

Chemical heredity without genetic templates

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20 October 2026

Abstract

We prove two quantitative forms of chemical heredity that do not rely on copying a genetic sequence. In the first model, a population of four compartments carries three prescribed catalyst types that compete for a shared local fuel. Two types are present from the start; a third is supplied for a fixed seeding period, after which all catalyst input stops. We prove that, with failure probability at most 0.004517975690072193, a single finite-stock process acquires the third type and keeps all three present in both daughters of every division, delivers three million catalytic conversion credits on a fixed per-site schedule, and rebuilds the population’s copy reserves eleven times, interleaved with ten divisions. The proof controls acquisition, catalytic service, recovery and resource consumption on one stochastic history; it does not assume that a compartment stays replenished between recovery and division. In the second model, an imperfect parent copies its majority conformation into an independent racemic blank, after which both outputs undergo cooperative restoration while spontaneous errors continue. At 4,000 molecules per compartment, 100 successive copying cycles succeed with probability at least 0.999, uniformly over a box of rate parameters. Complete conventional proofs and explicit finite certificates are supplied. The central finite-stock theorem, though not the restoration result, also has a machine-checked Lean formalization. A converse for random division explains why unlinked information units and linked carriers face different loss mechanisms. These results establish acquisition and retained function under prescribed chemistry and operations; chemically generated successive innovations and open-ended evolution require additional mechanisms.

1 Introduction

A chemical system can transmit a useful state without copying a genetic sequence. The difficult part is making that state survive the very events that reproduce it. Molecular partition can erase a catalyst, fuel limitation can silence a surviving repertoire, and replacement can destroy another compartment. A convincing inheritance claim must therefore follow the descendants and the chemistry that make the trait useful, while paying for the operations that sustain them.

Compositional-genome models introduced a concrete route to information transfer through growth and splitting in mutually catalytic assemblies [1]. Analyses of GARD found important limits to evolvability in the dynamics studied [2], while later models identified viable autocatalytic cores whose acquisition can support accumulated adaptations [3]. Other work established selection responses under suitable catalytic interactions [4] and quasispecies-like error thresholds in compositional assemblies [5]. Template-free adaptation is consequently a question with qualified positive answers, rather than a missing existence result.

Different physical mechanisms realize different aspects of that possibility. Inhibition-based reaction systems exhibit hereditary states, competition and adaptation [6]; related reaction–diffusion systems

adapt to time-dependent environments [7]. Vesicle populations can respond to imposed selection for turbidity [8]. These results motivate a sharper distinction between transmitting a state, maintaining useful chemistry, and acquiring a new function that can itself be inherited.

We prove a finite acquisition-and-retention theorem that joins the first two tasks on one stochastic history and adds a controlled form of the third. Four compartments each contain copies of three prescribed catalyst types and a shared local fuel token. Two types are prepared initially. A third, the candidate, immigrates during a fixed seeding phase, and all catalyst immigration then stops at time 30. The expanded repertoire must then continue to perform distributed conversion tasks through time 60, survive every intervening division in both daughters, and repeatedly rebuild the population’s copy reserves. A single finite allocation pays for all accepted inputs, fuel consumption, precursor entries and compartment replacements. The joint failure bound is 0.004517975690072193 (Theorem 1).

A second result addresses reliable copying under continual error. The carrier is a compartment of molecules in two conformations whose hereditary state is the majority conformation. An imperfect parent is brought into contact with a racemic blank compartment and drives it toward the parent’s majority; after separation, cooperative conversion restores both outputs. The parent continues to fluctuate during contact, and spontaneous errors continue throughout restoration. Theorem 2 gives a bound that is uniform over a box of rate parameters, and a 4,000-molecule, 100-cycle witness. Its two outputs are a retained parent and a new copy, whereas the first model produces the two complementary daughters of a partition. The second protocol does use conformational templating: the title excludes genetic templates, not every interaction that can transmit a molecular state.

The proofs are the main contribution of this article. Their ingredients are an explicit construction of the marked jump process; a composition rule that charges each failure at the stopping time where it is first exposed, instead of conditioning the future on eventual success; service and recovery bounds proved on the full shared-fuel law; an explicit rational certificate for the cost of divisions that interrupt a recovery; and a coupling that transfers the result from unlimited to finite stocks on the same history. The central finite-stock theorem also has an independently stated, machine-checked formalization in Lean [11]; the restoration theorem has a complete conventional proof but is not formalized. Supplement S1 contains the complete proof records, the rational data and a standalone verifier [9, 10].

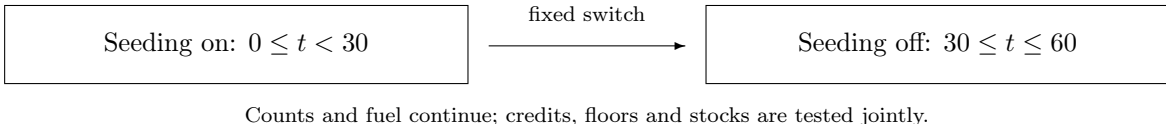
The scope is deliberate. Catalytic specificity, founders, compartment handling and phase control are supplied. Useful function means counted substrate-conversion events in the declared law. Neither theorem asserts a ten-generation lineage, an origin of catalytic specificity or autonomous open-ended evolution. Section 8 connects these results to information loss, initiation, physical reproduction and observation, and states what must change to obtain stronger claims. Appendices A–D contain the complete analytical arguments and their finite proof inputs.

