

Inscribed Square Problem and an Embedding of Möbius Band in a Family of Solid Toruses

Faris Elmikdad

Email: almokda20faris@gmail.com

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Abstract

In this paper, we relate the space of line segments induced by a Jordan curve in the plane to the space of line segments induced by the plane by embedding the Möbius band associated with the curve into the family of toruses associated with the plane; we further demonstrate how the geometric "square condition" of the curve translates into a topological property through this embedding.

Keywords: Möbius band, Torus, Rotation, Square.

1 Introduction

In 1911, Otto Toeplitz posed the following problem: Does every plane simple closed curve contain all four vertices of some square [4]. Arnold Emch (1916) showed that piecewise analytic curves always have inscribed squares. In particular this is true for polygons. Emch's proof considers the curves traced out by the midpoints of secant line segments to the curve, parallel to a given line. He shows that, when these curves are intersected with the curves generated in the same way for a perpendicular family of secants, there are an odd number of crossings. Therefore, there always exists at least one crossing, which forms the center of a rhombus inscribed in the given curve. By rotating the two perpendicular lines continuously through a right angle, and applying the intermediate value theorem, he shows that at least one of these rhombi is a square [5]. In 1977 Herbert Vaughan proved that any Jordan curve admits an inscribed rectangle by reducing the problem to the non-embeddability of the projective plane in R^3 ; his proof was published by Meyerson about 1980 [2] and 1981 [3]. In this paper we start in the second section by begin with Herbert Vaughan's work by defining the spaces of non-ordered line segments assigned with any Jordan curve and the plane and prove that any homeomorphism between two Jordan curves induces a homeomorphism between their respective segment spaces. In

the third section, we define a segment-rotation function and prove it to be continuous and injective and define the embedding of any Jordan curve segment space into the plane's segment space and specifically for the unite circle, examine how the "square conditions" is represented through intersections under rotation, and prove a property concerning homeomorphisms on any Jordan curve within the boundaries of the topological spaces under study. In the final section, we study the square inscribed in an ellipse, show that the intersections corresponding to this square are transversal, and prove a fundamental theorem regarding the impossibility of resolution of transversal intersections within the spaces under consideration.

2 Preparations

Lets have $\mathbb{R}^2$ plane equipped with Cartesian coordinates and Euclidean metric $||\cdot||_2$. Let have the unite circle with the center on the origin point of the plane. By the work of Herbert Vaughan [2] and [3] the non-ordered line segments of any Jordan curve including our unite circle is homeomorphic with Möbius band and the boundary of the band which represent the segments of length zero is corresponding to the curve homeomorphically. Lets give the symbol $M(S^1)$ for the non-ordered line segments space of the unite circle and $M(\gamma)$ for non-ordered line segments space for any other Jordan curve

$$M(\gamma) = \{[A, B] = [B, A]; A, B \in \gamma\} \cong \text{Möbius band} \quad (1)$$

For the space of non-ordered line segments of the plane we distinguish them by the coordinates of the middle point with the angle made between the segment and the positive direction of x -axis and half the length of the segment:

$$\begin{aligned} &\{(x_m, y_m, \theta, t); (x_m, y_m) \in \mathbb{R}^2, 0 \leq \theta < 2\pi, 0 < t < \infty\} \cup \\ &\{(x_m, y_m, t); (x_m, y_m) \in \mathbb{R}^2, t = 0\} \end{aligned} \quad (2)$$

but for non-ordered line segments we take quotient space for the angles $S^1/(\theta \sim \pi + \theta) \cong \mathbb{RP}^1$ which is $\mathbb{RP}^1 \cong S^1$ [1] but with range $[0, \pi)$ instead of $[0, 2\pi)$:

$$\begin{aligned} M(\mathbb{R}^2) = &\{(x_m, y_m, \theta, t); (x_m, y_m) \in \mathbb{R}^2, 0 \leq \theta < \pi, 0 < t < \infty\} \cup \\ &\{(x_m, y_m, t); (x_m, y_m) \in \mathbb{R}^2, t = 0\} \cong \mathbb{R}^2 \times (\mathbb{R}^2)^* \sqcup \mathbb{R}^2 \end{aligned} \quad (3)$$

which $(\mathbb{R}^2)^*$ is the punctured plane equipped with polar coordinates. The punctured plane is homeomorphic with the open on the circle $t = 0$ by $0 < t$ and infinity on the other side cylinder so the first parts of the equation (3) is homeomorphic to $\mathbb{R}^2 \times \mathbb{RP}^1 \times \mathbb{R}^{0<}$.

