m_j(x) = j \int_1^x m(t) \log^{j-1}(x/t) \frac{dt}{t} \quad (j \geq 1),$$

which proves (12). For $j \geq 2$ it also gives

$$m_j(x) = j(j-1) \int_1^x m_1(t) \log^{j-2}(x/t) \frac{dt}{t}.$$

The constant part $m_1 = 1$ contributes $j \log^{j-1} x$, and $\int_1^\infty |m_1(t) - 1| dt/t < \infty$ bounds the remainder by $Jj(j-1) \log^{j-2} x$. The same J works for every degree. $\square$

4 A finite descent uniform in the degree

This section keeps the degree variable. Its constants come only from the elementary estimates established in Section 3, with its fixed cited preliminary input; in particular they do not depend on a degree or on a short-interval parameter. Put

$$a_i(n) = i(\log n)^{i-1}, \quad r_i(n) = \Lambda_i(n) - a_i(n), \quad A_i(s, t) = \sum_{t < n \leq s} a_i(n),$$

with $a_1(1) = 1$. Differences of R_i are sums of r_i also when $i = 1$, since the constant 2γ in R_1 cancels in a difference. The finite convolution identities give

$$\Lambda_{i+1} = (\log) \Lambda_i + \Lambda * \Lambda_i, \quad \Lambda_i \geq 0, \quad \Lambda_i(1) = 0, \quad (17)$$

$$r_{i+1} = (\log) r_i + \Lambda * r_i + \mathbf{c}_i, \quad \mathbf{c}_i = i\Lambda * \log^{i-1} - \log^i. \quad (18)$$

The first identity was proved by logarithmically differentiating $\mu * \mathbf{1} = e_0$; subtracting the integer main rows proves the second. Only $\mathbf{c}_1$ vanishes. For example, $\mathbf{c}_i(p) = -(\log p)^i$ for a prime p and $i \geq 2$.

Fix constants $C_M, C_\psi \geq 1$ and a fixed lower threshold so that (10) and Lemma 3.1 give

$$|P(t) - \log t| \leq C_M \quad (t \geq 1), \quad \psi(s) - \psi(t) \leq C_\psi(s - t) \quad (s - t \geq s^{2/3}). \quad (19)$$

The second assertion is required only above that fixed threshold. Both assertions include prime powers and real endpoints. Also put

$$J = \int_1^\infty |m_1(t) - 1| \frac{dt}{t} < \infty.$$

The integral formula proved with (13) gives the explicit all-order estimate

$$|m_i(t) - i(\log t)^{i-1}| \leq Ji(i-1)(\log t)^{i-2} \quad (i \geq 2, t \geq 1). \quad (20)$$

Indeed the constant part of $i(i-1) \int_1^t m_1(v) \log^{i-2}(t/v) dv/v$ is $i(\log t)^{i-1}$; the absolute integral of the remaining factor is at most J . At the unit endpoint the assertion follows by continuity when necessary. Thus no fixed-degree O_i is being used for a growing degree.

Lemma 4.1 (A positive harmonic bound with its integer endpoints). *For every real $y \geq 1$ and integer $i \geq 1$,*

$$\sum_{d \leq y} \Lambda_i(d) \lfloor y/d \rfloor = \sum_{n \leq y} (\log n)^i, \quad \sum_{d \leq y} \frac{\Lambda_i(d)}{d} \leq 2(\log y)^i. \quad (21)$$

Proof. The exact identity is the finite sum of $\Lambda_i * \mathbf{1} = \log^i$. For $d \leq y$ we have $\lfloor y/d \rfloor \geq y/(2d)$. Positivity and $\sum_{n \leq y} (\log n)^i \leq y(\log y)^i$ prove the inequality. The term at $d = 1$ is zero, and the floors were retained before taking the inequality. $\square$

Here is a finite form of the comparison that will be used below. Its normalization $\mathcal{D}_i$ measures the local remainder on a common interval class.

Lemma 4.2 (Uniform finite comparison). *There are fixed constants $V_0, C_a, C_H > 0$ with the following property. Let $q \in [1/2, 1)$, $\kappa = \log(1/(1-q))$, $T \geq 3$ be an integer, $0 < \eta < 1/4$, $0 < \theta \leq 1/6$, and $u \geq V$. Suppose*

$$\kappa \leq \min(V^{1-\eta}, V/6), \quad V^{1-\eta} \geq 2T, \quad \eta(T+1) > 3, \quad (22)$$

$$V \geq \max\{V_0, 100(1+J)T^2, 12C_M T\}, \quad \theta V \geq \kappa + \log u, \quad W := V/\theta \leq u. \quad (23)$$

For $V \leq S \leq u$ put $z = e^S$, $H = (1-q)z$, and

$$\mathcal{D}_i(S) = \frac{|R_i(z) - R_i(qz)|}{HS^{i-1}}.$$

Set $G_T = 15 + 2JT(T-1)$ and $H_T = C_H(1+J)T^4$. On the entire common domain $V \leq S \leq u$,

$$\mathcal{D}_T(S) \leq \frac{G_T}{S}, \quad \mathcal{D}_i(S) \leq 8T \quad (1 \leq i \leq T). \quad (24)$$

On $W \leq S \leq u$, for every $2 \leq i \leq T$,

$$\boxed{\mathcal{D}_i(S) \leq C_a \theta^i + \frac{H_T}{V} + \frac{16T^3}{i(i-1)} \left(\frac{W}{S}\right)^{(i-1)/2}}. \quad (25)$$

All constants are independent of q, T, η, θ, V, u .

Proof. We give the arithmetic recurrence and its corrections before making the comparison.

The terminal seed and a common coarse bound. The hypotheses imply $H = e^{S-\kappa} \geq e^{5S/6}$ and $qz \geq 1$. In Lemma 2.3 the actual logarithmic interval parameter is $\kappa \leq S^{1-\eta}$. Since $E_T(h)$ there is an increasing polynomial for $h \geq 0$ and $S^{1-\eta} \geq 2T$,

$$|U_T(z) - U_T(qz)| \leq 15HS^{(1-\eta)(T+1)}.$$

After division by HS^{T-1} this is at most $15/S$, because $2 - \eta(T+1) < -1$. There are at most $H+1 \leq 2H$ integers in the interval. Equation (20) therefore bounds the remaining smooth-row correction by $2JT(T-1)/S$ in this normalization. This proves the terminal estimate in (24), including both counting errors in the two-lattice estimate.

Positivity in (17) gives, for $n > 1$,

$$\Lambda_i(n) \leq \frac{\Lambda_T(n)}{(\log n)^{T-i}} \quad (1 \leq i \leq T).$$

As $\log n \geq S - \log 2$ on the present interval, its positive mass satisfies

$$\frac{\sum_{qz < n \leq z} \Lambda_i(n)}{HS^{i-1}} \leq \left(\frac{S}{S - \log 2}\right)^{T-i} \left(2T + \frac{G_T}{S}\right) \leq 5T. \quad (26)$$

Here the $2T$ retains the possible extra integer in the main row; $V \geq 100(1+J)T^2$ makes $G_T/S \leq 1$ and the displayed power at most 2. The i th integer main row costs at most $2i \leq 2T$. Consequently $\mathcal{D}_i \leq 8T$ on the same domain, without any successive extension of that domain. No interval here contains the unit, but the original definitions at the unit have not been changed.

The complete arbitrary-cut recurrence. Write $b = e^{(1-\theta)S}$ and $\Delta R_i(s, t) = R_i(s) - R_i(t)$. For every $i \geq 1$, finite summation and a split of the convolution in (17) give the exact identity

$$\begin{aligned} \Delta R_{i+1}(z, qz) &= S\Delta R_i(z, qz) + \sum_{d \leq b} \Lambda(d) \Delta R_i(z/d, qz/d) + F_i + \mathsf{T}_i + M_i, \\ F_i &= \sum_{qz < n \leq z} (\log n - S) \Lambda_i(n), \\ \mathsf{T}_i &= \sum_{m \leq e^{\theta S}} \Lambda_i(m) \sum_{\max(b, qz/m) < d \leq z/m} \Lambda(d), \\ M_i &= SA_i(z, qz) + \sum_{d \leq b} \Lambda(d) A_i(z/d, qz/d) - A_{i+1}(z, qz). \end{aligned} \tag{27}$$

Every lower endpoint in these integer sums is strict and every upper endpoint is inclusive. These three rows reorganize the correction $\mathbf{c}_i$ in (18); they do not drop that correction.

