

WHICH STRING VIBRATIONS BECOME WHICH GENERATIONS ON THE DWORK QUINTIC

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ABSTRACT. We determine, for the heterotic string on the one-parameter family of quintic threefolds, which vibration modes become which families of charged particles. The hundred and one families are labelled by the characters of a symmetry group of order one hundred and twenty-five. Every Yukawa coupling that this symmetry allows is a power of the modulus, and every other coupling vanishes. The invariant deformation pairs each family with exactly one partner, so that fifty families are massless at the Fermat point and none anywhere else, and at the Fermat point the masses are given in closed form by the Gamma function. On the three other one-parameter families of the same type, in weighted projective spaces, the symmetry no longer tells the families apart and some families stay massless at every member. We also prove that the Geometric Selection Conjecture is equivalent to the existence of one viable vacuum fixed by mirror symmetry, and that a free quotient of a Calabi–Yau complete intersection of any of the thirteen one-parameter hypergeometric types never carries three families with the standard embedding; on seven of these types, among them the sextic and dextric pencils, no finite group acts freely at all.

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1. INTRODUCTION

A closed string moving in ten dimensions has a tower of vibration modes, and the massless ones of the heterotic string on a product of four-dimensional space-time with a compact six-manifold X are the fields of a

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four-dimensional gauge theory. When X is a Calabi–Yau threefold and the gauge connection is identified with the Levi-Civita connection of its Ricci-flat metric, the gauge group is E_6 and the charged chiral multiplets come in two conjugate representations whose numbers are the two Hodge numbers of X . This was established in 1985 by Candelas, Horowitz, Strominger and Witten [1] and by Witten [2], and the couplings of three charged fields were identified by Strominger [3] with an integral over X that Candelas [4] computed on the quintic threefold. The question in the title is the one that this dictionary leaves open: a vibration mode is a cohomology class on X , a particle generation is an irreducible piece of the four-dimensional spectrum, and one wants to know which class carries which generation, with what mass, and how this changes as X moves in its moduli space.¹

We answer it completely on one family, the Dwork pencil of quintics

$$(1) \quad X_\psi = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0\} \subset \mathbb{P}^4,$$

smooth for $\psi^5 \neq 1$, which is the family on which mirror symmetry was first tested by Candelas, de la Ossa, Green and Parkes [5] and on which the Yukawa couplings were counted by Doran, Greene and Jukes [6]. The group $\Gamma \cong (\mathbb{Z}/5)^3$ of diagonal phases preserving the equation acts on every member of the pencil, and the hundred and one classes in $H^{2,1}(X_\psi)$, which are the hundred and one copies of the **27**, are the monomials $x^{\mathbf{b}}$ with exponent vector $\mathbf{b} \in \{0, 1, 2, 3\}^5$ of total degree five. The results are the following.

Theorem 1 (Labels). *The hundred and one classes carry hundred and one distinct characters of Γ , each with multiplicity one, so every generation is labelled by its character. Of the hundred and twenty-five characters of Γ the twenty-four that do not occur are exactly the classes of the permutations of $(0, 1, 2, 3, 4)$. The permutation group S_5 of the coordinates permutes the labels in five orbits of sizes 1, 20, 20, 30 and 30, the five types of generation.*

Theorem 2 (Couplings). *Let $\psi^5 \neq 1$. The cubic form of the pencil, normalised so that the monomial $x_1^3 \cdots x_5^3$ has value one, takes on three generations $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the value ψ^r when $\mathbf{a} + \mathbf{b} + \mathbf{c} \equiv (3 - r)\mathbf{1} \pmod{5}$ for some $r \in \{0, 1, 2, 3\}$, and zero otherwise. There are 1431 unordered triples with a nonzero coupling, 566, 425, 280 and 160 of them with $r = 0, 1, 2, 3$, and all 1431 are nonzero at every smooth member with $\psi \neq 0$.*

Theorem 3 (Mass map). *The invariant class $h = x_1 x_2 x_3 x_4 x_5$, the direction of the pencil itself, pairs each generation $\mathbf{b}$ with exactly one generation $\sigma(\mathbf{b})$, where σ is an involution of the labels with one fixed point. The coupling of h , $\mathbf{b}$ and $\sigma(\mathbf{b})$ equals 1 for the fifty-one labels with all entries at most two, ψ for the thirty labels of type $(3, 1, 1, 0, 0)$ and ψ^2 for the twenty labels of type $(3, 2, 0, 0, 0)$. Hence the mass matrix induced by a vacuum value of the **27** labelled by h has full rank one hundred and one for every $\psi \neq 0$, $\psi^5 \neq 1$, its determinant is ψ^{70} , and at the Fermat point $\psi = 0$ its rank is fifty-one with the fifty generations containing an exponent three massless. The spectrum consists of one singlet and fifty doublets, and the Γ -invariant metric is diagonal in the labels, so the fifty doublet masses are, in units of the vacuum value, $|\psi|^{r(\mathbf{b})} (g_{\mathbf{b}} g_{\sigma(\mathbf{b})})^{-1/2}$ with g the Hodge norms of the classes.*

Theorem 2 is stronger than the count of Doran, Greene and Jukes [6], who found by a search that 1431 couplings are allowed by the symmetry; it shows that the symmetry is the only source of zeros, and gives every allowed coupling in closed form. The number 1431 is recovered by hand in Section 5.1. The theorems are proved in Sections 3 to 6 after the dictionary between vibrations and particles is set out, with complete proofs, in Section 2. Section 7 evaluates the Hodge norms at the Fermat point in terms of the Gamma function, which turns the mass formula of Theorem 3 into numbers there.

The fourth result concerns the choice of the vacuum. A conjecture of Bhattacharjee, Bhattacharya and Ghosh [33], the Geometric Selection Conjecture, asks for a functional on the heterotic landscape, built from

¹The word generation is used here in the sense of the E_6 theory: one copy of the representation **27**, which contains one family of quarks and leptons together with its Higgs and singlet partners. The three observed families are three such copies after the further breaking of E_6 ; the paper does not touch that breaking, which depends on discrete Wilson lines and on a quotient of X that the quintic does not admit in a form that reduces a hundred and one copies to three, so that the results here concern the full E_6 spectrum of the covering space.

the Weil–Petersson metric, that is proper, has a unique minimum, takes that minimum at a phenomenologically viable vacuum and is invariant under mirror symmetry. The precise statement is recalled in Section 9.

Theorem 4 (Selection). *The Geometric Selection Conjecture holds if and only if there is a viable vacuum, in the sense of conditions (C1) to (C9) of [33], that is not moved by the mirror involution. On the Dwork pencil the mass condition (C8) holds in the precise form that the mass map of the invariant direction has full rank with entries 1, ψ and ψ^2 , at every member with $\psi \neq 0$; the three-family condition (C2) fails on every smooth quintic and on every quotient of a smooth quintic by a group acting freely.*

The first part says that the functional carries no information beyond the vacuum it selects: once a viable vacuum is given, a functional with all the required properties is written down in Section 9 from the metric alone, and conversely the minimum of any such functional is a viable vacuum. What remains of the conjecture is therefore the existence of one viable vacuum, which is where its content lies. The second part locates the Dwork pencil with respect to that question: its mass spectrum has the full rank and the hierarchy of scales that condition (C8) asks for, and the obstruction is the number of families, a hundred and one on the quintic and $100/|H|$ on a free quotient by a group H , never three.

The same questions can be asked of the three other one-parameter families of Fermat type, the pencils

$$(2) \quad X_\psi = \left\{ \sum_{i=1}^5 x_i^{n_i} - d\psi x_1 x_2 x_3 x_4 x_5 = 0 \right\} \subset \mathbb{P}(w), \quad n_i = \frac{d}{w_i},$$

of sextics in $\mathbb{P}(1, 1, 1, 1, 2)$, octics in $\mathbb{P}(1, 1, 1, 1, 4)$ and dectics in $\mathbb{P}(1, 1, 1, 2, 5)$, whose periods and mirror maps were computed by Morrison [24], Klemm and Theisen [25] and Font [26]. Here $d = \sum_i w_i$ is 6, 8 and 10, each member with $\psi^d \delta \neq 1$, where $\delta = \prod_i w_i^{w_i}$ is 4, 256 and 12500, is smooth, and the group Γ of diagonal phases preserving the pencil has order 108, 128 and 100. The answer is different there.

Theorem 5 (Weighted pencils). *Let X_ψ be the sextic, octic or dectic pencil (2) with $\psi^d \delta \neq 1$. Then X_ψ is a smooth Calabi–Yau threefold with $h^{1,1} = 1$ and $h^{2,1} = 103, 149$ and 145 , and the following hold.*

- (i) *The generations are the monomials x^b of weighted degree d with $0 \leq b_i \leq n_i - 2$, but they carry only 96, 122 and 100 distinct characters of Γ : 7, 27 and 42 classes of two or three generations share a character.*
- (ii) *The cubic form, normalised so that $\prod_i x_i^{n_i-2}$ has value one, takes on three generations **a, b, c** the value $\psi^r \prod_i w_i^{k_i}$ when $a_i + b_i + c_i = n_i - 2 - r + n_i k_i$ for every i , with $r \in \{0, \dots, d-1\}$ and integers $k_i \geq 0$, and zero otherwise; for $\psi \neq 0$ every coupling allowed by Γ is nonzero.*
- (iii) *For $\psi \neq 0$ the mass map of the invariant direction has rank 99, 125 and 109, so that 4, 24 and 36 generations stay massless at every member; at $\psi = 0$ its rank is 44, 85 and 72.*
- (iv) *No member, and no quotient of a member by a group acting freely, carries three families with the standard embedding.*

So the two properties that make the Dwork pencil transparent are special to it. On the weighted pencils a shift of the exponents at the coordinates of weight one can be balanced by a shift at the coordinate of larger weight, the characters of Γ no longer label the generations, and the massless generations all lie in the span of the classes that share a character. The precise form of the mass condition (C8) in Theorem 4 therefore fails on all three pencils, and so does the three-family condition (C2). Theorem 5 is proved in Section 8, and Figure 4 compares the four pencils.

The count of families reaches beyond the four pencils. Of the fourteen one-parameter families of Calabi–Yau threefolds with hypergeometric Picard–Fuchs equations, thirteen are complete intersections in weighted projective space, and Proposition 27 shows that a finite group acting freely on a smooth member has order divisible only by the primes that divide the greatest common divisor g_X of the holomorphic Euler characteristics $\chi(X, \mathcal{O}_X(k))$, $k \geq 1$. This number is 1 for seven of the thirteen types, the sextic and the dectic among them, which therefore have no free quotient at all, and no free quotient of any of the thirteen carries three families.

The constructions behind the four theorems were organised on the Anaxia engine of the BLLM programme [36, 37].²

History. The dictionary between the vibrations of the string and the fields in four dimensions on a Calabi–Yau background was written down in [1]; the decomposition of the adjoint of E_8 under $E_6 \times SU(3)$ and the identification of the massless charged fields with the Dolbeault cohomology of the tangent bundle are there, and the identification of the number of generations with half the Euler characteristic. Witten [2] discussed the breaking of E_6 by Wilson lines and the pattern of masses from vacuum values of the **27**, which is the mechanism behind Theorem 3. Strominger [3] identified the cubic couplings of the **27** with the cubic form on $H^1(X, T_X)$ given by the holomorphic three-form, and the normalisation by the Kähler metric of the matter fields was clarified by Dixon, Kaplunovsky and Louis [8] and by Candelas and de la Ossa [9]. Candelas [4] computed the couplings on the Fermat quintic through the Jacobian ring, with the Gamma-function normalisation of the Hodge norms at that point; Greene and Plesser [7] introduced the quotient of the Dwork pencil by Γ as the mirror of the quintic, and [5] computed the coupling of h with itself along the pencil, $5/(1 - \psi^5)$ in their normalisation, which is the case $\mathbf{a} = \mathbf{b} = \mathbf{c} = h$ of Theorem 2 together with Remark 8. Doran, Greene and Judes [6] counted by computer the couplings not forced to vanish by Γ , found 1431, and observed that the count is close to the total divided by the order of the group. The Jacobian-ring description of $H^{2,1}$ of a hypersurface and of its cubic form is Griffiths’ [10], and the perfectness of the residue pairing with the Hessian in the socle is Macaulay’s theorem in the form given by Carlson and Griffiths [11]. On the physical side, the holomorphic Yukawa couplings of heterotic line-bundle models were computed by Blesneag, Buchbinder, Candelas and Lukas [14], and the physical, metric-normalised couplings of the quintic at the Fermat point by Butbaia, Mayorga Peña, Tan, Berglund, Hübsch, Jejjala and Mishra [15] with numerical Ricci-flat metrics, whose Gamma-function value for the invariant direction agrees with Section 7. Heterotic standard models with three families on other threefolds were constructed by Braun, He, Ovrut and Pantev [20], Bouchard and Donagi [21] and Anderson, Gray, Lukas and Palti [22, 23]; the question of which classes carry which families is posed there in a form that depends on the choice of bundle, and the present paper answers it in the one case where every piece of the data is explicit. Distler and Greene [16] computed the normalised couplings of the Gepner model at the Fermat point exactly, and Section 7 recovers their constant from the Hodge norms. The monodromy and intersection numbers used there go back to Pham [17], Sebastiani and Thom [18] and Sakamoto [19]. Two further aspects of the same compactifications, the suppression of the electroweak scale by singularities of the threefold and fibrations of Calabi–Yau manifolds by circles, are treated in [34, 35]. The weighted pencils of Section 8 are the one-parameter models whose Picard–Fuchs equations, periods and mirror maps were found by Morrison [24], Klemm and Theisen [25] and Font [26], and the Hodge theory of quasi-smooth hypersurfaces in weighted projective space used there is due to Steenbrink [27] and Dolgachev [28].

²Anaxia runs a failure-driven loop: each strategy for a target is a list of checks, each check is a computation or a proof with a named source, a failed check is recorded together with the change of strategy it forces, and the label of a target (open, candidate, proved) is computed by the engine from the record, never assigned by hand. For this paper the engine ran five targets. For the couplings, the first strategy was the count of couplings allowed by the symmetry, as in [6]; its check that the allowed couplings are nonzero failed, because a symmetry count only bounds the number of nonzero couplings from above, and the change it forced was to evaluate the socle functional of Section 4 on every monomial of degree fifteen, which passed. For the mass map the single strategy, the pairing by the invariant class, passed at once. For the Fermat point the engine checked the Gamma-function norms against the constant of Distler and Greene [16]. For the selection conjecture the first strategy was to look for the viable vacuum on the quintic itself; its check of the number of families failed for the quintic and for each of its free quotients, and the change it forced was to separate the functional from the vacuum, which gives Theorem 4; the check of the existence of a viable vacuum then failed on the literature, and that target stays open. For the weighted pencils the first strategy was to carry the proof of Theorem 1 over; its check that the characters are distinct failed at once on the sextic, where 103 generations carry 96 characters, and the change it forced was to classify the generations that share a character by the wrapping of Lemma 13 and to compute the mass map block by block, which gives Theorem 5. The record of all five targets was made on release 3.1.0 of the engine. The engine proposes no mathematics of its own here: every statement is proved in the text, and the record of the loop is part of the archive of the paper [38].

2. THE DICTIONARY

Throughout, X is a compact Kähler threefold with holonomy exactly $SU(3)$, so that X has a Ricci-flat Kähler metric g by Yau's theorem, a nowhere vanishing holomorphic three-form Ω , and $h^{1,0} = h^{2,0} = 0$. The tangent bundle T_X with the Levi-Civita connection is a holomorphic $SU(3)$ -bundle with a Hermitian Yang–Mills connection; this is the standard embedding.

2.1. Spinors and the Dolbeault complex. On a Kähler manifold of complex dimension n with trivial canonical bundle the spinor bundle is $S = \bigoplus_q \Lambda^{0,q} T_X^*$, and the Dirac operator of the Levi-Civita connection is $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$, with the chirality grading given by the parity of q (Hitchin [12]). If V is a holomorphic Hermitian bundle with a connection whose curvature is of type $(1, 1)$, the Dirac operator twisted by V is $\sqrt{2}(\bar{\partial}_V + \bar{\partial}_V^*)$ on $\Lambda^{0,\bullet} \otimes V$, and its kernel is the harmonic representative of $\bigoplus_q H^{0,q}(X, V)$. This is the only input from spin geometry.

2.2. The charged spectrum.

Theorem 6 (Spectrum of the standard embedding). *Let X be as above with the standard embedding. The massless charged chiral multiplets of the $E_8 \times E_8$ heterotic string on $\mathbb{R}^{3,1} \times X$ in the observable sector transform under E_6 as*

$$h^{2,1}(X) \text{ copies of } \mathbf{27} \oplus h^{1,1}(X) \text{ copies of } \overline{\mathbf{27}},$$

with the $\mathbf{27}$'s in natural bijection with $H^1(X, T_X) \cong H^{2,1}(X)$ and the $\overline{\mathbf{27}}$'s with $H^1(X, T_X^*) \cong H^{1,1}(X)$. The net number of generations is $|h^{2,1} - h^{1,1}| = \frac{1}{2}|\chi(X)|$.