2 Defining inheritance and its cost

2.1 Trait, preparation and descendants

A hereditary claim begins with a chemical state and a trait read from it. A trait might be a majority composition, the presence of every catalyst in a repertoire, or a function of several compartments. Let X_t denote the complete physical state, with any required observational history carried explicitly. A label w may have a set of admissible parent states $\mathcal{R}_w$. A copying statement then specifies an

A. Acquired catalytic repertoire: one finite-stock history



Counts and fuel continue; credits, floors and stocks are tested jointly.

B. Majority copying: an operated cycle with continual errors

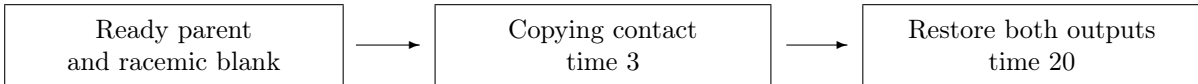


Figure 1: Two inheritance mechanisms and their operation schedules. (A) Two complementary daughters replace the dividing parent and one other compartment while local chemistry continues; the observation ledgers are reset at the recovery and division times, but physical stocks are not replenished. (B) A retained parent and newly formed copy are separated before restoration. Both protocols are operated, and their different descendants and resource statistics are kept distinct.

operation and a success event such as

$$\{X_{\text{out}}^{(1)} \in \mathcal{R}_w, X_{\text{out}}^{(2)} \in \mathcal{R}_w\}.$$

The operation matters. Two complementary daughters created by partition of one parent are dependent. A retained parent and a new compartment prepared from blank molecules are a different pair of outputs. Neither can be replaced by two independent samples from a favorable marginal law.

A repertoire trait is coarser than a compositional label. For a required catalyst set J and copy floor K , define

$$\mathcal{F}_J(K) = \{x : a_{ij}(x) \geq K \text{ for every occupied site } i \text{ and every } j \in J\}. \quad (1)$$

Keeping a population in this set preserves the required types. It need not preserve their ratios, distinguish many alternative labels, or identify the molecules ancestral to a later catalyst. Those are separate observations and require separate results.

Preparation is part of the claim. A theorem from a prepared founder, a theorem from food alone, and a theorem with a supplied catalyst-seeding phase answer different questions. The finite example below uses prepared founders for two old types and supplied seeding for the candidate. Its achievement is that the expanded repertoire persists and works after seeding stops.

2.2 Success over a mission

A mission specifies the tested descendants and a horizon. For a chosen lineage of D reproduction events, a uniform ready-parent failure bound ε gives failure at most $D\varepsilon$ by conditioning successively on the exposed past. For a full binary family of depth g , the corresponding union has $2^g - 1$ reproduction events. A population-renewal mission has a different event: it tests whichever compartments occupy the population at selected stopping times. Neither event implies that one named

lineage survives throughout.

Uniformity over admissible parents is essential. An estimate for an average initial parent does not automatically apply to a rare state produced after an unfavorable division. Likewise, an estimate conditional on future survival cannot simply be inserted into a proof that is supposed to establish survival. Appendix A states the conditional-composition argument used here without that circular conditioning.

2.3 What is paid?

We keep a vector of resource quantities rather than one unqualified cost. It can include initial catalysts, peak inventory, accepted feed entries, catalyst seeds, precursors, substrate conversions, division operations, time and external control. Expected consumption, a high-probability path ceiling and a physical minimum are different quantities. A sufficient feed allocation divided by a guaranteed output is a useful engineering certificate; it is not a measurement of consumption or a theorem of minimum dissipation.

External operations also belong in the account. Supplied compartments, selective catalysts, food activities, phase timing, mixing and scission can isolate a chemical-information problem without making the resulting system autonomous. Accounting for their presence is compatible with proving a substantial theorem under those assumptions.

3 Acquisition and retention in a catalytic population

3.1 The physical count law

Informally, the model is a population of four compartments. Each holds a pool of catalyst molecules of three prescribed types and a single fuel token. A fueled catalyst can make a copy of itself, be destroyed, or perform its assigned conversion job; each of these consumes the token, which is then recharged from an external feed. Catalysts also decay without fuel. Before time 30, copies of every type also immigrate from outside at a small rate; this seeding is what establishes the candidate. Compartments divide by splitting their contents at random between two daughters, which replace the parent and one other compartment, so the population always contains exactly four.

Formally, there are four occupied spatial sites. Each site i carries three catalyst counts $a_{ij} \in \{0, \dots, N\}$, with $N = 2048$, and a binary fuel state $R_i \in \{0, 1\}$ shared by the three types. Time is measured in units of the basal division time. For a selected count a and fuel bit R , the chemical rates are given in Table 1.

An empty fuel site recharges at rate β ; an occupied site discharges unproductively at rate β . Each site initiates basal division at rate one, and catalyst j initiates conversion-triggered division at rate $R_i a_{ij}/N$. A trigger consumes fuel and earns its assigned conversion credit. The central coefficient is $\beta = 4,000,000$.

The rates separate three time scales by orders of magnitude. Fuel is recharged and consumed millions of times per unit time, so the fuel bit equilibrates almost instantly to the current catalyst load. Individual catalyst copies are made and lost hundreds of times per unit time. Compartments divide roughly once per unit time. The fast fuel cycle is what couples the three types: every job, birth and active loss competes for the same token, so a heavy load of one type slows the others.

Table 1: Chemical channels at one site, for one catalyst type with count a and the site’s fuel bit R . The seeding rate $s(t)$ follows the schedule (3).