The hypotheses imply

$$\frac{H}{b} = e^{\theta S - \kappa} \geq u \geq S.$$

For $m \leq e^{\theta S}$, put $s = z/m$. Since $\log s \geq (1-\theta)S \geq 5S/6$ and $\kappa \leq S/6$, $H/m = se^{-\kappa} \geq s^{2/3}$. All these upper endpoints exceed the fixed sieve threshold once V_0 is fixed. Positivity permits the inner interval in T_i to be enlarged to $(qz/m, z/m]$. Lemma 4.1 and (19) thus give

$$0 \leq \mathsf{T}_i \leq C_\psi H \sum_{m \leq e^{\theta S}} \frac{\Lambda_i(m)}{m} \leq 2C_\psi H(\theta S)^i. \tag{28}$$

Also, with the exact integer main row retained,

$$|F_i| \leq (\log 2) \{ |\Delta R_i(z, qz)| + i(H+1)S^{i-1} \}. \tag{29}$$

The missing mean and all endpoint terms. Let $E_P(t) = P(t) - \log t$, so $E_P(1) = 0$ and $|E_P(t)| \leq C_M$. For $i \geq 2$ define the positive operator

$$(\mathcal{P}_i f)(S) = \frac{1}{S^i} \sum_{d \leq b} \frac{\Lambda(d)}{d} (S - \log d)^{i-1} f(S - \log d).$$

For $0 \leq \xi < i-1$, direct partial summation gives

$$\frac{\mathcal{P}_i[(W/S)^\xi]}{(W/S)^\xi} = \frac{1 - \theta^{i-\xi}}{i - \xi} + \Xi_{i,\xi}(S), \tag{30}$$

$$\Xi_{i,\xi}(S) = S^{\xi-i} \left\{ (\theta S)^{i-1-\xi} E_P(b) + (i-1-\xi) \int_0^{(1-\theta)S} E_P(e^v) (S-v)^{i-2-\xi} dv \right\},$$

$$|\Xi_{i,\xi}(S)| \leq \frac{C_M}{S}. \tag{31}$$

For the last bound, the endpoint plus the integral of the absolute derivative of $(S - v)^{i-1-\xi}$ is exactly $S^{i-1-\xi}$. The lower endpoint term vanishes because $E_P(1) = 0$; the upper cutoff term $E_P(b)$ is displayed explicitly. Thus this bound is uniform also when i and ξ grow. Put $\Xi_i = \Xi_{i,0}$ for $i \geq 2$, and put $\Xi_1 = E_P(b)/S$; then the corresponding mass identity at $i = 1$ is $P(b)/S = 1 - \theta + \Xi_1$.

Define the exact integer errors

$$\epsilon_i(s, t) = A_i(s, t) - i(s - t)(\log s)^{i-1}.$$

Substituting these definitions and the mass identity in M_i gives, for every $i \geq 1$,

$$\begin{aligned} \frac{M_i}{HS^i} &= -\theta^i + i\Xi_i(S) \\ &\quad + \frac{S\epsilon_i(z, qz) + \sum_{d \leq b} \Lambda(d)\epsilon_i(z/d, qz/d) - \epsilon_{i+1}(z, qz)}{HS^i}. \end{aligned} \quad (32)$$

In particular the omitted mean is exactly $-\theta^i$, not zero.

Here are bounds for the displayed endpoint terms. If $s/2 \leq t < s$, $L_s = \log s \geq 1$, and $h = s - t \geq L_s$, then, for $i \geq 2$,

$$|\epsilon_i(s, t)| \leq iL_s^{i-1} + i(i-1)(\log 2)(h+1)L_s^{i-2} \leq 2i^2hL_s^{i-2}. \quad (33)$$

Indeed $|\lfloor s \rfloor - \lfloor t \rfloor - h| \leq 1$ and the mean value theorem bounds the change in $(\log n)^{i-1}$. For $i = 1$ retain instead the exact floor error $\epsilon_1(s, t) = \lfloor s \rfloor - \lfloor t \rfloor - (s - t)$, of absolute value at most 1. The hypotheses $H/d \geq H/b \geq S \geq \log(z/d)$ verify every condition of (33) at every daughter. For $i \geq 2$,

$$\sum_{d \leq b} \Lambda(d)|\epsilon_i(z/d, qz/d)| \leq 2i^2HS^{i-2}P(b) \leq Ci^2HS^{i-1}.$$

The two other integer errors in (32) have the same normalized bound. At $i = 1$ the convolution of floor errors is at most $\psi(b) \ll b \leq H/S$, and the other two errors are covered by the same conclusion. Thus for all $i \geq 1$ the absolute value of the second line of (32) is at most $C(i+1)^2/S$, with a fixed C .

Transpose (27), divide by HS^i , and use (24) for the frozen row $\mathcal{D}_i(S)/S$ in (29). Equations (28)–(32) prove, simultaneously for $2 \leq i < T$,

$$\mathcal{D}_i(S) \leq \mathcal{D}_{i+1}(S) + (\mathcal{P}_i \mathcal{D}_i)(S) + E_i, \quad E_i = C\theta^i + \frac{CT^2}{V}. \quad (34)$$

Here and below C is one fixed sufficiently large constant. The inequality is used only for $S \geq W$, so all daughters lie in the established common domain: a nonzero term has $d \geq 2$ and

$$V \leq \theta S \leq S - \log d \leq S - \log 2. \quad (35)$$

In particular the frozen row has not been silently absorbed into a degree-dependent constant or assumed small before the coarse bound.

A finite simultaneous comparison. Set $\varepsilon = C_M/V$, $\xi_i = (i-1)/2$, and $J_i(S) = (W/S)^{\xi_i}$. The exact masses imply the upper bounds

$$\mathcal{P}_i 1 \leq q_i := \frac{1}{i} + \varepsilon, \quad \mathcal{P}_i J_i \leq q_i^* J_i, \quad q_i^* := \frac{2}{i+1} + \varepsilon.$$

They are strictly below 1. For $2 \leq i \leq k \leq T-1$, the ideal resolvent products telescope:

$$\prod_{j=i}^k (1 - 1/j)^{-1} = \frac{k}{i-1}, \quad \prod_{j=i}^k (1 - 2/(j+1))^{-1} = \frac{k(k+1)}{i(i-1)}.$$

Each logarithmic perturbation caused by ε is at most 6ε . This follows from $1 - 2/(j+1) \geq 1/3$ and $\varepsilon \leq 1/(12T)$. Since $6T\varepsilon \leq 1/2 < \log 2$, each perturbed product is at most twice its displayed ideal value.

Define nonnegative numbers backwards by

$$a_T = G_T/V, \quad b_T = 8T, \quad a_i = \frac{a_{i+1} + E_i}{1 - q_i}, \quad b_i = \frac{b_{i+1}}{1 - q_i^*} \quad (2 \leq i < T).$$

We claim

$$\mathcal{D}_i(S) \leq a_i + b_i J_i(S) \quad (W \leq S \leq u). \quad (36)$$

The terminal degree follows from the seed. Descend one degree at a time, and at each degree induct over the consecutive half-open strips $[W + n \log 2, W + (n+1) \log 2)$. Every same-degree daughter lies in an earlier strip by (35). Daughters in $[V, W)$ are covered by $\mathcal{D}_i \leq 8T \leq b_i J_i$. For the already treated upper degree, $J_{i+1}(S) \leq J_i(S)$ when $S \geq W$. Substitution of these bounds in (34) gives precisely $a_i + b_i J_i(S)$ by the definitions of a_i, b_i . These finitely many strips cover every finite upper endpoint u ; at a strip boundary its daughters are still in earlier strips. This proves the comparison without an infinite recursion or a regularity assumption on the step functions R_i .

Finally, unfolding the finite recurrences and using the telescoping products gives

$$a_i \leq \frac{2}{i-1} \left\{ \frac{(T-1)G_T}{V} + \sum_{j=i}^{T-1} j E_j \right\}, \quad b_i \leq \frac{16T^3}{i(i-1)}.$$

For $0 < \theta \leq 1/2$ and $i \geq 2$,

$$\frac{1}{i-1} \sum_{j=i}^{\infty} j \theta^j = \frac{\theta^i}{i-1} \left\{ \frac{i}{1-\theta} + \frac{\theta}{(1-\theta)^2} \right\} \leq 6\theta^i.$$

It follows that $a_i \leq 12C\theta^i + (2TG_T + CT^4)/V$. Choose $C_a = 12C$ and a fixed C_H with $2TG_T + CT^4 \leq C_H(1+J)T^4$. Together with (36) this proves (25); the terminal degree obeys the same enlarged bound. All sums and comparisons in this proof are finite, and each endpoint correction was retained before it was bounded. $\square$

5 Fixed-cutoff smoothing and one signed energy step

We prove only the strict degree-one coefficient needed in the final exponential sum. The cutoff fraction in this section is one fixed number; neither an iteration of strict prime estimates nor a local decay theorem is used. All constants denoted by C below are absolute, unless their dependence on this fixed cutoff is displayed.