Proof. The ten-dimensional gaugino is a Majorana–Weyl spinor with values in the adjoint of E_8 . Under $E_8 \supset E_6 \times SU(3)$,

$$\mathbf{248} = (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{27}, \mathbf{3}) \oplus (\overline{\mathbf{27}}, \overline{\mathbf{3}}),$$

and the standard embedding identifies the $SU(3)$ factor with the holonomy group acting on T_X and sends $\mathbf{3}$ to T_X , $\overline{\mathbf{3}}$ to T_X^* . A ten-dimensional Weyl spinor on $\mathbb{R}^{3,1} \times X$ decomposes as a four-dimensional Weyl spinor times a six-dimensional one of correlated chirality, so the four-dimensional massless chiral fermions in the $\mathbf{27}$ are the zero modes of the Dirac operator on X twisted by T_X , counted with their six-dimensional chirality; the $SU(3)$ holonomy makes the twisted operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ on $\Lambda^{0,\bullet} \otimes T_X$, and its chiral zero modes are $H^{0,1}(X, T_X) \oplus H^{0,3}(X, T_X)$ and $H^{0,0}(X, T_X) \oplus H^{0,2}(X, T_X)$ for the two chiralities. Since Ω trivialises $\Lambda^3 T_X^*$, contraction with Ω gives $H^{0,q}(X, T_X) \cong H^{2,q}(X)$ and $H^{0,q}(X, T_X^*) \cong H^{1,q}(X)$. With $h^{2,0} = h^{1,0} = 0$ and Serre duality, $H^{0,0}(T_X) = H^{2,0} = 0$, $H^{0,3}(T_X) = H^{2,3} = \overline{H^{1,0}}^* = 0$, $H^{0,2}(T_X) = H^{2,2} \cong \overline{H^{1,1}}$, and $H^{0,1}(T_X) = H^{2,1}$. A four-dimensional Weyl fermion in the $\mathbf{27}$ of one chirality and one in the $\overline{\mathbf{27}}$ of the opposite chirality are related by conjugation, so the chiral multiplets of definite four-dimensional chirality are $h^{2,1}$ copies of $\mathbf{27}$ from $H^{0,1}(T_X)$ and $h^{1,1}$ copies of $\overline{\mathbf{27}}$ from $H^{0,2}(T_X)$. The index of the twisted Dirac operator is $h^{2,1} - h^{1,1}$, which equals $-\chi(X)/2$ because $\chi = 2(h^{1,1} - h^{2,1})$ on a threefold with $h^{1,0} = h^{2,0} = 0$. $\square$

2.3. The cubic form. Let $V = H^1(X, T_X)$. The composition of cup product and contraction,

$$V \otimes V \otimes V \rightarrow H^3(X, \Lambda^3 T_X) = H^3(X, K_X^{-1}) \cong H^3(X, \mathcal{O}_X) \cong \mathbb{C},$$

where the last two isomorphisms use Ω , is a symmetric trilinear form Y on V , the cubic form of X . In terms of the variation of Hodge structure, with ∇ the Gauss–Manin connection and Ω extended holomorphically over the moduli space,

$$(3) \quad Y(a, b, c) = \int_X \Omega \wedge \nabla_a \nabla_b \nabla_c \Omega,$$

and this is the coupling of three $\mathbf{27}$'s [3]. Changing Ω to $f\Omega$ multiplies Y by f^2 , so only the cubic form up to a nowhere vanishing holomorphic function is intrinsic, and the physical couplings are Y divided by the square roots of the Hodge norms of the three classes and by $\langle \Omega, \overline{\Omega} \rangle$ [8, 9].

the curve $z_1^5 + z_2^5 = 1$ on the Fermat quintic

Hodge numbers $h^{p,q}$ (height $\sim \log$)

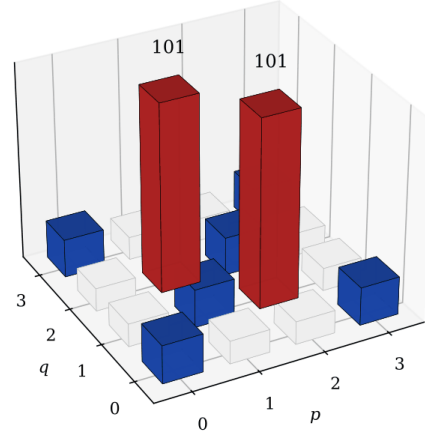
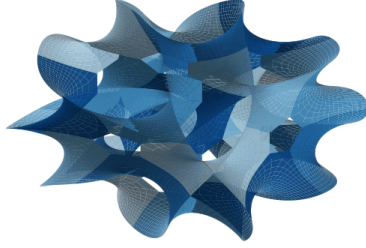


FIGURE 1. Left: the intersection of the Fermat quintic with the plane $x_3 = x_4 = 0$, which in an affine chart is the curve $z_1^5 + z_2^5 = 1$ in $\mathbb{C}^2$, drawn with the parametrisation of Hanson [13] and projected to three dimensions along $(\operatorname{Re} z_1, \operatorname{Re} z_2, \cos \alpha \operatorname{Im} z_1 + \sin \alpha \operatorname{Im} z_2)$. It is assembled from twenty-five patches, one for each pair of fifth roots of unity; Γ preserves the curve and acts on it through $\mu_5 \times \mu_5$, permuting the patches. Right: the Hodge numbers $h^{p,q}$ of a smooth quintic threefold, with height proportional to $1 + \log_2 h^{p,q}$; the six blue bars have height one, the two red bars are $h^{2,1} = h^{1,2} = 101$, and the grey cells are zero.

2.4. Mass terms. A vacuum value v of the scalar in the **27** labelled by a class $h \in V$ in a direction that breaks E_6 gives, through the superpotential $Y(h, \cdot, \cdot)$, a Dirac mass term $vY(h, b, c)$ between the fermions labelled by b and c [2]. The physical mass matrix is

$$(4) \quad M_{bc} = v \frac{Y(h, b, c)}{\sqrt{g_b g_c}}$$

in an orthogonal basis of V , where $g_b = \langle \omega_b, \overline{\omega_b} \rangle$ is the Hodge norm of the harmonic representative, and the masses are its singular values. We call $Y(h, \cdot, \cdot)$ the mass map of h .

2.5. Hypersurfaces. For a smooth hypersurface $X = \{F = 0\} \subset \mathbb{P}^4$ of degree five let $J_F = (\partial_1 F, \dots, \partial_5 F)$ be the Jacobian ideal and $R = \mathbb{C}[x_1, \dots, x_5]/J_F$ the Jacobian ring, graded by degree. Griffiths' theorem [10] identifies $H_{\text{prim}}^{3-q,q}(X)$ with $R_{5(q+1)-5}$ through residues: for a homogeneous P of degree $5q$,

$$\omega_P = \operatorname{Res}_X \frac{P \Omega_0}{F^{q+1}}, \quad \Omega_0 = \sum_i (-1)^i x_i dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_5,$$

so $H^{3,0} = R_0$, $H^{2,1} = R_5$, $H^{1,2} = R_{10}$, $H^{0,3} = R_{15}$, and every class of a quintic threefold is primitive in degree three. The ring R is a graded complete intersection with Hilbert series $(1 + t + t^2 + t^3)^5$, so $\dim R_5 = 101$, $\dim R_{15} = 1$, and R_{15} is the socle. Macaulay's theorem, as stated in [11], says that the Hessian $\operatorname{Hess} F = \det(\partial_i \partial_j F)$ has nonzero image in R_{15} and that the cubic form is multiplication in R followed by the residue functional $\lambda_F : R_{15} \rightarrow \mathbb{C}$, $\lambda_F(\operatorname{Hess} F) = 1$, up to a constant depending only on the normalisation of Ω :

$$(5) \quad Y(\omega_P, \omega_Q, \omega_S) = c \lambda_F(PQS).$$

Figure 1 shows a curve on the Fermat quintic and the Hodge numbers of a quintic threefold: by Theorem 6 the hundred and one classes of type $(2,1)$ are the generations and the single class of type $(1,1)$ is the antigeneration.

3. LABELS

Let $\zeta = e^{2\pi i/5}$ and let $K = \{a \in (\mathbb{Z}/5)^5 : \sum a_i = 0\}$ act on $\mathbb{P}^4$ by $x_i \mapsto \zeta^{a_i} x_i$. The diagonal subgroup D acts trivially on $\mathbb{P}^4$, and $\Gamma = K/D \cong (\mathbb{Z}/5)^3$ preserves $F_\psi = \sum x_i^5 - 5\psi \prod x_i$ for every ψ . The characters of Γ are the characters of K trivial on D , that is the classes

$$\widehat{\Gamma} = \{\mathbf{b} \in (\mathbb{Z}/5)^5 : \sum b_i = 0\} / \langle \mathbf{1} \rangle, \quad |\widehat{\Gamma}| = 125,$$

with $\mathbf{b}$ acting by $a \mapsto \zeta^{a \cdot \mathbf{b}}$. A monomial $x^{\mathbf{b}}$ transforms under $a \in K$ by $\zeta^{a \cdot \mathbf{b}}$, so it lies in the isotypic component of the class of $\mathbf{b}$.

Let B be the set of $\mathbf{b} \in \{0, 1, 2, 3\}^5$ with $|\mathbf{b}| = \sum b_i = 5$. The monomials $x^{\mathbf{b}}$, $\mathbf{b} \in B$, are a basis of R_5 for $\psi^5 \neq 1$, because the Jacobian ideal of F_ψ is generated by $x_i^4 - \psi \prod_{j \neq i} x_j$, and modulo these every monomial reduces to one with all exponents at most three; the reduced monomials of degree five are the 101 elements of B , which matches $\dim R_5$. The count is the number of compositions of five into five parts, $\binom{9}{4} = 126$, less the twenty with a part equal to four and the five with a part equal to five.

Proof of Theorem 1. Two elements $\mathbf{b}, \mathbf{b}' \in B$ define the same character when $\mathbf{b}' \equiv \mathbf{b} + k\mathbf{1} \pmod{5}$ for some k . Since the entries of $\mathbf{b}'$ lie in $\{0, \dots, 4\}$, $\mathbf{b}'$ is the vector of residues of $\mathbf{b} + k\mathbf{1}$ in that range: each b'_i is $b_i + k$ or $b_i + k - 5$, and $\sum b'_i = 5$. For $k = 1$ the residues are $b_i + 1 \leq 4$ unless $b_i = 4$, impossible, so all residues are $b_i + 1$, with sum $10 \neq 5$. For $k = 2$ the residues are $b_i + 2$ if $b_i \leq 2$ and $b_i - 3$ if $b_i = 3$; the sum is $15 - 5m$ where m is the number of threes, so $m = 2$, but two threes already sum to six. For $k = 3$ the residues are $b_i + 3$ if $b_i \leq 1$ and $b_i - 2$ if $b_i \geq 2$, sum $20 - 5m'$ with m' the number of entries at least two, so $m' = 3$, three entries at least two summing to at most five, impossible. For $k = 4$ the residues are $b_i + 4$ if $b_i = 0$ and $b_i - 1$ otherwise, sum $25 - 5m''$ with m'' the number of nonzero entries, so $m'' = 4$: four nonzero entries summing to five means $(2, 1, 1, 1, 0)$ up to order, and then $\mathbf{b}' = (1, 0, 0, 0, 4)$ has a four. So only $k = 0$ works and the 101 characters are distinct; being characters of an abelian group acting on R_5 , they are then the full decomposition, each isotypic component being one-dimensional.

The group $\widehat{\Gamma}$ has 125 elements and B realises 101 of them, so exactly 24 are missed. A permutation of $(0, 1, 2, 3, 4)$ has sum ten, so its class lies in $\widehat{\Gamma}$, and adding $\mathbf{1}$ to it gives another permutation of $(0, 1, 2, 3, 4)$ modulo five; the hundred and twenty permutations therefore fall into 24 classes of five. None of these classes meets B , because every representative with entries in $\{0, \dots, 4\}$ is a permutation of $(0, 1, 2, 3, 4)$ and contains a four. Hence the 24 missed classes are exactly the classes of the permutations.

Finally S_5 permutes the exponent vectors, and the orbits in B are the multisets of entries: $\{1, 1, 1, 1, 1\}$ (one element), $\{2, 1, 1, 1, 0\}$ (twenty), $\{3, 2, 0, 0, 0\}$ (twenty), $\{2, 2, 1, 0, 0\}$ (thirty) and $\{3, 1, 1, 0, 0\}$ (thirty). $\square$

Figure 2 shows the hundred and one labels.

4. THE SOCLE FUNCTIONAL

Fix ψ with $\psi^5 \neq 1$, write $R = R_{F_\psi}$ and $m_0 = x_1^3 x_2^3 x_3^3 x_4^3 x_5^3$, the only monomial of degree fifteen with all exponents at most three. Define the sector of a monomial $x^{\mathbf{e}}$ of degree fifteen as follows: $\mathbf{e}$ is in sector $r \in \{0, 1, 2, 3, 4\}$ if $e_i \equiv 3 - r \pmod{5}$ for every i ; otherwise $\mathbf{e}$ is in no sector. Sector 4 is empty, because its exponents are at least four each.

Theorem 7 (Socle functional). *For $\psi^5 \neq 1$, $m_0 \neq 0$ in R_{15} , and the linear functional λ on R_{15} with $\lambda(m_0) = 1$ satisfies, for every monomial $x^{\mathbf{e}}$ of degree fifteen,*

$$\lambda(x^{\mathbf{e}}) = \begin{cases} \psi^r & \text{if } \mathbf{e} \text{ is in sector } r, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover $\lambda(\text{Hess } F_\psi) = 3\,200\,000(1 - \psi^5)$.

Proof. The relations $x_i^4 \equiv \psi \prod_{j \neq i} x_j$ in R give, for every exponent vector $\mathbf{e}$ with $e_i \geq 4$,

$$(6) \quad x^{\mathbf{e}} \equiv \psi x^{\mathbf{e}'}, \quad \mathbf{e}' = \mathbf{e} + \mathbf{1} - 5\mathbf{e}_i,$$

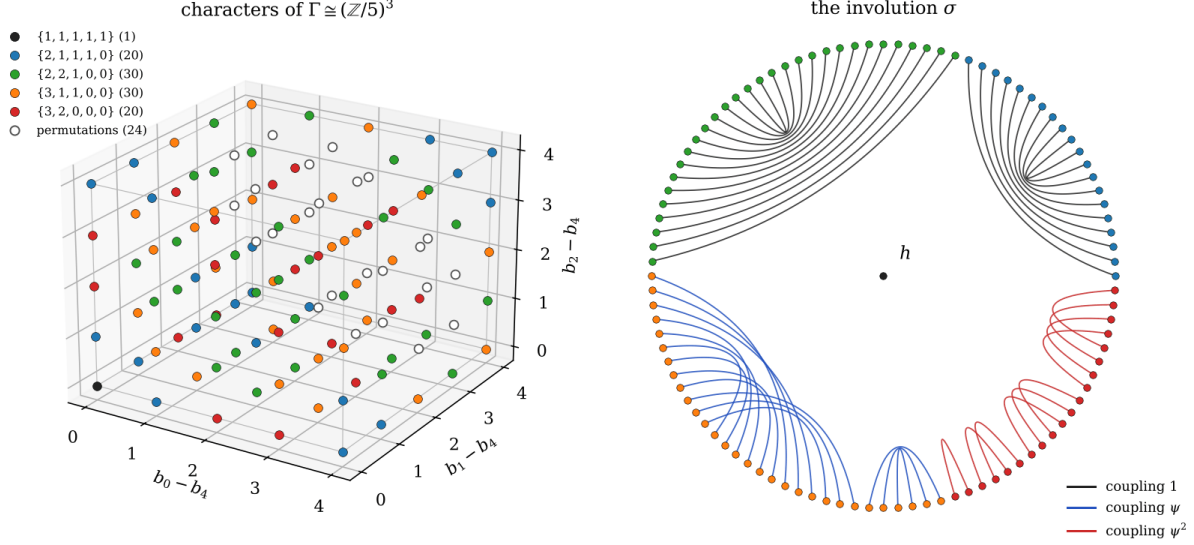


FIGURE 2. The hundred and one generations of the Dwork quintic as points of the character group of Γ . Left: $\widehat{\Gamma} \cong (\mathbb{Z}/5)^3$ drawn as a cube of 125 points, the class of $\mathbf{b}$ placed at $(b_0 - b_4, b_1 - b_4, b_2 - b_4) \bmod 5$; the 101 labels are filled and coloured by type (the five S_5 -orbits), and the 24 classes of permutations of $(0, 1, 2, 3, 4)$ are hollow. Right: the involution σ of Theorem 3, drawn as chords between paired labels, with the chord colour giving the power of ψ of the pairing: black for 1, blue for ψ , red for ψ^2 ; the fixed label is the invariant class h at the centre.

where $\mathbf{e}_i$ is the i -th unit vector. If $\mathbf{e}$ is in sector r then $\mathbf{e}'$ is in sector $r - 1$: its i -th entry is $e_i - 4 \equiv 3 - (r - 1)$ and the other entries are $e_j + 1 \equiv 3 - (r - 1)$. The sectors are listed by hand. In sector 0 the entries are congruent to 3 and sum to fifteen, so they are all equal to three and $\mathbf{e} = (3, 3, 3, 3, 3)$. In sector 1 the entries are congruent to two, so each is 2, 7 or 12; a twelve would leave three for four entries of at least two, so the entries are 2 or 7, and a total of fifteen forces exactly one seven: $\mathbf{e}$ is a permutation of $(7, 2, 2, 2, 2)$, and (6) applied to the seven gives ψm_0 . In sector 2 the entries are 1, 6 or 11, and the total forces two sixes and three ones (an eleven would leave four for the other four entries, which are congruent to one and so are all one, a total of fifteen; but then $\mathbf{e} = (11, 1, 1, 1, 1)$, and (6) applied to the eleven gives $(7, 2, 2, 2, 2)$, in sector 1 as well). In sector 3 the entries are multiples of five, so $\mathbf{e}$ is a permutation of $(15, 0, 0, 0, 0)$, $(10, 5, 0, 0, 0)$ or $(5, 5, 5, 0, 0)$, and one reduction at an entry of at least five lands in sector 2. By induction on r , every monomial in sector r satisfies $x^{\mathbf{e}} \equiv \psi^r m_0$ in R .³

If $\mathbf{e}$ is in no sector, the monomial $x^{\mathbf{e}}$ has a nontrivial character of Γ : its character is the class of $\mathbf{e}$ modulo $\mathbf{1}$, which is trivial exactly when $\mathbf{e} \equiv c\mathbf{1}$ for some c , that is, when $\mathbf{e}$ is in a sector. The ideal J_{F_ψ} is Γ -stable, so R is graded by the characters of Γ and $x^{\mathbf{e}}$ lies in a nontrivial isotypic component of R_{15} . Since R_{15} is one-dimensional it carries a single character, and m_0 carries the trivial one; so if $m_0 \neq 0$, every nontrivial component vanishes and $x^{\mathbf{e}} \equiv 0$.

It remains to show that $m_0 \neq 0$. The Hessian $\text{Hess } F_\psi$ has degree fifteen and is Γ -invariant, so its monomials in no sector vanish in R and the others are multiples of m_0 : $\text{Hess } F_\psi \equiv H(\psi) m_0$, with $H(\psi)$ the sum over the sector monomials of the Hessian of their coefficients times ψ^r . By Macaulay's theorem $\text{Hess } F_\psi \neq 0$ in R_{15} , so $m_0 \neq 0$ and $H(\psi) \neq 0$. The Hessian matrix has diagonal entries $20x_i^3$ and off-diagonal entries $-5\psi \prod_{k \neq i, j} x_k$, each of degree three; an off-diagonal entry omits exactly the two coordinates of its row and

³The reduction (6) uses only the relation $\partial_i F_\psi / 5 = x_i^4 - \psi \prod_{j \neq i} x_j$. It lowers the i -th exponent by four and raises each of the other four by one, so it preserves the degree and moves the class of $\mathbf{e}$ modulo $\mathbf{1}$ by $-5\mathbf{e}_i + \mathbf{1} \equiv \mathbf{1}$; this is why the sectors, which are the classes $(3 - r)\mathbf{1}$, are permuted cyclically and why each reduction costs exactly one factor of ψ .

TABLE 1. The Hessian of F_ψ in the socle, by cycle type of the permutation. Each row gives the number of permutations, their sign, the monomial they contribute, its sector r , and the total contribution to $H(\psi)$ after the reduction $x^e \equiv \psi^r m_0$.

type	count	sign	monomial	r	contribution
1^5	1	+	$(3, 3, 3, 3, 3)$	0	3 200 000
$2 1^3$	10	−	$(5, 5, 5, 0, 0)$	3	$-2\,000\,000 \psi^5$
$3 1^2$	20	+	$(6, 6, 1, 1, 1)$	2	$-1\,000\,000 \psi^5$
$4 1$	30	−	$(7, 2, 2, 2, 2)$	1	$-375\,000 \psi^5$
5	24	+	$(3, 3, 3, 3, 3)$	0	$-75\,000 \psi^5$
$2^2 1$	15	+	$(7, 2, 2, 2, 2)$	1	$187\,500 \psi^5$
$3 2$	20	−	$(3, 3, 3, 3, 3)$	0	$62\,500 \psi^5$

column. The determinant is the sum over permutations π of the sign of π times $\prod_i (\text{entry}(i, \pi(i)))$, and the monomial of π depends only on its cycle type: a fixed point a contributes x_a^3 from the diagonal and x_a from every off-diagonal factor, and a moved coordinate m contributes x_m from every off-diagonal factor whose row and column both differ from m . Table 1 lists the seven cycle types; the derivation of each row is in Appendix A. Summing the last column,

$$\begin{aligned}
 H(\psi) &= 3\,200\,000 - (2\,000\,000 + 1\,000\,000 + 375\,000 \\
 &\quad + 75\,000 - 187\,500 - 62\,500) \psi^5 \\
 &= 3\,200\,000 (1 - \psi^5).
 \end{aligned}$$

□

Remark 8 (Normalisation). With the residue normalisation $\lambda_F(\text{Hess} F) = 1$ of (5), the cubic form of the pencil is $\lambda_{F_\psi} = \lambda / (3\,200\,000(1 - \psi^5))$. For $\mathbf{a} = \mathbf{b} = \mathbf{c} = h$ the monomial is $x^{3\mathbf{1}} = m_0$, so $Y(h, h, h)$ is a constant times $1/(1 - \psi^5)$, which is the coupling $5/(1 - \psi^5)$ of [5] in the gauge in which Ω is the residue of Ω_0/F_ψ ; the factor five is the constant c of (5) in that gauge. All ratios of couplings at fixed ψ are independent of these constants, and Theorem 2 is stated for λ .