By [1, section 1.2, Exercise 9, page 53] there is retraction r retracting the torus to the smaller circle of its two circle (the circle which not rotate around the hole of the torus). Which every horizontal circle retracting to the point of intersection with the vertical circle of image:

$$r : T \rightarrow S_V^1 \quad (4)$$

V for vertical. By extending the domain of retracting function to open solid torus we find:

$$r : e^2 \times S^1 \rightarrow e^2 \cong \mathbb{R}^2 \quad (5)$$

e^2 is the 2-cell or the open 2-disk. Lets equip e^2 with the coordinates (x_m, y_m) and the circle S^1 with $0 \leq \theta < \pi$, so the retracting map in (5) be:

$$r : (x_m, y_m, \theta) \rightarrow (x_m, y_m) \quad (6)$$

We can consider the set of zero length segments (the second parts of the equation (3)) as the image of the retracting solid torus $e^2 \times S^1 \times 0 \cong \mathbb{R}^2 \times \mathbb{R}P^1 \times 0$, by (5):

$$r : (x_m, y_m, \theta, 0) \rightarrow (x_m, y_m, 0) \quad (7)$$

By attaching each closer point from the cylinders in $\mathbb{R}^2 \times \mathbb{R}P^1 \times \mathbb{R}^{0<}$ with the point from $\mathbb{R}^2 \times \mathbb{R}P^1 \times 0$ with the same coordinates of (x_m, y_m) and θ we got the space which lets call it the Blow up of $M(\mathbb{R}^2)$:

$$BM(\mathbb{R}^2) = \mathbb{R}^2 \times \mathbb{R}P^1 \times \mathbb{R}^{0\leq} \quad (8)$$

By extending the domain of retracting function in (7) by attaching it to id for the points with $0 < t$ it give the retracting map from the blow up to $M(\mathbb{R}^2)$:

$$r : BM(\mathbb{R}^2) \rightarrow M(\mathbb{R}^2) \quad (9)$$

which is homeomorphism for non zero segments and surjective for all the points of $M(\mathbb{R}^2)$.

For any Jordan curve there is a homeomorphism between the unit circle and the curve $\varphi : S^1 \rightarrow \mathbb{R}^2$ [1] and [7] which $Im(\varphi) = \gamma$ so there exist a homeomorphism between any two Jordan curves (by taking the inverse of one with circle then applied the second one). Lets define the max metric on the set of the homeomorphisms on Jordan curve $d(\varphi_1, \varphi_2) = \max(\|\varphi_1(A) - \varphi_2(A)\|_2; A \in S^1)$ as S^1 is a compact space and both homeomorphisms and the Euclidean metric are continuous the image of all the points of the circle is compact in $\mathbb{R}$ [6] and by Heine-Borel theorem is a closed and bounded set which mean the maximum value is exist and the metric is well defined, for the conditions of the metric: positivity and commutativity are clear, if the metric is zero will give zero for all points giving match for the two homeomorphisms the inverse is clear and for trigonometry inequality we have:

$$\begin{aligned} \max(\|\varphi_1(A) - \varphi_2(A)\|_2; A \in S^1) &= \|\varphi_1(A) - \varphi_3(A) + \varphi_3(A) - \varphi_2(A)\|_2 \leq \\ &\|\varphi_1(A) - \varphi_3(A)\|_2 + \|\varphi_3(A) - \varphi_2(A)\|_2 \leq \\ &\|\varphi_1(B) - \varphi_3(B)\|_2 + \|\varphi_3(C) - \varphi_2(C)\|_2 = \\ &d(\varphi_1, \varphi_3) + d(\varphi_2, \varphi_3) \end{aligned} \quad (10)$$

which A,B and C are the maximum points for the metric for the homeomorphisms (1,2),(1,3) and (2,3) in the same order.

