

Point separation for adjoint line bundles

Brian Nugent

Abstract

Let X be a smooth complex projective variety of dimension n and let L be an ample line bundle. Using the freeness and minimization inputs specified in the introduction, we prove that $K_X + mL$ separates distinct points for every integer $m \geq n + 2$, the sharp bound predicted by Fujita’s very ampleness conjecture. Exponential-moment convergence produces point-centred minimizing valuations at both points in a common degree. Strict convexity then gives a divisor with discrepancy equality at one point and strict excess at the other. A rational perturbation on a resolution, followed by Kawamata–Viehweg vanishing and descent, gives the required sections.

1 Introduction

Fujita’s conjecture [Fuj87] predicts that an ample line bundle L on a smooth projective n -fold X has globally generated adjoints $K_X + mL$ for $m \geq n + 1$ and very ample adjoints for $m \geq n + 2$. We prove separation of distinct points at the latter bound.

Standing hypothesis. Throughout the paper we assume the freeness theorem and the valuation minimization and tie-breaking inputs from OpenAI’s preprint *Fujita’s freeness conjecture* [OAI], dated September 23, 2026. The manuscript is unrefereed; the precise results on which we rely are listed in Appendix A. We prove the additional estimates needed to apply those constructions in the range $n + 1 < t < n + 2$.

Theorem A. *Let X be a smooth complex projective variety of dimension $n \geq 1$ and let L be an ample line bundle. For every integer $m \geq n + 2$ and every pair of distinct closed points $x, y \in X$, the evaluation map*

$$H^0(X, K_X + mL) \longrightarrow (K_X + mL) \otimes k(x) \oplus (K_X + mL) \otimes k(y)$$

is surjective.

The bound is sharp: for $X = \mathbf{P}^n$ and $L = \mathcal{O}_{\mathbf{P}^n}(1)$, the bundle $K_X + (n + 1)L = \mathcal{O}_X$ separates no pair of points. Separation of tangent vectors remains the additional requirement for very ampleness.

Together with global generation, Theorem A shows that $|K_X + mL|$ defines a finite morphism, injective on closed points and birational onto its image. Indeed, its fibres are finite, and in characteristic zero injectivity forces degree one on each image component. The morphism is also a homeomorphism onto its image, since it is finite and bijective.

The sharp bound is classical for curves, and Reider’s theorem gives very ampleness of $K_X + mL$ for $m \geq 4$ on surfaces [Rei88]. On threefolds, Ein–Lazarsfeld proved global generation for $m \geq 4$ [EL93, Corollary 2*]. For separation of distinct points, Fujita obtained $m \geq 6$ [Fuj94, Theorem (0.11)]; the sharp bound $m \geq 5$ follows from Ghidelli–Lacini’s Theorem 5.2 and their computation $F(3, 2) < 5$ [GL24, Table 1 and Corollary 5.3]. Their theorem concerns every reduced pair of points, although the corollary records only birationality. In any dimension, very ampleness at $m \geq n + 2$ is also known when L is ample and globally generated [AS95, Section 11]. Theorem A imposes no global-generation hypothesis on L .

For arbitrary ample L , several general bounds apply directly to distinct-point separation. Angehrn–Siu proved that $K_X + mL$ separates any r distinct points when

$$m \geq \frac{n^2 + 2rn - n + 2}{2},$$

so their bound for a pair is $m \geq (n^2 + 3n + 2)/2$ [AS95, Corollary 0.4]. Ghidelli–Lacini obtained a sufficient bound of order $n \log \log n$:

$$m \geq n + 1 + \sqrt{2}n(\log \log n + 2.34) \quad (n \geq 2),$$

by setting $r = 2$ in their Theorem 1.3 [GL24]. More recently, Han’s preprint gives the sufficient bound

$$m \geq \lceil C_0 n + 3\sqrt{2en} \rceil, \quad C_0 = 1.77629988\dots,$$

as the ample case of a theorem for nef and big divisors away from the augmented base locus [Han26, Theorem 5.4]. These are bounds for separation of arbitrary distinct points; separation of tangent vectors is a further issue. Theorem A reaches the sharp threshold $n + 2$ under the standing hypothesis.

The parallel development of freeness bounds supplies much of the method. Angehrn–Siu obtained global generation for $m \geq n(n + 1)/2 + 1$ [AS95, Corollary 0.2]. Combining their method with Helmke’s multiplicity estimates [Hel97, Hel99], Heier obtained

$$m \geq \left(e + \frac{1}{2}\right)n^{4/3} + \frac{1}{2}n^{2/3} + 1$$

as a sufficient freeness bound [Hei02, Theorem 1.4]. Ghidelli–Lacini subsequently gave $m \geq \max\{n + 1, n(\log \log n + 2.34)\}$ for $n \geq 2$ [GL24, Theorem 1.1], and Han gave $m \geq \lceil C_0 n \rceil$ [Han26, Theorem 1.1]. These freeness statements provide sections nonzero at a prescribed point; the separation statements above control their values at two points.

The geometric strategy has its roots in the use of singular divisors and vanishing by Ein–Lazarsfeld [EL93], the multiplier-ideal construction of Angehrn–Siu [AS95], and Kollár’s algebraic treatment [Kol97, Sections 5–6]. One constructs a divisor with controlled singularities, isolates a zero-dimensional centre, and uses vanishing to produce a section with a prescribed value. Helmke’s estimates control the multiplicities of the centres arising in the induction; Ghidelli–Lacini optimize the accumulated cost of cutting them down. Our final lifting step follows the discrepancy calculation on a resolution used by Fujita [Fuj93, Section 1, (1.6)]. Within the valuation minimization framework assumed here, the additional ingredients are convergence of the exponential moments, which supplies a common degree for two prescribed points, and a strict-convexity comparison, which produces discrepancy equality over one point and strict excess over the other. These are the steps that turn the freeness construction into point separation at the bound $n + 2$.