5.1 The raw interval bounds and the integer interpolant

The proof of Lemma 3.1, before any fixed-margin specialization, gives the following useful uniform form. If $z = e^s$, $q \in [1/2, 1)$, $\kappa = \log(1/(1-q)) \leq s/3$ and $H = (1-q)z \geq 3$, then

$$\frac{|R(z) - R(qz)|}{H} \leq 1 + C \left\{ \frac{\kappa}{s} + \frac{\log s}{s} + s^2 e^{\kappa-s/2} + e^{\kappa-s} \right\}. \quad (37)$$

Indeed, the displayed prime-counting sieve there bounds the prime sum divided by H by $2 + C\{\kappa/s + \log s/s + s^2 e^{\kappa-s/2}\}$. The proper-prime-power term is the third term here. Subtracting $\lfloor z \rfloor - \lfloor qz \rfloor = H + O(1)$, and using the nonnegativity of the prime sum for the opposite inequality, proves (37). Thus its constant is independent of a shrinking margin. We also use the absolute bound $|R(z) - R(qz)| \leq CH$ when $H \geq z^{2/3}$, and the elementary bounds proved above:

$$\begin{aligned} R(z) = O(z), \quad \sum_{n \leq z} \frac{|\omega(n)|}{n} &\leq C(1 + \log z), \quad \sum_{n \leq z} |\omega(n)| \leq Cz, \\ \left| \sum_{n \leq z} \frac{\omega(n)}{n} \right| &\leq C, \quad \omega(n) = \Lambda(n) - 1. \end{aligned} \quad (38)$$

Here $\omega(1) = -1$.

For $11/12 \leq p < 1$, take integer knots

$$b_0 = 3, \quad b_{j+1} = b_j + \lfloor b_j e^{-(\log b_j)^p} \rfloor,$$

and let S_0 interpolate $(b_j, R(b_j))$ linearly. Its derivative at a knot is the left derivative. All increments are positive, and $b_{j+1} \leq 2b_j$. In particular, for every integer $n \geq 4$,

$$S'_0(n) = S_0(n) - S_0(n-1), \quad S'_0(t) = S'_0(n) \quad (n-1 < t \leq n). \quad (39)$$

If $Z = \log b_j$ and $b_j e^{-Z^p} \geq 2$, the interval length $\Delta = b_{j+1} - b_j$ obeys $\Delta \geq b_j e^{-Z^p}/2$. Consequently its relative logarithmic width is at most $Z^p + \log 4$. Applying (37) to the whole interval, provided $Z^{p-1} \leq 1/8$ and Z exceeds a fixed absolute lower bound, gives

$$|S'_0(t)| \leq 1 + C \left\{ Z^{p-1} + \frac{\log Z}{Z} + Z^2 e^{-Z/3} \right\} \quad (b_j < t \leq b_{j+1}). \quad (40)$$

This statement has no fixed- p implied constant; its stated conditions will be checked on the common schedule below.

There is also a pointwise chord bound requiring no prime asymptotic:

$$E(z) := R(z) - S_0(z) = O\left(ze^{-(\log(z/2))^p} \log z\right) \quad (z > 3). \quad (41)$$

On the block containing z , both deviations from $R(b_j)$ are bounded by $\sum_{b_j < n \leq b_{j+1}} (\Lambda(n) + 1)$, which is at most $\Delta(\log b_{j+1} + 1)$. Since $b_j \geq z/2$, this proves (41) with a constant independent of p .

5.2 The complete fixed-cutoff smoothing identity

Fix $u > v > 1$, $q \in [1/2, 1)$ and $0 < \theta \leq 1/6$. Set

$$f(t) = \mathbf{1}_{(v,u]}(t) \frac{R(e^t) - R(qe^t)}{(1-q)e^t}, \quad g(t) = \mathbf{1}_{(v,u]}(t) S'_0(e^t), \quad h(t) = \frac{1}{t} \int_0^t f(t-s)g(s) \, ds,$$

with $h(0) = 0$. Suppose $|f| \leq A_f$, $|g| \leq A_g$, where $A_f, A_g \leq 2$. At an equation endpoint $t \in [w, u]$, use

$$x = e^t, \quad H = (1 - q)x, \quad a = e^{\theta t}, \quad B = x/a, \quad \Delta_n R = R(x/n) - R(qx/n).$$

Assume $\theta t \geq 2v$, $a \geq 4$, $\kappa \leq (1 - \theta)t/3$, and the fixed elementary lower bounds. Then, with an absolute constant independent of p, q, θ , the EXACT equation has the form

$$f(t) + h(t) = d_2(t, q) + r_\theta(t, q), \quad d_2(t, q) = \frac{R_2(e^t) - R_2(qe^t)}{(1 - q)e^t t}, \quad (42)$$

where the complete remainder satisfies

$$|r_\theta(t, q)| \leq C \left\{ (1 + A_f A_g) \theta + \frac{1}{t} + \frac{(1 + A_g)e^{\kappa - \theta t}}{t} + \frac{e^{\kappa - t/2}}{t} + \frac{e^{\kappa + \theta t - t}}{t} + e^{\kappa - t} + t \exp[\kappa - (\theta t - \log 4)^p] \right\}. \quad (43)$$

We prove this, recording the exact differences before bounding them.

The $i = 1$ instance of (27) is

$$t \Delta_1 R + \sum_{n \leq B} \Lambda(n) \Delta_n R = \Delta R_2 - F_1 - T_1 - M_1. \quad (44)$$

Its three correction rows retain the original strict lower and inclusive upper endpoints. Their bounds can be seen directly here. The frozen-log row is at most $(\log 2) \sum_{qx < n \leq x} \Lambda(n) = O(H)$, by the absolute interval sieve. In the tail, $m \leq a$ and every full inner interval has length $H/m \geq (x/m)^{2/3}$. Its prime sum is at most CH/m , even when the actual interval is clipped. The exact harmonic bound $\sum_{m \leq a} \Lambda(m)/m \leq 2 \log a$ therefore gives $T_1 \leq CH\theta t$. For the main row, (32) retains the missing mean $-\theta Ht$ and the harmonic endpoint $HD_P(B)$, where $|D_P| \leq C_M$. To check its remaining integer terms directly, write $\epsilon_i(s, z) = A_i(s, z) - i(s - z)(\log s)^{i-1}$. Then $|\epsilon_1| \leq 1$, and for $s/2 \leq z < s$, $|\epsilon_2(s, z)| \leq C\{s - z + \log s\}$. Consequently their unnormalized contribution is at most $C(H + t + B)$. These facts give

$$\frac{|F_1 + T_1 + M_1|}{Ht} \leq C \left\{ \theta + \frac{1}{t} + e^{\kappa - t} + \frac{e^{\kappa - \theta t}}{t} \right\}. \quad (45)$$

Thus neither the missing mean nor the weighted-mass endpoint has been removed by an unproved signed cancellation.

For constant centering the exact identity is

$$\sum_{n \leq x} \Delta_n R = \tilde{R}(x) - \tilde{R}(qx) - 2\gamma([\![x]\!] - [\![qx]\!]). \quad (46)$$

The last term uses $R(z) = 2\gamma$ below one. In particular,

$$\begin{aligned} C_{\text{cen}} &:= \sum_{n \leq B} \Lambda(n) \Delta_n R - \sum_{a < n \leq B} \omega(n) \Delta_n R \\ &= \sum_{n \leq a} \omega(n) \Delta_n R + \tilde{R}(x) - \tilde{R}(qx) - 2\gamma([\![x]\!] - [\![qx]\!]) - \sum_{B < n \leq x} \Delta_n R. \end{aligned} \quad (47)$$

The first sum is at most $CH(1 + \theta t)$: every daughter satisfies the absolute interval sieve, and (38) applies. For the last sum, expand the jumps first; then

$$\sum_{B < n \leq x} |\Delta_n R| \leq \sum_{m \leq a} |\omega(m)|(H/m + 1) \leq C\{H(1 + \theta t) + a\}.$$

The displayed +1 is the retained integer-count error. Using $\tilde{R}(x) = O(\sqrt{x})$ and the floor term in (47) proves

$$\frac{|C_{\text{cen}}|}{Ht} \leq C \left\{ \theta + \frac{1}{t} + \frac{e^{\kappa+\theta t-t}}{t} + \frac{e^{\kappa-t/2}}{t} + e^{\kappa-t} \right\}. \quad (48)$$

In particular the unit in the first sum is present.

Put $c(n) = \omega(n) - S'_0(n)$ and $Q_a = \exp[-(\log(a/4))^p]$. The exact cumulative coefficient error is

$$K_a(d) := \sum_{a < n \leq d} c(n) = E(\lfloor d \rfloor) - E(\lfloor a \rfloor) = O(dQ_at) \quad (a < d \leq x). \quad (49)$$

Set $K_a(d) = 0$ when $d \leq a$. Both floor arguments are retained; if they coincide the difference is zero. Otherwise they are at least $a/2$, and (41) proves the bound. For $z = x, qx$ the finite jump expansion is exactly

$$\sum_{a < n \leq B} c(n)R(z/n) = \sum_{m \leq z/a} \omega(m)K_a(\min(B, z/m)) + 2\gamma K_a(B).$$

The constant is displayed before subtracting the two values of z . It follows from (38) that

$$C_{\text{coef}} := \sum_{a < n \leq B} [\omega(n) - S'_0(n)]\Delta_n R = O(xQ_at^2). \quad (50)$$

There is a separate, smaller discrete-to-integral difference

$$C_{\text{cell}} := \sum_{a < n \leq B} S'_0(n)\Delta_n R - \int_a^B S'_0(z)\{R(x/z) - R(qx/z)\} dz = O(A_g x/a). \quad (51)$$

Indeed (39) is exact on every complete unit cell. The change of $R(z/n)$ on that cell is bounded by $\sum_{z/n < m \leq z/(n-1)} |\omega(m)|$. These half-open reciprocal intervals are disjoint as n varies, so Chebyshev bounds their total by Cx/a for $z = x, qx$. The two incomplete endpoint cells have the same bound, by $R(z) = O(z)$. Their logarithms lie in the support where A_g applies, since $\theta t \geq 2v$. No step function is differentiated.