Theorem 1: Let $\varphi : S^1 \rightarrow \mathbb{R}^2$ be a homeomorphism between the unit circle (or any Jordan curve) with a Jordan curve γ then the induced map between the spaces of non-ordered line segments $\tilde{\varphi} : M(S^1) \rightarrow M(\gamma)$ defined as $\tilde{\varphi} : [A, B] \rightarrow [\varphi(A), \varphi(B)]$ is a homeomorphism.

Proof: For any open set of the segment $[\varphi(A), \varphi(B)]$ there are open neighborhoods $U(\varphi(A))$ and $U(\varphi(B))$ for both $\varphi(A)$ and $\varphi(B)$ (inverse is true too any open neighborhoods for the end points can construct open neighborhood of the segment) which is homeomorphic with the product of them [1] (as non-ordered) by taking the inverse for $[\varphi(A), \varphi(B)]$ to $[A, B]$ and because φ is continuous the inverse of the two open neighborhoods $U(\varphi(A))$ and $U(\varphi(B))$ is open and the product of them is open which equal the inverse of their product so the map is continuous. For the inverse map by the same method is continuous too. Lets take two different segments $[A, B], [C, D] \in M(S^1)$ (at least one of C or D is different point from A and B) if $[\varphi(A), \varphi(B)] = [\varphi(C), \varphi(D)]$ will give at least two points have the same image-for example $\varphi(A) = \varphi(C)$ -but φ is injective so contradiction and $\tilde{\varphi}$ is injective. For any $[A, B] \in M(\gamma)$ there exist the segment $[\varphi(A)^{-1}, \varphi(B)^{-1}] \in M(S^1)$ which $\tilde{\varphi}([\varphi(A)^{-1}, \varphi(B)^{-1}]) = [A, B]$ so it surjective and the map is homeomorphism. $\square$

3 The Embedding Map

Lets define the embedding for unite circle non-ordered segments space $i : M(S^1) \hookrightarrow BM(\mathbb{R}^2)$ as follow

$$i : [A, B] \rightarrow (x_m, y_m, \theta, t) \quad (11)$$

Which $A, B \in S^1$ and (x_m, y_m) is the coordinates of the middle of the segment $[A, B]$, θ and t are defined as in section 2. For the boundary of the Möbius band of S^1 which represents the points of the curve which $t = 0$ the angle θ will be the angle of inclination of the tangent line in $BM(S^1)$:

$$i : \partial M(S^1) \hookrightarrow \mathbb{R}^2 \times \mathbb{R}P^1 \times \{0\} \quad (12)$$

$$i : [A, A] \rightarrow (\cos(\theta_A), \sin(\theta_A), \frac{\pi}{2} + \theta_A, 0) \quad (13)$$

which the middle point of the segments is the point A from the circle itself and θ_A is the central angle between the line from $(0,0)$ to A and the positive direction of the x -axis and $\frac{\pi}{2} + \theta_A$ is the angle of inclination of the tangent line at A . θ_A is going from 0 to 2π and the full turn in $BM(\mathbb{R}^2)$ is π around $\mathbb{R}P^1$, the boundary of the band will embed as it (the circle curve itself will turn twice around $\mathbb{R}P^1$) with out deformations or self intersections. For segments with half length $0 < t < 1$, $(ls(t))$ the embedding will be:

$$i : ls(t) \rightarrow \{(\cos(\theta_m)\sqrt{1-t^2}, \sin(\theta_m)\sqrt{1-t^2}, \frac{\pi}{2} + \theta_m, t); 0 \leq \theta_m < 2\pi\} \quad (14)$$

which θ_m is the central angle of the middle point of the segment, the embedding will be a curve in the solid torus $\mathbb{R}^2 \times \mathbb{R}P^1 \times \{t\}$ rotate twice around $\mathbb{R}P^1$ without self intersect. For $t = 1$ we get a representation for the diameters of the unite circle as follow:

$$i : ls(1) \rightarrow \{(0, 0, \frac{\pi}{2} + \theta_m, 1); 0 \leq \theta_m < 2\pi\} \quad (15)$$