Outline of the proof. Fix $n + 1 < t < n + 2$, and write $N_k = h^0(X, kL)$. The auxiliary degree k is independent of the adjoint coefficient m . For a test valuation v with log discrepancy $A(v)$ and closed centre $c_X(v)$, and a basis $\mathbf{s}$ of $H^0(X, kL)$, set

$$F_k(v, \mathbf{s}) = \frac{1}{N_k} \sum_i \exp\left(\frac{A(v)}{t} - \frac{v(s_i)}{k}\right), \quad f_k(x) = \inf_{\substack{x \in c_X(v) \\ \mathbf{s} \text{ a basis}}} F_k(v, \mathbf{s}).$$

The centre of a test entering this infimum may have positive dimension.

Sections 4–5 prove that $f_k(x)$ converges to a positive limit and that, in every sufficiently large degree, the infimum is attained only by tests with centre $\{x\}$. The convergence argument applies a

uniform estimate of Blum–Jonsson to exponential moments on an Okounkov body. It supplies the common degree needed for two prescribed points.

Assume without loss of generality that $f_k(x) \geq f_k(y)$, and choose minimizing tests v, w at x, y with a common adapted basis. The Gibbs coefficients of v define

$$b_i = \frac{t}{k} \frac{e^{-v(s_i)/k}}{\sum_h e^{-v(s_h)/k}}, \quad D_v = \sum_i b_i \operatorname{div}(s_i) \equiv tL.$$

Scale stationarity gives $v(D_v) = A(v)$. The strict-convexity argument in Section 3 gives $w(D_v) > A(w)$, even when the two minimum values coincide.

Section 6 perturbs the pullback of D_v on a resolution, preserving strict excess at y and producing a reduced divisor over x along which discrepancy equality holds. Vanishing lifts a nonzero value from this divisor, and descent gives a section nonzero at x and zero at y . Global generation supplies a second section nonzero at y . Thus $t > n + 1$ eliminates positive-dimensional centres, while $t < n + 2$ leaves the positivity needed for vanishing.

2 Valuations and the minimization problem

We work over $\mathbf{C}$ and write line bundles additively. Unless stated otherwise, X is smooth, connected and projective of dimension $n \geq 2$, and L is ample. Write

$$R_k = H^0(X, kL), \quad N_k = \dim R_k = \frac{L^n}{n!} k^n + O(k^{n-1}).$$

The intersection number L^n is a positive integer. We consider only degrees for which $N_k > 0$.

2.1 Tests and centres

A *test* is a nonzero quasi-monomial valuation represented on a smooth projective birational model $\pi : Y \rightarrow X$ as follows. Choose coordinate divisors $E_1, \dots, E_r$ with simple normal crossings (SNC), compatible with $\operatorname{Supp} K_{Y/X}$ near an irreducible component Z of their intersection. Assign positive real weights $u_1, \dots, u_r$ to their local equations and weight zero to the remaining parameters. The valuation takes the least weighted order in these equations. Its discrepancy and closed centre are

$$A(v) = \sum_j u_j A(E_j), \quad c_X(v) = \pi(\overline{Z}), \quad A(E) = 1 + \operatorname{ord}_E(K_{Y/X}). \quad (2.1)$$

These are the tests of the freeness construction, expressed in the usual quasi-monomial formalism [JM12]. A *test at x* satisfies $x \in c_X(v)$; it is *point-centred at x* if $c_X(v) = \{x\}$.

For a nonzero section s of a line bundle, compute $v(s)$ in a local trivialization at the generic point of $c_X(v)$. This value is nonnegative, and it is positive exactly when s vanishes identically on the centre. Extend valuation orders to effective real divisors by linearity. Positive rescaling preserves the class of tests, with $A(\lambda v) = \lambda A(v)$.

We will use retraction to an SNC divisor. Let $\pi : Y \rightarrow X$ resolve effective divisors $D_1, \dots, D_N$, and let E_j be the components of

$$\operatorname{Exc}(\pi) \cup \operatorname{Supp} \pi^* \left(\sum_i D_i \right).$$

The weights $u_j = v(E_j)$ define a nonzero monomial retraction $r(v)$ whenever at least one is positive. It satisfies

$$r(v)(D_i) = v(D_i), \quad A(r(v)) \leq A(v), \quad c_X(r(v)) \supseteq c_X(v). \quad (2.2)$$

Each π^*D_i is locally a monomial times a unit, so retraction preserves its value. The discrepancy inequality is the SNC retraction inequality [JM12, Corollaries 4.8 and 5.4]; see also Appendix A.

We also use retraction for an effective rational divisor Γ on a smooth model $\rho : T \rightarrow X$, where $\Sigma = \text{Exc}(\rho) \cup \text{Supp } \Gamma$ is SNC. Write $\gamma_F = \text{coeff}_F \Gamma$ for its components, allowing zero coefficients on exceptional primes. The same local calculation gives

$$w(\Gamma) = \sum_{F \subset \Sigma} w(F) \gamma_F, \quad A(w) \geq \sum_{F \subset \Sigma} w(F) A(F). \quad (2.3)$$

Here orders on T are computed at the centre of w on T . If w is a test at y and $w(\Gamma) > A(w)$, some component F satisfies $y \in \rho(F)$ and $\gamma_F > A(F)$.

2.2 Adapted bases

A test defines a decreasing filtration

$$\mathcal{F}_v^\lambda R_k = \{s \in R_k : v(s) \geq \lambda\} \cup \{0\}.$$

List its jumping numbers with multiplicity in decreasing order as $a_{k,1}(v) \geq \dots \geq a_{k,N_k}(v)$. A basis is *adapted* if every filtered subspace is spanned by the basis vectors it contains.

Lemma 2.1. *If a basis is ordered so that $v(s_1) \geq \dots \geq v(s_{N_k})$, then $v(s_i) \leq a_{k,i}(v)$ for every i , with equality throughout exactly when the basis is adapted. Any two finite filtrations of a finite-dimensional vector space admit a common adapted basis.*

Proof. The first i basis vectors lie in $\mathcal{F}_v^{v(s_i)} R_k$, so this subspace has dimension at least i . Hence $v(s_i) \leq a_{k,i}(v)$. Equality for every i says precisely that each filtered subspace contains enough basis vectors to span it.