Finally substitution $z = e^s$ in the integral gives its normalized value

$$\frac{1}{t} \int_{\theta t}^{(1-\theta)t} g(s)f(t-s) ds.$$

Both factors there lie above their lower support. Completing this to $h(t)$ adds the EXACT term

$$C_{\text{end}} = \frac{1}{t} \left\{ \int_v^{\theta t} + \int_{(1-\theta)t}^{t-v} \right\} g(s)f(t-s) ds, \quad |C_{\text{end}}| \leq 2A_f A_g \theta.$$

Thus the precise remainder in (42) is

$$r_\theta = -\frac{F_1 + T_1 + M_1 + C_{\text{cen}} + C_{\text{coef}} + C_{\text{cell}}}{Ht} + C_{\text{end}}.$$

Equations (45), (48)–(51) prove (43). It includes both completed end ranges, both floor chords, the constant 2γ , all reciprocal jumps, all prime powers and every count correction.

5.3 A finite real-convolution inequality

Lemma 5.1 (Finite Wirsing inequality; Lu’s Lemma 3). *There are absolute constants $C_W > 0$ and an integer $N_W \geq 2$ with the following property. Let $X > 2Y > 0$ and let real measurable f, g on $[0, X]$ satisfy*

$$\int_0^t f^2 \leq Ft, \quad \int_0^t g^2 \leq Gt \quad (0 < t \leq X), \quad F, G > 0.$$

Put $h(t) = t^{-1} \int_0^t f(t-s)g(s) ds$. For an integer $N \geq N_W$ write $\epsilon = 1/N$. If

$$\left| \int_0^t h(s) ds \right| \leq \frac{\epsilon^3}{4} \sqrt{FG} t \quad (Y \leq t \leq X),$$

then

$$\int_Y^X h^2 \leq (\tfrac{1}{2} + C_W/N) FG(X - Y). \quad (52)$$

The constants are independent of N, X, Y, F, G, f, g .

Source and normalization. This is Lu [4, §4, pp. 991–992, Lemma 3], based on Wirsing [5, proof of Lemma 1, pp. 2–6]. Lu states the result for reciprocal sufficiently large integers and explicitly makes the error constant absolute. The source’s large- X, Y convention causes no restriction here: replacing $f(t), g(t)$ by $f(t/a), g(t/a)$ on $[0, aX]$ gives $h(t/a)$, and multiplies every integral and its corresponding interval length by a . Apply the source at large aY and divide by a . The initial-interval energy hypotheses and the factor $X - Y$ are unchanged.

The initial strip in the source’s geometric-block argument can be kept explicit. Write its absolute block constant as C_b (Lu’s equation (36)). Complete blocks of ratio $1 + 1/N$ ending at X leave a first point a with $Y \leq a < (1 + 1/N)Y$. Cauchy–Schwarz gives $|h| \leq \sqrt{FG}$ from the initial-interval hypotheses, so

$$\int_Y^X h^2 \leq FG\{(a - Y) + (\tfrac{1}{2} + C_b/N)(X - a)\} \leq (\tfrac{1}{2} + (C_b + 1)/N) FG(X - Y),$$

using $a - Y \leq Y/N$ and $Y < X - Y$. Thus one may take $C_W = C_b + 1$; the unpaired strip is included, and no short-interval energy bound is assumed. $\square$

Fix once and for all an integer

$$N_0 \geq \max\{N_W, 2, \lceil 400C_W/9 \rceil\}. \quad (53)$$

Thus $1/2 + C_W/N_0 \leq 209/400$. This integer is absolute and does not vary with the growing degree, the interval margin or the outer X .

5.4 The signed primitive and a quantitative single step

The primitive uses the arithmetic signs, not an absolute variation bound. Before truncation put $F_q(t) = [R(e^t) - R(qe^t)]/[(1 - q)e^t]$ for $t \geq 0$. Finite jump integration gives, for every $T \geq 0$,

$$\int_0^T F_q(t) dt = \sum_{n \leq qe^T} \frac{\omega(n)}{n} + \frac{1}{1 - q} \sum_{qe^T < n \leq e^T} \omega(n)(1/n - e^{-T}). \quad (54)$$

To verify it, integrate the contribution of each jump over $\log n \leq t < \log(n/q)$, clipped to $[0, T]$; the integral of $e^{-t}/(1 - q)$ is $1/n$ for a complete jump and is the displayed coefficient for a partial one.

This includes $n = 1$ at the lower endpoint. The partial coefficient is between zero and $1/n$. The first sum is bounded by (38); the absolute partial sum is bounded on $(qe^T, e^T]$ by Chebyshev, since $q \geq 1/2$. Therefore the whole primitive is at most an absolute C_* in magnitude. For the truncated f ,

$$F(T) := \int_0^T f(t) dt = \mathbf{1}_{T>v} \left\{ \int_0^{\min(T,u)} F_q(t) dt - \int_0^v F_q(t) dt \right\}, \quad |F(T)| \leq K_f := 2C_*. \quad (55)$$

Thus the lower-support subtraction has not been dropped.

Let $C(T) = \int_0^T (g * f)(t) dt$. Finite Fubini yields $C(T) = \int_0^T g(s)F(T-s) ds$. Both factors vanish below v , so $C(T) = 0$ for $T \leq 2v$, and $|C(T)| \leq A_g K_f (T - 2v)$ otherwise. Integration by parts retains the upper endpoint:

$$\int_0^T h(t) dt = \frac{C(T)}{T} + \int_{2v}^T \frac{C(t)}{t^2} dt, \quad T > 2v, \quad (56)$$

$$\left| \int_0^T h(t) dt \right| \leq A_g K_f \log(T/(2v)). \quad (57)$$

For $T \leq 2v$ this integral is zero. The last inequality uses the exact equality $(T - 2v)/T + \int_{2v}^T (t - 2v)t^{-2} dt = \log(T/(2v))$. No primitive estimate for g is needed.

Lemma 5.2 (One finite strict energy step). *Suppose $u > 2w$, $|f|, |g| \leq A$, and the exact equation $f + h = e$ holds on $[w, u]$, with $|e| \leq E$. For an integer $N \geq N_W$, assume*

$$K_f \frac{\log u}{w} \leq \frac{A}{4N^3}. \quad (58)$$

Then

$$|f(u)| \leq \sqrt{\frac{1}{2} + C_W/N} A^3 + (1 + A)E + A^2 \sqrt{w/u}. \quad (59)$$

In particular, for

$$A = \frac{257}{256}, \quad N = N_0, \quad E \leq \frac{1}{256}, \quad \sqrt{w/u} \leq \frac{1}{256}, \quad (60)$$

one has $|f(u)| < 3/4$.

Proof. Use Lemma 5.1 with $X = u, Y = w, F = G = A^2$. Equations (57) and (58) imply its primitive hypothesis, because $AK_f \log u/w \leq A^2/(4N^3)$. Put $\kappa_N = 1/2 + C_W/N$. The conclusion is $\int_w^u h^2 \leq \kappa_N A^4(u - w)$. The L^2 triangle inequality in the exact equation, restoring the entire initial interval $[0, w]$, gives

$$\frac{\|f\|_{L^2(0,u)}}{\sqrt{u}} \leq \sqrt{\kappa_N} A^2 + E + A \sqrt{w/u}.$$

Cauchy–Schwarz at u gives $|h(u)| \leq A\|f\|_2/\sqrt{u}$; the pointwise arithmetic equation at u proves (59), also when an endpoint is an integer or prime power. For $N = N_0$, $\kappa_N \leq 209/400$ and $\sqrt{209/400} < 29/40$. The resulting rational upper bound is

$$\frac{29}{40} \left(\frac{257}{256} \right)^3 + \frac{1 + 257/256}{256} + \frac{(257/256)^2}{256} = \frac{500158277}{671088640} < \frac{3}{4}.$$

□

5.5 The common growing-degree schedule

We now verify all preceding finite conditions on exactly the family needed later. Fix $K > 400$. Let X be the outer variable, and write

$$L = \log X, \quad r = \log L, \quad Y_0 = \frac{\sqrt{L}}{2}, \quad \lambda = \frac{K \log r}{r}, \quad \delta = \lambda/2, \quad T = \left\lceil \frac{24}{\lambda} \right\rceil, \quad \eta = \lambda/8 = \delta/4. \quad (61)$$

All these parameters remain fixed during this outer proof. Eventually $0 < \delta \leq 1/4$. At each root logarithm

$$Y_0 \leq u \leq L + 1, \quad q \in [1/2, 1), \quad \kappa = \log(1/(1 - q)) \leq u^{1-\delta}, \quad (62)$$

put

$$\alpha_0 = 1 - \frac{3\delta}{4}, \quad \beta = \frac{1 - \delta}{1 - \delta/2}, \quad V = u^{\alpha_0}, \quad v = u^\beta, \quad W = V/\theta, \quad w = 2v/\theta, \quad p = 1 - \delta/3. \quad (63)$$

Here θ will be one fixed number, independent of X, u, q .