Channel	Rate	Count and fuel update
Seed entry	$s(t)(1 - a/N)$	$a \mapsto a + 1$; fuel unchanged
Passive loss	a	$a \mapsto a - 1$; fuel unchanged
Catalytic birth	$800Ra(1 - a/N)$	$a \mapsto a + 1$; fuel becomes zero
Active loss	$400Ra$	$a \mapsto a - 1$; fuel becomes zero
Ordinary assigned conversion	$\beta Ra/N$	Count unchanged; one credit; fuel becomes zero

At division, the initiating parent and a uniformly chosen distinct victim site are replaced by two daughters. Conditional on the complete predivision state and chosen mark, independently across species,

$$K_j \sim \text{Binomial}(a_{ij}, 1/2), \quad \text{daughter counts } K_j \text{ and } a_{ij} - K_j. \quad (2)$$

The two daughters are complementary within each type: what one receives, the other lacks. At a basal division the parent’s fuel token, if present, goes to one daughter chosen by a fair coin independent of the species split; a triggered division consumes the token first, so both daughters start without fuel. The victim’s contents are discarded and charged to the resource account. A division whose outcome happens to leave the counts unchanged still counts as a division.

All seeding follows the deterministic schedule

$$s(t) = 6 \quad (t < 30), \quad s(t) = 0 \quad (t \geq 30). \quad (3)$$

The switch is deterministic and is never conditioned on success; jobs, chemistry and divisions continue through it. Since all rates are uniformly bounded, the process is a nonexplosive marked jump process whose state consists of the finite count/fuel configuration together with unbounded observation ledgers (credits earned, inputs accepted, divisions completed).

3.2 The preparation and operating obligations

Each old type starts with at least 512 copies per site. Candidate counts and fuel are arbitrary. The required types are the old pair before time 30 and all three afterward. The service obligation is local and continuous: every required type at every site must earn 500 new credits in every fixed block of length 0.1 through time 60, including while the population is recovering from a division.

Divisions repeatedly deplete the population: each one halves the dividing parent’s counts and discards the victim. The renewal obligation asks that the population rebuild its reserves after each division. Let Ready_{512} mean that all twelve current counts are at least 512, and let τ_0 be the first time at or after 30 at which Ready_{512} holds. For $r = 1, \dots, 10$, let σ_r be the first division strictly after τ_{r-1} , and τ_r the first time at or after the completion of that division at which Ready_{512} holds again. The recovery interval from σ_r to τ_r includes all intervening chemistry and any further divisions.

3.3 The finite mission

Theorem 1 (Finite-stock acquisition and retained function). *Under the physical law and preparation above, one finite-stock process satisfies all five properties below with failure probability at most*

$$\delta_* = \frac{4517975690072193}{10^{18}} < 0.004518. \quad (4)$$

1. *The old types remain at least 16 at every site through time 30; the candidate is at least 16 everywhere at time 30; all three types remain at least 16 everywhere through time 60. These floors are tested at every instant, including immediately after each division or loss event.*
2. *Every division in $[30, 60]$ gives both daughters at least 16 copies of every type. Each initiating parent therefore contains at least 32 copies of each type immediately before that division.*
3. *Every applicable type/site/block service quota is met. Each site initiates at least five divisions in $[30, 60]$.*
4. *Each of the eleven recoveries takes at most 0.5, and each of the ten waits for the next division takes at most 2. Hence*

$$\tau_{10} \leq 30 + 11(1/2) + 10(2) = 55.5 < 60. \quad (5)$$

5. *Through time 60 there are at most 800 million accepted feed entries, 5000 accepted seed entries and 1200 divisions, together with the stock ceilings listed in Section 6.*

Appendices A–C give the complete proof; the proof records and all finite inputs are in Supplement S1 [9]. The full statement is also machine checked (Section 9). The prepared-founder hypothesis is explicit; no arbitrary-start or successive-innovation theorem is asserted.

The service total is the sum of two obligations (Table 2):

Table 2: Service quotas and guaranteed conversion credits by phase.

Phase	Type/site/block quotas	Credits per quota	Guaranteed credits
Acquisition, 0–30	$2 \times 4 \times 300 = 2400$	500	1,200,000
Maintenance, 30–60	$3 \times 4 \times 300 = 3600$	500	1,800,000
Full mission	6000	500	3,000,000

Three million is the full-mission total, not the postwithdrawal total. The renewal deadline 55.5 does not shorten the service horizon 60. These credits count modeled assigned conversion firings; the model does not calibrate a particular chemical product yield or convert a credit into a joule.

4 Why the expanded repertoire is hereditary

4.1 Both daughters, directly from one parent

On the event that the floors of Theorem 1 hold, immediately after each division the two installed counts are K_j and $a_{ij} - K_j$. Both receiving sites belong to the same population-wide floor test.

Therefore both counts are at least 16, and their sum forces the predivision parent to contain at least 32. This is a pathwise consequence of the division rule and the floor event; it does not treat the sisters as independent, and it does not allow a failed daughter to be recolonized later.

The 32-copy requirement is necessary, not sufficient, for a favorable random split. The probability argument bounds the chance that any floor ever fails; it does not assert that the next division succeeds from every possible 16-copy state. Nor does Ready_{512} at τ_{r-1} imply that the parent is still Ready_{512} when it later divides.

4.2 Immigration has genuinely stopped

Fix a catalyst j and consider the face

$$Z_j = \{x : a_{ij} = 0 \text{ at every site } i\}. \quad (6)$$

After time 30, every channel that could increase a j -count either is a seed channel, whose rate is zero, or requires an existing j -catalyst. A division from a parent with zero copies has $\text{Binomial}(0, 1/2)$ allocation and gives zero to both daughters. Other species and fuel cannot create j . Thus Z_j is invariant under the postwithdrawal law, including after a finite natural-history stopping time. Appendix A gives the complete marked-path proof.