For common adaptedness, regard the filtrations as finite increasing chains and induct on the length of the first. Choose a common adapted basis of its largest proper subspace U . Extend it along the second chain, adding at each step $G_{j-1} \subset G_j$ lifts of a basis of $G_j/(G_{j-1} + U \cap G_j)$. This preserves adaptedness to both chains. $\square$

2.3 The objective and its Gibbs distribution

For a fixed parameter $t > 0$, define

$$F_k(v, \mathbf{s}) = \frac{1}{N_k} \sum_{i=1}^{N_k} \exp \left(\frac{A(v)}{t} - \frac{v(s_i)}{k} \right), \quad (2.4)$$

$$F_k(v) = \min_{\mathbf{s}} F_k(v, \mathbf{s}), \quad f_k(x) = \inf_{x \in c_X(v)} F_k(v). \quad (2.5)$$

We suppress t from the notation. The objective decreases strictly as any section order increases, so Lemma 2.1 gives

$$F_k(v) = \frac{1}{N_k} \sum_i \exp \left(\frac{A(v)}{t} - \frac{a_{k,i}(v)}{k} \right). \quad (2.6)$$

The minimizing bases are exactly those adapted to v .

A pair $(v, \mathbf{s})$ is *minimizing at x* if v is a test at x and $F_k(v, \mathbf{s}) = f_k(x)$. This means that v attains the infimum and $\mathbf{s}$ is adapted to its filtration. For any pair $(v, \mathbf{s})$, define its Gibbs probabilities and coefficients by

$$p_v(i) = \frac{e^{-v(s_i)/k}}{\sum_h e^{-v(s_h)/k}}, \quad b_i = \frac{t}{k} p_v(i). \quad (2.7)$$

Their dependence on k and $\mathbf{s}$ will be understood.

Lemma 2.2 (Scale stationarity). *If $(v, \mathbf{s})$ is minimizing, then*

$$\sum_i p_v(i) \frac{v(s_i)}{k} = \frac{A(v)}{t}, \quad \sum_i b_i v(s_i) = A(v), \quad k \sum_i b_i = t. \quad (2.8)$$

Proof. The function $\lambda \mapsto F_k(\lambda v, \mathbf{s})$ has a minimum at $\lambda = 1$. Differentiating there and dividing by the positive exponential sum gives the first equality. The other two follow from (2.7). $\square$

3 The two-point comparison

The comparison rests on the strict inequality $e^a > 1 + a$ for $a \neq 0$. Its geometric input is that a common adapted basis cannot have the same valuation orders at two points separated by kL .

Lemma 3.1 (Strict two-point excess). *Suppose kL separates $x \neq y$ and $f_k(x) \geq f_k(y)$. Let v, w be minimizing tests at x, y , with $c_X(v) = \{x\}$, and choose a common adapted basis $\mathbf{s}$. The Gibbs coefficients of v satisfy*

$$\sum_i b_i w(s_i) > A(w). \quad (3.1)$$

Proof. Write $v_i = v(s_i)$, $w_i = w(s_i)$ and

$$\Delta_i = \frac{A(w) - A(v)}{t} - \frac{w_i - v_i}{k}.$$

We first show that the Δ_i are not all zero. Otherwise $v_i - w_i$ would be constant. Since the basis is adapted and kL is generated at both points, $\min_i v_i = \min_i w_i = 0$, so this constant would be zero. The $N_k - 1$ basis vectors of positive v -value span $H^0(X, kL \otimes \mathfrak{m}_x)$. They would all have positive w -value and hence vanish at y , contradicting separation of the two points.

All Gibbs probabilities are positive. Strict convexity and scale stationarity therefore give

$$\begin{aligned} 1 &\geq \frac{f_k(y)}{f_k(x)} = \sum_i p_v(i) e^{\Delta_i} > 1 + \sum_i p_v(i) \Delta_i \\ &= 1 + \frac{1}{t} \left(A(w) - \sum_i b_i w_i \right). \end{aligned}$$

This is precisely (3.1). $\square$

Remark 3.2. The excess also has the exact expression

$$\sum_i b_i w(s_i) - A(w) = t \left(\log \frac{f_k(x)}{f_k(y)} + \text{KL}(p_v \| p_w) \right), \quad \text{KL}(p \| q) = \sum_i p(i) \log \frac{p(i)}{q(i)}.$$

This follows by substituting the Gibbs probabilities and using scale stationarity. The proof above obtains the needed strict inequality directly.

4 Bounds and convergence

Fix $x \in X$ and $n + 1 < t < n + 2$. Our goal is convergence of $f_k(x)$ to a positive limit. We first bound the discrepancies of near-minimizers, then use these bounds to pass from convergence for each fixed valuation to convergence of the infima.

4.1 Sublevel bounds

Choose regular parameters at x and order Taylor monomials first by total degree, then lexicographically. Let $H_k \subset \mathbf{Z}_{\geq 0}^n$ be the set of initial exponents of sections in R_k . The initial-exponent construction gives $\#H_k = N_k$, and sections representing the distinct exponents form a basis. It also gives a constant $C = C(X, L, x)$ such that

$$\text{mult}_x(s) \leq Ck, \quad v(s) \leq A(v) \text{mult}_x(s) \quad (4.1)$$

for every nonzero $s \in R_k$ and every test v at x . Appendix A records the source of these estimates.