Let C_a be the absolute cutoff-head constant in Lemma 4.2, and let C_r cover the θ terms in (43) when $A_f, A_g \leq 2$. Choose $0 < \theta \leq 1/32$ once, small enough that

$$16C_a\theta^2 \leq \frac{1}{64}, \quad C_a\theta^2 + C_r\theta \leq \frac{1}{1024}. \quad (64)$$

The first allowance is kept for the final exponential sum; it is not claimed to vanish. The second supplies room in the present finite energy test.

Proposition 5.3 (Uniform strict coefficient on the required family). *For all sufficiently large X , with a threshold depending only on K, θ and the fixed elementary constants, every interval in (62) satisfies*

$$|R(e^u) - R(qe^u)| \leq \frac{3}{4}(1 - q)e^u. \quad (65)$$

All bounds are uniform in $u \in [Y_0, L + 1]$ and in q .

Proof. First verify the complete arithmetic certificate. The exact identities

$$\alpha_0(1 - \eta) = 1 - \delta + 3\delta^2/16, \quad \beta - \alpha_0 = \delta \left(\frac{3}{4} - \frac{1}{2 - \delta} \right) \geq \delta/7 \quad (66)$$

show that $\kappa \leq V^{1-\eta}$. Also $\eta(T + 1) > 3$, $T \leq 25/\lambda = O_K(r/\log r)$, and $V \geq Y_0^{13/16}$. This exponential growth in r implies $V^{1-\eta} \geq 2T$, $V \geq 100(1 + J)T^2$ and $V \geq 12C_M T$ after one fixed threshold. The remaining cutoff tests follow from

$$\frac{\theta V}{u^{1-\delta}} = \theta u^{\delta/4} \longrightarrow \infty, \quad \log u \leq \log(L + 1) = O(r),$$

uniformly, so $\kappa \leq V/6$ and $\theta V \geq \kappa + \log u$. Also $W/u = \theta^{-1}u^{-3\delta/4} \rightarrow 0$ uniformly, and $w/W \geq 2$ by (66). These verify every finite size and mass condition in Lemma 4.2; the additional $w \leq u$ needed below follows from the displayed ratio w/u shortly.

For all $w \leq t \leq u$, its degree-two conclusion gives

$$|d_2(t, q)| \leq C_a\theta^2 + \frac{C_H(1 + J)T^4}{V} + 8T^3\sqrt{W/t}. \quad (67)$$

The entire boundary below W is included in the last term. Because $W/w = V/(2v)$, (66) gives

$$T^3 \sqrt{W/w} \leq CT^3 e^{-\delta \log u/14}.$$

Eventually $\log Y_0 \geq r/3$, whence $\delta \log u \geq K \log r/6$. Consequently

$$T^3 \sqrt{W/w} = O_K(r^{3-K/84}) = o(1). \quad (68)$$

The shared T^4/V row also tends uniformly to zero. The only nonvanishing arithmetic term in (67) is the explicit $C_a \theta^2$ allowance.

Next check the raw amplitudes, without using any strict prime result. For $v \leq t \leq u$,

$$\frac{\kappa}{v} \leq u^{-\delta(1-\delta)/(2-\delta)} \leq u^{-3\delta/7} = O_K(r^{-K/14}).$$

Equation (37) therefore gives $|f| \leq 1 + o(1)$ uniformly; its logarithmic, proper-power and floor rows vanish as well. In the integer mesh defining g , every relevant lower knot has logarithm $Z \geq v - \log 2$. Since $\beta \geq 6/7$,

$$Z^{p-1} = Z^{-\delta/3} = O_K(r^{-K/21}).$$

The unrounded length $\exp(Z - Z^p)$ is eventually at least $e^{Z/2}$ uniformly, justifying the retained factor $1/2$ after taking its floor. The mesh upper endpoint differs from the evaluation logarithm by at most $\log 2$. Equation (40) now proves $|g| \leq 1 + o(1)$ uniformly. Hence both actual amplitudes are at most $A = 257/256$ after one common threshold. This uses only the elementary interval sieve, including its proper-prime-power and integer errors.

The smoothing geometry also holds throughout $[w, u]$. In fact $\theta w = 2v$ and

$$\frac{w}{u} = \frac{2}{\theta} e^{-\delta \log u/(2-\delta)} \rightarrow 0, \quad \sqrt{w/u} = O_{K,\theta}(r^{-K/24}).$$

Thus $u > 2w$, $\sqrt{w/u} \leq 1/256$ and $\kappa \leq (1 - \theta)w/3$ eventually. Moreover

$$\beta p - (1 - \delta) = \frac{(1 - \delta)\delta}{6(1 - \delta/2)} \geq \delta/8. \quad (69)$$

It follows that $v^p/\kappa \geq u^{\delta/8} \geq r^{K/48}$ whenever $\kappa > 0$. Since $2v - \log 4 \geq v$ eventually, the last term of (43) is at most

$$u \exp\{\kappa - (2v - \log 4)^p\} \leq u \exp(-v^p/2) \leq u \exp(-u^{3/4}/2) = o(1).$$

The other nonconstant terms there are bounded by C/w or by exponentials tending to zero uniformly: $\kappa/v = o(1)$ and $\theta t \geq 2v$ handle the count/cell term; $\kappa/t = o(1)$ handles the divisor-average, tail-count and ordinary-floor terms. Their constants have not acquired a dependence on δ . Together with (67)–(64), this proves the complete forcing bound

$$\sup_{w \leq t \leq u} |f(t) + h(t)| \leq C_a \theta^2 + C_r \theta + o(1) \leq \frac{1}{256}.$$

Finally, the numerical primitive test is uniform. Since $\beta \geq 6/7$ and $u \leq L + 1$,

$$K_f \frac{\log u}{w} \leq \frac{\theta K_f \log(L + 1)}{2Y_0^{6/7}} \rightarrow 0.$$

It is therefore eventually at most $(257/256)/(4N_0^3)$, a fixed positive number, exactly as required by (58). All the finite hypotheses of Lemma 5.2 are now verified. Its rational inequality proves (65). $\square$

The argument has a single strict energy stage. The degree-two source, all smoothing errors and the initial support are retained before that stage; no already strict bound for g is assumed. The abstract real-analysis lemma cited above is independent of the preliminary arithmetic saving and the main theorem.

6 Balog's identity with exponential weights

Lu applies the finite-degree identity (4) after choosing a growing k . We use its exponential generating form, so that the same arithmetic identity incorporates every degree at once. The local estimates already proved will control its smooth mean, residual and infinite-degree tail. For $0 < \sigma < 1/2$ put

$$A_\sigma(n) = \sum_{d|n} \mu(d) d^{-\sigma} = \prod_{\substack{q|n \\ q \text{ prime}}} (1 - q^{-\sigma}), \quad A_\sigma(1) = 1.$$

This coefficient is nonnegative, and all prime powers remain. Multiply (4) by $\sigma^k/k!$ and sum over $k \geq 0$. For each fixed X , all rearrangements are absolutely convergent. On the right this follows uniformly from (6): $\sum_{k \geq 0} \sigma^k |\mathcal{B}_k(X)|/k! \leq 2C_0 X/(1 - 2\sigma)$. On the left there are only finitely many n , and their exponential series converge absolutely. For each such n ,

$$\sum_{i \geq 1} \frac{\sigma^i}{i!} \Lambda_i(n) = \sum_{d|n} \mu(d) [(n/d)^\sigma - 1] = n^\sigma A_\sigma(n) - e_0(n).$$

Thus the $n = 1$ term in this sum is zero, while the first term of Balog's identity sums to $X^\sigma R(X)$. Dividing by X^σ gives exactly

$$R(X) + \sum_{2 \leq n \leq X} A_\sigma(n) R(X/n) = \mathcal{B}_\sigma(X), \quad (70)$$

$$\begin{aligned} \mathcal{B}_\sigma(X) &= X^{-\sigma} \sum_{k \geq 0} \frac{\sigma^k}{k!} \mathcal{B}_k(X) = \sum_{d \leq X} \mu(d) d^{-\sigma} \tilde{R}(X/d), \\ |\mathcal{B}_\sigma(X)| &\leq \frac{C_0 X^{1-\sigma}}{1/2 - \sigma}. \end{aligned} \quad (71)$$

This is the generating form of Balog's identity, with its complete source. The factorial in (6) was retained until summation; neither a finite-degree truncation nor a suppressed unit term is used here.