Consequently, retained candidate function depends on earlier acquired inventory and its amplification and transmission. A single empty site can still receive copies from a positive parent elsewhere; the proposition concerns absence from the whole population. It establishes a causal dependence on prior inventory, not fidelity of molecular ancestry, a many-label information capacity, or an autonomous origin of the supplied catalyst specificity.

4.3 Useful service continues through turnover

Credits are attributed to their initiating species and spatial site. Replacement does not erase previous work, and a division does not duplicate a parent's past credits into two new outputs. The fixed local quotas therefore test continuing productive service during population turnover. They are stronger than a large pooled total that could hide an inactive site, but they are different from assigning a fixed output history to a named daughter lineage.

The construction joins acquisition and retained function under one law. It does not establish that the candidate was chemically invented, that its presence was favored by a measured reproductive advantage over a competing repertoire, or that further innovations can be generated indefinitely. Those omissions identify the next interfaces rather than diminishing the stated finite theorem.

5 Proof architecture

The principal difficulty is maintaining several properties on the same history. Shared fuel couples jobs to amplification and loss. Division couples two daughters to one parent and overwrites another compartment. The law changes at time 30. A recovery estimate that suspends service, assumes a fresh supply, or restarts only from an idealized state would not prove this mission.

Three parts of the argument prevent those substitutions. First, potentials of the count/fuel state control acquisition and the all-time floors without assuming that the candidate is already replenished at time 30 (Appendix B). Second, an exponential tilt of each site’s credit count accounts for every other fuel-consuming channel and every division that touches the tested site, so the service bound holds on the full shared-fuel law (Appendix B). Third, a recovery potential is combined with a division clock that is itself controlled by the fuel count, which bounds how many divisions can interrupt one recovery (Appendix C). All three give estimates that hold uniformly from every admissible state exposed at a stopping time. The final composition (Appendix A) charges each first failure where it is exposed, without conditioning the future dynamics on eventual global success.

Seed withdrawal needs its own generator check. Removing immigration is not automatically favorable for a potential that is not monotone in the counts, so a certificate proved with seeds cannot simply be reused after time 30. The proof checks the seed-free generator rows separately, with the same physical division law. Likewise, the resource tails are proved for the switched law itself rather than inferred from a monotone comparison between seeded and seed-free populations.

The resulting failure allocation is

$$\frac{9545923}{906250000} + \frac{11}{100000} + 10e^{-8} + 3e^{-20} \leq \delta_*. \quad (7)$$

The first term protects acquisition, every required floor, all fixed-grid service and the per-site division count. Eleven local recovery allowances and ten waiting allowances follow. Three global resource tails are charged once. No independence between these events is asserted.

6 Resources and operating comparisons

6.1 A same-path finite supply certificate

The model has intrinsic capacity $4 \times 3 \times 2048 = 24576$ catalyst slots. The prepared old inventory is separate from later inputs. It is at most 16384 copies when the candidate starts absent, or 24576 when its initial counts are arbitrary. At most four initially occupied fuel tokens are supplied. Table 3 lists sufficient ceilings on everything consumed through time 60.

Table 3: Sufficient cumulative stock ceilings through time 60.

Cumulative quantity through 60	Sufficient ceiling
Accepted feed/recharge entries	800,000,000
Consumed fuel, including discarded victim fuel	800,000,004
Accepted catalyst seeds	5000
Catalytic-birth plus seed precursor entries	800,005,004
Complete division operations	1200
Catalyst-position handling in overwritten compartments	7,372,800
Discarded victim fuel (already charged above)	1,200

Handling-slot restoration supplies capacity, not catalyst molecules. All new catalysts still enter by seeds before withdrawal or paid catalytic birth. Declared job substrates can be debited once per conversion; their sufficient allocation is bounded by the consuming-fuel ceiling under that convention. This is effective-model accounting, not an elemental mass balance for unspecified chemical structures or a price for scission work.

The feed-cap/output-floor ratio is

$$\frac{800,000,000}{3,000,000} = \frac{800}{3} \simeq 266.7. \quad (8)$$

It compares a sufficient ceiling with a guaranteed floor. Using only the 1.8 million postwithdrawal credits in the denominator would answer a different question. Neither ratio is measured efficiency, minimum fuel demand or dissipated energy.

6.2 Comparing operating points under one declared event

The same model with seeding continued through time 60, whose proof records are retained in S1 [9], has certified operating points at $\beta = 8,000,000$, $\beta = 4,500,000$ and $\beta = 4,000,000$ (Table 4). The $\beta = 8,000,000$ point was certified with a recovery allowance of 0.2; since that event is contained in the event with allowance 0.5, all three points can be compared on one common event: fixed-grid service through 60, eleven recoveries of at most 0.5, ten waits of at most 2, and completion by 55.5. This containment changes no physical rate.

Table 4: Certified operating points of the continuously seeded model, compared on one common event.

Continuously seeded model	$\beta = 8,000,000$	$\beta = 4,500,000$	$\beta = 4,000,000$
Certified recovery allowance	0.2	0.5	0.5
Certified renewal deadline	52.2	55.5	55.5
Credits by 60	3,000,000	3,000,000	3,000,000
Accepted-feed ceiling	1,558,500,000	900,000,000	800,000,000
Feed ceiling/guaranteed credits	519.5	300	266.7
Failure ceiling	<0.006	≤ 0.005464632462 485986	≤ 0.004517975690 072193

Theorem 1 is the $\beta = 4,000,000$ point with the additional seed withdrawal at time 30, at the same guarantee; that switch is not a feature of the rows in this table. The table shows certified operating tradeoffs under an explicit common event; it does not prove optimality or stochastic ordering in beta. Different certificates can produce a smaller failure bound at a lower rate without proving that lowering the rate always improves the physical process.