Lemma 4.1 (Counting initial exponents). *For a finite set $H \subset \mathbf{Z}_{\geq 0}^n$ with $\#H = N$,*

$$\sum_{\gamma \in H} \|\gamma\|_1 \geq \frac{n}{n+1} (n!N)^{1/n} N - \frac{n}{2} N.$$

Consequently

$$\liminf_{k \rightarrow \infty} \frac{1}{kN_k} \sum_{\gamma \in H_k} \|\gamma\|_1 \geq \frac{n}{n+1} (L^n)^{1/n} \geq \frac{n}{n+1}.$$

Proof. The union $H + [0, 1)^n$ has volume N . Among measurable subsets of the positive orthant of this volume, the simplex $\{u : \|u\|_1 \leq b\}$, where $b^n/n! = N$, minimizes the integral of $\|u\|_1$. That integral is $nbN/(n+1)$, whereas the integral over the unit cube based at γ is $\|\gamma\|_1 + n/2$. Summing over H proves the first inequality. The Hilbert asymptotic and $L^n \geq 1$ give the second. $\square$

Proposition 4.2 (Uniform sublevel bounds). *There are constants $c > 0$, $0 < \delta < 1$ and $0 < a_- \leq a_+ < \infty$ such that, for all sufficiently large k ,*

$$c \leq f_k(x) \leq 1 - \delta.$$

In these degrees, every pair at x with $F_k(v, \mathbf{s}) \leq 1 - \delta/2$ satisfies $a_- \leq A(v) \leq a_+$. Consequently $f_k(x)$ is attained for all sufficiently large k .

Proof. Since $t > n+1$, we may choose $\epsilon > 0$ with $3\epsilon = n/(n+1) - n/t$. Lemma 4.1 shows that, for all large k , a basis with distinct initial exponents satisfies

$$\frac{1}{N_k} \sum_i \frac{\text{mult}_x(s_i)}{k} \geq \frac{n}{t} + 2\epsilon.$$

Let E be the exceptional divisor of the blow-up at x , so $A(\text{ord}_E) = n$, and put $\theta_i = n/t - \text{mult}_x(s_i)/k$. These numbers have mean at most -2ϵ and, by (4.1), absolute value at most $n+C$. For $0 < \lambda \leq 1/(n+C)$, the elementary bound $e^u \leq 1 + u + u^2$ for $|u| \leq 1$ gives

$$F_k(\lambda \text{ord}_E, \mathbf{s}) \leq 1 - 2\epsilon\lambda + \lambda^2(n+C)^2.$$

Fixing $\lambda > 0$ sufficiently small gives the uniform upper bound $f_k(x) \leq 1 - \delta$.

To bound discrepancies, we need a positive proportion of sections of small ordinary order. Choose ℓ with ℓL very ample, and sections $\sigma_0, \dots, \sigma_n$ such that $\sigma_0(x) \neq 0$ and the ratios $z_i = \sigma_i/\sigma_0$ form regular parameters at x . For each residue class r modulo ℓ , choose h_r in that class with $h_r L$ globally generated and a section τ_r nonzero at x . For large k in this class, write $k = q\ell + h_r$. The sections

$$\tau_r \sigma_0^{q - \|\alpha\|_1} \sigma^\alpha, \quad \|\alpha\|_1 \leq r_k := \min\{q, \lfloor k/(2t) \rfloor\},$$

have germs equal to a common unit times z^α . Every nonzero linear combination has multiplicity at most $r_k \leq k/(2t)$. Since r_k grows linearly with k , their span has dimension at least κN_k for some fixed $0 < \kappa < 1$. Hence

$$\text{codim}_{R_k} \{s : \text{mult}_x(s) > k/(2t)\} \geq \kappa N_k \quad (4.2)$$

for all large k .

Write $a = A(v)$. By (4.1), the subspace $\{s : v(s) > ak/(2t)\}$ is contained in the subspace in (4.2). At least κN_k vectors in any basis therefore satisfy $v(s_i) \leq ak/(2t)$, giving

$$F_k(v, \mathbf{s}) \geq \kappa e^{a/(2t)}. \quad (4.3)$$

This proves $f_k(x) \geq \kappa$ and bounds a above whenever $F_k(v, \mathbf{s}) \leq 1 - \delta/2$. To bound a below on the same sublevel, use $v(s_i)/k \leq Ca$ to obtain $F_k(v, \mathbf{s}) \geq e^{-Ca}$. Choose $a_- > 0$ with $e^{-Ca_-} > 1 - \delta/2$.

These bounds are exactly the hypotheses of the attainment argument in Appendix A, item 3. Closed realizability of normalized section orders and retraction to a fixed SNC model then turn a minimizing sequence into a minimizing monomial pair. $\square$

4.2 Exponential moments on an Okounkov body

Choose an auxiliary rank- n valuation ord_z with one-dimensional leaves as in [BJ20, Section 2], and let $\Delta \subset \mathbf{R}^n$ be the associated Okounkov body of L . Set

$$\Lambda_k = \text{ord}_z(R_k \setminus \{0\}), \quad V = \text{vol}(\Delta) = \frac{L^n}{n!}, \quad \rho_k = k^{-n} \sum_{\alpha \in \Lambda_k} \delta_{\alpha/k}.$$

The one-dimensional leaves give $\#\Lambda_k = N_k$. The measures ρ_k converge weakly to Lebesgue measure ρ on Δ , with total masses $N_k/k^n \rightarrow V$.

For a test v , the valuation filtration is multiplicative and linearly bounded by (4.1). Denote its concave transform by $G_v : \Delta \rightarrow \mathbf{R}_{\geq 0}$, and define the exponential moments and limiting objectives

$$\begin{aligned} Z_k(v) &= \frac{1}{N_k} \sum_i e^{-a_{k,i}(v)/k}, & Z_\infty(v) &= \frac{1}{V} \int_\Delta e^{-G_v} d\rho, \\ F_\infty(v) &= e^{A(v)/t} Z_\infty(v), & f_\infty(x) &= \inf_{x \in c_X(v)} F_\infty(v). \end{aligned}$$

The finite-degree objective is $F_k(v) = e^{A(v)/t} Z_k(v)$.

Lemma 4.3 (Exponential-moment convergence). *For each fixed test v , one has $Z_k(v) \rightarrow Z_\infty(v)$. Moreover, for every $\epsilon > 0$ and all sufficiently large k ,*

$$Z_k(v) \geq Z_\infty(v) - \epsilon \quad (4.4)$$

uniformly over all tests v .