5. COUPLINGS

Proof of Theorem 2. By (5) and Theorem 7, $Y(x^{\mathbf{a}}, x^{\mathbf{b}}, x^{\mathbf{c}}) = \lambda(x^{\mathbf{a}+\mathbf{b}+\mathbf{c}})$ up to the normalisation of Remark 8, and $\mathbf{a} + \mathbf{b} + \mathbf{c}$ has degree fifteen; the value is ψ^r if $\mathbf{a} + \mathbf{b} + \mathbf{c} \equiv (3 - r)\mathbf{1}$ and zero otherwise. For $\psi \neq 0$ every sector value is nonzero. The count of the unordered triples is in Section 5.1. □

5.1. The count. A triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ of elements of B , with repetitions allowed, has a nonzero coupling exactly when $\mathbf{a} + \mathbf{b} + \mathbf{c} \equiv c\mathbf{1} \pmod{5}$ for some c , and then $c \equiv 3 - r$ with r the sector. We count ordered triples first. The congruence says that the character of $\mathbf{c}$ is the inverse of the character of $\mathbf{a} + \mathbf{b}$ in $\widehat{\Gamma}$. Every character occurs in B at most once (Theorem 1), so for each ordered pair $(\mathbf{a}, \mathbf{b})$ there is at most one $\mathbf{c}$, and there is one exactly when the class of $-(\mathbf{a} + \mathbf{b})$ is one of the hundred and one realised classes, that is, when it is not the class of a permutation of $(0, 1, 2, 3, 4)$. The negative of a permutation of $(0, 1, 2, 3, 4)$ is, modulo five, a permutation of $(0, 1, 2, 3, 4)$ again, so this fails exactly when $\mathbf{a} + \mathbf{b}$ has five distinct residues modulo five.

Excluded pairs. The entries of $\mathbf{a} + \mathbf{b}$ lie in $\{0, \dots, 6\}$ and sum to ten. Five distinct residues are the residues $0, 1, 2, 3, 4$, whose sum is already ten, so no entry can exceed four and $\mathbf{a} + \mathbf{b}$ is a permutation of $(0, 1, 2, 3, 4)$. Fix the permutation, so that the coordinate k of $\mathbf{a} + \mathbf{b}$ equals k for $k = 0, \dots, 4$ after renaming. Write p_k for the part given to $\mathbf{a}$ in coordinate k ; then $0 \leq p_k \leq \min(k, 3)$ and $k - p_k \leq 3$, so $p_0 = 0$, $p_1 \in \{0, 1\}$, $p_2 \in \{0, 1, 2\}$, $p_3 \in \{0, 1, 2, 3\}$ and $p_4 \in \{1, 2, 3\}$, and $|\mathbf{a}| = 5$ means $p_1 + p_2 + p_3 + p_4 = 5$. The number of solutions is the coefficient of t^5 in

$$(1+t)(1+t+t^2)(1+t+t^2+t^3)(t+t^2+t^3),$$

which is 16.⁴ With 120 permutations there are 1920 excluded ordered pairs, so the number of ordered triples with a nonzero coupling is $101^2 - 1920 = 8281$.

Repetitions. An ordered triple $(\mathbf{a}, \mathbf{a}, \mathbf{a})$ has a nonzero coupling when $3\mathbf{a} \equiv c\mathbf{1}$, that is, when all entries of $\mathbf{a}$ are congruent modulo five; entries in $\{0, 1, 2, 3\}$ summing to five then force $\mathbf{a} = \mathbf{1}$. For a triple $(\mathbf{a}, \mathbf{a}, \mathbf{c})$ the partner $\mathbf{c}$ exists unless $2\mathbf{a}$ has five distinct residues; the residues of $2a_i$ for $a_i = 0, 1, 2, 3$ are 0, 2, 4, 1, and five distinct residues would need five distinct entries of $\mathbf{a}$ in a set of four values, which is impossible. So every $\mathbf{a}$ has a partner $\mathbf{c}$, and $\mathbf{c} = \mathbf{a}$ only for $\mathbf{a} = \mathbf{1}$. There are therefore 100 unordered triples of the form $\{\mathbf{a}, \mathbf{a}, \mathbf{c}\}$ with $\mathbf{c} \neq \mathbf{a}$, giving 300 ordered triples, one triple $\{\mathbf{1}, \mathbf{1}, \mathbf{1}\}$, and $8281 - 300 - 1 = 7980$ ordered triples with three distinct entries. The number of unordered triples is

$$\frac{7980}{6} + 100 + 1 = 1330 + 101 = 1431.$$

Sectors. In sector r the sum $\mathbf{e} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ is one of the vectors listed in the proof of Theorem 7 with entries at most nine: $(3, 3, 3, 3, 3)$ for $r = 0$, the five permutations of $(7, 2, 2, 2, 2)$ for $r = 1$, the ten permutations of $(6, 6, 1, 1, 1)$ for $r = 2$ and the ten permutations of $(5, 5, 5, 0, 0)$ for $r = 3$. The number $O(\mathbf{e})$ of ordered triples with sum $\mathbf{e}$ is the coefficient of $x^5 y^5$ in $\prod_{i=1}^5 P_{e_i}(x, y)$, where $P_e(x, y) = \sum x^a y^b$ over the pairs $0 \leq a, b \leq 3$ with $0 \leq e - a - b \leq 3$. These coefficients are

$$\begin{aligned} O(3, 3, 3, 3, 3) &= 3391, & O(7, 2, 2, 2, 2) &= 480, \\ O(6, 6, 1, 1, 1) &= 162, & O(5, 5, 5, 0, 0) &= 87, \end{aligned}$$

so the ordered counts by sector are 3391, 2400, 1620 and 870, with total 8281. The hundred unordered triples $\{\mathbf{a}, \mathbf{a}, \mathbf{c}\}$ with $\mathbf{c} \neq \mathbf{a}$ split as 0, 50, 20 and 30 by sector (the sector of $2\mathbf{a} + \mathbf{c}$), and $\{\mathbf{1}, \mathbf{1}, \mathbf{1}\}$ lies in sector 0, so the unordered counts are

$$N_r = \frac{O_r - 3D_r - T_r}{6} + D_r + T_r = 566, 425, 280, 160,$$

with O_r, D_r, T_r the ordered, repeated and constant counts of sector r . The coefficient extractions are written out in Appendix B.

6. THE MASS MAP

Proof of Theorem 3. For $\mathbf{b} \in B$ the coupling $\lambda(x^{\mathbf{b}+\mathbf{c}+\mathbf{1}})$ is nonzero exactly when $\mathbf{b} + \mathbf{c} + \mathbf{1} \equiv (3-r)\mathbf{1}$, that is $\mathbf{c} \equiv (2-r)\mathbf{1} - \mathbf{b} \pmod{5}$, with $\mathbf{c} \in B$. For each r the residues of $(2-r)\mathbf{1} - \mathbf{b}$ are determined, and $\mathbf{c}$ exists in B exactly when those residues, taken in $\{0, 1, 2, 3\}$ (a residue 4 excludes), sum to five. Go through the types. If all $b_i \leq 2$ then $r = 0$ gives $\mathbf{c} = 2\mathbf{1} - \mathbf{b} \in \{0, 1, 2\}^5$ of sum five: this is the partner. For $r = 1$ the residues of $\mathbf{1} - \mathbf{b}$ are 1, 0, 4 for $b_i = 0, 1, 2$, so a two in $\mathbf{b}$ excludes, and $\mathbf{b} \in \{0, 1\}^5$ of sum five means $\mathbf{b} = \mathbf{1}$, whose $r = 1$ residues are all zero, sum zero: excluded. For $r = 2$ the residues of $-\mathbf{b}$ are 0, 4, 3: a one excludes, and $\mathbf{b} \in \{0, 2\}^5$ has even sum: excluded. For $r = 3$ the residues of $-\mathbf{1} - \mathbf{b}$ are 4, 3, 2: a zero excludes, so $\mathbf{b} \in \{1, 2\}^5$ of sum five means $\mathbf{b} = \mathbf{1}$, residues all 3, sum fifteen: excluded. So a label with entries at most two has exactly one partner, $2\mathbf{1} - \mathbf{b}$, with coupling 1; this covers the fifty-one labels of the types $\{1, 1, 1, 1, 1\}$, $\{2, 1, 1, 1, 0\}$ and $\{2, 2, 1, 0, 0\}$, and $\sigma(\mathbf{b}) = 2\mathbf{1} - \mathbf{b}$ is of the same type, with the single fixed point $\mathbf{b} = \mathbf{1}$.

If $\mathbf{b}$ has type $\{3, 1, 1, 0, 0\}$: $r = 0$ gives residue 4 at the three, excluded; $r = 1$ gives residues 3, 0, 0, 1, 1, sum five, so $\mathbf{c}$ is the label with 3 where $\mathbf{b}$ has 3, zeros where $\mathbf{b}$ has ones and ones where $\mathbf{b}$ has zeros, of the same type, with coupling ψ ; $r = 2$ gives residues 2, 4, ..., excluded by the ones; $r = 3$ gives residues 1, 3, 3, 4, 4, excluded. If $\mathbf{b}$ has type $\{3, 2, 0, 0, 0\}$: $r = 0$ excluded by the three; $r = 1$ gives residues 3, 4, ..., excluded by the two; $r = 2$ gives residues 2, 3, 0, 0, 0, sum five, so $\mathbf{c}$ swaps the three and the two of $\mathbf{b}$, of the same type, coupling ψ^2 ; $r = 3$ gives residues 1, 2, 4, 4, 4, excluded. This proves that every label has exactly one partner, that σ is an involution preserving type with the one fixed point $\mathbf{1}$, and the three values of the coupling. The matrix $M = (\lambda(x^{\mathbf{b}+\mathbf{c}+\mathbf{1}}))_{\mathbf{b}, \mathbf{c}}$ is therefore a monomial matrix with entries 1 (fifty-one times),

⁴Expand $(1+t)(1+t+t^2) = 1+2t+2t^2+t^3$ and $(1+t+t^2+t^3)(t+t^2+t^3) = t+2t^2+3t^3+3t^4+2t^5+t^6$. The coefficient of t^5 in the product is $1 \cdot 2 + 2 \cdot 3 + 2 \cdot 3 + 1 \cdot 2 = 16$, pairing the terms of degree 0, 1, 2, 3 of the first factor with those of degree 5, 4, 3, 2 of the second.

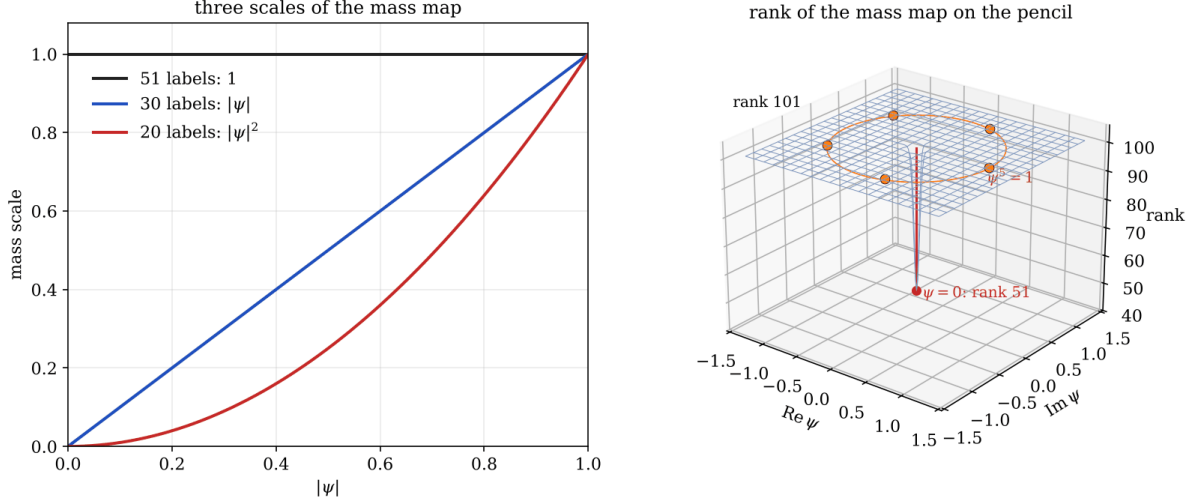


FIGURE 3. The mass spectrum of Theorem 3 along the pencil. Left: the three mass scales 1, $|\psi|$ and $|\psi|^2$ against $|\psi|$ for $0 \leq |\psi| < 1$, with the multiplicities of the labels (fifty-one, thirty, twenty) that carry them; the singlet h sits in the first group. Right: the rank of the mass map over the ψ -plane, fifty-one at the Fermat point and a hundred and one elsewhere (the drop at $\psi = 0$ is drawn with a small width); the conifold points $\psi^5 = 1$, where the Jacobian ring degenerates, are marked on the unit circle.

ψ (thirty times) and ψ^2 (twenty times), so its determinant is $\psi^{30+40} = \psi^{70}$, the sign of σ , a product of fifty transpositions, being $+1$, and its rank is 101 for $\psi \neq 0$. At $\psi = 0$ the rows of the fifty labels containing a three vanish and the remaining fifty-one rows are a permutation matrix on the fifty-one labels with entries at most two: rank fifty-one.

The Γ -invariant Hermitian metric on $H^{2,1}$, the Hodge metric, pairs the isotypic components of distinct characters to zero because a unitary representation of a finite abelian group is an orthogonal sum of its isotypic components; by Theorem 1 the components are the lines spanned by the $x^{\mathbf{b}}$, so the metric is diagonal with entries $g_{\mathbf{b}}$. The mass matrix (4) is then ν times a monomial matrix with entries $\psi^{r(\mathbf{b})}(g_{\mathbf{b}}g_{\sigma\mathbf{b}})^{-1/2}$ at $(\mathbf{b}, \sigma\mathbf{b})$. The singular values of a symmetric monomial matrix with an involution are the absolute values of its entries, each two-cycle giving a doubly degenerate pair and the fixed point a singlet; this is the stated spectrum. $\square$

Remark 9 (Degenerations). The determinant $\pm\psi^{70}$ shows that the mass map of the invariant direction degenerates on the pencil only at the Fermat point, where the group of automorphisms of X_ψ jumps from $\Gamma \rtimes S_5$ to the full group of order $5^4 \cdot 120$ of the Fermat quintic. The fifty massless generations at $\psi = 0$ are the labels with an exponent three. The larger group acts on a monomial through its full exponent vector, not only through its class modulo $\mathbf{1}$, so at $\psi = 0$ a coupling of h , $\mathbf{b}$ and $\mathbf{c}$ survives only if $\mathbf{1} + \mathbf{b} + \mathbf{c} = (3, 3, 3, 3, 3)$; for a label with an exponent three the required partner $2\mathbf{1} - \mathbf{b}$ has a negative entry. At the conifold points $\psi^5 = 1$ the matrix M has nonzero determinant, but the normalisation of Remark 8 has a pole: the physical couplings diverge as $1/(1 - \psi^5)$ in the gauge of Ω chosen there, and the Hodge norm of h diverges logarithmically, as in [5].

7. THE FERMAT POINT

At $\psi = 0$ the Hodge norms are explicit.

TABLE 2. Hodge norms and masses at the Fermat point, from (7) and Theorem 3. Norms are in units of C and masses in units of v/C ; the types with an exponent three are massless at $\psi = 0$.

type	count	$g_{\mathbf{b}}/C$	value	mass
$\{1, 1, 1, 1, 1\}$	1	$\gamma(2/5)^5$	7.33179	0.136392
$\{2, 1, 1, 1, 0\}$	20	$\gamma(2/5)^2 \gamma(1/5)$	8.74859	0.114304
$\{2, 2, 1, 0, 0\}$	30	$\gamma(1/5)^2 / \gamma(2/5)$	10.4392	0.095793
$\{3, 1, 1, 0, 0\}$	30	$\gamma(2/5)^2 \gamma(1/5)$	8.74859	0
$\{3, 2, 0, 0, 0\}$	20	$\gamma(1/5)^2 / \gamma(2/5)$	10.4392	0

Proposition 10 (Hodge norms at the Fermat point). *Let X_0 be the Fermat quintic and $\omega_{\mathbf{b}}$ the class of $x^{\mathbf{b}}$, $\mathbf{b} \in B$. Write $k_i = b_i + 1$ and $\gamma(t) = \Gamma(t)/\Gamma(1-t)$. There is a constant $C > 0$ independent of $\mathbf{b}$ such that*

$$(7) \quad g_{\mathbf{b}} = \langle \omega_{\mathbf{b}}, \overline{\omega_{\mathbf{b}}} \rangle = C \prod_{i=1}^5 \gamma(k_i/5).$$

Proof. The full group μ_5^5/μ_5 of diagonal phases acts on X_0 , and $\omega_{\mathbf{b}}$ spans the eigenline of the character $\mathbf{k} = (k_i)$, with $1 \leq k_i \leq 4$ and $\sum k_i = 10$. Eigenlines of distinct characters are orthogonal for the Hodge metric, so $g_{\mathbf{b}}$ is the only entry of the metric on that line, and it is computed from the periods of $\omega_{\mathbf{b}}$ by the second Riemann bilinear relation: if P is an integral cycle and $P_{\mathbf{k}}$ its component in the eigenspace of $\mathbf{k}$ in $H_3(X_0, \mathbb{C})$, then $g_{\mathbf{b}} = |\int_P \omega_{\mathbf{b}}|^2 / |P_{\mathbf{k}} \cdot P_{-\mathbf{k}}|$ up to a factor depending only on the Hodge type. For the cycle of Pham, the join of the five sets of fifth roots of unity, the period is a product of beta integrals, and the intersection number is a product of factors $1 - \zeta^{k_i}$; combining them with $\Gamma(t)\Gamma(1-t) = \pi/\sin \pi t$ gives (7). This is the computation of Candelas [4] on the Fermat quintic and, in the language of the Gepner model, of Distler and Greene [16]; the steps are written out in Appendix C. $\square$

The proposition fixes the ratios of the Hodge norms inside $H^{2,1}$, which is all that the masses of Theorem 3 need. Since $\gamma(1-t) = 1/\gamma(t)$, every norm is a monomial in the two numbers $\gamma(1/5) = 3.94325$ and $\gamma(2/5) = 1.48951$. For the singlet, $\mathbf{k} = (2, 2, 2, 2, 2)$ and $g_1 = C\gamma(2/5)^5$; for the type $\{2, 1, 1, 1, 0\}$, $\mathbf{k}$ is a permutation of $(3, 2, 2, 2, 1)$ and $g_{\mathbf{b}} = C\gamma(2/5)^2 \gamma(1/5)$; for the type $\{2, 2, 1, 0, 0\}$, $\mathbf{k}$ is a permutation of $(3, 3, 2, 1, 1)$ and $g_{\mathbf{b}} = C\gamma(1/5)^2 / \gamma(2/5)$. The ratio of the second to the first is $\Gamma(3/5)^3 \Gamma(1/5) / \Gamma(2/5)^3 \Gamma(4/5) = 1.19324$, the square of the constant $\kappa = 1.09236$ in which Distler and Greene [16] expressed the normalised couplings of the Gepner model (see [15, eq. (4.3)]). The norm of Ω , the class of 1 with $\mathbf{k} = (1, 1, 1, 1, 1)$, has the same shape with a different constant, and the Weil–Petersson metric of the pencil at the Fermat point, computed from the periods by Candelas, de la Ossa, Green and Parkes [5], is

$$G_{\psi\bar{\psi}}(0) = 25 \frac{\Gamma(2/5)^5 \Gamma(4/5)^5}{\Gamma(1/5)^5 \Gamma(3/5)^5} = 0.19226,$$

the value that Butbaia and collaborators [15] reproduce with numerical Ricci-flat metrics.⁵

Inserting (7) into Theorem 3 gives the fifty-one nonzero masses at the Fermat point. Each label of the types $\{2, 1, 1, 1, 0\}$ and $\{2, 2, 1, 0, 0\}$ is paired with a label of the same type and the same multiset of k_i , so $g_{\mathbf{b}} = g_{\sigma\mathbf{b}}$ and its mass is $v/g_{\mathbf{b}}$; the singlet has mass v/g_1 . Table 2 lists the values. In units of the singlet mass the two doublet masses are 0.83805 (twenty labels, ten doublets) and 0.70234 (thirty labels, fifteen doublets). The fifty generations with an exponent three are massless there, and acquire masses proportional to $|\psi|$ (thirty) and $|\psi|^2$ (twenty) as the modulus moves off the Fermat point.