7 Majority copying and restoration with continual errors

The second model concerns a different carrier: a compartment of $n = 100m$ molecules, each in one of two conformations, whose hereditary state is the majority conformation. Three processes act on the molecules. Cooperative restoration converts a molecule to the conformation of two others it encounters, so the majority amplifies itself; a reverse reaction with relative rate r undoes conversions;

and spontaneous flips at rate κ per molecule introduce errors at all times. A parent with at most 15% wrong molecules is placed in contact with a racemic blank compartment for time 3, during which parent molecules template the blank’s molecules toward their own conformation. The two compartments are then separated and each restores on its own for time 20. The quantitative bounds below assume integer $m \geq 10$, unit restoration and copying rates, fixed values $9/200 \leq r \leq 11/200$ and $9/1000 \leq \kappa \leq 11/1000$, and a blank whose initial composition is binomial and independent of the parent. Appendix D gives the exact channel law and derives the bound uniformly on this parameter box. The outputs are a retained parent and a new copy, not two daughters of a partition. The proof record is in Supplement S1 [10].

Theorem 2 (Noisy-parent copying and majority restoration). *For every ready parent (at most $15m$ wrong molecules), every parameter choice in the stated box, and an independent binomial racemic blank, contact for time 3 followed by restoration of both outputs for time 20 leaves at least one output not ready with probability at most*

$$\varepsilon_m = e^{-3m/10} + e^{-8m} + 2\rho_m, \quad \rho_m \leq 40(5/8)^m. \quad (9)$$

At $m = 40$, that is $n = 4000$ molecules per compartment, exact comparison gives $\varepsilon_{40} \leq 10^{-5}$. Using the newly formed copy as the parent of the next cycle, with a fresh independent blank each time, therefore gives success probability at least 0.999 over 100 cycles.

Appendix D proves the return and noisy-parent bounds uniformly over the parameter box, then the finite comparison and the conditional iteration. The coarser estimate $\varepsilon_m \leq 2e^{-0.003n}$, which also holds, would not by itself justify the 0.999 witness. The iterated lineage follows a chosen physical output at each cycle, namely the newly formed copy.

The resource account here is an expected count rather than a high-probability ceiling. Expected fuelled reaction events are at most $15.4n$ per cycle, counting both directions; expected spontaneous flips are $46\kappa n$, which is 1840 at $\kappa = 0.01$ and $n = 4000$. The two output compartments contain 8000 molecules in total. These figures describe a different trait, operation and pair of outputs from the repertoire mission and are not directly comparable with its feed ceiling.

8 Connections, limits and successive innovation

The two constructions isolate complementary parts of chemical heredity. The repertoire theorem preserves the presence and activity of several catalyst types through physical division. The restoration theorem preserves a distinguishable majority state despite continued errors. Presence, label fidelity and catalytic service are different observables: a population can satisfy one while failing another. This distinction also clarifies how the results connect to related directions without merging distinct models into a single unproved composite.

8.1 Random division, independent units and linkage

A simple converse shows why the architecture carrying a trait matters. Consider k independently assorting information units, with alphabet sizes $a_i \geq 2$ and positive copy caps m_i . Each unit carries one letter as identical copies, and every admitted parent contains between one and m_i copies of that unit. Each copy goes to one daughter with probability $q \in (0, 1)$, independently across copies

and units. After partition, the same word-independent recovery kernel acts on each daughter, with the two recoveries conditionally independent given the complete partition. Subsequent branch evolutions use conditionally independent copies of the declared law. Assume recovery is equivariant under independent relabelling of each unit’s alphabet and that complete erasure leaves no accessible record of that unit’s letter. Collective codes, a controller retaining the word, and a hidden chemical record are excluded by this assumption. Success requires both daughters to recover the entire original word within the copy caps.

Proposition 1 (A restricted random-division converse). *Under these hypotheses, the uniform probability of one successful cycle is at most*

$$\rho_q = \prod_{i=1}^k \left[1 - (q^{m_i} + (1-q)^{m_i}) \left(1 - \frac{1}{a_i} \right) \right].$$

For D successive cycles along an actual selected output, each required to produce two successful daughters, the probability is at most ρ_q^D . For a complete binary family of depth g , it is at most $\rho_q^{2^g-1}$. These bounds require the same hypotheses at every admitted parent state.

Proof. If unit i has $n_i \leq m_i$ positive copies, one daughter receives none with probability $q^{n_i} + (1-q)^{n_i} \geq q^{m_i} + (1-q)^{m_i}$. The two erasure alternatives are disjoint. Relabelling symmetry and absence of another record make the erased letter uniform among its a_i possibilities; recovering it has probability at most $1/a_i$. Conditional on the set of erased units, independent relabellings give the product of these guessing factors even if recovery otherwise couples units. Independent assortment then yields the displayed product bound. Any additional chemical error or failure of the copy cap can only decrease success. Apply this uniform bound conditionally at each exposed cycle. For a full family, condition first on the root’s two outputs and then on the two descendant families; induction counts $2^g - 1$ reproduction events. $\square$

For fair allocation, $q = 1/2$, write $S = \sum_i m_i$ and $L_\eta = -\log(1-\eta)$, with $0 < \eta < 1$. Since $1 - 1/a_i \geq 1/2$,

$$\rho_{1/2} \leq \exp \left(- \sum_i 2^{-m_i} \right) \leq \exp \left(-k 2^{-S/k} \right).$$

A D -cycle success guarantee of at least $1 - \eta$ therefore requires

$$S \geq k \log_2 \left(\frac{kD}{L_\eta} \right).$$

The bound is informative only within its architecture. Linking several units changes the number of independently erased records; it does not prove a universal advantage of molecular order or sequences. Correlated partition, asymmetric recovery or collective coding requires a different analysis. The catalytic repertoire theorem also tests a different trait and permits recovery of the current population, so this label-converse is not a substitute for its service and renewal proof.