Proof. The limit-measure theorem of Boucksom–Chen [BC11, Theorem 1.11], in the form of [BJ20, Theorem 2.8], gives

$$k^{-n} \sum_i \delta_{a_{k,i}(v)/k} \longrightarrow (G_v)_* \rho.$$

For fixed v , these measures have uniformly bounded support. Integrating e^{-s} and normalizing by $N_k/k^n \rightarrow V$ proves convergence of $Z_k(v)$.

For the uniform lower bound, choose a basis adapted to both v and ord_z . Its ord_z -values enumerate Λ_k , while its v -values are the jumping numbers. If a basis section has values (α, a) , then α/k belongs to the Okounkov body of the level- a/k filtered series, so $G_v(\alpha/k) \geq a/k$. It follows that

$$Z_k(v) \geq \frac{1}{N_k} \sum_{\alpha \in \Lambda_k} e^{-G_v(\alpha/k)} = 1 - \frac{k^n}{N_k} \int_{\Delta} g_v d\rho_k, \quad (4.5)$$

where $g_v = 1 - e^{-G_v}$. This function is concave, since G_v is concave and $s \mapsto 1 - e^{-s}$ is increasing and concave; moreover, $0 \leq g_v \leq 1$.

The uniform estimate of Blum–Jonsson [BJ20, Lemma 2.2] states that, for every $\eta > 0$ and all sufficiently large k ,

$$\int_{\Delta} g d\rho_k \leq \int_{\Delta} g d\rho + \eta$$

for every concave function $0 \leq g \leq 1$. Since $k^n/N_k \rightarrow V^{-1}$ and $\int g d\rho \leq V$, choosing η small enough gives

$$\frac{k^n}{N_k} \int_{\Delta} g d\rho_k \leq \frac{1}{V} \int_{\Delta} g d\rho + \epsilon$$

for all large k , uniformly in g . Applying this to g_v in (4.5) proves (4.4). $\square$

Proposition 4.4 (Convergence of the minima). *For every $x \in X$,*

$$\lim_{k \rightarrow \infty} f_k(x) = f_{\infty}(x) > 0.$$

Proof. For any fixed test v at x , we have $f_k(x) \leq F_k(v) \rightarrow F_{\infty}(v)$. Taking the infimum over v yields $\limsup f_k(x) \leq f_{\infty}(x)$.

For the reverse inequality, choose minimizing tests v_k at x in all large degrees. Proposition 4.2 bounds their discrepancies by a_+ . For any $\epsilon > 0$, Lemma 4.3 therefore gives, for all sufficiently large k ,

$$f_k(x) = F_k(v_k) \geq F_{\infty}(v_k) - \epsilon e^{a_+/t} \geq f_{\infty}(x) - \epsilon e^{a_+/t}.$$

Taking the lower limit and then letting $\epsilon \rightarrow 0$ proves convergence. Proposition 4.2 makes the common limit positive. $\square$

The uniform lower bound (4.4) is the step that permits the minimizing test to vary with k . The argument requires neither a uniform upper bound nor attainment of the limiting infimum.

5 Point-centred minimizers in large degree

Call a degree k *good at x* if $f_k(x)$ is attained and every minimizing test at x has centre $\{x\}$. We have proved attainment in all large degrees. We now use convergence to rule out positive-dimensional minimizing centres in every sufficiently large degree.

5.1 The tangential moment estimates

The centre-elimination argument uses five estimates from the minimization construction. Fix an integer ℓ for which ℓL is very ample. For each $1 \leq d \leq n$, there are constants $C_d \geq 0$ and $H_d \geq 1$, depending only on (X, L, ℓ, d) , with the following property.

Suppose $(v, \mathbf{s})$ minimizes at x in degree k and its centre W has dimension $d > 0$. At a very general point of W , the construction gives initial-exponent sets

$$\Gamma_j \subset \mathbf{Z}_{\geq 0}^{n-d} \times \mathbf{Z}_{\geq 0}^d, \quad (\nu, \beta) \in \Gamma_j,$$

in local parameters on the model, with β recording the d tangential directions. The nonnegative weight $\ell_v(\nu)$ records the order under v . As j varies, we keep the valuation, coordinates and reference denominator k fixed. Define

$$\begin{aligned} Q_j &= \sum_{(\nu, \beta) \in \Gamma_j} e^{-\ell_v(\nu)/k}, \\ T_k &= \frac{1}{kQ_k} \sum_{(\nu, \beta) \in \Gamma_k} \|\beta\|_1 e^{-\ell_v(\nu)/k}, \\ B_j &= \frac{1}{d!} \sum_{\nu} e^{-\ell_v(\nu)/k} (d! \#\{\beta : (\nu, \beta) \in \Gamma_j\})^{(d+1)/d}. \end{aligned}$$

The estimates are

$$Q_k = N_k f_k(x) e^{-A(v)/t}, \quad (5.1)$$

$$Q_j \geq N_j f_j(x) e^{-jA(v)/(kt)}, \quad (5.2)$$

$$B_{j+h} \geq B_j + (d+1)(h - C_d)Q_j \quad (h \geq H_d), \quad (5.3)$$

$$kQ_k T_k + dQ_k \geq \frac{d}{d+1} B_k, \quad (5.4)$$

$$T_k \leq \frac{d}{t}. \quad (5.5)$$

The first four formulas come from the initial-exponent and restriction-space estimates listed in Appendix A. They hold for every $t > 0$ whenever a minimizing pair exists. The restriction estimate uses $L^d \cdot W \geq 1$, which makes the constants in (5.3) independent of W .

The upper bound (5.5) follows by varying the weights in the tangential directions. At a suitable point z on the model over a very general $y \in W$, retain the r positive weights of v , assign weight ϵ to the d tangential parameters, and assign weight ϵ^2 to the remaining $n - d - r$ parameters. The resulting valuation has discrepancy

$$A(v_\epsilon) = A(v) + d\epsilon + (n - d - r)\epsilon^2.$$

For the basis $\mathbf{s}'$ represented by Γ_k , the first variation of a section order is $\|\beta\|_1$. Specialization gives $f_k(y) \geq f_k(x)$, and local monomial approximation, justified below, gives

$$F_k(v_\epsilon, \mathbf{s}') \geq f_k(y) \geq f_k(x) = F_k(v, \mathbf{s}').$$

The right derivative at $\epsilon = 0$ must therefore be nonnegative. Dividing it by the common positive factor gives $d/t - T_k \geq 0$, as required.