⁵In [15] the value is given for the mirror quintic, whose complex structure moduli space is the ψ -line of the Dwork pencil divided by the symmetry $\psi \mapsto \zeta\psi$. The Weil–Petersson metric is $-\partial\bar{\partial} \log \int \Omega \wedge \bar{\Omega}$, and passing to the quotient by Γ multiplies $\int \Omega \wedge \bar{\Omega}$ by the constant $1/125$, which does not change the metric; the two numbers are therefore the same. Their machine-learned harmonic representatives give 0.1906 against the exact 0.1923.

TABLE 3. The four pencils of Fermat type. The rows give the degree and weights, the constant $\delta = \prod w_i^{w_i}$ of the discriminant $\psi^d \delta = 1$, the intersection numbers of the hyperplane class L , the Euler characteristic and $h^{2,1}$, the order of Γ , the number of distinct characters carried by the generations, the number of classes of two or three generations sharing a character, the number of characters of Γ carried by no generation, and the rank of the mass map of the invariant direction for $\psi \neq 0$ and at $\psi = 0$, with the number of generations it leaves massless.

	quintic	sextic	octic	dectic
$(d; w)$	$(5; 1, 1, 1, 1, 1)$	$(6; 1, 1, 1, 1, 2)$	$(8; 1, 1, 1, 1, 4)$	$(10; 1, 1, 1, 2, 5)$
δ	1	4	256	12500
$L^3, c_2 \cdot L$	5, 50	3, 42	2, 44	1, 34
$\chi, h^{2,1}$	-200, 101	-204, 103	-296, 149	-288, 145
$ \Gamma $	125	108	128	100
distinct characters	101	96	122	100
shared classes	0	7	27	42
characters not carried	24	12	6	0
rank for $\psi \neq 0$ (massless)	101 (0)	99 (4)	125 (24)	109 (36)
rank at $\psi = 0$ (massless)	51 (50)	44 (59)	85 (64)	72 (73)

8. WEIGHTED FERMAT PENCILS

8.1. **The pencils.** In this section $(d; w)$ is one of $(6; 1, 1, 1, 1, 2)$, $(8; 1, 1, 1, 1, 4)$ and $(10; 1, 1, 1, 2, 5)$, so that $d = \sum w_i$ and $n_i = d/w_i$ is an integer: $n = (6, 6, 6, 6, 3)$, $(8, 8, 8, 8, 2)$ and $(10, 10, 10, 5, 2)$. The polynomial $F_\psi = \sum_i x_i^{n_i} - d\psi P$, $P = x_1 x_2 x_3 x_4 x_5$, is weighted homogeneous of degree d for $\deg x_i = w_i$, and $X_\psi \subset \mathbb{P}(w)$ is its zero locus (2). We write

$$\delta = \prod_i w_i^{w_i} \in \{4, 256, 12500\},$$

and use the letters i, j, k, l for distinct indices among the coordinates of weight one. Table 3 collects the numbers proved in this section, with the quintic for comparison.⁶

Lemma 11. *If $\psi^d \delta \neq 1$, then X_ψ is quasi-smooth, it misses the singular points of $\mathbb{P}(w)$, and it is a smooth Calabi–Yau threefold with $h^{1,1} = 1$. Its hyperplane class $L = c_1(\mathcal{O}_X(1))$ generates $H^2(X, \mathbb{Z})$ modulo torsion, and*

$$(8) \quad \begin{aligned} L^3 &= \frac{d}{\prod_i w_i}, & c_2 \cdot L &= \sigma_2(w) L^3, \\ \chi &= (\sigma_3(w) - d \sigma_2(w)) L^3, \end{aligned}$$

with σ_k the elementary symmetric functions of the weights. If $\psi^d \delta = 1$, then X_ψ is singular.

Proof. The Euler relations $x_i \partial_i F_\psi = n_i x_i^{n_i} - d\psi P$ show that at a common zero of the partial derivatives $x_i^{n_i} = w_i \psi P$ for every i . If $P = 0$ this forces $x = 0$. Otherwise raise the i -th relation to the power w_i and multiply: since $\prod_i x_i^{n_i w_i} = P^d$, this gives $P^d = \psi^d \delta P^d$, so $\psi^d \delta = 1$. Hence X_ψ is quasi-smooth when $\psi^d \delta \neq 1$. Conversely, at $\psi = 2^{-1/3}$, $\psi = 1/2$ and $\psi = 2^{-1/5} 5^{-1/2}$ the points $(1, 1, 1, 1, 2^{1/3})$, $(1, 1, 1, 1, 2)$ and $(1, 1, 1, 2^{1/5}, 5^{1/2})$ satisfy $x_i^{n_i} = w_i \psi P = w_i$, so they are singular points of the sextic, octic and dectic; the other roots of $\psi^d \delta = 1$ are $\zeta \psi$ with $\zeta^d = 1$, and $x_1 \mapsto \zeta^{-1} x_1$ maps $X_{\zeta \psi}$ to X_ψ because $n_1 = d$.

⁶The three pencils are the weighted members of the family of one-parameter models whose periods and mirror maps were computed by Morrison [24], Klemm and Theisen [25] and Font [26]. In those papers the coefficient of P is normalised differently and the natural coordinate on the moduli space is ψ^d up to a constant; the discriminant $\psi^d \delta = 1$ below is the conifold locus of those papers, and the Fermat point $\psi = 0$ is the Gepner point.

The singular points of $\mathbb{P}(w)$ are $(0:0:0:0:1)$ for the sextic and the octic, with isotropy μ_2 and μ_4 , and $(0:0:0:1:0)$ and $(0:0:0:0:1)$ for the dextic, with isotropy μ_2 and μ_5 . Each monomial $x_i^{n_i}$ occurs in F_ψ with coefficient one, so F_ψ does not vanish at a coordinate point and X_ψ misses them all; a quasi-smooth hypersurface in the smooth locus of $\mathbb{P}(w)$ is smooth. By adjunction $K_X = \mathcal{O}_X(d - \sum w_i) = \mathcal{O}_X$, and the Lefschetz theorem for quasi-smooth hypersurfaces [28] gives $h^{1,0} = h^{2,0} = 0$ and $h^{1,1} = 1$.

The tangent bundle of $\mathbb{P}(w)$ on its smooth locus has total Chern class $\prod_i (1 + w_i H)$ by the weighted Euler sequence, and the normal bundle of X is $\mathcal{O}_X(d)$, so $c(T_X) = \prod_i (1 + w_i L)(1 + dL)^{-1}$. Expanding with $\sigma_1(w) = d$ gives $c_1 = 0$, $c_2 = \sigma_2 L^2$ and $c_3 = (\sigma_3 - d\sigma_2)L^3$, and $L^3 = d \int_{\mathbb{P}(w)} H^4 = d / \prod w_i$. For the three pencils $(\sigma_2, \sigma_3) = (14, 16)$, $(22, 28)$ and $(34, 52)$, which gives Table 3. If $L = mL'$ with L' integral, then m^3 divides $L^3 \in \{3, 2, 1\}$, which is cube-free, so $m = \pm 1$; as $H^2(X, \mathbb{Q})$ is spanned by L , the class L generates $H^2(X, \mathbb{Z})$ modulo torsion. $\square$

8.2. Labels and characters. The Jacobian ring $R = \mathbb{C}[x_1, \dots, x_5] / (\partial_1 F_\psi, \dots, \partial_5 F_\psi)$ is graded by $\deg x_i = w_i$. Quasi-smoothness says that the partial derivatives, of degrees $d - w_i$, form a regular sequence, so R is a graded complete intersection with Hilbert series $\prod_i (1 - t^{d-w_i}) / (1 - t^{w_i})$: it has dimension $\prod_i (n_i - 1)$, which is 1250, 2401 and 2916, its top degree is $\sum_i (d - 2w_i) = 3d$, and R_{3d} is one-dimensional and spanned by $\text{Hess } F_\psi$ by Macaulay's theorem, which holds for any regular sequence of weighted homogeneous polynomials. Griffiths' description of the Hodge structure extends to quasi-smooth hypersurfaces in weighted projective space, as shown by Steenbrink [27] and Dolgachev [28]: with

$$\Omega_0 = \sum_i (-1)^i w_i x_i dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_5,$$

the residue along X of $P\Omega_0 / F_\psi^{q+1}$, for P of degree qd , identifies R_{qd} with $H^{3-q,q}(X)$, so that $H^{2,1} = R_d$ and R_{3d} is the socle. The infinitesimal variation of the Hodge structure is again multiplication in R , and the cubic form is (5) with λ a functional on R_{3d} ; the derivation in [11] uses only the residue map and the residue pairing of R , both of which are available in the weighted setting.

Let B be the set of $\mathbf{b} \in \mathbb{Z}^5$ with $0 \leq b_i \leq n_i - 2$ and $\sum_i w_i b_i = d$. We call its elements the labels.

Lemma 12. *Let $\psi^d \delta \neq 1$. Every monomial $x^\mathbf{e}$ of degree d satisfies $x^\mathbf{e} \equiv c \psi^r x^\mathbf{b}$ in R with $\mathbf{b} \in B$, c a product of weights, and $r \leq 1$ for the sextic and $r \leq 2$ for the octic and the dextic. The monomials $x^\mathbf{b}$, $\mathbf{b} \in B$, are a basis of $R_d = H^{2,1}$, so $|B| = h^{2,1}$.*

Proof. The relations $\partial_i F_\psi / n_i = 0$ read

$$(9) \quad x_i^{n_i-1} \equiv w_i \psi \prod_{j \neq i} x_j.$$

In a monomial of degree d at most one exponent satisfies $e_i \geq n_i - 1$: two such exponents would carry weight at least $2d - w_i - w_j > d$. If $e_i \geq n_i - 1$, then $e_i \in \{n_i - 1, n_i\}$, and (9) replaces $x^\mathbf{e}$ by $w_i \psi x^\mathbf{e}'$ with $e'_i = e_i - n_i + 1$ and $e'_j = e_j + 1$ for $j \neq i$. If $e_i = n_i$, then $x^\mathbf{e} = x_i^{n_i}$ and $\mathbf{e}' = \mathbf{1}$. If $e_i = n_i - 1$, then $e'_i = 0$, the other coordinates carry weight w_i , and $e'_j \geq n_j - 1$ forces $w_j e_j \geq d - 2w_j$, hence $w_i + 2w_j \geq d$. For the sextic no pair $i \neq j$ has $w_i + 2w_j \geq 6$, and $\mathbf{1} \in B$ because every $n_j \geq 3$; so one step suffices. For the octic and the dextic the only j with $w_i + 2w_j \geq d$ for some $i \neq j$ is $j = 5$, where $w_5 = d/2$ and $n_5 = 2$; in that case, and in the case $\mathbf{e}' = \mathbf{1}$, the exponent of x_5 is one and a second step at x_5 gives an exponent vector with $e''_5 = 0$ and the other entries raised by one. A third step would need $w_5 + 2w_k \geq d$, that is $w_k \geq d/4$, for some $k \leq 4$, which fails. So at most two steps reach a vector with all $e_i \leq n_i - 2$, which is a label.

At $\psi = 0$ the ring is $\bigotimes_i \mathbb{C}[x_i] / (x_i^{n_i-1})$, whose monomials with $e_i \leq n_i - 2$ are a basis; so $|B| = \dim R_d$. The Hilbert series does not depend on ψ , and the labels span R_d for every ψ by the first part, so they are a basis. $\square$

The symmetry group. Let K be the group of $\varepsilon \in \prod_i \mu_{n_i}$ with $\prod_i \varepsilon_i = 1$, acting by $x_i \mapsto \varepsilon_i x_i$. It preserves F_ψ for every ψ . The elements $(\lambda^{w_1}, \dots, \lambda^{w_5})$ with $\lambda^d = 1$ lie in K and act trivially on $\mathbb{P}(w)$, and they are the only such elements; they form a subgroup isomorphic to μ_d because the weights are coprime. Let $\Gamma = K / \mu_d$. Since

$\text{lcm}(n_i) = d$, the product map $\prod_i \mu_{n_i} \rightarrow \mu_d$ is onto and $|\Gamma| = \prod_i n_i/d^2$, which is 108, 128 and 100. A character of $\prod_i \mu_{n_i}$ is a vector $\mathbf{b} \in \bigoplus_i \mathbb{Z}/n_i$, acting by $\varepsilon \mapsto \prod_i \varepsilon_i^{b_i}$; it is trivial on K exactly when it is a multiple of $\mathbf{1}$, and it is trivial on μ_d exactly when $\sum_i w_i b_i \equiv 0 \pmod{d}$, a condition that is well defined because $w_i n_i = d$. Hence

$$\hat{\Gamma} = \left\{ \mathbf{b} \in \bigoplus_i \mathbb{Z}/n_i : \sum_i w_i b_i \equiv 0 \pmod{d} \right\} / \langle \mathbf{1} \rangle.$$

A monomial x^ε of degree divisible by d transforms by the class $[\varepsilon]$ of its exponent vector, and Ω_0 transforms by $\prod_i \varepsilon_i = 1$, so the isomorphisms $R_{qd} \cong H^{3-q,q}$ are Γ -equivariant and R is graded by $\hat{\Gamma}$. The Hessian is invariant, since $\partial_i \partial_j F_\psi$ transforms by $\varepsilon_i^{-1} \varepsilon_j^{-1}$ and the determinant by $\prod_i \varepsilon_i^{-2} = 1$; so R_{3d} carries the trivial character. Permutations of coordinates of equal weight also preserve F_ψ and permute the labels.

Lemma 13 (Wrapping). *Two labels $\mathbf{b} \neq \mathbf{b}'$ carry the same character if and only if there is a shift $s \in \{1, \dots, d-1\}$ such that, with $a_i = \lfloor (b_i + s)/n_i \rfloor$,*

$$\begin{aligned} b'_i &= b_i + s - n_i a_i, & \sum_i a_i &= s, \\ b_i + s &\not\equiv n_i - 1 \pmod{n_i} & \text{for all } i. \end{aligned}$$

The shift from $\mathbf{b}'$ back to $\mathbf{b}$ is then $d - s$.

Proof. Both labels satisfy $\sum w_i b_i = d \equiv 0$, so they define the same character exactly when $\mathbf{b}' \equiv \mathbf{b} + s\mathbf{1}$ modulo $(n_1, \dots, n_5)$ for some s modulo d . As $0 \leq b'_i \leq n_i - 2$, the entry b'_i is the residue of $b_i + s$ in $\{0, \dots, n_i - 1\}$, which must differ from $n_i - 1$. Then $\sum_i w_i b'_i = \sum_i w_i b_i + sd - d \sum_i a_i$, which equals d exactly when $\sum_i a_i = s$. $\square$

Proposition 14. *Let $\psi^d \delta \neq 1$. The classes of labels that carry the same character are exactly the following, and every other label carries a character of its own.*

- (a) *Sextic: the 4 pairs $\{x_i^4 x_5, x_j^2 x_k^2 x_l^2\}$ at shift 2 and the 3 pairs $\{x_i^3 x_j^3, x_k^3 x_l^3\}$ at shift 3; 96 distinct characters.*
- (b) *Octic: the 12 pairs $\{x_i^6 x_j^2, x_j^4 x_k^2 x_l^2\}$ and the 12 pairs $\{x_i^6 x_j x_k, x_j^3 x_k^3 x_l^2\}$ at shift 2, and the 3 pairs $\{x_i^4 x_j^4, x_k^4 x_l^4\}$ at shift 4; 122 distinct characters.*
- (c) *Dectic: the 3 triples $\{x_i^8 x_4, x_j^2 x_k^2 x_4^3, x_i^4 x_j^4 x_k^4\}$; at shift 2 the 12 pairs $\{x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^3, x_1^{c_1+2} x_2^{c_2+2} x_3^{c_3+2}\}$ with $c_1 + c_2 + c_3 = 4$ and $\mathbf{c}$ not a permutation of $(2, 2, 0)$, the 6 pairs $\{x_i^8 x_j^2, x_j^4 x_k^2 x_4^2\}$ and the 3 pairs $\{x_i^8 x_j x_k, x_j^3 x_k^3 x_4^2\}$; at shift 4 the 6 pairs $\{x_i^6 x_j^2 x_4, x_j^6 x_k^4\}$, the 3 pairs $\{x_i^6 x_j x_k x_4, x_j^5 x_k^5\}$, the 6 pairs $\{x_i^7 x_j x_4, x_i x_j^5 x_k^4\}$ and the 3 pairs $\{x_i^6 x_4^2, x_j^4 x_k^4 x_4\}$; 100 distinct characters, so every character of Γ is carried by some generation.*

The characters carried by no label are the 12 classes of $(\pi(0, 1, 3, 4), 2)$ for the sextic and the 6 classes of $(\pi(0, 2, 4, 6), 1)$ for the octic, π running over the permutations of four entries. The set of characters carried by labels is closed under $\chi \mapsto -\chi$.

Proof. We use Lemma 13 and treat the shifts $s \leq d/2$; the others are their inverses. For the sextic, $b_i + s \leq 9 < 12$ for $i \leq 4$, so $a_i = 1$ exactly when $b_i \geq 6 - s$, and the residue $n_i - 1 = 5$ is excluded when $b_i = 5 - s$; for x_5 , $b_5 \leq 1$ and $a_5 = \lfloor (b_5 + s)/3 \rfloor$, with $b_5 + s \equiv 2 \pmod{3}$ excluded. At $s = 1$ we have $a_5 = 0$ and need one $b_i \geq 5$, which is impossible. At $s = 2$ the value $b_5 = 0$ is excluded, so $b_5 = 1$ and $a_5 = 1$; we need exactly one $b_i \geq 4$ among the first four, which sum to four, so $\mathbf{b} = (4, 0, 0, 0, 1)$ up to order, and its image is $(0, 2, 2, 2, 0)$. At $s = 3$ we have $a_5 = 1$ and need two entries $b_i \geq 3$ among the first four and none equal to two; two entries of at least three already have weight six, so $\mathbf{b} = (3, 3, 0, 0, 0)$ up to order, with image $(0, 0, 3, 3, 0)$. No label occurs at two different shifts, so the classes are the pairs listed, 14 labels in 7 classes and $103 - 7 = 96$ characters. The same analysis for the octic and the dectic is carried out in Appendix D. It uses two facts: for these pencils $n_5 = 2$ and $b_5 = 0$, so $a_5 = s/2$ and an odd shift is excluded by the residue of x_5 ; and the bounds $b_i \leq n_i - 2$ make each a_i either 0 or 1, except $a_4 \in \{0, 1, 2\}$ for the dectic.