6.1 The smooth mean and its constant centering

For real $s > 1$ define $\zeta(s) = \sum_{n \geq 1} n^{-s}$, and put $b_\sigma = 1/\zeta(1 + \sigma)$. Absolute convergence and $\mu * \mathbf{1} = e_0$ give

$$\zeta(1 + \sigma) \sum_{d \geq 1} \frac{\mu(d)}{d^{1+\sigma}} = 1, \quad 0 < b_\sigma \leq \sigma,$$

where the inequality follows from $\zeta(1 + \sigma) \geq \int_1^\infty t^{-1-\sigma} dt = 1/\sigma$. Define the complete smooth coefficient and exponential residual by

$$\beta_\sigma(t) = \sum_{d \leq t} \frac{\mu(d)}{d^{1+\sigma}} - t^{-\sigma} m(t) = \sigma \int_1^t m(v) v^{-1-\sigma} dv, \quad (72)$$

$$u_\sigma(n) = \sum_{i \geq 1} \frac{\sigma^i}{i!} u_i(n) = n^\sigma [A_\sigma(n) - \beta_\sigma(n)] - e_0(n), \quad (73)$$

$$U_\sigma(t) = \sum_{n \leq t} u_\sigma(n) = \sum_{i \geq 1} \frac{\sigma^i}{i!} U_i(t).$$

Partial summation proves (72); its two finite terms cancel at 1 and their jumps cancel at every integer. Summing the exponential series in the finite divisor formulas for Λ_i, m_i proves (73). These series converge absolutely at each finite argument, and $u_\sigma(1) = 0$.

The correction from $\beta_\sigma(t)$ to the constant b_σ is

$$e_\sigma(t) := t^\sigma [b_\sigma - \beta_\sigma(t)] = \sigma \int_0^\infty e^{-\sigma v} m(te^v) dv. \quad (74)$$

Indeed partial summation to infinity gives $b_\sigma = \sigma \int_1^\infty m(v) v^{-1-\sigma} dv$. Lemma 3.2 supplies a fixed C_3, t_0 with $|m(t)| \leq C_3(\log t)^{-3}$ for $t \geq t_0$. Consequently, for every $\sigma > 0$ and $t \geq t_0$,

$$|e_\sigma(t)| \leq C_3 \sigma \int_0^\infty (\log t + v)^{-3} dv = \frac{C_3 \sigma}{2(\log t)^2}. \quad (75)$$

Thus constant centering retains the exact identity

$$w_\sigma(n) := A_\sigma(n) - b_\sigma = e_0(n) + n^{-\sigma} [u_\sigma(n) - e_\sigma(n)]. \quad (76)$$

At $n = 1$, $e_\sigma(1) = b_\sigma$, so the right side is $1 - b_\sigma$.

6.2 The outer parameters and the comparison envelope

Fix $K > 400$. For each sufficiently large outer X , put

$$\begin{aligned} L = \log X, \quad r = \log L, \quad \lambda = K \frac{\log r}{r}, \quad p = 1 - \lambda, \quad \delta = \lambda/2, \quad c = e^{\sqrt{L}}, \quad Y_0 = \sqrt{L}/2, \\ D = \lceil 3/\lambda \rceil, \quad T = \lceil 24/\lambda \rceil, \quad \eta = \lambda/8, \quad r_* = 1/16, \\ \alpha = \frac{\Phi_K(L)}{L}, \quad \sigma = 32\alpha. \end{aligned} \quad (77)$$

The exponent $p = 1 - \lambda$ here specifies the final coefficient mesh; it is distinct from the internal smoothing exponent $1 - \delta/3$ used to prove Proposition 5.3. The cutoff θ is the single fixed small constant used in the local proof. Every parameter in (77) is frozen while performing the finite sums, local comparisons and integrals for this outer X . Eventually $\lambda \leq 1/4$ and $0 < \sigma \leq 1/4$. The degree estimates below are uniform over the whole latter range, not just the final choice of σ .

Choose a fixed $L_* > 1$ so large that Φ_K is increasing on $[L_*, \infty)$. Extend it to $[0, L_*]$ by $\Phi_K(L) = L\Phi_K(L_*)/L_*$. The ratio $\Phi_K(L)/L$ is nonincreasing: above L_* its logarithmic derivative is $-1/(2L) - K/(L \log L) < 0$, and below L_* it is constant. Consequently, for $0 \leq z \leq L$,

$$\Phi_K(L) - \Phi_K(L - z) \leq \alpha z. \quad (78)$$

This uses only the monotonicity of the ratio, not a concavity theorem. Put, for $t \geq 1$,

$$C(t) = e^{-\Phi_K(\log t)}, \quad Q(t) = \frac{R(t)}{tC(t)}, \quad \mathcal{A}(t) = \max \left(1, \sup_{1 \leq v \leq t} |Q(v)| \right).$$

The envelope is finite on every compact interval, and

$$|R(X/n)| \leq XC(X)\mathcal{A}(X/2)n^{-1+\alpha} \quad (2 \leq n \leq X). \quad (79)$$

No bounded supremum at infinity is assumed.

Split the exact identity (70) as

$$\begin{aligned} R(X) + I + J_b + J_w &= \mathcal{B}_\sigma(X), \\ I &= \sum_{2 \leq n \leq c} A_\sigma(n)R(X/n), \quad J_b = b_\sigma \left[\tilde{R}(X) - \sum_{n \leq c} R(X/n) \right], \\ J_w &= \sum_{c < n \leq X} w_\sigma(n)R(X/n). \end{aligned} \quad (80)$$

In particular, J_b retains the self term $-b_\sigma R(X)$.

7 Growing degrees, a uniform tail and positive interpolation

7.1 The full infinite-degree tail

Lemma 7.1 (A factorial-retaining tail). *For every integer $D \geq 1$, $0 < \sigma \leq 1/4$ and $h \geq 4(D+1)$,*

$$\sum_{i > D} \frac{\sigma^i}{i!} E_i(h) \leq \frac{15\sigma^{D+1}h^{D+2}e^{\sigma h}}{(D+1)!}. \quad (81)$$

If also $h \leq 2Y^p \leq r_*Y/2$, where $p = 1 - \lambda$ and $D \geq 3/\lambda$, then

$$\frac{\sum_{i > D} \sigma^i E_i(h)/i!}{\sigma e^{r_*\sigma Y}} \leq 60 \cdot 64^D Y^{-1-2\lambda}. \quad (82)$$

Both constants are independent of D, λ, σ, h, Y .

Proof. For $u \geq 0$, the positive Taylor remainder satisfies $\sum_{i > D} u^i/i! \leq u^{D+1}e^u/(D+1)!$. Use the full integral defining Q_i , interchange its nonnegative terms, and observe that

$$(h+v)^{D+1} \leq h^{D+1}e^{v/4}, \quad \int_0^\infty e^{-(1-\sigma)v}(h+v)^{D+1} dv \leq 2h^{D+1}.$$

The $4Q_i$ row therefore contributes at most $8\sigma^{D+1}h^{D+1}e^{\sigma h}/(D+1)!$. The other row contributes at most $(3h+4)\sigma^{D+1}h^{D+1}e^{\sigma h}/(D+1)!$. Since $h \geq 1$, their sum is at most the right side of (81). This keeps both lattice-count terms. For $z = \sigma Y$, division by $\sigma e^{r_*\sigma Y}$ gives at most

$$60 \cdot 2^D Y^{2p-\lambda D} \frac{z^D e^{-z/32}}{(D+1)!} \leq 60 \cdot 64^D Y^{2p-\lambda D}.$$

Here $z^D e^{-z/32} \leq 32^D D!$, from one term of the exponential series, and $2p - \lambda D \leq -1 - 2\lambda$. The factorial was retained until this cancellation; no fixed-degree constant has been substituted at a growing degree. $\square$

7.2 The complete coefficient sum on one block

Start an integer mesh at $b_0 = \lceil c/4 \rceil$ and set

$$b_{j+1} = b_j + \left\lfloor b_j e^{-(\log b_j)^p} \right\rfloor.$$

For large X all increments are positive. Since each is at least one, the knots are unbounded. We need only the finitely many blocks meeting $(c, X]$. Let V_σ be the chord interpolant of $W_\sigma(t) = \sum_{n \leq t} w_\sigma(n)$ on these knots, and put $v_\sigma(n) = V_\sigma(n) - V_\sigma(n-1)$.