For completeness, positive weights ω_i in arbitrary regular parameters at a closed point z of a smooth model $\pi : Y \rightarrow X$ define a local monomial valuation with

$$A_X(v_\omega) = \sum_i \omega_i + v_\omega(K_{Y/X});$$

see [JM12, Proposition 3.1, formula (5.1), Remarks 5.5–5.6]. Positive rational weights give scaled divisorial tests with centre $\{\pi(z)\}$, realizable on projective models; see [JM12, Remark 3.9] and [KM98,

Lemma 2.45]. For a fixed local equation, least weighted order is continuous in the positive weights: locally, only finitely many monomials can achieve the minimum. Approximating ω by rational vectors thus gives convergence of the finitely many section orders and of the discrepancy, proving $f_k(\pi(z)) \leq F_k(v_\omega, \mathbf{s}')$. In the variation above, z avoids the components of $K_{Y/X}$ other than those carrying the original positive weights, which gives the stated discrepancy formula.

5.2 Eliminating positive-dimensional centres

Theorem 5.1. *For every $x \in X$ and $n+1 < t < n+2$, every sufficiently large degree is good at x .*

Proof. If the assertion fails, infinitely many degrees k admit a minimizing pair with positive-dimensional centre. Pass to a subsequence on which the centre dimension is a fixed $d \geq 1$. Apply the tangential moment estimates in each of these degrees; the minimizing test and local coordinates may vary with k .

Fix an integer $a \geq 2$ and set $j_i = \lfloor ki/a \rfloor$ for $1 \leq i \leq a$, so $j_a = k$. For large k , every increment $j_{i+1} - j_i$ exceeds both H_d and C_d . Iterating (5.3), starting from $B_{j_1} \geq 0$, gives

$$B_k \geq (d+1) \sum_{i=1}^{a-1} (j_{i+1} - j_i - C_d) Q_{j_i}.$$

Together with (5.4), this yields

$$T_k + \frac{d}{k} \geq d \sum_{i=1}^{a-1} \frac{j_{i+1} - j_i - C_d}{k} \frac{Q_{j_i}}{Q_k}. \quad (5.6)$$

By (5.1)–(5.2),

$$\frac{Q_{j_i}}{Q_k} \geq \frac{N_{j_i}}{N_k} \frac{f_{j_i}(x)}{f_k(x)} \exp \left(\left(1 - \frac{j_i}{k} \right) \frac{A(v)}{t} \right) \geq \frac{N_{j_i}}{N_k} \frac{f_{j_i}(x)}{f_k(x)}.$$

Keep a fixed and let k tend to infinity along the chosen subsequence. Then

$$\frac{j_{i+1} - j_i - C_d}{k} \rightarrow \frac{1}{a}, \quad \frac{N_{j_i}}{N_k} \rightarrow \left(\frac{i}{a} \right)^n, \quad \frac{f_{j_i}(x)}{f_k(x)} \rightarrow 1.$$

The last limit follows from Proposition 4.4 and the positivity of $f_\infty(x)$. Using these limits in (5.6) and applying (5.5), we obtain

$$\frac{1}{t} \geq \frac{1}{a} \sum_{i=1}^{a-1} \left(\frac{i}{a} \right)^n.$$

As $a \rightarrow \infty$, the sum tends to $\int_0^1 u^n du = 1/(n+1)$. Thus $1/t \geq 1/(n+1)$, contradicting $t > n+1$. $\square$

Convergence is what makes this contradiction work along any sequence of degrees with positive-dimensional minimizing centres. The resulting conclusion in all large degrees supplies the common degree needed for the two-point comparison.

6 Perturbation and lifting

We carry out the rational perturbation on a resolution and apply vanishing there. The construction must preserve two features: discrepancy equality over x and strict excess at y .

Definition 6.1. Let $\rho : T \rightarrow X$ be a smooth projective birational model, and let Γ be an effective rational divisor with $\Sigma = \text{Exc}(\rho) \cup \text{Supp } \Gamma$ SNC. A nonzero reduced divisor $S \leq \Sigma$ is an *equality configuration at x* if every component F of S maps to x and has coefficient $A(F)$ in Γ , while every component of Σ outside S that meets S has coefficient strictly less than $A(F)$.

Proposition 6.2 (Rational perturbation). *Let k be good at x , let $(v, \mathbf{s})$ minimize there, and let $\mathbf{b}$ be its Gibbs coefficients. Fix an integer $m > t$, a neighbourhood U of $\mathbf{b}$, and $\eta_0 > 0$. There exist a smooth projective birational model $\rho : T \rightarrow X$, an effective integral divisor G , and data*

$$\mathbf{b}^0 \in U \cap \mathbf{Q}_{>0}^{N_k}, \quad \eta \in \mathbf{Q}, \quad 0 < \eta < \min\{1, \eta_0\}, \quad M \in \mathbf{Z}_{>0},$$

such that

$$\Gamma = (1 - \eta) \sum_i b_i^0 \rho^* \text{div}(s_i) + \frac{\eta}{M} G \quad (6.1)$$

has an equality configuration S at x , and $m\rho^*L - \Gamma$ is nef and big.

Proof. We apply the supporting-data and tie-breaking construction of [OAI, Sections 6.1–6.2]. We describe the data and check the positivity needed in our parameter range.

Resolve the divisors $D_i = \text{div}(s_i)$ on $\pi : Y \rightarrow X$, including the exceptional locus in the SNC divisor with components E_j . Retraction preserves the section orders and decreases discrepancy, so minimality forces equality of discrepancies. Some section order is positive, since otherwise rescaling toward zero decreases the objective. Thus the retraction is nonzero, and goodness forces its centre to be $\{x\}$. We may replace v by this retraction without changing the Gibbs coefficients.