The characters not carried by a label are $108 - 96 = 12$, $128 - 122 = 6$ and $100 - 100 = 0$ in number. For the sextic, adding $s\mathbf{1}$ to $(\pi(0, 1, 3, 4), 2)$ gives a vector with an entry 5 for $s \not\equiv 0 \pmod{3}$, and the vector

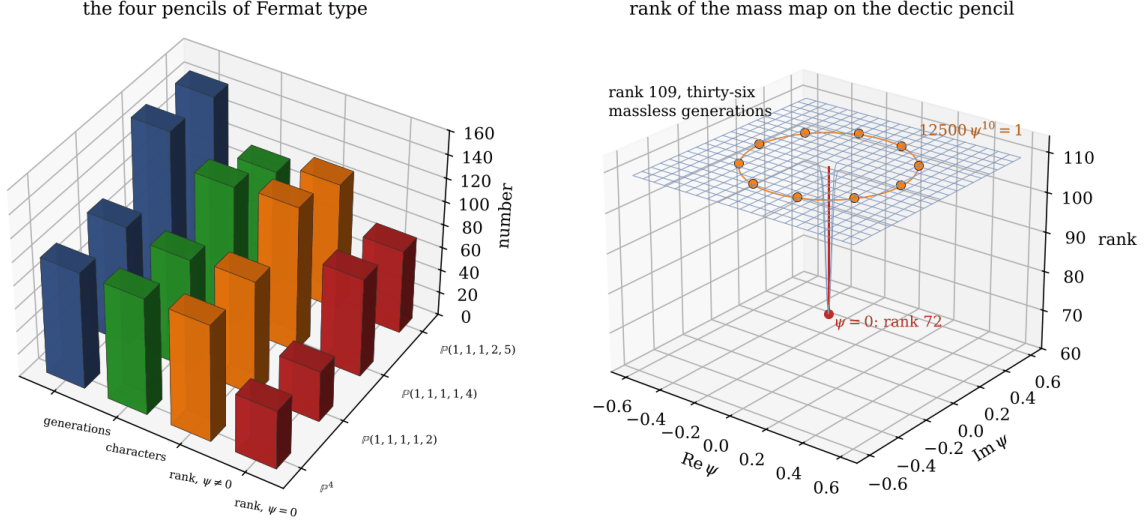


FIGURE 4. Left: for the quintic in $\mathbb{P}^4$ and the sextic, octic and dectic pencils in $\mathbb{P}(1, 1, 1, 1, 2)$, $\mathbb{P}(1, 1, 1, 1, 4)$ and $\mathbb{P}(1, 1, 1, 2, 5)$, the number of generations (101, 103, 149, 145), the number of distinct characters they carry (101, 96, 122, 100), and the rank of the mass map of the invariant direction for $\psi \neq 0$ (101, 99, 125, 109) and at $\psi = 0$ (51, 44, 85, 72). Only on the quintic do the first three bars agree. Right: the rank of the mass map over the ψ -plane of the dectic pencil, 109 off the Fermat point and 72 at $\psi = 0$ (the drop is drawn with a small width); the ten points of the circle $12500\psi^{10} = 1$ are the singular members.

$(\pi'(0, 1, 3, 4), 2)$ for $s = 3$, where the last entry is $n_5 - 1$. So every representative of these classes has an entry $n_i - 1$, and none is a label; each class contains two of the 24 vectors $(\pi(0, 1, 3, 4), 2)$, which gives 12 classes, and these are all the missing characters. For the octic, adding $s\mathbf{1}$ to $(\pi(0, 2, 4, 6), 1)$ gives an entry 7 when s is odd and the last entry $1 = n_5 - 1$ when s is even; each class contains four of the 24 vectors, which gives 6 classes. Finally $-(\pi(0, 1, 3, 4), 2) = (\pi(0, 5, 3, 2), 1)$, which becomes $(\pi(1, 0, 4, 3), 2)$ after adding $\mathbf{1}$, and $-(\pi(0, 2, 4, 6), 1) = (\pi(0, 6, 4, 2), 1)$; so the missing characters are closed under negation, and so are the carried ones. $\square$

Figure 4 compares the four pencils.

8.3. The socle functional. Let $m_0 = x^{\mathbf{n}-2\mathbf{1}}$, the monomial with exponents $n_i - 2$; it has degree $\sum_i w_i(n_i - 2) = 3d$. A monomial $x^{\mathbf{e}}$ of degree $3d$ is in sector $r \in \{0, \dots, d-1\}$ if $e_i \equiv n_i - 2 - r \pmod{n_i}$ for every i . We then write

$$(10) \quad e_i = n_i - 2 - r + n_i k_i;$$

each k_i is a nonnegative integer, since $k_i \leq -1$ would give $e_i \leq -2 - r$, and $\sum_i k_i = r$, by comparing degrees: $\sum_i w_i e_i = 5d - 2d - rd + d \sum_i k_i = 3d$.

Theorem 15 (Weighted socle functional). *Let $\psi^d \delta \neq 1$. Then m_0 spans R_{3d} , and the functional λ on R_{3d} with $\lambda(m_0) = 1$ satisfies, for every monomial $x^{\mathbf{e}}$ of degree $3d$,*

$$\lambda(x^{\mathbf{e}}) = \begin{cases} \psi^r \prod_i w_i^{k_i} & \text{if } \mathbf{e} \text{ is in sector } r, \text{ with } k \text{ as in (10),} \\ 0 & \text{otherwise.} \end{cases}$$

A monomial of degree $3d$ is in a sector if and only if its character is trivial.

Proof. The character of $x^{\mathbf{e}}$ is trivial when $\mathbf{e} \equiv c\mathbf{1}$ modulo $(n_1, \dots, n_5)$ for some integer c ; taking $r \equiv -2 - c \pmod{d}$ and using $n_i \mid d$ this is the condition $e_i \equiv n_i - 2 - r$, so the monomials with trivial character are those

in a sector. Since R_{3d} is one-dimensional with trivial character, every monomial with a nontrivial character vanishes in R .

Let $\mathbf{e}$ be in sector $r \geq 1$. We claim that some i has $n_i k_i \geq r + 1$. Otherwise $k_i \leq r/n_i = rw_i/d$ for every i , so $r = \sum_i k_i \leq r \sum_i w_i/d = r$, with equality only if $n_i k_i = r$ for every i ; then every n_i divides r , and so does their least common multiple d , which is impossible for $1 \leq r < d$. For such an i we have $e_i \geq n_i - 1$, and (9) gives $x^{\mathbf{e}} \equiv w_i \psi x^{\mathbf{e}'}$ with $e'_i = e_i - n_i + 1$ and $e'_j = e_j + 1$ for $j \neq i$; by (10) $\mathbf{e}'$ is in sector $r - 1$ with $k'_i = k_i - 1$ and the other k_j unchanged. After r steps we reach sector 0, where all k_i vanish and $\mathbf{e} = \mathbf{n} - 2\mathbf{1}$. Collecting the factors, $x^{\mathbf{e}} \equiv \psi^r \prod_i w_i^{k_i} m_0$. Finally, R_{3d} is spanned by monomials, each of which is zero or a multiple of m_0 , and $R_{3d} \neq 0$; so m_0 spans R_{3d} . $\square$

Corollary 16. *Let $\psi^d \delta \neq 1$. The cubic form takes on three labels $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the value $\lambda(x^{\mathbf{a}+\mathbf{b}+\mathbf{c}})$, up to the normalisation of λ : this is $\psi^r \prod_i w_i^{k_i}$ when $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is in sector r , and zero when the character of $x^{\mathbf{a}+\mathbf{b}+\mathbf{c}}$ is nontrivial. In particular, for $\psi \neq 0$ every coupling allowed by Γ is nonzero. This is Theorem 5(ii).*

8.4. The mass map. The only label with trivial character is $h = \mathbf{1}$ for the sextic and $h = (2, 2, 2, 2, 0)$ for the octic and the dectic: a label $\mathbf{b} \equiv c\mathbf{1}$ has $b_i = c$ for the coordinates of weight one, and the condition on the weight leaves only these. So the invariant part of R_d is spanned by x^h , the invariant direction. For $\psi \neq 0$ it contains the tangent vector $\partial_\psi F_\psi = -dP$ of the pencil: for the sextic $P = x^h$, and for the octic and the dectic $P = x^1 \equiv w_5 \psi x^h$ by (9) at x_5 , whose exponent one equals $n_5 - 1$. By Corollary 16 the mass map of the invariant direction in the basis of labels is

$$M_{\mathbf{bc}} = \lambda(x^{h+\mathbf{b}+\mathbf{c}}), \quad \mathbf{b}, \mathbf{c} \in B.$$

Since h has trivial character, $M_{\mathbf{bc}} \neq 0$ requires $[\mathbf{c}] = -[\mathbf{b}]$. For a character χ let B_χ be the set of labels carrying it; then M is the direct sum of the blocks on $B_\chi \cup B_{-\chi}$, one for each pair $\{\chi, -\chi\}$ of characters carried by labels (Proposition 14 shows that $-\chi$ is carried whenever χ is).

Lemma 17. *Let $\psi \neq 0$ and $\psi^d \delta \neq 1$. Every entry of a block is $\psi^r \prod_i w_i^{k_i} \neq 0$. For $\mathbf{b}, \mathbf{b}' \in B_\chi$ and $\mathbf{c}, \mathbf{c}' \in B_{-\chi}$ let r, r', ρ, ρ' be the sectors of $h + \mathbf{b} + \mathbf{c}$, $h + \mathbf{b}' + \mathbf{c}'$, $h + \mathbf{b} + \mathbf{c}'$ and $h + \mathbf{b}' + \mathbf{c}$, and $\Delta = r + r' - \rho - \rho'$. Then $\Delta \in \{-d, 0, d\}$, and*

$$\begin{aligned} M_{\mathbf{bc}} M_{\mathbf{b}'\mathbf{c}'} - M_{\mathbf{bc}'} M_{\mathbf{b}'\mathbf{c}} \\ = M_{\mathbf{bc}'} M_{\mathbf{b}'\mathbf{c}} ((\psi^d \delta)^{\Delta/d} - 1), \end{aligned}$$

which vanishes exactly when $\Delta = 0$.

Proof. The first statement is Theorem 15, since each $h + \mathbf{b} + \mathbf{c}$ in a block has trivial character. The two sums $(h + \mathbf{b} + \mathbf{c}) + (h + \mathbf{b}' + \mathbf{c}')$ and $(h + \mathbf{b} + \mathbf{c}') + (h + \mathbf{b}' + \mathbf{c})$ are equal, so by (10), with k, k', κ, κ' the vectors of the four monomials, $n_i(k_i + k'_i - \kappa_i - \kappa'_i) = \Delta$ for every i . Hence every n_i divides Δ , so d does, and $|\Delta| < 2d$ because the sectors lie in $\{0, \dots, d-1\}$. The quotient of the two products is $\psi^\Delta \prod_i w_i^{\Delta w_i/d} = (\psi^d \delta)^{\Delta/d}$. $\square$

Proposition 18. *Let $\psi \neq 0$ and $\psi^d \delta \neq 1$. The blocks of the mass map that involve labels sharing a character are those of Table 4. The mass map has rank 99, 125 and 109 for the sextic, octic and dectic, its kernel has dimension 4, 24 and 36 and lies in the span of the labels that share a character, and the rank is the same at every member with $\psi \neq 0$ and $\psi^d \delta \neq 1$. At $\psi = 0$ the mass map is the matrix of the partial involution $\mathbf{b} \mapsto \mathbf{n} - 2\mathbf{1} - h - \mathbf{b}$ on the labels with $\mathbf{b} \leq \mathbf{n} - 2\mathbf{1} - h$, of rank 44, 85 and 72.*

Proof. The classes of Proposition 14 and their negatives give the rows of Table 4: for one representative of each orbit the character $-\chi$ is computed from Lemma 13, and the entries from Theorem 15; Appendix E lists the sectors and the vectors k . For example, for the sextic block with $i = 1$, $h + (3, 1, 1, 1, 0) + (4, 0, 0, 0, 1) = (8, 2, 2, 2, 2)$ is in sector 2 with $k = (1, 0, 0, 0, 1)$, which gives $2\psi^2$, and $h + (3, 1, 1, 1, 0) + (0, 2, 2, 2, 0) = (4, 4, 4, 4, 1) = \mathbf{n} - 2\mathbf{1}$ gives 1. A block in which one of $B_\chi, B_{-\chi}$ is a single label has the form $\begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}$ with A a row or a column of nonzero entries, so it has rank two and leaves $|B_\chi| + |B_{-\chi}| - 2$ generations massless, in the span of the class of two or three labels. For a block with 2×2 matrix A , Lemma 17 decides the rank: $\Delta = 0$ for the second octic row and the sixth and seventh dectic rows, for instance $6 + 0 - 2 - 4 = 0$ in the

TABLE 4. The blocks of the mass map of the invariant direction that involve labels sharing a character. Each row gives the labels of B_χ and of $B_{-\chi}$ in the notation of Proposition 14 (one representative of the orbit under permutations of the coordinates of weight one), the number of such blocks, the entries $M_{\mathbf{bc}}$ with $\mathbf{b} \in B_\chi$ in the order listed, and the number of generations each block leaves massless for $\psi \neq 0$; “self” means $-\chi = \chi$. All other blocks are a single label with $\chi = -\chi$ or a pair of single labels, and have full rank.

pencil	B_χ	$B_{-\chi}$	entries	blocks	massless
sextic	$x_i^4 x_j x_k x_l^2, x_j^2 x_k^2 x_l^2$	$x_i^3 x_j x_k x_l$	$2\psi^2; 1$	4	1
	$x_i^3 x_j^3, x_k^3 x_l^3$	self	$(2\psi^3; 1; 1, 2\psi^3)$	3	0
octic	$x_i^6 x_j x_k, x_j^3 x_k^3 x_l^2$	$x_i^4 x_j x_k x_l^2$	$4\psi^2; 1$	12	1
	$x_i^6 x_j^2, x_j^4 x_k^2 x_l^2$	$x_j^6 x_i^2, x_i^4 x_k^2 x_l^2$	$(16\psi^4, 4\psi^2; 4\psi^2, 1)$	6	2
	$x_i^4 x_j^4, x_k^4 x_l^4$	self	$(16\psi^4; 1; 1, 16\psi^4)$	3	0
dectic	$x_i^2 x_j x_k x_l^3, x_i^4 x_j^3 x_k^3$	$x_i^2 x_j^3 x_k^3 x_l^4$	$10\psi^2; 1$	3	1
	$x_i^3 x_j x_k^3, x_i^5 x_j^3 x_k^2$	$x_i x_j^3 x_k^4 x_l^4$	$10\psi^2; 1$	6	1
	$x_i^8 x_j x_k, x_j^3 x_k^3 x_l^2$	$x_i^4 x_j x_k x_l^2$	$50\psi^4; 10\psi^2$	3	1
	$x_i^7 x_j x_k, x_i x_j^5 x_k^4$	$x_i^5 x_j x_k^4 x_l^4$	$50\psi^4; 1$	6	1
	$x_i^8 x_k, x_j^2 x_k^3 x_l^3, x_i^2 x_j^4 x_k^4$	$x_i^4 x_j^2 x_k^2 x_l^4$	$50\psi^4; 10\psi^2; 1$	3	2
	$x_i^4 x_k^3, x_j^6 x_k^2 x_l^2$	$x_i^6 x_k^2, x_j^4 x_k^4 x_l^4$	$(500\psi^6, 10\psi^2; 50\psi^4, 1)$	3	2
	$x_i^8 x_j^2, x_j^4 x_k^2 x_l^2$	$x_j^8 x_i^2, x_i^4 x_k^2 x_l^2$	$(250\psi^6, 50\psi^4; 50\psi^4, 10\psi^2)$	3	2
	$x_i^6 x_j^2 x_k, x_j^6 x_k^4$	$x_i^6 x_k^2 x_l^4, x_k^6 x_j^4$	$(50\psi^4; 1; 1, 250\psi^6)$	3	0
	$x_i^6 x_j x_k x_l^4, x_j^5 x_k^5$	self	$(50\psi^4; 1; 1, 250\psi^6)$	3	0

sixth, so A has rank one and two generations on each side of the block are massless; $\Delta = \pm d$ for the eighth dectic row and the three self-dual rows, whose determinants are $4\psi^6 - 1$, $256\psi^8 - 1$ and $12500\psi^{10} - 1$ up to nonzero monomials, so these blocks have full rank. A single label with $\chi = -\chi$, and a pair of single labels, give a block of full rank by the first part of Lemma 17. Adding the last column of Table 4: $4 \cdot 1 = 4$, $12 \cdot 1 + 6 \cdot 2 = 24$ and $3 + 6 + 3 + 6 + 3 \cdot 2 + 3 \cdot 2 + 3 \cdot 2 = 36$. None of the entries or determinants depends on ψ beyond these monomials and the factor $\psi^d \delta - 1$, so the rank is constant on the stated set.

At $\psi = 0$ every sector $r \geq 1$ gives zero, so $M_{\mathbf{bc}} \neq 0$ exactly when $h + \mathbf{b} + \mathbf{c} = \mathbf{n} - 2\mathbf{1}$. The vector $\mathbf{c} = \mathbf{n} - 2\mathbf{1} - h - \mathbf{b}$ has degree $3d - 2d = d$ and entries at most $n_i - 2$, so it is a label exactly when $\mathbf{b} \leq \mathbf{n} - 2\mathbf{1} - h$. Here $\mathbf{n} - 2\mathbf{1} - h = (3, 3, 3, 3, 0)$, $(4, 4, 4, 4, 0)$ and $(6, 6, 6, 1, 0)$, and the labels below these vectors are counted by the coefficient of t^6 in $(1 + t + t^2 + t^3)^4$, which is 44, of t^8 in $(1 + t + \dots + t^4)^4$, which is 85, and by $36 + 36 = 72$ solutions of $b_1 + b_2 + b_3 + 2b_4 = 10$ with $b_i \leq 6$ and $b_4 \leq 1$. $\square$

8.5. Three families.

Proposition 19. *Let X be a member of the sextic, octic or dectic pencil with $\psi^d \delta \neq 1$, and H a finite group acting freely on X . With the standard embedding, X/H does not have three net families.*

Proof. By Theorem 6 the number of net families on $Y = X/H$ is $|\chi(Y)|/2 = |\chi(X)|/(2|H|)$, and three families need $|H| = |\chi(X)|/6$, that is 34 for the sextic and 48 for the dectic; for the octic $296/6$ is not an integer. Suppose H has this order. The projection $\pi: X \rightarrow Y$ is étale of degree $|H|$, $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)/|H| = 0$ and $c_1(Y)$ is torsion. As $H^2(X, \mathbb{Q}) = \mathbb{Q}L$ and L generates $H^2(X, \mathbb{Z})$ modulo torsion (Lemma 11), every line bundle D on Y has $\pi^*D \equiv mL$ numerically for an integer m , and these m form a subgroup of $\mathbb{Z}$. It contains $|H|$: the line bundle $\bigotimes_{g \in H} g^* \mathcal{O}_X(1)$ is linearised by permuting its factors, so it descends to Y , and it is numerically $|H|L$ because H acts trivially on $H^2(X, \mathbb{Q})$. Let $m_0 > 0$ generate the subgroup and D realise it. Then m_0

divides $|H|$, and

$$D^3 = \frac{m_0^3 L^3}{|H|}, \quad c_2(Y) \cdot D = \frac{m_0 c_2 \cdot L}{|H|},$$

$$\chi(Y, D) = \frac{D^3}{6} + \frac{c_2(Y) \cdot D}{12},$$

the last by Riemann–Roch with $c_1(Y) \equiv 0$ and $\chi(\mathcal{O}_Y) = 0$. All three numbers are integers. For the sextic, $D^3 = 3m_0^3/34$ forces $34 \mid m_0$, so $m_0 = 34$, $D^3 = 3468$, $c_2(Y) \cdot D = 42$ and $\chi(Y, D) = 578 + 7/2$. For the dctic, $D^3 = m_0^3/48$ forces $12 \mid m_0$ and $c_2(Y) \cdot D = 17m_0/24$ forces $24 \mid m_0$, so $m_0 \in \{24, 48\}$, with $\chi(Y, D) = 48 + 17/12$ and $384 + 17/6$. None is an integer, a contradiction. $\square$

Proof of Theorem 5. Smoothness and $h^{2,1}$ are Lemma 11, part (i) is Proposition 14, part (ii) is Corollary 16, part (iii) is Proposition 18, and part (iv) is Proposition 19 together with $|\chi(X)|/2 \in \{102, 148, 144\}$ for X itself. $\square$

Remark 20 (Why the quintic is different). On the quintic all n_i equal five, so a shift moves every exponent by the same amount and the wrapping count $\sum_i a_i = s$ of Lemma 13 has no solution: this is the proof of Theorem 1. When the n_i differ, a wrap at a coordinate with large n_i can be balanced by wraps at the coordinate with $n_5 = 3$ or $n_5 = 2$, and the shared classes of Proposition 14 are exactly these balances. The mass map then has blocks larger than 1×1 whose entries, all nonzero, satisfy $\Delta = 0$, and these blocks carry the massless generations. Among the four pencils, the quintic is therefore the only one on which the characters of the symmetry group label the generations and the invariant deformation gives every generation a mass.

