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Vanishing of Higher Magnitude Homology for Finite Prefix Models

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Abstract

Bradley and Vigneaux define a generalized distance on a finite prefix poset using next-token probabilities. They conjecture that its higher magnitude homology vanishes. Motivated by their expression of magnitude as a weighted sum of homological Euler characteristics, we verify this conjecture for their explicit complex, in which both endpoint faces are zero. For each finite length, the homology groups in degrees two and above are zero. The degree-one homology group is free abelian, with one generator for each positive-probability edge of that length. The proof constructs a contracting homotopy for each component whose endpoints have an interior prefix. This homotopy inserts the least interior prefix. Edge probabilities zero and one are both allowed.

1 Introduction

Bradley–Vigneaux [1, §§2, 3.13] model next-token probabilities by an enriched category of texts and express its magnitude as a weighted sum of Euler characteristics. They describe the degree-zero and degree-one homology groups [1, p. 22], following the low-degree calculations of Leinster–Shulman [2, Theorems 4.1 and 4.3], and conjecture that the higher groups vanish. An identity of alternating sums alone does not imply this vanishing. Theorem 2 verifies their conjecture for the finite prefix model and the explicit complex of [1, §3.13].

Our contraction is the elementary cone contraction on each nonempty open prefix interval, in keeping with the interval order-complex viewpoint of Kaneta–Yoshinaga [3, §4.3]. We give it directly at the chain level to include asymmetric distances and zero-length edges, with the endpoint convention specified below.

2 Prefix order and the probabilistic distance

Let M be a monoid. Define $a \preceq b$ if $at = b$ for some $t \in M$. This relation is a preorder. For a free monoid, this is the prefix order. Fix a finite alphabet A, distinct beginning and end symbols $\perp, \dagger \notin A$, and an integer $N \geq 1$. Put $\widetilde{A} = A \sqcup \{\dagger\}$ and equip

$$L = \{\perp a \mid a \in A^*,\ |a| \leq N-1\} \sqcup \{\perp a\dagger \mid a \in A^*,\ |a| \leq N-2\}$$

with the prefix order. A set with a negative length bound is understood to be empty. For $x \in L$, let $|x|$ denote its length, including the beginning and end symbols. Let $I \subseteq L$ be the set of nonmaximal elements. The elements covering $x \in I$ are precisely xc for $c \in \widetilde{A}$. If $x < y$, the interval $[x, y]$ is the unique chain

$$x = z_0 < z_1 < \cdots < z_m = y, \qquad z_i = z_{i-1}c_i, \quad m = |y| - |x|.$$

For each $x \in I$, choose a probability distribution $p_x \colon \widetilde{A} \to [0,1]$, so that $\sum_{c \in \widetilde{A}} p_x(c) = 1$. Following Bradley–Vigneaux [1, §2], define

$$\pi(y \mid x) = \begin{cases} 1 & x = y, \\ \displaystyle\prod_{i=1}^{m} p_{z_{i-1}}(c_i) & x < y, \\ 0 & x \not\leq y. \end{cases}$$

For $x \le y \le z$ we have $\pi(z \mid x) = \pi(y \mid x)\pi(z \mid y)$, and for arbitrary x, y, z,

$$\pi(y \mid x)\pi(z \mid y) \le \pi(z \mid x).$$

Thus $\mathcal{L}(x, y) = \pi(y \mid x)$ defines a category enriched over $[0, 1]$ with its usual order and multiplication.

With the convention $-\log 0 = \infty$, put

$$\operatorname{ob}\mathcal{M} = L, \qquad d(x, y) = -\log \pi(y \mid x).$$

Then $d(x, x) = 0$ and $d(x, z) \le d(x, y) + d(y, z)$, so $\mathcal{M}$ is a generalized metric space in the sense of Lawvere [4], allowing asymmetric and infinite distances. See also [2, Section 2, Example 2.9]. For $x \le y \le z$ we moreover have

$$d(x, z) = d(x, y) + d(y, z). \tag{1}$$

An edge $x \to xc$ has *positive probability* if $p_x(c) > 0$; its length is $-\log p_x(c)$. Such an edge may have length zero.