this is a two overlapping circles in the solid torus $\mathbb{R}^2 \times \mathbb{R}P^1 \times \{1\}$. By the existing of the second derivative at any point on the unit circle the first derivative is continuous and any segment is continuously moving to any tangency line so embeddings defined by equations (12)-(15) embed the space of non-ordered line segments of the unite circle continuously into a set of curves one for each t which corresponding to a solid torus, each curve rotate twice around the hole of the solid torus without self intersect, for the diameters we get two overlapping circles. By theorem 1 for any homeomorphism φ between the unite circle and any Jordan curve and by applying the induced homeomorphism on the embedding image of $M(S^1)$ we can define the embedding for any $M(\gamma)$ in $BM(\mathbb{R}^2)$ by $(\tilde{\varphi} \circ i)(M(S^1))$.

Lets define rotation by $\pi/2$ map on space of non-ordered line segments of unit circle or any Jordan curve $R : Im_{\tilde{\varphi} \circ i}(M(S^1)) \rightarrow BM(\mathbb{R}^2)$ as rotate the segment by $\pi/2$ in the positive direction around the middle point of the segments with the map of the embedding we have :

$$R : (x_m, y_m, \theta, t) \rightarrow (x_m, y_m, \theta + \frac{\pi}{2}, t) \quad (16)$$

For any open neighborhood for any point $A \in BM(\mathbb{R}^2)$ we take the inverse function which it is the function itself and if $R(U(A)) \cap Im_{\tilde{\varphi} \circ i}(M(S^1))$ is empty (which is open in Euclidean metric space) or has a point from the continuous surface $Im_{\tilde{\varphi} \circ i}(M(S^1))$ in it and the intersection equal to open set on the surface which meaning R is continuous, the injectivity is obvious.

As we can see the effect of R on $BM(\mathbb{R}^2)$ is taking the point to the opposite side of the solid torus because the full turn on it is π . If we look at the square geometrically we see the two diameters of it has the same half length and middle point and each one is from the other by rotation $\frac{\pi}{2}$, these Properties for a Jordan curve with inscribed square meaning the existing of a point with $0 < t$ and $x \in Im_{\tilde{\varphi} \circ i}(M(S^1))$ which $R(x) \in Im_{\tilde{\varphi} \circ i}(M(S^1))$ too, we have the following lemma

Lemma 1: *For any Jordan curve $\varphi : S^1 \rightarrow \mathbb{R}^2$, there exist inscribed square (non trivial) on it if and only if there exist a point not on the boundary of the band $A \in Im_{\tilde{\varphi} \circ i}(M(S^1))$ which $R(A) \in Im_{\tilde{\varphi} \circ i}(M(S^1))$ or in equivalence meaning $Im_{\tilde{\varphi} \circ i}(M(S^1)) \cap R(Im_{\tilde{\varphi} \circ i}(M(S^1))) \neq \emptyset$. Each pair $\{A, R(A)\}$ correspond to one inscribed square and vice versa.*

Proof: If Jordan curve has an inscribed square on it the two diameters of

the square will give the points for some θ_0 and $t_0 \neq 0$, $(x_m, y_m, \theta_0, t_0)$ and $(x_m, y_m, \theta_0 + \frac{\pi}{2}, t_0) = R((x_m, y_m, \theta_0, t_0))$ in $Im_{\tilde{\varphi} \circ i}(M(S^1))$ because for non-trivial segments the retracting map (9) is id which its inverse is exist away from the zero length segments in $M(\mathbb{R}^2)$ to $BM(\mathbb{R}^2)$ which prove the first half. If there exist $A, R(A) \in int(Im_{\tilde{\varphi} \circ i}(M(S^1)))$ and by applying the retracting map (9) on $BM(\mathbb{R}^2)$ to $M(\mathbb{R}^2)$ the two points still the same, then for some θ_0 and $t_0 \neq 0$ $A = (x_m, y_m, \theta_0, t_0)$ and $R(A) = (x_m, y_m, \theta_0 + \frac{\pi}{2}, t_0)$ represents two line segments which their ends points on the curve and have the same middle point and same half length and each one is from the other by rotation $\frac{\pi}{2}$ these are the conditions of a square so we have inscribed square on the curve. For any different in the intersection points will correspond to a different in middle point coordinate or angle of segment or the half length which geometrically give different squares and vice versa for any two deferent square proving the second statement of the lemma. $\square$

In the above lemma we put a condition for the intersection be not on the boundary, actually if there an intersection it is impossible to it to be on the boundary of the band as the following lemma tell us.