9. THE SELECTION CONJECTURE

9.1. The landscape and its metric. We use the setting of [33]. A heterotic vacuum is a pair (X, V) of a Calabi–Yau threefold X and a slope-polystable holomorphic bundle V with $c_1(V) = 0$ for which $c_2(T_X) - c_2(V)$ is effective; the standard embedding is the case $V = T_X$. The landscape $\mathcal{L}$ is the set of isomorphism classes of vacua. It is the disjoint union of the components $\mathcal{L}_\tau$ of fixed topological type τ , and each component carries the Riemannian metric $g = G^{\text{WP}} \oplus G^V$ made of the Weil–Petersson metric of the complex structures and the L^2 metric of the bundle moduli.⁷ Mirror symmetry enters through an involution μ of a union $\mathcal{L}^\mu$ of components, which sends a vacuum to its mirror and is an isometry of g , as [33] records for the Weil–Petersson metric. A vacuum is *viable* when it satisfies the nine conditions (C1)–(C9) of the phenomenological closure theorem of [33]: anomaly cancellation and stability, three net families, a Wilson line breaking to the Standard Model group, three families without chiral exotics, masses for the vectorlike exotics, doublet–triplet splitting, stabilised moduli, an acceptable rank and hierarchy of the Yukawa masses, and consistent supersymmetry breaking.

Conjecture 21 (Geometric Selection Conjecture [33]). There is a functional $\mathcal{S} : \mathcal{L} \rightarrow [0, \infty)$, constructed from the metric g , such that

- (G1) $\mathcal{S}$ is smooth and proper on each component $\mathcal{L}_\tau$;
- (G2) $\mathcal{S}$ has a unique global minimum $\mathcal{V}^* \in \mathcal{L}$;
- (G3) $\mathcal{V}^*$ is viable;
- (G4) $\mathcal{S} \circ \mu = \mathcal{S}$ on $\mathcal{L}^\mu$.

⁷By the theorem of Bogomolov, Tian and Todorov the complex structures of X are unobstructed, and the bundle moduli of a stable bundle are unobstructed when $H^2(X, \text{End}_0 V) = 0$. Where a component is an orbifold or has singular points, everything below is applied on its smooth locus, or on orbifold charts; the only tools used are partitions of unity, geodesic balls and cut-off functions, and all three exist in that generality. As in [33] we take each $\mathcal{L}_\tau$ to be a connected manifold of finite dimension with countably many components in all.

9.2. The functional adds nothing.

Theorem 22. *Conjecture 21 holds if and only if there is a viable vacuum $\mathcal{V}^*$ with $\mu(\mathcal{V}^*) = \mathcal{V}^*$ whenever $\mathcal{V}^* \in \mathcal{L}^\mu$.*

Proof. Suppose $\mathcal{S}$ exists, and let $\mathcal{V}^*$ be its minimum. It is viable by (G3). If $\mathcal{V}^* \in \mathcal{L}^\mu$, then $\mathcal{S}(\mu\mathcal{V}^*) = \mathcal{S}(\mathcal{V}^*)$ by (G4), so $\mu\mathcal{V}^*$ is also a global minimum, and $\mu\mathcal{V}^* = \mathcal{V}^*$ by (G2).

Conversely, let $\mathcal{V}^*$ be viable and fixed by μ when μ is defined there, and let $\mathcal{L}_{\tau^*}$ be its component. The functional is built in three steps.

A proper function on every component. A second countable manifold M has a smooth proper function $\rho : M \rightarrow [0, \infty)$: take compact sets $K_1 \subset K_2 \subset \dots$ with K_j inside the interior of K_{j+1} and union M , a partition of unity (ϕ_j) with ϕ_j supported in the interior of K_{j+1} minus K_{j-1} , and $\rho = \sum_j j\phi_j$; then $\rho \geq j$ outside K_j . Choose such a ρ_τ on every component. The involution μ permutes the components of $\mathcal{L}^\mu$, so on each orbit of components we may replace ρ by $\rho + \rho \circ \mu$, which is smooth, proper (it is at least ρ) and μ -invariant. Call the result ρ .

A quadratic well at $\mathcal{V}^*$. Choose $r > 0$ smaller than the injectivity radius of g at $\mathcal{V}^*$ and such that the closed geodesic ball $\bar{B}$ of radius r about $\mathcal{V}^*$ is compact. On the open ball B the function $f = d_g(\mathcal{V}^*, \cdot)^2$ is smooth, since it is the squared norm of the inverse of the exponential map. If $\mathcal{V}^* \in \mathcal{L}^\mu$, then μ is an isometry fixing $\mathcal{V}^*$, so it preserves B and $f \circ \mu = f$.

Gluing. Let $\chi : [0, \infty) \rightarrow [0, 1]$ be smooth, equal to 1 on $[0, r^2/4]$ and to 0 on $[r^2/2, \infty)$, and extend $\chi \circ f$ by zero outside B ; the extension is smooth because $\chi \circ f$ vanishes on the part of B at distance more than $r/\sqrt{2}$ from $\mathcal{V}^*$. Put $c = r^2/4$ and

$$\mathcal{S} = (\chi \circ f)f + (1 - \chi \circ f)(c + \rho)$$

on $\mathcal{L}_{\tau^*}$, and $\mathcal{S} = c + \rho$ on every other component. Then $\mathcal{S}$ is smooth and μ -invariant, because every ingredient is. It vanishes at $\mathcal{V}^*$ and nowhere else: where $\chi \circ f = 1$ it equals f , which vanishes only at $\mathcal{V}^*$; where $0 < \chi \circ f < 1$ it is a convex combination of $f \geq r^2/4$ and $c + \rho \geq c$, both positive; and where $\chi \circ f = 0$ it equals $c + \rho \geq c$. So $\mathcal{V}^*$ is the unique global minimum. Finally a sublevel set $\{\mathcal{S} \leq t\}$ of $\mathcal{L}_{\tau^*}$ lies in $\bar{B} \cup \{\rho \leq t - c\}$, a union of two compact sets, and it is closed, so it is compact; on the other components the sublevel sets of $c + \rho$ are compact. Thus $\mathcal{S}$ satisfies (G1) to (G4). $\square$

Figure 5 draws the functional of the proof on one component.

Remark 23 (Constructed from the metric). The phrase “constructed from the metric” in Conjecture 21 is not a mathematical condition, and the strongest condition it can be read as is invariance under a group of isometries of $(\mathcal{L}, g)$. The proof gives that reading as well: if G is a compact group of isometries containing μ , averaging ρ over G with its Haar measure gives a G -invariant smooth proper function (if $\int \rho(hx) dh \leq t$ then $\rho(hx) \leq t$ for some h , so x lies in the compact set $G \cdot \{\rho \leq t\}$), and the well at a G -fixed vacuum is G -invariant. So the conjecture in that reading is equivalent to the existence of a viable vacuum fixed by G . In every reading the functional is determined by the vacuum it selects and contributes nothing else.

Remark 24 (Mirror pairs). A viable vacuum outside the standard embedding lies outside $\mathcal{L}^\mu$ when μ is the mirror map of $(2, 2)$ vacua, so the condition on μ in Theorem 22 is empty for it. If mirror pairs are identified in $\mathcal{L}$, as one point each, on the ground that they define the same superconformal field theory, the condition is empty for every vacuum and the conjecture is equivalent to the existence of one viable vacuum.

9.3. The Dwork pencil.

Proposition 25. (i) *For $\psi \neq 0$ and $\psi^5 \neq 1$, the mass map of the invariant direction h on X_ψ has rank 101, its entries in the basis of labels are 1, ψ and ψ^2 , on 51, 30 and 20 labels, and the masses of Theorem 3 are $|\psi|^{r(b)} \text{ times the Hodge factor } (g_{\text{bgsb}})^{-1/2}$.*

(ii) *Let X be a smooth quintic threefold and H a finite group acting freely on X . The standard embedding on X/H has $100/|H|$ net families. In particular condition (C2) fails on X/H for every such H , and no vacuum of the form $(X/H, T_{X/H})$ is viable.*

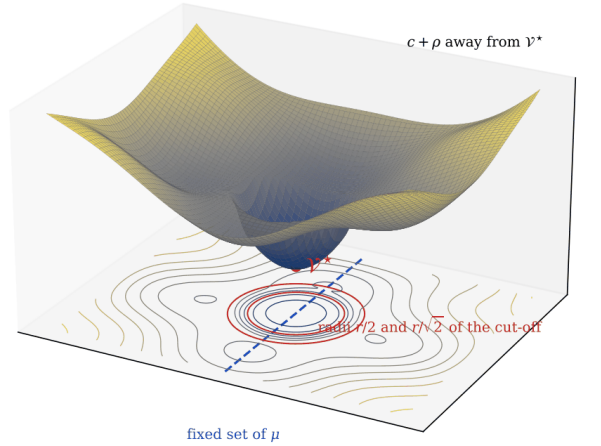


FIGURE 5. The functional $\mathcal{S}$ of the proof of Theorem 22 on a two-dimensional component $\mathcal{L}_{\tau^*}$ on which μ acts by a reflection (dashed line: its fixed set). Inside the radius $r/2$ about $\mathcal{V}^*$ the functional is the squared distance f ; outside the radius $r/\sqrt{2}$ it is $c + \rho$ with ρ proper and μ -invariant; between the two circles it is a convex combination of the two. The contour lines on the floor are the level sets of $\mathcal{S}$; all of them are compact, and the only zero is $\mathcal{V}^*$.

(iii) *On the sextic, octic and dextic pencils of Section 8, with $\psi \neq 0$ and $\psi^d \delta \neq 1$, the mass map of the invariant direction has a kernel of dimension 4, 24 and 36, and condition (C2) fails on every member and on every quotient of a member by a group acting freely.*

Proof. Part (i) is Theorem 3. For (ii), a smooth quintic has $h^{1,1} = 1$ and $h^{2,1} = 101$, so $\chi(X) = -200$. A free action multiplies the Euler characteristic of the quotient by $1/|H|$, so $\chi(X/H) = -200/|H|$, and by Theorem 6 the net number of families is $|\chi(X/H)|/2 = 100/|H|$. Wilson lines on X/H break the gauge group but do not change this index. The equation $100/|H| = 3$ has no solution in integers, so (C2) fails. Part (iii) is Propositions 18 and 19. $\square$

The proposition shows where the Dwork pencil stands with respect to the conjecture. The condition on masses is met in the strongest form one can ask of a single modulus: the mass matrix has full rank and three scales $1, |\psi|, |\psi|^2$, which separate as $\psi \rightarrow 0$. The pencil fails on the count of families, and it fails there for every free quotient, so no choice of Wilson lines rescues it. Three-family vacua satisfying (C1) to (C4) exist on other threefolds: the heterotic standard models of Braun, He, Ovrut and Pantev [20], of Bouchard and Donagi [21], and the line-bundle models of Anderson, Gray, Lukas and Palti [22, 23], which also address (C5) and (C6). For none of them are (C7), (C8) and (C9) established, and the words “controlled”, “acceptable” and “consistent” in those conditions are fixed by comparison with measurement, not by a definition. By Theorem 22 this is the whole of the open part of the Geometric Selection Conjecture. The weighted pencils fail on both counts: they too never carry three families, and their mass map leaves generations massless at every member.

9.4. Free quotients of the one-parameter families. The four pencils are four of the fourteen one-parameter families of Calabi–Yau threefolds with $h^{1,1} = 1$ whose Picard–Fuchs equations are hypergeometric, listed by Doran and Morgan [31]. Thirteen of them are complete intersections $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(w)$ of degrees $d_1, \dots, d_c$ with $\sum_j d_j = \sum_i w_i$, whose general member is quasi-smooth and misses the singular points of $\mathbb{P}(w)$; they are the rows of Table 5. The fourteenth, $X_{2,12} \subset \mathbb{P}(1, 1, 1, 1, 4, 6)$, is not of this kind: its quadric involves only the four coordinates of weight one, so it is not quasi-smooth at the point of $\mathbb{P}(4, 6)$ where the equation

TABLE 5. The thirteen one-parameter complete intersections. The columns give the type, the weights, the intersection numbers, the Euler characteristic, the values $\chi(X, kL) = k^3 L^3/6 + k c_2 \cdot L/12$ for $k = 1, 2, 3$, their greatest common divisor g_X , and the order $|\chi|/6$ that a free action giving three families would need.

type	w	L^3	$c_2 \cdot L$	χ	$\chi(kL)$	g_X	$ \chi /6$
X_5	1^5	5	50	-200	5, 15, 35	5	—
X_6	$1^4, 2$	3	42	-204	4, 11, 24	1	34
X_8	$1^4, 4$	2	44	-296	4, 10, 20	2	—
X_{10}	$1^3, 2, 5$	1	34	-288	3, 7, 13	1	48
$X_{3,3}$	1^6	9	54	-144	6, 21, 54	3	24
$X_{4,2}$	1^6	8	56	-176	6, 20, 50	2	—
$X_{3,2,2}$	1^7	12	60	-144	7, 26, 69	1	24
$X_{2,2,2,2}$	1^8	16	64	-128	8, 32, 88	8	—
$X_{4,4}$	$1^4, 2^2$	4	40	-144	4, 12, 28	4	24
$X_{6,6}$	$1^2, 2^2, 3^2$	1	22	-120	2, 5, 10	1	20
$X_{4,3}$	$1^5, 2$	6	48	-156	5, 16, 39	1	26
$X_{6,4}$	$1^3, 2^2, 3$	2	32	-156	3, 8, 17	1	26
$X_{6,2}$	$1^5, 3$	4	52	-256	5, 14, 31	1	—

of degree twelve vanishes, and we leave it aside. For the thirteen, the count of families of Proposition 25 becomes a statement about every free quotient, through one arithmetic invariant.

Lemma 26 (Primes of a free action). *Let X be a compact Kähler threefold with $K_X \cong \mathcal{O}_X$, $h^{1,0} = h^{2,0} = 0$, $b_2 = 1$ and $\pi_1(X) = 0$, let L be an ample line bundle on X , and let g be an automorphism of prime order p without fixed points. Then p divides $\chi(X, kL)$ for every $k \geq 1$.*

Proof. Since X is simply connected, $H^2(X, \mathbb{Z})$ has no torsion, and since $h^{0,1} = h^{0,2} = 0$ the first Chern class identifies $\text{Pic } X$ with $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. Let A be its ample generator, so that $L = A^a$ with $a \geq 1$. The bundle g^*A is again an ample generator, so $g^*A \cong A$. The obstruction to lifting the action of the cyclic group $G = \langle g \rangle$ to A lies in $H^2(G, \mathbb{C}^*)$, which vanishes for a cyclic group, so A and all its powers are G -linearised and G acts on $H^q(X, A^m)$. For $m \geq 1$ these groups vanish in positive degree by Kodaira's theorem, as $A^m = K_X \otimes A^m$ with A^m ample. For $1 \leq j < p$ the automorphism g^j has no fixed points, and the holomorphic Lefschetz formula of Atiyah and Bott [29] gives $\sum_q (-1)^q \text{tr}(g^j | H^q(X, A^m)) = 0$, that is, $\text{tr}(g^j | H^0(X, A^m)) = 0$. A representation of $\mathbb{Z}/p$ whose character vanishes off the identity is a multiple of the regular representation, so p divides $h^0(X, A^m) = \chi(X, A^m)$. Taking $m = ka$ gives the statement. $\square$

Proposition 27. *Let $X \subset \mathbb{P}(w)$ be a quasi-smooth complete intersection of one of the thirteen types of Table 5 that misses the singular points of $\mathbb{P}(w)$, and let H be a finite group acting freely on X . Then every prime divisor of $|H|$ divides g_X . Hence H is trivial for $X_6, X_{10}, X_{3,2,2}, X_{6,6}, X_{4,3}, X_{6,4}$ and $X_{6,2}$; it is a 5-group for X_5 , a 3-group for $X_{3,3}$, and a 2-group for $X_8, X_{4,2}, X_{2,2,2,2}$ and $X_{4,4}$. With the standard embedding, no free quotient of X carries three net families.*

Proof. As in Lemma 11, X is a smooth threefold with $K_X \cong \mathcal{O}_X(\sum_j d_j - \sum_i w_i) = \mathcal{O}_X$, and the Lefschetz theorem for quasi-smooth complete intersections [28] gives $h^{1,0} = h^{2,0} = 0$ and $b_2 = 1$. For π_1 , let $C \subset \mathbb{C}^N$ be the affine cone over X , the zero set of the c weighted homogeneous equations. Quasi-smoothness says that C is a complete intersection of dimension four with an isolated singular point at the origin, so by Hamm's theorem [30] its link is 2-connected; the weighted action of $\mathbb{R}_{>0}$ meets every sphere about the origin once, so $C \setminus \{0\}$ is homotopy equivalent to the link and is simply connected. As X misses the points with nontrivial isotropy, $\mathbb{C}^*$ acts freely on $C \setminus \{0\}$ with quotient X , and the exact sequence of homotopy groups of this fibration gives $\pi_1(X) = 0$.

Let p be a prime divisor of $|H|$ and $g \in H$ an element of order p . It has no fixed points, so by Lemma 26 with $L = \mathcal{O}_X(1)$, p divides $\chi(X, kL)$ for all $k \geq 1$. By Riemann–Roch, with $c_1(X) = 0$ and $\chi(\mathcal{O}_X) = 0$, $\chi(X, kL) = k^3 L^3/6 + k c_2 \cdot L/12$, a polynomial that takes integer values and vanishes at $k = 0$; writing it as $\alpha_1 \binom{k}{1} + \alpha_2 \binom{k}{2} + \alpha_3 \binom{k}{3}$ with integers α_i , its values at all $k \geq 1$ have the same greatest common divisor as $\alpha_1, \alpha_2, \alpha_3$, which is that of its values at $k = 1, 2, 3$, the number g_X . The intersection numbers are computed as in Lemma 11, from $c(T_X) = \prod_i (1 + w_i L) \prod_j (1 + d_j L)^{-1}$ and $L^3 = \prod_j d_j / \prod_i w_i$, and give the table.