3 The magnitude complex

We assume throughout that the length parameter ℓ satisfies $0 \le \ell < \infty$. For $k \ge 0$, let

$$G_{k,\ell} = \left\{ (x_0, \dots, x_k) \in L^{k+1} \;\middle|\; x_i \ne x_{i+1} \ (0 \le i < k),\ \sum_{i=0}^{k-1} d(x_i, x_{i+1}) = \ell \right\},$$
$$MC_{k,\ell}(\mathcal{M}) = \mathbb{Z}[G_{k,\ell}],$$

where $\mathbb{Z}[S]$ denotes the free abelian group on a set S. For $k = 0$ the sum is zero, so $MC_{0,0} = \mathbb{Z}[L]$ and $MC_{0,\ell} = 0$ for $\ell > 0$.

Lemma 1. *For $k \ge 1$, a tuple $(x_0, \dots, x_k)$ belongs to $G_{k,\ell}$ if and only if*

$$x_0 < x_1 < \cdots < x_k, \qquad \pi(x_k \mid x_0) > 0, \qquad d(x_0, x_k) = \ell.$$

Every interior point of such a tuple satisfies

$$d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1}).$$

Proof. The condition $d(u, v) < \infty$ is equivalent to $u \le v$ and $\pi(v \mid u) > 0$. Let $(x_0, \dots, x_k) \in G_{k,\ell}$. Each adjacent distance is finite. Since adjacent points are distinct, we have $x_0 < x_1 < \cdots < x_k$. By (1), its total length equals the distance between its endpoints. The converse and the assertion about interior points follow from the same additivity. □

By Lemma 1, the magnitude differential of [1, §3.13], specialized to $\mathcal{M}$, takes the form

$$\partial(x_0, \dots, x_k) = \sum_{i=1}^{k-1} (-1)^i (x_0, \dots, \widehat{x_i}, \dots, x_k). \tag{2}$$

For $k \le 1$, set $\partial = 0$. In particular, *both endpoint faces are always zero.* Deleting an interior point preserves the endpoints and total length and leaves a strict chain. Deleting two interior points in either order gives the same tuple. The corresponding terms have opposite signs, so they cancel. Thus $\partial^2 = 0$. We write

$$H_{k,\ell}(\mathcal{M}) = \frac{\ker(\partial\colon MC_{k,\ell} \to MC_{k-1,\ell})}{\operatorname{im}(\partial\colon MC_{k+1,\ell} \to MC_{k,\ell})}, \qquad MC_{-1,\ell} = 0.$$

4 Vanishing by contraction of intervals

Theorem 2. *Let*

$$E_\ell^+ = \{\, x \to xc \mid x \in I,\ c \in \widetilde{A},\ p_x(c) > 0,\ -\log p_x(c) = \ell \,\}$$

be the set of positive-probability edges of length ℓ. For every $0 \le \ell < \infty$,

$$\begin{aligned} &H_{k,\ell}(\mathcal{M}) = 0 && (k \ge 2),\\ &H_{1,\ell}(\mathcal{M}) \cong \mathbb{Z}[E_\ell^+], \\ &H_{0,0}(\mathcal{M}) \cong \mathbb{Z}[L], \qquad H_{0,\ell}(\mathcal{M}) = 0 && (\ell > 0). \end{aligned}$$

The degree-one isomorphism identifies an edge $x \to xc$ with the homology class of the cycle (x, xc).