8.2 A synthesis of distinct hereditary tasks

Table 5 separates results established here from interfaces needed for stronger chemical systems. The related directions include multibit carriers sharing food, physically modelled scission, food-only initiation, ancestral inference, and transitions to sequence copying. They motivate further theorems;

Task	Established here	Additional interface
Presence and useful function	Acquired three-type repertoire, both partition daughters and new local conversion credits	Selectable chemically generated variants with further useful functions
Label fidelity	Uniform noisy-parent copying and restoration of both operated outputs	A compatible carrier that preserves several labels under shared chemistry and resources
Information loss	Proposition 1 under independent assortment, symmetry and erasure	Explicit linkage, collective codes or an accessible record outside that architecture
Initiation and reproduction	Prepared founders, a declared seeding phase and paid compartment replacement	Food-only catalytic initiation and a justified membrane/scission mechanism
Memory and observation	Actual daughter states, spatial-slot credits and current-population renewal	Ancestral reconstruction from specified records; causal identification under controlled observations
Successive adaptation	Finite acquisition followed by retained function after withdrawal	Variation, selection and joint retention over successive useful innovations

Table 5: Hereditary tasks that must be connected rather than identified. The first three rows contain results proved in this article. The remaining interfaces specify stronger modelling and experimental requirements; no unproved composition of companion models is claimed.

their coexistence in a research programme does not establish that their mechanisms compose on one common law.

Food-only initiation is stronger than starting from prepared founders: it must specify a route that makes an initially absent catalytic type. Conversely, successful initiation need not give long-term inheritance under a different loss mechanism. A membrane model must connect its complete division marks, daughter allocation and leakage to the same chemistry used for service. Neither a geometric possibility of scission nor a separate reliability bound automatically supplies that connection.

Observation is another distinct task. Retaining the current repertoire does not identify which molecules are ancestral, and recovering a founder label from a population record need not mean that every lineage copied it faithfully. Wash, synthesis and division records can answer different causal questions even for the same endpoint composition. The tests in this article avoid those substitutions by reading actual immediate daughters, counting newly earned credits, and distinguishing first readiness hits from later dividing states.

8.3 A scalable target for successive useful innovation

A stronger adaptation theorem should fix generative chemical rules and a declared reproduction mechanism. For acquisition count r and failure tolerance η , it must state what may grow: molecular inventory, population, duration, fuel/material throughput, chemical diversity and rule-description length. Starting from a declared preparation, useful variants must arise through the allowed chemistry, establish under a specified selection mechanism, retain earlier functions and permit further beneficial descendants. Supplied catalyst specificity, designed substrate catalogues and environmental schedules belong in that account.

This target builds on existing positive results, including autocatalytic-core acquisition [3], composi-

tional adaptation [6] and temporal environmental learning [7]. It asks which additional events can be proved jointly and how their resource requirements scale. A finite-stock theorem has a finite horizon; a family redesigned for each r and a continuing-input system are different constructions. Neither must be dismissed, but neither by itself demonstrates indefinite novelty in a fixed closed finite system.

9 Formal verification and applicability

Theorem 1 is supported by two complementary presentations: the conventional marked-process and finite-certificate proof in Appendices A–C, and an independently stated formalization of the full switched finite-stock event [11]. The formal statement uses the same prepared old-founder condition, arbitrary allowed candidate counts and fuel, physical daughter updates, service schedule, population-renewal event and seven stock ceilings. The formal construction realizes the process from explicit driving randomness and reads every physical observation from the stopped finite-stock trajectory; failure events are counted, not conditioned away.

The formal evidence consists of strict Lean checking, an exact comparison of the independently specified statement and its substantive definitions against the model of Section 3, and separate acceptance by Lean replay and by the NanoDa and con-ron checkers. The proof sources and the fixed snapshot are identified in Supplement S1. The formal scope is Theorem 1 alone: it does not extend to the operating comparisons of Section 6, to Proposition 1, or to Theorem 2, whose complete conventional proof is Appendix D.

The recovery certificate of Appendix C is also independently reproducible. Supplement S1 supplies the rational table and a standalone verifier that checks, in exact arithmetic, 496 low-count pairs, all 4,066 count/fuel residual rows and 2,035,153 binomial summands, obtaining the displayed outward bounds. These checks certify the fixed witness used in the analytical argument; the soundness argument that this witness bounds the physical reset gain is given in Appendix C.

The effective population law still assumes prescribed catalyst specificity and jobs, supplied compartments, a controlled seeding schedule and externally implemented division/replacement. The numerical rates and stock ceilings are sufficient conditions. They are not physical minima, laboratory calibrations or an account of membrane-scission work. The restoration example uses maintained chemical activities and timed contact/separation, with expected event counts rather than a high-probability finite-stock bound.

Together the results give a precise affirmative conclusion: useful catalytic function can be acquired and maintained through actual reproduction after catalyst input stops, and a hereditary majority can be copied and restored despite continuing errors. Their guarantees depend on explicit carriers, preparations and operations. The next task is to connect those mechanisms to chemically generated variation and selection while preserving earlier functions on the same law and resource account.