By Theorem 6 the quotient X/H carries $|\chi(X)|/(2|H|)$ net families, so three families need $|H| = |\chi(X)|/6$. This is not an integer for $X_5, X_8, X_{4,2}, X_{2,2,2,2}$ and $X_{6,2}$. In the remaining eight rows it is 34, 48, 24, 24, 24, 20, 26 or 26, and each of these has a prime factor not dividing g_X : 17, 2, 2, 2, 3, 2 and 2 (twice). $\square$

The proposition contains Theorem 5(iv) and Proposition 25(ii) and strengthens them: the sextic and dextic pencils admit no free action at all, and on a quintic every free action is by a 5-group, so that only the orders 5 and 25 divide $|\chi| = 200$ and a free quotient of a quintic has 20 or 4 families; the quotient by $\mathbb{Z}/5 \times \mathbb{Z}/5$ with four families is already in [1]. For the five complete intersections in ordinary projective space the free actions were classified by Braun [32] with a computer search; the proposition uses only the holomorphic Lefschetz formula and Riemann–Roch.⁸

10. CONCLUSION

On the Dwork pencil the question of the title has a complete answer. The hundred and one generations are labelled by characters of the symmetry group Γ , each character occurring once; the couplings of three generations are the values of one functional on the socle of the Jacobian ring, equal to a power of the modulus when the symmetry allows them and to zero otherwise; the invariant deformation pairs the generations by an involution with a single fixed point, so its mass matrix is monomial with determinant ψ^{70} ; and at the Fermat point the masses are explicit values of the Gamma function. The proofs use two facts only, the reduction (6) and the grading of the Jacobian ring by characters, together with the general dictionary of Section 2.

Two questions are not settled by the geometry of the pencil. The breaking of E_6 to the Standard Model needs a free quotient and Wilson lines, and on a quintic the number of families of a free quotient is $100/|H|$, never three (Proposition 25), and the same holds for every free quotient of the thirteen one-parameter complete intersections (Proposition 27). The selection of the vacuum is reduced by Theorem 22 to the existence of one viable vacuum, and that existence depends on conditions (C7) to (C9), whose thresholds are set by measurement. The same two facts drive Section 8, where the three other one-parameter families of Fermat type are treated. There the characters no longer separate the generations, the invariant deformation leaves 4, 24 and 36 of them massless at every member, and no member or free quotient carries three families. Among the four pencils of Fermat type, the quintic is the only one on which the symmetry labels every generation and the invariant deformation gives every generation a mass.

DATA AND CODE AVAILABILITY

The programs that repeat the finite computations of this paper are in its archive, release 1.2.0, <https://github.com/creelie/strings/releases/tag/v1.2.0> [38], which also holds the figures and the record of the engine runs. They are written in C (the label and coupling counts), C++ (the expansion of the Hessian by permutations), Python (all counts, the intersection numbers of Appendix C, the Gamma values, and the labels, characters, couplings and mass maps of the weighted pencils, and the intersection numbers, values $\chi(X, kL)$ and invariants g_X of Table 5), Julia (the Gamma values and the intersection numbers), Macaulay2 (the Jacobian ring of the quintic at two rational values of ψ , with the socle functional on all 3876 monomials of degree fifteen and the Hessian, and the Jacobian rings of the three weighted pencils at two rational values of ψ , with the socle functional on every monomial of degree $3d$) and Lean 4 (the counts, the

⁸On a quintic the group $\mathbb{Z}/5 \times \mathbb{Z}/5$ of [1], generated by the cyclic permutation of the coordinates and the diagonal phases $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$, acts on the five coordinates through a central extension of order 125, the Heisenberg group, and $H^0(X, \mathcal{O}_X(1)) = \mathbb{C}^5$ is its irreducible representation of dimension five; each cyclic subgroup of order five acts on it by the regular representation, as Lemma 26 requires.

arithmetic of the Hessian and the numbers of the weighted pencils, checked by the kernel), with one shell script that runs all 215 checks. No proof in the paper depends on them. The constructions were organised on the Anaxia engine [36], described in [37]; its record for the five targets of this paper is part of the archive. The programs and the engine runs were prepared by Deep Bhattacharjee with AI assistance from Claude (Anthropic).

ETHICAL DECLARATIONS

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APPENDIX A. THE HESSIAN IN THE SOCLE

The Hessian matrix of F_ψ has entries

$$H_{ii} = 20x_i^3, \quad H_{ij} = -5\psi \prod_{k \neq i,j} x_k \quad (i \neq j).$$

For a permutation π of $\{1, \dots, 5\}$ with m moved points and $f = 5 - m$ fixed points, the product $\prod_i H_{i\pi(i)}$ is $20^f (-5\psi)^m$ times a monomial whose exponents are read off as follows. A fixed coordinate a receives 3 from its diagonal entry and 1 from each of the m off-diagonal entries, since an off-diagonal entry $(i, \pi(i))$ omits only the moved coordinates i and $\pi(i)$. A moved coordinate c is omitted by exactly two off-diagonal entries, those with $i = c$ and with $\pi(i) = c$, which are different because $\pi(c) \neq c$; it receives 1 from each of the other $m - 2$. Hence the monomial of π is

$$\prod_{a \text{ fixed}} x_a^{3+m} \prod_{c \text{ moved}} x_c^{m-2},$$

of degree $f(3+m) + m(m-2) = 15$, and it depends only on the number of moved points. The five values of m give:

- $m = 0$: the monomial $m_0 = (3, 3, 3, 3, 3)$, sector 0;
- $m = 2$: $(5, 5, 5, 0, 0)$ up to order, sector 3, so $\lambda = \psi^3$;
- $m = 3$: $(6, 6, 1, 1, 1)$ up to order, sector 2, so $\lambda = \psi^2$;
- $m = 4$: $(7, 2, 2, 2, 2)$ up to order, sector 1, so $\lambda = \psi$;
- $m = 5$: $(3, 3, 3, 3, 3)$, sector 0.

The contribution of a permutation to $H(\psi) = \lambda(\text{Hess } F_\psi)$ is $\text{sgn}(\pi) 20^f (-5\psi)^m \psi^{r(m)}$ with $r(0) = r(5) = 0$, $r(2) = 3$, $r(3) = 2$, $r(4) = 1$, so $m + r(m) = 5$ for every $m \geq 2$ and every non-identity permutation contributes a multiple of ψ^5 . The cycle types are

$$\begin{aligned} 1^5 &: 1 \text{ permutation, } m = 0, +20^5 = 3\,200\,000; \\ 2\,1^3 &: 10, \text{ odd, } m = 2: 10 \cdot (-1) \cdot 20^3 \cdot 25 = -2\,000\,000; \\ 3\,1^2 &: 20, \text{ even, } m = 3: 20 \cdot 20^2 \cdot (-125) = -1\,000\,000; \\ 4\,1 &: 30, \text{ odd, } m = 4: 30 \cdot (-1) \cdot 20 \cdot 625 = -375\,000; \\ 2^2\,1 &: 15, \text{ even, } m = 4: 15 \cdot 20 \cdot 625 = 187\,500; \\ 5 &: 24, \text{ even, } m = 5: 24 \cdot (-3125) = -75\,000; \\ 3\,2 &: 20, \text{ odd, } m = 5: 20 \cdot (-1) \cdot (-3125) = 62\,500, \end{aligned}$$

where the last six numbers are the coefficients of ψ^5 . The counts $1 + 10 + 20 + 30 + 15 + 24 + 20$ add up to 120. The coefficients of ψ^5 add up to $-3\,200\,000$, which gives $H(\psi) = 3\,200\,000(1 - \psi^5)$.

APPENDIX B. THE COUPLING COUNT

B.1. Ordered triples by sector. For $\mathbf{e}$ of degree fifteen, $O(\mathbf{e})$ is the number of ordered triples $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in B^3$ with $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{e}$. We count it by the third label: for each $\mathbf{c} \in B$ with $\mathbf{c} \leq \mathbf{e}$, put $\mathbf{s} = \mathbf{e} - \mathbf{c}$, a vector of degree ten, and count the $\mathbf{a} \in B$ with $\mathbf{s} - \mathbf{a} \in B$, that is, with

$$\max(0, s_i - 3) \leq a_i \leq \min(3, s_i) \quad \text{for all } i, \quad \sum a_i = 5.$$

This number $N(\mathbf{s})$ is the coefficient of t^5 in $\prod_i (t^{\max(0, s_i - 3)} + \dots + t^{\min(3, s_i)})$. Write $[n] = 1 + t + \dots + t^{n-1}$.

Sector 0, $\mathbf{e} = (3, 3, 3, 3, 3)$. Here $\mathbf{s} = 3\mathbf{1} - \mathbf{c}$ has entries in $\{0, \dots, 3\}$, so $N(\mathbf{s})$ is the coefficient of t^5 in $\prod_i [s_i + 1]$. The multiset of entries of $\mathbf{s}$ is the complement of the type of $\mathbf{c}$:

type of $\mathbf{c}$	entries of $\mathbf{s}$	# $\mathbf{c}$	$N(\mathbf{s})$
$\{3, 2, 0, 0, 0\}$	$\{3, 3, 3, 1, 0\}$	20	24
$\{3, 1, 1, 0, 0\}$	$\{3, 3, 2, 2, 0\}$	30	28
$\{2, 2, 1, 0, 0\}$	$\{3, 3, 2, 1, 1\}$	30	38
$\{2, 1, 1, 1, 0\}$	$\{3, 2, 2, 2, 1\}$	20	44
$\{1, 1, 1, 1, 1\}$	$\{2, 2, 2, 2, 2\}$	1	51

The values of N are central coefficients. For $\{3, 3, 3, 1, 0\}$, $[4]^3 = 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 12t^5 + \dots$ and the factor $[2]$ gives $12 + 12 = 24$. For $\{3, 3, 2, 2, 0\}$, $[4]^2 = (1, 2, 3, 4, 3, 2, 1)$ and $[3]^2 = (1, 2, 3, 2, 1)$, so the coefficient is $2 + 6 + 12 + 6 + 2 = 28$. For $\{3, 3, 2, 1, 1\}$, $[3][2]^2 = (1, 3, 4, 3, 1)$ and the coefficient is $2 + 9 + 16 + 9 + 2 = 38$. For $\{3, 2, 2, 2, 1\}$, $[3]^3 = (1, 3, 6, 7, 6, 3, 1)$ and $[4][2] = (1, 2, 2, 2, 1)$, so the coefficient is $3 + 12 + 14 + 12 + 3 = 44$. For $\{2, 2, 2, 2, 2\}$ the coefficient of t^5 in $(1 + t + t^2)^5$ is 51. Hence

$$O(3, 3, 3, 3, 3) = 20 \cdot 24 + 30 \cdot 28 + 30 \cdot 38 + 20 \cdot 44 + 51 = 3391.$$

(The table for $\mathbf{e} = (3, 3, 3, 3, 3)$ also shows that every $\mathbf{c} \in B$ occurs, since $\mathbf{s} = 3\mathbf{1} - \mathbf{c}$ always admits some $\mathbf{a}$.)

Sector 1, $\mathbf{e} = (7, 2, 2, 2, 2)$. Since $a_1 + b_1 \leq 6$, $c_1 \geq 1$, and $c_i \leq 2$ for $i \geq 2$. With $\mathbf{s} = \mathbf{e} - \mathbf{c}$, the entry $s_1 = 7 - c_1$ forces $a_1 \geq 4 - c_1$, and the remaining entries $s_i = 2 - c_i \leq 2$ impose only $a_i \leq s_i$. Grouping $\mathbf{c}$ by c_1 and by the multiset of $(c_2, \dots, c_5)$:

c_1	$(c_2, \dots, c_5)$	# $\mathbf{c}$	$N(\mathbf{s})$	total
1	$\{1, 1, 1, 1\}$	1	6	6
1	$\{2, 1, 1, 0\}$	12	4	48
1	$\{2, 2, 0, 0\}$	6	3	18
2	$\{1, 1, 1, 0\}$	4	14	56
2	$\{2, 1, 0, 0\}$	12	10	120
3	$\{1, 1, 0, 0\}$	6	26	156
3	$\{2, 0, 0, 0\}$	4	19	76

For instance, when $c_1 = 3$ and $(c_2, \dots, c_5) = (2, 0, 0, 0)$, a_1 ranges over $\{1, 2, 3\}$ and the other entries satisfy $a_i \leq (0, 2, 2, 2)$ with sum $5 - a_1$; the coefficients of t^4, t^3, t^2 in $[3]^3$ are 6, 7, 6, which add to 19. The total is $O(7, 2, 2, 2, 2) = 480$.

Sector 2, $\mathbf{e} = (6, 6, 1, 1, 1)$. Here $c_3, c_4, c_5 \leq 1$, and the same grouping gives

$\{c_1, c_2\}$	(c_3, c_4, c_5)	# $\mathbf{c}$	$N(\mathbf{s})$	total
$\{1, 1\}$	$\{1, 1, 1\}$	1	2	2
$\{2, 0\}$	$\{1, 1, 1\}$	2	1	2
$\{2, 1\}$	$\{1, 1, 0\}$	6	4	24
$\{2, 2\}$	$\{1, 0, 0\}$	3	10	30
$\{3, 0\}$	$\{1, 1, 0\}$	6	2	12
$\{3, 1\}$	$\{1, 0, 0\}$	6	8	48
$\{3, 2\}$	$\{0, 0, 0\}$	2	22	44

and $O(6, 6, 1, 1, 1) = 162$.

Sector 3, $\mathbf{e} = (5, 5, 5, 0, 0)$. Here $c_4 = c_5 = 0$ and $c_1 + c_2 + c_3 = 5$. For $\{c_1, c_2, c_3\} = \{2, 2, 1\}$ the entries of $\mathbf{s}$ are $(3, 3, 4)$, the bounds are $a_i \leq 3$ on the first two and $1 \leq a_3 \leq 3$, and the coefficient of t^5 in $[4]^2(t + t^2 + t^3)$ is $3 + 4 + 3 = 10$; for $\{3, 1, 1\}$ the entries are $(2, 4, 4)$ and the coefficient of t^5 in $[3](t + t^2 + t^3)^2$ is $2 + 3 + 2 = 7$; for $\{3, 2, 0\}$ the entries are $(2, 3, 5)$ and the coefficient of t^5 in $[3][4](t^2 + t^3)$ is $3 + 3 = 6$. With three, three and six choices of $\mathbf{c}$, $O(5, 5, 5, 0, 0) = 30 + 21 + 36 = 87$.

The ordered counts by sector are $O_0 = 3391$, $O_1 = 5 \cdot 480 = 2400$, $O_2 = 10 \cdot 162 = 1620$ and $O_3 = 10 \cdot 87 = 870$, with total 8281 in agreement with Section 5.1.

B.2. Repeated labels. For $\mathbf{a} \in B$ the partner $\mathbf{c}$ with $2\mathbf{a} + \mathbf{c}$ in a sector is found type by type, as in the proof of Theorem 3. For $\mathbf{a} = \mathbf{1}$, $\mathbf{c} = \mathbf{1}$ in sector 0. For $\mathbf{a}$ of type $\{2, 1, 1, 1, 0\}$, sector 1 gives $c_i \equiv 2 - 2a_i$, that is $\mathbf{c}$ with a three where $\mathbf{a}$ has its two, a two where $\mathbf{a}$ has its zero and zeros elsewhere, of sum five. For type $\{3, 1, 1, 0, 0\}$, sector 1 gives a one where $\mathbf{a}$ has its three, zeros where it has ones and twos where it has zeros. For type $\{3, 2, 0, 0, 0\}$, sector 2 gives zero, two and ones. For type $\{2, 2, 1, 0, 0\}$, sector 3 gives ones at the twos, a three at the one and zeros elsewhere. In each case the other sectors produce a residue four or a total different from five. So the hundred triples $\{\mathbf{a}, \mathbf{a}, \mathbf{c}\}$ with $\mathbf{c} \neq \mathbf{a}$ lie 20 + 30 = 50 in sector 1, 20 in sector 2 and 30 in sector 3, and none in sector 0, which is the list $D_r = 0, 50, 20, 30$ used in Section 5.1.

APPENDIX C. HODGE NORMS AT THE FERMAT POINT

This appendix proves Proposition 10. Write $G = \mu_5^5 / \mu_5$ for the group of diagonal phases of $\mathbb{P}^4$, which acts on X_0 . For $\mathbf{k} \in \{1, 2, 3, 4\}^5$ with $\sum k_i \equiv 0 \pmod{5}$ let $\chi_{\mathbf{k}}(a) = \zeta^{a \cdot \mathbf{k}}$.

C.1. Eigenlines. The character $\chi_{\mathbf{k}}$ occurs in $H^3(X_0, \mathbb{C})$ exactly once, and these are all the characters that occur: the number of such $\mathbf{k}$ is $\frac{1}{5} \sum_{j=0}^4 \left(\sum_{k=1}^4 \zeta^{jk} \right)^5 = \frac{1}{5} (4^5 - 4) = 204 = b_3(X_0)$, and each occurs at least once because the residue $\text{Res}(x^{\mathbf{k}-1} \Omega_0 / F^{q+1})$, $5(q+1) = \sum k_i$, is a nonzero class of character $\chi_{\mathbf{k}}$ (Griffiths' theorem and the fact that $x^{\mathbf{k}-1}$, with exponents at most three, is nonzero in the Jacobian ring $\mathbb{C}[x]/(x_1^4, \dots, x_5^4)$ of the Fermat quintic). In particular $\omega_{\mathbf{b}}$ spans the line of $\mathbf{k} = \mathbf{b} + \mathbf{1}$.

C.2. The bilinear relation. Let Q be the intersection form on $H_3(X_0, \mathbb{Z})$, extended to complex coefficients, and identify H^3 with H_3 by Poincaré duality D , so that $\int_c \omega = Q(c, D\omega)$ and $\int_{X_0} \omega \wedge \eta = Q(D\omega, D\eta)$. Since Q is G -invariant, it pairs the eigenline L of χ only with the eigenline L' of $\bar{\chi}$. Let e be an integral cycle and u its component in L' ; its component in L is $\bar{u}$, because e is real. If $D\omega_{\mathbf{b}} = \alpha \bar{u}$ with $\bar{u} \neq 0$, then

$$\int_e \omega_{\mathbf{b}} = Q(e, \alpha \bar{u}) = \alpha Q(u, \bar{u}), \quad \int_{X_0} \omega_{\mathbf{b}} \wedge \overline{\omega_{\mathbf{b}}} = |\alpha|^2 Q(\bar{u}, u),$$

and therefore

$$(11) \quad g_{\mathbf{b}} = \left| \int_{X_0} \omega_{\mathbf{b}} \wedge \overline{\omega_{\mathbf{b}}} \right| = \frac{|\int_e \omega_{\mathbf{b}}|^2}{|Q(u, \bar{u})|}.$$

It remains to compute both sides for one cycle.

C.3. The cycle of Pham. On the affine chart $x_5 = 1$ put $y_i = x_i / x_5$; the open part of X_0 is $U = \{f = 0\}$, $f = y_1^5 + y_2^5 + y_3^5 + y_4^5 + 1$, and μ_5^4 acts by $y_i \mapsto \zeta^{a_i} y_i$. A character $\mathbf{k}$ of G restricts to the character (k_1, k_2, k_3, k_4) of μ_5^4 , and $k_5 \equiv -(k_1 + k_2 + k_3 + k_4)$. Pham [17] showed that U retracts onto the join of four copies of the set of fifth roots of -1 , and that $H_3(U, \mathbb{Z})$ is generated over $\mathbb{Z}[\mu_5^4]$ by the cycle

$$e = (1 - T_1)(1 - T_2)(1 - T_3)(1 - T_4) \Delta,$$

where Δ is the real simplex $\{y_i = \varepsilon v_i, v_i \geq 0, \sum v_i^5 = 1\}$ with $\varepsilon^5 = -1$, and T_i multiplies y_i by ζ .