Proof. The differential preserves endpoints. Hence the complex in degrees at least one splits as a direct sum of components $C(x,y)$. The sum runs over all pairs (x,y) satisfying $x < y$, $\pi(y \mid x) > 0$, and $d(x,y) = \ell$. For $q \geq 1$, a basis of $C(x,y)_q$ consists of the tuples

$$\sigma = (x, z_1, \ldots, z_{q-1}, y), \qquad x < z_1 < \cdots < z_{q-1} < y,$$

and we set $C(x,y)_0 = 0$. Let V_ℓ be the complex with $\mathbb{Z}[L]$ in degree zero when $\ell = 0$ and zero in all other cases. Its differential is zero. The full decomposition is

$$MC_{*,\ell}(\mathcal{M}) \cong V_\ell \oplus \bigoplus_{\substack{x<y,\ \pi(y|x)>0 \\ d(x,y)=\ell}} C(x,y). \tag{3}$$

If the open interval $(x,y) = \{z \mid x < z < y\}$ is empty, then $y = xc$. The component $C(x,y)$ consists solely of $\mathbb{Z}(x,y)$ in degree one, contributing one copy of $\mathbb{Z}$ for the corresponding edge.

Suppose instead that (x,y) is nonempty, and let v be its least element. For each $q \geq 1$, we define a homomorphism $h : C(x,y)_q \to C(x,y)_{q+1}$. This map raises the homological degree by one. On basis elements, we set

$$h(\sigma) = \begin{cases} -(x, v, z_1, \ldots, z_{q-1}, y) & v \notin \{z_1, \ldots, z_{q-1}\}, \\ 0 & v \in \{z_1, \ldots, z_{q-1}\}. \end{cases}$$

Inserting v leaves a strict chain because v is the least interior prefix. It also preserves the total length, which depends only on the endpoints.

If v does not occur in σ, the term of $\partial h(\sigma)$ obtained by deleting v is $+\sigma$: the minus sign in h cancels the boundary sign $(-1)^1$. The term obtained by deleting z_j has sign $(-1)^j$, whereas its counterpart in $h\partial(\sigma)$ has sign $(-1)^{j+1}$. These terms cancel, giving $\partial h(\sigma) + h\partial(\sigma) = \sigma$. This includes $q = 1$, since $\partial(-(x,v,y)) = (x,y)$.

If v occurs in σ, then $z_1 = v$ and $h(\sigma) = 0$. Deleting v from σ gives the term $-(x, z_2, \ldots, z_{q-1}, y)$ in $\partial(\sigma)$, and h sends this term to σ. Every other term retains v and is sent to zero by h. Thus in both cases

$$\partial h + h\partial = \mathrm{id}_{C(x,y)}\,.$$

Every cycle z in this component therefore satisfies $z = \partial h(z)$, so the component has zero homology in every degree.

Taking the direct sum, only the positive-probability edges contribute in degree one, and every summand has zero homology in degrees at least two. The assertion in degree zero follows from the remaining summand. □

Remark 3 (Weights and zero distances). The proof does not use the normalization of the probabilities, so the same theorem holds for arbitrary edge weights in $[0,1]$. An edge of probability one remains a generator of $H_{1,0}$. In the presence of such edges, the complex used here must be distinguished from the magnitude complex for general enriched categories of Leinster–Shulman [2, Definition 5.7]. In the latter complex, an endpoint face can survive when the corresponding end edge has length zero. For example, on the two-point prefix space $x < y$ with $d(x,y) = 0$, one has $\partial^{\mathrm{LS}}(x,y) = (y) - (x)$. If every edge probability is strictly less than one, all distances between distinct points are positive or infinite, and the two complexes coincide.

References

[1] T.-D. Bradley and J. P. Vigneaux. The magnitude of categories of texts enriched by language models. arXiv:2501.06662v2, 2025. `https://arxiv.org/abs/2501.06662v2`.

[2] T. Leinster and M. Shulman. Magnitude homology of enriched categories and metric spaces. *Algebraic & Geometric Topology*, 21:2175–2221, 2021. `https://arxiv.org/abs/1711.00802`.

[3] R. Kaneta and M. Yoshinaga. Magnitude homology of metric spaces and order complexes. *Bulletin of the London Mathematical Society*, 53:893–905, 2021. `https://arxiv.org/abs/1803.04247`.

[4] F. W. Lawvere. Metric spaces, generalized logic and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 43:135–166, 1973. Reprinted in *Reprints in Theory and Applications of Categories* **1** (2002), 1–37. `https://tac.mta.ca/tac/reprints/articles/1/tr1abs.html`.