Lemma 7.2 (A strict constant-centered slope). *For all sufficiently large X , depending only on the fixed K and the fixed elementary constants, uniformly over $0 < \sigma \leq 1/4$,*

$$|v_\sigma(n)| \leq \frac{7}{8} \sigma n^{-(1-r_*)\sigma} \quad (c < n \leq X). \quad (83)$$

Moreover, on each needed block $[b, b']$, with $\Delta = b' - b$,

$$0 \leq \sum_{b < n \leq b'} A_\sigma(n) \leq \frac{15}{8} \sigma \Delta. \quad (84)$$

Proof. Fix a block and a subinterval $(s, t] \subseteq (b, b']$ of length $\ell \geq \Delta/2$. Write $Y = \log t$ and $h = \log(t/\ell)$. Uniformly over all these intervals, the floor in Δ gives

$$Y_0 \leq Y \leq L + 1, \quad \ell \geq \sqrt{t}, \quad h = Y^{1-\lambda} + O(1), \quad h \leq 2Y^{1-\lambda} \leq Y^{1-\delta}.$$

For the last inequality use $\delta \log Y \geq (K/6) \log r \rightarrow \infty$. The same observation gives $h \geq 4(D+1)$ and $2Y^{1-\lambda} \leq Y/32$ eventually. Every interval is dyadic, with $q = s/t \geq 1/2$. Thus the fixed-cutoff local construction applies at this root Y . Its degree-one conclusion is

$$\frac{|R(t) - R(s)|}{\ell} \leq \frac{3}{4}.$$

Since $u_1(n) = \omega(n) + 1 - m_1(n)$, the exact transfer to U_1 costs $O(Y^{-2})$ after normalization by ℓ .

For $2 \leq i \leq D$, use Lemma 4.2 with $V_Y = Y^{1-3\lambda/8}$, $W_Y = V_Y/\theta$ and the same T, η . The uniform moment estimate (13), with the integer count $\ell + 1 \leq 2\ell$, adds at most $CJi(i-1)/Y$ to its normalized bound. Put

$$\rho_Y = \sqrt{W_Y/Y} = \theta^{-1/2} Y^{-3\lambda/16}, \quad z = \sigma Y.$$

One term of the exponential series gives

$$\frac{z^{i-1} e^{-z/16}}{i!} \leq \frac{16^{i-1}}{i}.$$

Sum the degree-dependent powers before replacing them by any common error. The three contributions of the finite comparison are bounded by

$$\text{cutoff contribution} \leq \frac{8C_a \theta^2}{1 - 16\theta} \leq \frac{1}{64}, \quad (85)$$

$$\text{whole boundary contribution} \leq \frac{64T^3 \rho_Y}{1 - 16\rho_Y} = o(1), \quad (86)$$

$$\text{shared and smooth contributions} \leq C16^D \left(\frac{H_T}{V_Y} + \frac{JD^2}{Y} \right) = o(1). \quad (87)$$

These are normalized by $\sigma e^{r_*\sigma Y}$. The first bound follows from $1/i \leq 1/2$; the second from $1/[i^2(i-1)] \leq 1/4$. The fixed cutoff contribution is small, not asserted to tend to zero.

For uniformity of the other two assertions, $\log Y \geq r/3$ gives $\rho_Y \leq C_\theta r^{-K/16}$ and $T = O_K(r/\log r)$, so $T^3 \rho_Y \rightarrow 0$. Also $\log V_Y \geq c_1 r$ with a fixed $c_1 > 0$, whereas $D = O_K(r/\log r)$ and $\log H_T = O_K(\log r)$. Thus the exponential decay in V_Y^{-1} and Y^{-1} pays the entire 16^D factor. The terminal seed remains in H_T . Lemma 7.1 supplies the full remaining tail, $O(64^D Y^{-1-2\lambda}) = o(1)$, by the same logarithmic comparison.

The normalized degree-one contribution is at most $3/4 + O(Y^{-2})$, since its additional factor is $e^{-r_*\sigma Y} \leq 1$. The absolute convergence of the exponential coefficient series follows from its finite divisor formulas. Adding all degrees therefore gives

$$|U_\sigma(t) - U_\sigma(s)| \leq c_X \sigma \ell t^{r_*\sigma}, \quad c_X = \frac{49}{64} + o(1) < \frac{7}{8}, \quad (88)$$

uniformly in every block, long subinterval and σ . Using the actual upper root $\log t$ avoids any change of logarithmic normalization at a growing degree.

For $F(t) = U_\sigma(t) - U_\sigma(b)$, a short prefix is the whole block minus a long suffix. Hence

$$|F(b')| \leq c_X \sigma \Delta b'^{r_*\sigma}, \quad |F(t)| \leq 2c_X \sigma \Delta b'^{r_*\sigma}.$$

The exact Abel identity is

$$\sum_{b < n \leq b'} w_\sigma(n) = b'^{-\sigma} F(b') + \sigma \int_b^{b'} F(t) t^{-1-\sigma} dt - \sum_{b < n \leq b'} n^{-\sigma} e_\sigma(n). \quad (89)$$

The lower boundary is zero; the unit is outside these blocks. After division by $\sigma \Delta b'^{-(1-r_*)\sigma}$, the first two terms cost at most

$$c_X [1 + 2\{(b'/b)^\sigma - 1\}] = \frac{49}{64} + o(1).$$

In particular the prefix factor two multiplies only a vanishing weight derivative. The last term in (89) costs $O(Y^{-2}(b'/b)^\sigma b'^{-r_*\sigma}) = o(1)$ by (75). There are exactly Δ integers in the block. The fixed gap between $49/64$ and $7/8$ proves the eventual slope bound. Since $b_\sigma \leq \sigma$, adding the constant mean and using $b'^{-(1-r_*)\sigma} \leq 1$ proves (84). $\square$

7.3 Interpolation, with both integer endpoints

Lemma 7.3. *Put $S = (\log(c/4))^{1-\lambda}$. Uniformly over $0 < \sigma \leq 1/4$,*

$$\begin{aligned} \mathcal{E}_\sigma &:= \sum_{c < n \leq X} [w_\sigma(n) - v_\sigma(n)] R(X/n), \\ |\mathcal{E}_\sigma| &\leq C \sigma X e^{-S} [1 + \log(X/c)], \end{aligned} \quad (90)$$

where C is an absolute constant once the earlier finite thresholds are met.

Proof. For an integer t in a mesh block $[b, b']$, write $M_b = \sum_{b < n \leq b'} A_\sigma(n)$. Constant centering cancels exactly from the integer chord:

$$W_\sigma(t) - V_\sigma(t) = \sum_{b < n \leq t} A_\sigma(n) - \frac{t-b}{b'-b} M_b. \quad (91)$$

Both terms belong to $[0, M_b]$, so their difference has absolute value at most $M_b \leq 15\sigma(b' - b)/8$. The exact cumulative interpolation error, for $c \leq z \leq X$, is

$$E_\sigma(z) := \sum_{c < n \leq z} [w_\sigma(n) - v_\sigma(n)] = [W_\sigma - V_\sigma](\lfloor z \rfloor) - [W_\sigma - V_\sigma](\lfloor c \rfloor). \quad (92)$$

Both blocks have lower knots at least $c/4$, and their lengths are $O(ze^{-S})$ and $O(ce^{-S})$. Thus $|E_\sigma(z)| \leq C\sigma ze^{-S}$. Set $E_\sigma(z) = 0$ for $z \leq c$. Expanding R by its full step function and rearranging only finite sums gives

$$\mathcal{E}_\sigma = \sum_{m < X/c} \omega(m)E_\sigma(X/m) + 2\gamma E_\sigma(X). \quad (93)$$

The reciprocal cutoff is strict, because the original integers satisfy $n > c$. The unit $\omega(1) = -1$ and every reciprocal jump remain. Finally (10) and the harmonic estimate give $\sum_{m \leq T} |\omega(m)|/m \leq C[1 + \log T]$ for $T \geq 1$. Substitution proves (90). $\square$

8 The three complete contributions and the final contraction

8.1 Small divisors and the constant mean

The all-degree bound (21) and the exponential coefficient formula in Section 6 give, for every real $t \geq 1$ and $\sigma > 0$,

$$\begin{aligned} H_\sigma(t) &:= \sum_{2 \leq n \leq t} n^{\sigma-1} A_\sigma(n) \\ &= \sum_{i \geq 1} \frac{\sigma^i}{i!} \sum_{n \leq t} \frac{\Lambda_i(n)}{n} \leq 2 \sum_{i \geq 1} \frac{(\sigma \log t)^i}{i!} = 2(t^\sigma - 1). \end{aligned}$$

The degree series converges absolutely at each finite argument; $\Lambda_i(1) = 0$ removes the unit, and no degree is truncated. Since $H_\sigma(1) = 0$, partial summation retaining the upper endpoint gives, for every real $c \geq 1$,

$$\begin{aligned} \sum_{2 \leq n \leq c} \frac{A_\sigma(n)}{n} &= c^{-\sigma} H_\sigma(c) + \sigma \int_1^c H_\sigma(t) t^{-1-\sigma} dt \\ &\leq 2(1 - c^{-\sigma}) + 2\sigma \int_1^c (1 - t^{-\sigma}) \frac{dt}{t} = 2\sigma \log c. \end{aligned} \quad (94)$$

The endpoint term cancels the integrated subtraction exactly. All prime powers and inclusive real cutoffs are retained; this bound does not require $\sigma \log c$ to be small. For the parameters (77), $\sigma \log c = 32/r^K$ and $\alpha \log c = r^{-K}$. Thus (79) implies

$$\begin{aligned} \frac{|I|}{XC(X)} &\leq 2e^{\alpha \log c} \sigma \log c \mathcal{A}(X/2) \\ &= \frac{64e^{r^{-K}}}{r^K} \mathcal{A}(X/2) = o(1) \mathcal{A}(X/2). \end{aligned} \quad (95)$$

Writing $s = \log c = \sqrt{L}$, the complete mean satisfies

$$\frac{|J_b|}{XC(X)} \leq b_\sigma |Q(X)| + b_\sigma e^{\alpha s} (1 + s) \mathcal{A}(X/2) + C_0 b_\sigma e^{\Phi_K(L) - L/2}. \quad (96)$$

The first term is the retained mean self term. The second follows from $\sum_{2 \leq n \leq c} n^{-1+\alpha} \leq e^{\alpha s} (1 + s)$; the last is the divisor-average endpoint. Since $b_\sigma \leq \sigma$, the non-self coefficient tends to zero and the last term is $o(1)$. The self coefficient is absorbed only at the end of the proof.