Author contributions

Michael Mislan directed the research and preparation of this article. Mathematical development, formalization, source checking and manuscript preparation used AI assistance under his direction.

AI assistance and acknowledgments

AI systems assisted mathematical exploration, proof development, auditing, drafting and editorial revision. Mechanical verification is described above and in Supplement S1; no independent human peer review is asserted. The formalization uses Lean, Mathlib and the identified verification implementations, whose licences and source notices are retained with the supporting material.

Data and code availability

The published manuscript and source are available at [doi:10.5281/zenodo.23535782](https://doi.org/10.5281/zenodo.23535782). The public Supplement S1 and formalization archive accompany that record. The central finite-stock theorem is registered in Palomar, version 1.

Supplement S1 contains the literal rational table, portable exact-arithmetic verifier, result file, complete mathematical source records and retained operating-comparison support [9, 10]. Its provenance crosswalk separates original sources from supplied editions and maps development identifiers to the article’s proofs. The full central finite-stock formalization and reproduction instructions are available in the versioned public repository [11]; the immutable snapshot is identified in S1. Original code/proofs use Apache-2.0 and original prose/data CC-BY-4.0, with embedded third-party terms retained. No additional companion manuscript is needed to follow the proofs given here.

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A Physical inheritance, restart and whole-mission composition

A.1 Complementary daughters and the population floor

Proposition A1. In the model of Section 3, fix an interval I and required set J . Suppose every complete state in I belongs to $\mathcal{F}_J(16)$. At every division in I , both daughters contain at least 16 copies of every $j \in J$, directly from the initiating parent. The parent’s predivision count is at least 32 for each such type.

Proof. The complete division mark installs K_j at the parent site and $a_{ij}(t-) - K_j$ at the distinct victim site. Both are counts tested by the same postmark population floor. Their sum is the parent’s premark inventory. No intervening seed or catalytic-birth mark occurs within this complete update. This proves both statements. It also shows why a division from a parent with 16–31 copies of some required type cannot occur on the floor event. $\square$

Proposition A2. Under the seed-free law after time 30, the face Z_j in (6) is invariant, including after any almost-surely finite stopping time of the natural marked history at which the state belongs to Z_j .

Proof. At a state in Z_j , seed birth has zero rate. Every other chemical channel that changes a j -count or earns its conversion credit has a factor a_{ij} and thus zero rate. Other-species reactions and fuel recharge/discharge cannot create j . At division, Binomial(0, 1/2) is zero almost surely, so both daughters and all untouched sites still have zero copies. Every positive-rate complete mark therefore maps Z_j into itself. Bounded rates imply finitely many marks on each bounded interval. Induction over marks proves invariance there; a countable union extends the conclusion to all later times. Restart at a finite natural-history stopping time gives the conditional version by A.2 below. The face need not be a single absorbing physical state, because other types and fuel can still evolve. $\square$

A.2 The marked law at exposed stopping times

Choose a uniform rate exceeding the total marked-event rate at every physical state in either phase. For the present model the rate 93515864 suffices. Independent exponential waits at this rate and

independent marks implement each physical channel, its victim and full complementary allocation, with a residual dummy proposal. Carry cumulative observations with the state. A dummy proposal is not material input or division; a state-null actual division remains a counted physical mark.

At deterministic times, Poisson independent increments and the exponential lack of memory give the same future construction from the completed present state. This extends to grid-valued stopping times by splitting over their possible values. For a finite stopping time σ , approximate from above by $\sigma_m = 2^{-m} \lceil 2^m \sigma \rceil$. On bounded intervals only finitely many proposals occur almost surely, so the right-continuous state at σ_m eventually equals the completed state at σ . Future cylinder laws converge; bounded convergence and a monotone-class argument extend the identity to bounded future-path functionals. Localization removes a bound on σ .

Consequently a pointwise local estimate valid from every admissible completed state applies conditionally on the complete past at such a start. The filtration reveals no future wait or mark. For starts after time 30, the restarted physical law is homogeneous and seed-free. Service-grid position, cumulative stock counters and previous success flags remain carried history; they are not replenished.

For a finite physical projection, put $P = I + L/\Lambda$. Its Poisson-mixture law is $p_t = e^{-\Lambda t} \sum_{n \geq 0} (\Lambda t)^n p_0 P^n / n!$. The series converges absolutely and solves $p'_t = p_t L$ with initial value p_0 ; uniqueness of the finite linear system identifies it with $p_0 e^{tL}$. This supplies the finite-semigroup reading used by the restoration example. Observation counters remain marked-path quantities rather than being falsely declared finite-state. For a bounded-rate set of channels, the firing count minus the integral of its instantaneous rate is an integrable compensated-count martingale on finite horizons. Taking expectations identifies the expected count with that rate integral, including actual marked firings whose physical projection is unchanged.

This constructs the complete marked law used in the conventional arguments.

A.3 First failures rather than future-success conditioning

Let G protect acquisition, all required floors, fixed-grid service and initiating-division counts through 60. Suppose $\mathbb{P}(G^c) \leq a$. At every admissible recovery start after time 30, suppose

$$\mathbb{P}(H > h, D > h \mid \mathcal{F}_\sigma) \leq b, \quad (\text{A1})$$

where H is the first simultaneous Ready₅₁₂ time and D the first floor exit in the freshly stopped local run. At every finite exposed start suppose the wait for the next division exceeds w with conditional probability at most c . Define the eleven recoveries and ten waits as in Section 3.