Lemma 2: *For any Jordan curve $\varphi : S^1 \rightarrow \mathbb{R}^2$ and if we have $Im_{\tilde{\varphi} \circ i}(M(S^1)) \cap R(Im_{\tilde{\varphi} \circ i}(M(S^1))) \neq \emptyset$, then any point belong to the intersection has $0 < t$.*

Proof: Let A be a point belong to the intersection as $t_A = 0$ then $R(A)$ belong to the intersection too with $t_R = 0$ and $(x_R, y_R) = (x_A, y_A)$ by equation (16). But by (12), (13) and the argument above every point from the curve embed 1-1 to the boundary of the band and so to the embedding in the torus $t = 0$ and when $t = 0$ the middle point of the segments will be the points on the curve itself so (x_R, y_R) and (x_A, y_A) are coordinates of points on the curve but we find above they are on the same horizontal circle and by applying the retracting map (9) the two points will have the same image at $\mathbb{R}^2$ so the Jordan curve cut itself but this is a contradiction with the definition of the homeomorphism $\varphi : S^1 \rightarrow \mathbb{R}^2$, the lemma follow. $\square$

Corollary 1: *The condition in the second half of lemma 1 (the intersection point not on the boundary of the band) can be omitted, then if we have any intersection $Im_{\tilde{\varphi} \circ i}(M(S^1)) \cap R(Im_{\tilde{\varphi} \circ i}(M(S^1))) \neq \emptyset$ the corresponding inscribed square (squares) is non-trivial.*

For example for the unit circle curve there exit inscribed square only when the half length of the segments is equal to 1 and the middle point on $(0,0)$ and for any diameter we can construct a square by rotation $\frac{\pi}{2}$ degree around $(0,0)$ this is corresponding to equation (8) which every pair of opposite points of the flat circle (8) correspond to a square on the unit circle and we have opposite sides points along $[0, \frac{\pi}{2}]$ in (8) which mean we have inscribed square geometrically along segments of quarter of the unit circle which is actually the case.

4 Intersections and The Ellipse Curve

It known that any ellipse $\varphi_{E(a,b)} : S^1 \rightarrow \mathbb{R}^2$ (non circle or degenerate) has only one inscribed square on it so by Lemma 1 for any $\varphi_{E(a,b)}$ (for simplification we will write φ) the intersection $Im_{\tilde{\varphi}oi}(M(S^1)) \cap R(Im_{\tilde{\varphi}oi}(M(S^1))) = \{A, R(A)\}$ which $A = (x \text{ of the middle of the ellips}, y \text{ of the middle of the ellips}, \theta_0, \frac{ab\sqrt{2}}{\sqrt{a^2+b^2}})$ this is true for any ellipse arbitrary close (we can use the metric defined in the second section) to the unite circle. For example by applying horizontal translate T (which is a homeomorphism) on unite circle points to get the ellipse $\frac{x^2}{(1+\epsilon)^2} + \frac{y^2}{1^2} = 1$ for any $0 < \epsilon$ and we get $d(id, T) = \epsilon$. From being homeomorphisms space of Jordan curves connected (for any two homeomorphisms there is homotopy (path) connecting them) the induced homeomorphisms space on the bands is connected too so we can chose an ellipse curve which its $int(Im_{\tilde{\varphi}oi}(M(S^1)))$ is very close to the equations (14) around $t = \frac{ab\sqrt{2}}{\sqrt{a^2+b^2}}$. Now we have the following lemma about the two intersection points assigned with an ellipse.