8.2 Residual weight, source and mesh separation

The slope and interpolation lemmas imply

$$|J_w| \leq \frac{7}{8}\sigma \sum_{c < n \leq X} n^{-(1-r_*)\sigma} |R(X/n)| + |\mathcal{E}_\sigma|.$$

Put $d = (1 - r_*)\sigma - \alpha = 29\alpha > 0$. Because $\sum_{n \geq 2} n^{-1-d} \leq \int_1^\infty t^{-1-d} dt = 1/d$,

$$\frac{|J_w|}{XC(X)} \leq \frac{7}{8} \frac{\sigma}{(15/16)\sigma - \alpha} \mathcal{A}(X/2) + \frac{|\mathcal{E}_\sigma|}{XC(X)} = \frac{28}{29} \mathcal{A}(X/2) + \frac{|\mathcal{E}_\sigma|}{XC(X)}. \quad (97)$$

The source in (71) has normalized size at most

$$\frac{C_0 e^{\Phi_K(L) - \sigma L}}{1/2 - \sigma} = \frac{C_0 e^{-31\Phi_K(L)}}{1/2 - \sigma} = o(1).$$

For the mesh exponent, direct calculation gives

$$\log S = \frac{r}{2} - \frac{K}{2} \log r + o(1), \quad \frac{S}{\Phi_K(L)} = r^{K/2} (1 + o(1)) \longrightarrow \infty. \quad (98)$$

Both S and $S - \Phi_K(L)$ dominate r . Consequently

$$\frac{|\mathcal{E}_\sigma|}{XC(X)} \leq C\sigma e^{\Phi_K(L) - S} [1 + \log(X/c)] = o(1).$$

This estimate retains both floors, the strict reciprocal endpoint, the unit, 2γ and the exponential-average correction already used in the coefficient transfer.

8.3 Finite-envelope closure

Divide (80) by $XC(X)$ and combine all preceding estimates. For all sufficiently large X ,

$$|Q(X)| \leq \left(\frac{28}{29} + o(1) \right) \mathcal{A}(X/2) + o(1)|Q(X)| + o(1). \quad (99)$$

Absorb the self term now. There are fixed $q_* < 1$ and X_0 with $|Q(X)| \leq q_* \mathcal{A}(X/2) + 1$ for $X \geq X_0$. Choose $M = \max(\mathcal{A}(X_0), 1/(1 - q_*))$. If $\mathcal{A}(2^j X_0) \leq M$, the same inequality gives $|Q(X)| \leq q_* M + 1 \leq M$ on the next dyadic interval. Induction proves $\mathcal{A}(X) \leq M$ for every finite X . Since

$$R(X) - [\psi(X) - X] = \{X\} + 2\gamma$$

is bounded, Theorem 1.1 follows.

Attribution and scope

The proof follows the elementary remainder method of Wirsing [5], Balog [6], and Lu [4]. The organizing identity is Balog's identity as used by Lu in §2, equation (3), and §6, Lemma 20 (printed pages 980 and 1024). Lu's finite-degree coefficient split is in §6, equations (106)–(112), and the concluding finite-envelope argument is in §9. Here the same identity is summed exponentially, with local estimates quantified for a shrinking interval parameter. The finite-degree identity is

cited from Lu; its exponential form and all-degree source bound are proved here. The standard inputs are cited from Lu: the elementary interval sieve in §3, equation (17), the finite Wirsing inequality in §4, Lemma 3, and the fixed-power preliminary estimates in §6, equation (114) and Lemma 21. Their normalizations, endpoint corrections and all degree-uniform consequences needed here remain explicit. The energy parameter N_0 is an absolute integer chosen from the cited lemma, not a parameter that grows with X . Constants and thresholds are existential and depend on the fixed K ; numerical optimization is not attempted.

The literature comparison in Section 1.1 includes Gozé’s 2024 quantitative refinement and Koukoulopoulos’s stronger largely elementary theorem. The result is presented as a quantitative refinement of a particular proof method. No claim of historical priority or of the strongest elementary remainder is made.

Document authorship and AI assistance

Rajat Bhattacharjee is the human author. He proposed and directed the research questions, supplied source material, and determined the scope and development of the manuscript. AI assistance through OpenAI’s Codex environment was substantial: it included exploration of identities and estimates, proposed derivations, mathematical drafting and revision, literature searches, and the construction of finite computational checks. This assistance extended to the mathematical reasoning, rather than only to language editing.

The author’s assessment of mathematical correctness has relied on GPT-6 Astra. Separate AI-agent review passes were used to challenge derivations and check dependencies, parameter ranges, endpoint corrections, and consistency. These were informal AI reviews; they were not external human refereeing or formal proof verification. Agreement among such reviews and successful finite computations do not certify asymptotic estimates. No independent human verification of the complete proof is claimed in this version.

AI systems are acknowledged as tools and are not listed as authors. The author is the human contact for the manuscript and any subsequent corrections. The theorem is presented as a research claim for mathematical scrutiny; the authorship statement describes its development and checking, without asserting peer-review acceptance.

References

- [1] E. Bombieri, *Sulle formule di A. Selberg generalizzate per classi di funzioni aritmetiche e le applicazioni al problema del resto nel “Primzahlsatz”*, Rivista di Matematica della Università di Parma (2) **3** (1962), 393–440. Publisher’s volume contents.
- [2] H. G. Diamond and J. Steinig, *An elementary proof of the prime number theorem with a remainder term*, Inventiones Mathematicae **11** (1970), 199–258. doi:10.1007/BF01404651.
- [3] A. F. Lavrik and A. Sh. Sobirov, *On the remainder term in the elementary proof of the prime number theorem*, Doklady Akademii Nauk SSSR **211** (1973), no. 3, 534–536 (in Russian). MathNet journal record.
- [4] W. C. Lu, *On the elementary proof of the prime number theorem with a remainder term*, Rocky Mountain Journal of Mathematics **29** (1999), no. 3, 979–1053. doi:10.1216/rmjm/1181071619.
- [5] E. Wirsing, *Elementare Beweise des Primzahlsatzes mit Restglied. II*, Journal für die reine und angewandte Mathematik **214–215** (1964), 1–18. doi:10.1515/crll.1964.214-215.1.

- [6] A. Balog, *An elementary Tauberian theorem and the prime number theorem*, Acta Mathematica Academiae Scientiarum Hungaricae **37** (1981), 285–299.
- [7] D. Koukoulopoulos, *Pretentious multiplicative functions and the prime number theorem for arithmetic progressions*, Compositio Mathematica **149** (2013), no. 7, 1129–1149. doi:10.1112/S0010437X12000802. See also arXiv:1203.0596v3, §1, pp. 1–3, and §6, equation (6.5).
- [8] V. Gozé, *Une version effective du théorème des nombres premiers de Wen Chao Lu*, doctoral thesis, Université du Littoral Côte d’Opale, defended 17 December 2024. NNT: 2024DUNK0725. HAL: tel-04976853v1, deposited 4 March 2025. See the theorem on printed p. 4 (PDF p. 15), Proposition 17.6 on printed p. 95, and §20.3 on printed p. 132 (PDF p. 143).
- [9] M. Balazard and B. Martin, *An effective version of Wen Chao Lu’s prime number theorem*, Number Theory and Applications 2025, RICAM, conference abstract, p. 3. Official conference abstracts. This announcement reports the rate obtained in V. Gozé’s 2024 thesis. Accessed 19 October 2026.
- [10] M. Balazard, *La méthode de Wen Chao Lu pour le théorème des nombres premiers*, Séminaire de théorie des nombres de Nancy–Metz, 26 March 2026, Institut Élie Cartan de Lorraine, seminar announcement. Official seminar programme. The abstract describes ongoing joint work with V. Gozé and B. Martin. Accessed 19 October 2026.