Set $a_j = A(E_j)$ and $z_j = (\text{ord}_{E_j}(D_i)/a_j)_i$. For each intersection stratum whose image contains x , take the convex hull of the z_j for its crossing components. These rational polytopes parametrize normalized section orders of monomial tests at x . The vector $q = (v(s_i)/A(v))_i$ belongs to their union, but to none indexed by a stratum with positive-dimensional image: such a representation would give another minimizer with that larger centre. Scale stationarity and variation at fixed discrepancy give

$$\mathbf{b} \cdot q = 1, \quad \mathbf{b} \cdot q' \leq 1 \quad \text{on every polytope containing } q.$$

The rational supporting-data lemma [OAI, Lemma 6.1] gives rational $q^0, \mathbf{b}^0$, arbitrarily close to $q, \mathbf{b}$, with the same polytope incidences and supporting inequalities. Choose $\mathbf{b}^0 \in U$ with $k \sum_i b_i^0 < m$. A rational monomial ray representing q^0 extracts a prime divisor F_0 over x with normalized section-order vector q^0 and equality in the retraction discrepancy inequality.

Put $I = \{i : q_i^0 > 0\}$, choose $M > 0$ with M/q_i^0 integral for $i \in I$, and form the ideal

$$\mathfrak{c} = \sum_{i \in I} \mathcal{O}_X(-(M/q_i^0)D_i). \quad (6.2)$$

Choose $u > 0$ so that $\mathfrak{c}(uL)$ is globally generated. On a resolution $\rho : T \rightarrow X$ dominating Y and retaining F_0 , write $\mathfrak{c}\mathcal{O}_T = \mathcal{O}_T(-G)$ and arrange that G , the ρ^*D_i and the exceptional locus have SNC support. The divisor $u\rho^*L - G$ is globally generated.

For every rational $0 < \eta < 1$, the coefficient comparison of [OAI, Lemma 6.2] gives an equality configuration for (6.1). Explicitly, S is the reduced union of primes F whose images contain x , whose normalized section-order vector is q^0 , and whose discrepancy equals that of their retraction to Y . It contains F_0 , and the preserved polytope incidences force every component to map to x . The formula

$$\frac{\text{ord}_F(G)}{M} = \min_{i \in I} \frac{\text{ord}_F(D_i)}{q_i^0}$$

preserves discrepancy equality along S . The supporting inequalities and positivity of every b_i^0 make equality on a prime meeting S possible only when that prime also belongs to S . These conclusions are independent of t .

Finally,

$$\begin{aligned} m\rho^*L - \Gamma &\equiv \left(m - (1 - \eta)k \sum_i b_i^0 - \frac{\eta u}{M} \right) \rho^*L \\ &\quad + \frac{\eta}{M}(u\rho^*L - G). \end{aligned} \tag{6.3}$$

The first coefficient is positive for sufficiently small η , since $k \sum_i b_i^0 < m$. Choose such a rational $\eta < \eta_0$. The first summand is nef and big and the second is nef, proving the assertion. $\square$

The construction allows U to be fixed first and η to be chosen last. Thus a strict inequality at a second point survives without any bound on the auxiliary integers M, u .

Proposition 6.3 (Lifting a separating section). *Let $x \neq y$ and $m \in \mathbf{Z}$, and let Γ on $\rho : T \rightarrow X$ have an equality configuration S at x . Suppose $m\rho^*L - \Gamma$ is nef and big, and a test w at y satisfies $w(\Gamma) > A(w)$. Then $K_X + mL$ has a section nonzero at x and zero at y .*

Proof. Put $P = K_X + mL$ and define the integral divisors

$$D = K_T + m\rho^*L - [\Gamma], \quad C = D + S - \rho^*P = K_{T/X} - [\Gamma] + S.$$

The fractional part $\{\Gamma\}$ has SNC support, and

$$D - (K_T + \{\Gamma\}) = m\rho^*L - \Gamma$$

is nef and big. Kawamata–Viehweg vanishing [Fuj09, Theorem 0.1] gives $H^1(T, \mathcal{O}_T(D)) = 0$. Restriction to S is therefore surjective:

$$H^0(T, \mathcal{O}_T(D + S)) \longrightarrow H^0(S, \mathcal{O}_S(D + S)). \tag{6.4}$$

Along a component F of S , discrepancy equality and the integrality of $A(F)$ give

$$\text{coeff}_F C = (A(F) - 1) - A(F) + 1 = 0.$$

Every other component meeting S has $\text{coeff}_F \Gamma < A(F)$, hence $\text{coeff}_F C \geq 0$. Thus C is effective near S and contains no component of S .

Since S is reduced and every component maps to x , the morphism $\rho|_S$ factors through x , so $\rho^*P|_S \simeq P_x \otimes \mathcal{O}_S$. Choose a nonzero element of P_x and multiply the corresponding constant section on S by the canonical section of $\mathcal{O}_T(C)$ near S . This gives a section of $\mathcal{O}_S(D + S)$ nonzero at the generic point of every component of S . Lift it by (6.4) to a section $\tilde{s}$ of $\mathcal{O}_T(\rho^*P + C)$.

The positive part of C is exceptional: on a nonexceptional prime outside S , its coefficient is $-\lfloor \text{coeff}_F \Gamma \rfloor \leq 0$, and on S it is zero. Viewed as a rational section of P , the lift therefore has no pole along any prime divisor of X . Normality of X makes it a regular section $s \in H^0(X, P)$. Its generic nonvanishing along S , where C has coefficient zero, implies $s(x) \neq 0$.