The period. By Griffiths' residue formula, $\int_e \omega_{\mathbf{b}} = \frac{1}{2\pi i} \int_{\tau(e)} x^{\mathbf{b}} \Omega_0 / F^2$ with τ the tube map. On the chart this is the integral of $y^{\mathbf{b}'} dy / f^2$, $\mathbf{b}' = (b_1, \dots, b_4)$. Put $f_s = \sum y_i^5 + s$; then $y^{\mathbf{b}'} dy / f_s^2 = -\partial_s(y^{\mathbf{b}'} dy / f_s)$, and

$\frac{1}{2\pi i} \int_{\tau(e)} y^{\mathbf{b}'} dy / f_s = \int_{e_s} y^{\mathbf{b}'} dy / df_s$ with the Gelfand–Leray form. The substitution $y = s^{1/5} u$ shows that the last integral is $s^A P$ with $A = \frac{1}{5} \sum_{i \leq 4} k_i - 1$ and P its value at $s = 1$. Differentiating at $s = 1$,

$$\int_e \omega_{\mathbf{b}} = -AP.$$

On Δ , with $t_i = v_i^5$, the Gelfand–Leray form is $5^{-4} \prod_i t_i^{1/5-1} dt$ on the simplex $\sum t_i = 1$, and $y^{\mathbf{b}'} = \varepsilon^{|\mathbf{b}'|} \prod t_i^{b_i/5}$; Dirichlet’s integral gives $\int_{\Delta} = \varepsilon^{|\mathbf{b}'|+4} 5^{-4} \prod_{i \leq 4} \Gamma(k_i/5) / \Gamma(\sum_{i \leq 4} k_i/5)$, and each factor $1 - T_i$ multiplies it by $1 - \zeta^{k_i}$. Since $\sum_{i \leq 4} k_i = 10 - k_5$, $A = 1 - k_5/5$ and $\Gamma(A+1) = A\Gamma(A)$, so

$$(12) \quad \left| \int_e \omega_{\mathbf{b}} \right| = 5^{-4} \prod_{i=1}^4 |1 - \zeta^{k_i}| \frac{\prod_{i=1}^4 \Gamma(k_i/5)}{\Gamma(1 - k_5/5)}.$$

The intersection number. For one variable, the reduced homology of the fibre $\{y^5 = -1\}$ is the augmentation ideal I of $\mathbb{Z}[t]/(t^5 - 1)$, the monodromy is multiplication by t , and the variation operator is multiplication by $t - 1$. The Seifert form $L(x, z) = \langle x, \text{Var}^{-1} z \rangle$ is therefore, in terms of the values $\hat{x}(k) = x(\zeta^k)$,

$$L(x, z) = \frac{1}{5} \sum_{k=1}^4 \frac{\hat{x}(k) \hat{z}(-k)}{\zeta^{-k} - 1}.$$

By the theorem of Sebastiani and Thom [18], in the form of Sakamoto [19] for Seifert forms, the Seifert form of $y_1^5 + \dots + y_4^5$ is, up to sign, the tensor product of four copies, and the intersection form on the three-dimensional fibre is $Q = -L + L^T$. On the eigenline of $(k_1, \dots, k_4)$ this gives

$$Q(x, z) = \pm \frac{1}{5^4} \hat{x}(\mathbf{k}) \hat{z}(-\mathbf{k}) \left(\prod_{i \leq 4} \frac{1}{\zeta^{k_i} - 1} - \prod_{i \leq 4} \frac{1}{\zeta^{-k_i} - 1} \right).$$

With four factors, $\prod(\zeta^{k_i} - 1) = \zeta^{\sum k_i} \prod(\zeta^{-k_i} - 1)$, so the bracket has absolute value

$$\frac{|1 - \zeta^{k_1+k_2+k_3+k_4}|}{\prod_{i \leq 4} |1 - \zeta^{k_i}|^2} = \frac{|1 - \zeta^{k_5}|}{\prod_{i \leq 4} |1 - \zeta^{k_i}|^2}.$$

The cycle e has $\hat{e}(\mathbf{k}) = \prod_{i \leq 4} (1 - \zeta^{k_i})$ up to a unimodular factor, so its component u satisfies

$$(13) \quad |Q(u, \bar{u})| = 5^{-4} \prod_{i=1}^5 |1 - \zeta^{k_i}|.$$

This is nonzero, so $\bar{u} \neq 0$, as needed in (11).

C.4. Conclusion of the proof. Inserting (12) and (13) in (11),

$$g_{\mathbf{b}} = 5^{-4} \prod_{i=1}^4 |1 - \zeta^{k_i}| \Gamma(k_i/5)^2 \cdot \frac{1}{|1 - \zeta^{k_5}| \Gamma(1 - k_5/5)^2}.$$

With $|1 - \zeta^k| = 2 \sin(\pi k/5)$ and $\Gamma(t) \Gamma(1-t) = \pi / \sin \pi t$,

$$2 \sin(\pi t) \Gamma(t)^2 = 2\pi \gamma(t), \quad \frac{1}{2 \sin(\pi t) \Gamma(1-t)^2} = \frac{\gamma(t)}{2\pi},$$

so $g_{\mathbf{b}} = 5^{-4} (2\pi)^3 \prod_{i=1}^5 \gamma(k_i/5)$, which is (7) with $C = (2\pi)^3/5^4$. The class of Ω , with $\mathbf{k} = \mathbf{1}$, has no derivative in the period computation, so $|\int_e \Omega| = |P|$ with $\Gamma(\sum_{i \leq 4} k_i/5) = \Gamma(4/5) = \Gamma(1 - k_5/5)$, and the same formula holds: $\|\Omega\|^2 = C \gamma(1/5)^5$. Since $\partial_{\psi} \text{Res}(\Omega_0/F_{\psi}) = \text{Res}(5x_1 \dots x_5 \Omega_0/F_{\psi}^2) = 5\omega_1$ and ω_1 is of type $(2, 1)$ at $\psi = 0$ (its character differs from that of Ω), the Weil–Petersson metric at the Fermat point is

$$\begin{aligned} G_{\psi \bar{\psi}}(0) &= \frac{25 g_1}{\|\Omega\|^2} = 25 \frac{\gamma(2/5)^5}{\gamma(1/5)^5} \\ &= 25 \frac{\Gamma(2/5)^5 \Gamma(4/5)^5}{\Gamma(1/5)^5 \Gamma(3/5)^5} = 0.19226. \end{aligned}$$

TABLE 6. The entries of the blocks of Table 4, for the representative with $(i, j, k, l) = (1, 2, 3, 4)$. Exponent vectors are written in full; the column r is the sector of $h + \mathbf{b} + \mathbf{c}$ and k is the vector of (10).

pencil	row	$\mathbf{b}$	$\mathbf{c}$	$h + \mathbf{b} + \mathbf{c}$	r	k	entry
sextic	1	(4, 0, 0, 0, 1)	(3, 1, 1, 1, 0)	(8, 2, 2, 2, 2)	2	(1, 0, 0, 0, 1)	$2\psi^2$
	1	(0, 2, 2, 2, 0)	(3, 1, 1, 1, 0)	(4, 4, 4, 4, 1)	0	(0, 0, 0, 0, 0)	1
	2	(3, 3, 0, 0, 0)	(3, 3, 0, 0, 0)	(7, 7, 1, 1, 1)	3	(1, 1, 0, 0, 1)	$2\psi^3$
	2	(3, 3, 0, 0, 0)	(0, 0, 3, 3, 0)	(4, 4, 4, 4, 1)	0	(0, 0, 0, 0, 0)	1
	2	(0, 0, 3, 3, 0)	(3, 3, 0, 0, 0)	(4, 4, 4, 4, 1)	0	(0, 0, 0, 0, 0)	1
	2	(0, 0, 3, 3, 0)	(0, 0, 3, 3, 0)	(1, 1, 7, 7, 1)	3	(0, 0, 1, 1, 1)	$2\psi^3$
octic	1	(6, 1, 1, 0, 0)	(4, 1, 1, 2, 0)	(12, 4, 4, 4, 0)	2	(1, 0, 0, 0, 1)	$4\psi^2$
	1	(0, 3, 3, 2, 0)	(4, 1, 1, 2, 0)	(6, 6, 6, 6, 0)	0	(0, 0, 0, 0, 0)	1
	2	(6, 2, 0, 0, 0)	(2, 6, 0, 0, 0)	(10, 10, 2, 2, 0)	4	(1, 1, 0, 0, 2)	$16\psi^4$
	2	(6, 2, 0, 0, 0)	(4, 0, 2, 2, 0)	(12, 4, 4, 4, 0)	2	(1, 0, 0, 0, 1)	$4\psi^2$
	2	(0, 4, 2, 2, 0)	(2, 6, 0, 0, 0)	(4, 12, 4, 4, 0)	2	(0, 1, 0, 0, 1)	$4\psi^2$
	2	(0, 4, 2, 2, 0)	(4, 0, 2, 2, 0)	(6, 6, 6, 6, 0)	0	(0, 0, 0, 0, 0)	1
	3	(4, 4, 0, 0, 0)	(4, 4, 0, 0, 0)	(10, 10, 2, 2, 0)	4	(1, 1, 0, 0, 2)	$16\psi^4$
	3	(4, 4, 0, 0, 0)	(0, 0, 4, 4, 0)	(6, 6, 6, 6, 0)	0	(0, 0, 0, 0, 0)	1
	3	(0, 0, 4, 4, 0)	(4, 4, 0, 0, 0)	(6, 6, 6, 6, 0)	0	(0, 0, 0, 0, 0)	1
	3	(0, 0, 4, 4, 0)	(0, 0, 4, 4, 0)	(2, 2, 10, 10, 0)	4	(0, 0, 1, 1, 2)	$16\psi^4$
dectic	1	(2, 1, 1, 3, 0)	(2, 3, 3, 1, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	1	(4, 3, 3, 0, 0)	(2, 3, 3, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	2	(3, 1, 0, 3, 0)	(1, 3, 4, 1, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	2	(5, 3, 2, 0, 0)	(1, 3, 4, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	3	(8, 1, 1, 0, 0)	(4, 1, 1, 2, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	3	(0, 3, 3, 2, 0)	(4, 1, 1, 2, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	4	(7, 1, 0, 1, 0)	(5, 1, 2, 1, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	4	(1, 5, 4, 0, 0)	(5, 1, 2, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	5	(8, 0, 0, 1, 0)	(4, 2, 2, 1, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	5	(0, 2, 2, 3, 0)	(4, 2, 2, 1, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	5	(2, 4, 4, 0, 0)	(4, 2, 2, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	6	(4, 0, 0, 3, 0)	(6, 0, 0, 2, 0)	(12, 2, 2, 7, 0)	6	(1, 0, 0, 2, 3)	$500\psi^6$
	6	(4, 0, 0, 3, 0)	(0, 4, 4, 1, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	6	(6, 2, 2, 0, 0)	(6, 0, 0, 2, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	6	(6, 2, 2, 0, 0)	(0, 4, 4, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	7	(8, 2, 0, 0, 0)	(2, 8, 0, 0, 0)	(12, 12, 2, 2, 0)	6	(1, 1, 0, 1, 3)	$250\psi^6$
	7	(8, 2, 0, 0, 0)	(4, 0, 2, 2, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	7	(0, 4, 2, 2, 0)	(2, 8, 0, 0, 0)	(4, 14, 4, 4, 0)	4	(0, 1, 0, 1, 2)	$50\psi^4$
	7	(0, 4, 2, 2, 0)	(4, 0, 2, 2, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	8	(6, 2, 0, 1, 0)	(6, 0, 2, 1, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	8	(6, 2, 0, 1, 0)	(0, 4, 6, 0, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	8	(0, 6, 4, 0, 0)	(6, 0, 2, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	8	(0, 6, 4, 0, 0)	(0, 4, 6, 0, 0)	(2, 12, 12, 2, 0)	6	(0, 1, 1, 1, 3)	$250\psi^6$
	9	(6, 1, 1, 1, 0)	(6, 1, 1, 1, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	9	(6, 1, 1, 1, 0)	(0, 5, 5, 0, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	9	(0, 5, 5, 0, 0)	(6, 1, 1, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	9	(0, 5, 5, 0, 0)	(0, 5, 5, 0, 0)	(2, 12, 12, 2, 0)	6	(0, 1, 1, 1, 3)	$250\psi^6$

C.5. Numerical values. With $\Gamma(1/5) = 4.59084$, $\Gamma(2/5) = 2.21816$, $\Gamma(3/5) = 1.48919$ and $\Gamma(4/5) = 1.16423$, $\gamma(1/5) = 3.94325$ and $\gamma(2/5) = 1.48951$; $\gamma(3/5)$ and $\gamma(4/5)$ are their inverses. The norms of the five types follow by multiplying the five factors, and the masses of Table 2 are their inverses.

APPENDIX D. SHARED CHARACTERS ON THE OCTIC AND THE DECTIC

This appendix completes the proof of Proposition 14. By Lemma 13 it suffices to find, for each shift $s \leq d/2$, the labels $\mathbf{b}$ with $b_i + s \not\equiv n_i - 1 \pmod{n_i}$ for every i and $\sum_i a_i = s$, and their images $\mathbf{b}'$; the shift $d - s$ only reverses these pairs, so two labels carry the same character exactly when they are joined by a chain of the pairs found. For both pencils $n_5 = 2$ and $b_5 = 0$, so the condition at x_5 excludes every odd shift, and for an even shift $a_5 = s/2$. In this appendix we drop the last coordinate, which is zero, and write labels as vectors of length four.

The octic. Here $b_i \leq 6$ and $b_1 + b_2 + b_3 + b_4 = 8$, and the shifts to consider are $s = 2$ and $s = 4$. Since $b_i + s \leq 10 < 16$, we have $a_i = 1$ exactly when $b_i \geq 8 - s$, and the excluded residue is $b_i = 7 - s$. So $\sum_i a_i = s$ asks for exactly $s/2$ entries with $b_i \geq 8 - s$.

At $s = 2$ exactly one entry equals six and none equals five, and the other three entries add up to two. Up to a permutation of the coordinates this leaves $(6, 2, 0, 0)$ and $(6, 1, 1, 0)$, with images $(0, 4, 2, 2)$ and $(0, 3, 3, 2)$; there are twelve labels of each kind, which gives the $12 + 12$ pairs of Proposition 14(b). At $s = 4$ exactly two entries are at least four and none equals three; two entries of at least four already add up to eight, so $\mathbf{b}$ is a permutation of $(4, 4, 0, 0)$, with image the complementary permutation $(0, 0, 4, 4)$, which gives three pairs.

The labels of the three lists are distinct: the sources at $s = 2$ have an entry six, their images have at most one entry larger than three, and the labels at $s = 4$ have two entries four and no entry six. So no label lies in two pairs, the 27 pairs are the classes, and the 149 labels carry $149 - 27 = 122$ characters.

The dectic. Here $b_i \leq 8$ for $i \leq 3$, $b_4 \leq 3$ and $b_1 + b_2 + b_3 + 2b_4 = 10$, and the shifts to consider are $s = 2$ and $s = 4$. For $i \leq 3$, $b_i + s \leq 12 < 20$, so $a_i = 1$ exactly when $b_i \geq 10 - s$, and the excluded residue is $b_i = 9 - s$. For x_4 we have $a_4 = \lfloor (b_4 + s)/5 \rfloor$ and the excluded residue is $b_4 + s \equiv 4 \pmod{5}$. The condition is

$$\#\{i \leq 3 : b_i \geq 10 - s\} + a_4 = s/2.$$

At $s = 2$ the value $b_4 = 2$ is excluded, $a_4 = 1$ exactly when $b_4 = 3$, and no b_i equals seven. Either one b_i equals eight and $b_4 \leq 1$, which leaves $(8, 0, 0, 1)$, $(8, 2, 0, 0)$ and $(8, 1, 1, 0)$ up to a permutation of the first three coordinates, with images $(0, 2, 2, 3)$, $(0, 4, 2, 2)$ and $(0, 3, 3, 2)$ under the same permutation; or no b_i is at least eight and $b_4 = 3$, which leaves the fifteen labels $(c_1, c_2, c_3, 3)$ with $c_1 + c_2 + c_3 = 4$, with images $(c_1 + 2, c_2 + 2, c_3 + 2, 0)$.

At $s = 4$ the value $b_4 = 0$ is excluded, so $b_4 \geq 1$ and $a_4 = 1$; hence exactly one b_i is at least six, and none equals five. The other entries satisfy $b_j + b_k + 2b_4 = 10 - b_i \leq 4$ with $b_4 \geq 1$, which leaves $(6, 2, 0, 1)$, $(6, 1, 1, 1)$, $(6, 0, 0, 2)$, $(7, 1, 0, 1)$ and $(8, 0, 0, 1)$ up to a permutation of the first three coordinates, with images $(0, 6, 4, 0)$, $(0, 5, 5, 0)$, $(0, 4, 4, 1)$, $(1, 5, 4, 0)$ and $(2, 4, 4, 0)$.

Sorting the labels of these lists by b_4 shows which of them coincide. With $b_4 = 3$ there are the sources $(c_1, c_2, c_3, 3)$ and the image $(0, 2, 2, 3)$, which is one of them, with $\mathbf{c} = (0, 2, 2)$. With $b_4 = 2$ there are $(0, 4, 2, 2)$, $(0, 3, 3, 2)$ and $(6, 0, 0, 2)$, all different. With $b_4 = 1$ there are $(8, 0, 0, 1)$, which occurs at both shifts, and $(6, 2, 0, 1)$, $(6, 1, 1, 1)$, $(7, 1, 0, 1)$ and $(0, 4, 4, 1)$, all different. With $b_4 = 0$ there are $(8, 2, 0, 0)$ and $(8, 1, 1, 0)$, the images $(c_1 + 2, c_2 + 2, c_3 + 2, 0)$, all of whose entries are at least two, and $(0, 6, 4, 0)$, $(0, 5, 5, 0)$, $(1, 5, 4, 0)$ and $(2, 4, 4, 0)$, of which only the last has all entries at least two; it is the image of $\mathbf{c} = (0, 2, 2)$. So the only chains longer than one pair are

$$(8, 0, 0, 1) \xrightarrow{s=2} (0, 2, 2, 3) \xrightarrow{s=2} (2, 4, 4, 0) \xleftarrow{s=4} (8, 0, 0, 1)$$

and its permutations, the three triples of Proposition 14(c). The remaining pairs are the 12 with $\mathbf{c}$ not a permutation of $(2, 2, 0)$ and the $6 + 3$ others at $s = 2$, and the $6 + 3 + 3 + 6$ at $s = 4$, which are the pairs listed there: 39 pairs and 3 triples, 87 labels in 42 classes, and $145 - 87 + 42 = 100$ characters.

APPENDIX E. THE BLOCKS OF THE MASS MAP

Table 6 gives the data behind Table 4. For each row we take the representative with $(i, j, k, l) = (1, 2, 3, 4)$ and list, for $\mathbf{b} \in B_\chi$ and $\mathbf{c} \in B_{-\chi}$ in the order of Table 4, the vector $h + \mathbf{b} + \mathbf{c}$, its sector r , the vector k of (10) and the entry $\psi^r \prod_i w_i^{k_i}$ of Theorem 15. The other blocks of each row follow by permuting the coordinates of

weight one. For the blocks of size 2×2 the number Δ of Lemma 17 is 6 for the self-dual sextic row, 0 and 8 for the second and third octic rows, and 0, 0, 10 and 10 for the sixth to ninth dectic rows; for instance in the eighth dectic row $\Delta = 4 + 6 - 0 - 0 = 10$, so the determinant is $50 \cdot 250 \psi^{10} - 1 = 12500 \psi^{10} - 1$.

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