Lemma 3: *For some ellipse close enough to the unit circle in the meaning of the metric define on the second section the intersection $Im_{\tilde{\varphi}oi}(M(S^1)) \cap R(Im_{\tilde{\varphi}oi}(M(S^1))) = \{A, R(A)\} \subset BM(\mathbb{R}^2)$ at both points is transversal.*

Proof: The expected dimension of intersections for our spaces is $dim(Im_{\tilde{\varphi}oi}(M(S^1))) + dim(R(Im_{\tilde{\varphi}oi}(M(S^1)))) - dim(BM(\mathbb{R}^2)) = 2 + 2 - 4 = 0$ thats meaning the existing of transversal intersections (can not be resolve Instantly by arbitrary small deformation look [1] and [7]). Lets take the normal direction on the intersection point A (being Möbuis band unorientable meaning non exiting of a global orientation but locally it exist look [1]). In the equation (8) the curve decrees in the first turn then increase in the second turn so for close enough ellipse to the circle the projection of normal vector on the solid torus of intersection (the torus assigned with the t of A and R(A)) on A and R(A) for $Im_{\tilde{\varphi}oi}(M(S^1))$ will have non zero opposite signs of component y, when we apply R to the surface the sign of component y of the normal vector on A for surface $Im_{\tilde{\varphi}oi}(M(S^1))$ will stay the same for the component y of the normal vector on R(A) for surface $R(Im_{\tilde{\varphi}oi}(M(S^1)))$ and vice versa, So for both intersection points the normal vectors for the two surfaces on each point has different signs at least on one component with no overlapping (for close enough to the band assigned with the unit circle) so there exit sharp angle between normal vectors at both points so the intersections at them are transversal [7]□.

Theorem 2: *For any Jordan curve $\varphi : S^1 \rightarrow \mathbb{R}^2$ the intersection $Im_{\tilde{\varphi}oi}(M(S^1)) \cap R(Im_{\tilde{\varphi}oi}(M(S^1))) \subset BM(\mathbb{R}^2)$ is not empty.*

Proof: Lets begin with an ellipse as described in the previous lemma which have two transversal intersection points thats mean for resolve them we must applying continuous homotopy $H(u); 0 \leq u \leq 1$ on the homeomorphism $\varphi_{E(a,b)} :$

$S^1 \rightarrow \mathbb{R}^2$ of the initial ellipse through the connected set of all Jordan curves by [1] and [7]. We have the following cases:

Case A: We can apply the homotopy on the ellipse curve such that the two intersection points continuously moving to each other then collide for form tangency intersection point then resolve the two surfaces by arbitrary small deformation but the two intersection points will be always on opposite sides by equation (16) of any solid torus they on, so all possible collide points are on the middle of the torus which is not belong to the space $BM(\mathbb{R}^2)$. Any sequence to collide for the intersection points does not has any limit points in $BM(\mathbb{R}^2)$, So impossible to resolve the intersection by collide method.

Case B: We can apply the homotopy on the ellipse curve such that the two intersection points continuously moving to the boundary of the two surfaces then resolve but as the homotopy $H(u)$ is continuous there exit time u_0 such that the two intersection points exit on the boundaries of the surfaces but by lemma 2 this is contradiction with the definition of Jordan curves.

Case C: For circle curve we have tangency points by equation (15) but for resolving them we need to push one of the surfaces up or down on y or t coordinate direction but by equation (16) the second surface will move simultaneously on these directions and the intersection will still intact. If there exit other homotopy start from the circle to a Jordan curve which have no intersection point we can find homotopy start from our ellipse to the same curve which by cases A and B will have intersection points, contradiction.

Case D: We can applying a homotopy as pushing the band to one side of the family of the toruses and Separating the two surfaces but this will lead to being the center circle contractible inside the solid toruses but the center circle is homotopic to the $\mathbb{R}P^1$ of the toruses which is not contractible in $BM(\mathbb{R}^2)$, contradiction.

By cases A,B,C and D the theorem follow $\square$.

Corollary 2: *For every Jordan curve $\varphi : S^1 \rightarrow \mathbb{R}^2$ there exit non-trivial inscribed square on it.*

Proof: By lemma 1, lemma 2, corollary 1 and theorem 2 the corollary follow $\square$.

Conflict of Interest

The author declare that he has no conflicts of interest to declare that are relevant to the content of this article. Author certify that he has no affiliations with or involvement in any organization or entity with any financial or non-financial interest in the subject matter or materials discussed in this manuscript.

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The author confirm that no artificial intelligence (AI) technologies were used in the making, generation, or editing of this submitted work.

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