It remains to check the value at y . By (2.3), the inequality $w(\Gamma) > A(w)$ gives a prime F with $y \in \rho(F)$ and $\text{coeff}_F \Gamma > A(F)$. This prime is not in S , since $x \neq y$, and

$$\text{coeff}_F C = A(F) - 1 - \lfloor \text{coeff}_F \Gamma \rfloor \leq -1.$$

Regularity of $\tilde{s}$ as a section of $\mathcal{O}_T(\rho^*P + C)$ forces ρ^*s to vanish along F . Hence s vanishes on $\rho(F)$, and in particular $s(y) = 0$. $\square$

7 Proof of Theorem A

We now combine the common-degree theorem, the two-point comparison and the lifting proposition.

Theorem A. *Let X be a smooth complex projective variety of dimension $n \geq 1$ and let L be an ample line bundle. For every integer $m \geq n + 2$ and every pair of distinct closed points $x, y \in X$, the evaluation map*

$$H^0(X, K_X + mL) \longrightarrow (K_X + mL) \otimes k(x) \oplus (K_X + mL) \otimes k(y)$$

is surjective.

Proof. Fix $x \neq y$ and an integer $m \geq n + 2$. Global generation separates points on different connected components, so we may assume X is connected. If $n = 1$, then $\deg(K_X + mL) \geq 2g + 1$, and the adjoint bundle is very ample. Assume henceforth that $n \geq 2$.

Choose t with $n + 1 < t < n + 2$. By Theorem 5.1, there is a degree k good at both points and large enough that kL is very ample. Without loss of generality, $f_k(x) \geq f_k(y)$. Choose minimizing tests v, w at x, y and a common adapted basis $\mathbf{s}$. If $\mathbf{b}$ denotes the Gibbs coefficients of v , Lemma 3.1 gives

$$\sum_i b_i w(s_i) > A(w).$$

By continuity, there are a neighbourhood U of $\mathbf{b}$ and $\eta_0 > 0$ such that

$$(1 - \eta) \sum_i b'_i w(s_i) > A(w) \quad \text{whenever } \mathbf{b}' \in U, \quad 0 < \eta < \eta_0. \quad (7.1)$$

Since $m > t$, Proposition 6.2 gives a resolution $\rho : T \rightarrow X$ and a divisor

$$\Gamma = (1 - \eta) \sum_i b_i^0 \rho^* \operatorname{div}(s_i) + \frac{\eta}{M} G, \quad \mathbf{b}^0 \in U, \quad 0 < \eta < \eta_0,$$

with an equality configuration at x and with $m\rho^*L - \Gamma$ nef and big. Effectiveness of G and (7.1) give

$$w(\Gamma) \geq (1 - \eta) \sum_i b_i^0 w(s_i) > A(w).$$

Proposition 6.3 therefore gives a section of $K_X + mL$ nonzero at x and zero at y . Global generation gives a second section nonzero at y . Their evaluations at (x, y) form a triangular matrix with nonzero diagonal, proving surjectivity. $\square$

Remark 7.1. The construction prescribes a value at x and forces vanishing at a distinct point; it does not prescribe arbitrary first jets at x . Tangent separation therefore remains outside the conclusion. The sharp numerical bound uses integrality through $L^n \geq 1$ in the sublevel estimate and $L^d \cdot W \geq 1$ in the restriction-space estimate.

A Inputs from the freeness construction

We list the results used from [OAI], retaining its numbering. The parameter in this paper is $n + 1 < t < n + 2$; the following list also records why the imported constructions apply in that range.

- (1) *Global generation.* Theorem 1.1 and Corollary 6.3 give global generation of $K_X + mL$ for every integer $m \geq n + 1$.

- (2) *Valuations and ordinary order.* Section 2.1, formula (2.2) and Lemmas 2.1–2.2 give the tests, discrepancy formula and retraction properties used in Section 2. Lemma 3.1, bound (3.1) and Lemma 4.1 give the initial-exponent construction and (4.1). These statements are independent of t .
- (3) *Closed realizability and attainment.* Lemma 4.3 asserts that, for fixed k and rational $\alpha_1 \geq \dots \geq \alpha_{N_k} \geq 0$, the locus of points admitting a basis and a divisorial test at that point with $v(s_i)/A(v) \geq \alpha_i$ is Zariski closed, as is the corresponding incidence locus with the complete flag variety of R_k . The proof of Proposition 4.4 then gives attainment whenever $f_k(x) \leq 1 - \delta$ and the discrepancies on $F_k \leq 1 - \delta/2$ lie in a compact interval in $(0, \infty)$. The source uses non-generation only to obtain these bounds; Proposition 4.2 supplies them here.
- (4) *Specialization and tangential moments.* Lemma 4.5 gives $f_k(y) \geq f_k(x)$ for a very general point y of an irreducible closed subvariety through x , for every $t > 0$. Lemma 5.2 gives the uniform restriction-space estimate. Sections 5.2–5.4 yield (5.1), (5.2), (5.3) and (5.4), corresponding respectively to source formulas (5.7), (5.19), (5.15) and (5.16). Our C_d, H_d are the source’s C'_d, H'_d . Although (5.19) is stated for large j , its proof compares the test $(j/k)v$ with an initial-exponent basis in R_j and works whenever $N_j > 0$. The first variation in (5.8)–(5.10) gives (5.5); Section 5 supplies the local approximation needed for that comparison. These arguments use minimality, specialization, the restriction estimate and Brunn–Minkowski, and apply for every $t > 0$.
- (5) *Rational perturbation.* Section 6.1 and Lemma 6.1 give the rational supporting data and extraction of a monomial ray. Formula (6.6) and Lemma 6.2 give the ideal and equality configuration in Proposition 6.2. Their hypotheses are scale stationarity, the supporting inequalities and exclusion of positive-dimensional minimizing centres. The source’s bound $t < n + 1$ serves only to keep its positivity coefficient positive; here $m > t$ gives (6.3). The lifting proof follows the restriction-and-descent construction of Section 6.3, with the additional negative coefficient over y forcing the descended section to vanish there.

The new numerical steps are the sublevel estimates for $t > n + 1$, exponential-moment convergence, and its use in centre elimination to obtain $1/t \geq 1/(n + 1)$ from any infinite sequence of bad degrees. Proposition 3.2 and the numerical choices in source formulas (5.17) and (5.21) are not used.

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