Charge each first failed episode only on histories where earlier deadlines succeeded and no earlier floor failure occurred. That prefix indicator is measurable at the exposed start. On G , a late recovery implies the local event $\{H > h, D > h\}$. Conditional expectation of its prefix-weighted indicator is therefore at most b times the probability of that prefix. Sum the eleven terms. Apply the same argument to the ten waits. Adding G^c gives

$$\mathbb{P}(G^c \text{ or a renewal deadline fails}) \leq a + 11b + 10c. \quad (\text{A2})$$

This does not condition the future law on G . Successful preceding deadlines keep all relevant starts inside the protected horizon when $30 + 11h + 10w \leq 60$. Four rate-one basal clocks ensure $c \leq e^{-4w}$,

since the next division is no later than their next firing, even while triggers and chemistry continue. At $h = 1/2, w = 2$, the deadline is 55.5 and the wait contribution is $10e^{-8}$.

A.4 Transfer to finite supplies

Construct the unlimited and finite-stock versions of the **same switched law** with identical proposal waits, channel decisions, victims and complete split marks. In the finite version stop at the first proposal that would exceed a declared cumulative allocation. Before that proposal the states, rates and ledgers agree by induction over proposals. On an event bounding every cumulative use through 60 by the allocation, no such excess occurs, so the entire histories agree through 60.

If the three resource-cap failures have total probability at most $3e^{-20}$ on the unlimited switched law, combining this coupling with (A2) costs that probability once. No local recovery receives a new budget. This proves the event-level finite-stock transfer rather than merely appending stock numbers to an unlimited-source theorem.

B Acquisition, all-time floors and continuing service

B.1 The precise certificate needed for establishment

For $N = 2048$ and $\beta = 4,000,000$, define

$$h_5(a) = \frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}, \quad C_0 = 1860480h_5(12)(4/3)^{12},$$

$$A(a) = \begin{cases} C_0(3/4)^a, & a \leq 12, \\ 1860480h_5(a), & a \geq 13, \end{cases} \quad g(0) = -A(0)/2, \quad g(a) = A(a) \min(6a, 70) \quad (a \geq 1),$$

$$W(a, 1) = A(a), \quad W(a, 0) = A(a) + g(a)/\beta. \quad (\text{B1})$$

Heuristically, W is a barrier function for the floor. The constant $1860480 = 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20$ normalizes $A(15) = 1$, and A is decreasing, so W is at least 1 wherever a count is below the floor 16 and is tiny wherever a count is at least 512. The inverse-polynomial shape is dictated by division: halving a count multiplies h_5 by about $2^5 = 32$ (this is the leading term of $J(a)$ below), and that penalty must be paid by the downward drift that catalytic growth supplies between divisions. The geometric prefix below 12 keeps W bounded as the count approaches zero, and the small correction g/β separates the fueled and unfueled values by the amount the recharge channel needs.

The finite rational branching certificate proves $0 \leq W \leq 80$, $W \geq 1$ below 16, and $W \leq 6/10^8$ above 512. For $\Phi_j = \sum_i W(a_{ij}, R_i)$, its required full-generator bounds are

$$L\Phi_j \leq -\frac{1}{2}\Phi_j + 4b_0 \quad (t < 30), \quad L\Phi_j \leq 4b_0 \quad (t \geq 30, \text{ all } j\text{-counts} \geq 16), \quad b_0 = 13/10^7. \quad (\text{B2})$$

The second assertion is only a live-region retention bound. It is not an all-state seed-free acquisition theorem. Both include the complete physical division and victim law. The concrete finite

row/moment obligations are identified in Appendix C.4; neither acquisition probability nor eventual mission success is assumed as an input.

The relation of the finite row check to the physical law is as follows. Write $u = a/N$, let $U_f \in [0, 2]$ be the other two species' total normalized count, and let $v \in [0, 1843200]$ be their combined active birth/death intensity. Their ordinary/active fuel loss is $H_f = \beta U_f + v$. Put $E_r(a) = \mathbb{E}[W(K, r)]$, $K \sim \text{Binomial}(a, 1/2)$. With seed coefficient $s \in \{0, 6\}$, the branching upper operator tested by the certificate is

$$\begin{aligned} \mathcal{S}_s W(a, 0) &= s(1-u)[W(a+1, 0) - W(a, 0)] + a[W(a-1, 0) - W(a, 0)] \\ &\quad + \beta[W(a, 1) - W(a, 0)] + 2E_0(a) - 2W(a, 0), \\ \mathcal{S}_s W(a, 1) &= s(1-u)[W(a+1, 1) - W(a, 1)] + a[W(a-1, 1) - W(a, 1)] \\ &\quad + 800a(1-u)[W(a+1, 0) - W(a, 1)] + 400a[W(a-1, 0) - W(a, 1)] \\ &\quad + [\beta(1+u) + H_f][W(a, 0) - W(a, 1)] \\ &\quad + E_1(a) + E_0(a) - 2W(a, 1) + (u + U_f)[2E_0(a) - W(a, 1)]. \end{aligned} \tag{B3}$$

At $a = 0$ the term $W(-1, \cdot)$ may be defined as zero because its coefficient vanishes; at $a = N$ the upward coefficients vanish. All binomial outcomes, including below-floor daughters, enter E_r . A basal division gives complementary sisters with the same marginal count law and one inherited fuel token when the parent is fueled; a triggered division consumes that token and has two fuel-zero outputs. Summing basal marks over the four possible parents gives two daughter gains, total parent loss $-\Phi_j$, and uniform-victim loss $-\Phi_j$: the latter follows because each site is a victim for three other parents with probability $1/3$. This is the basal $-2W$ in (B3). For triggers, dropping the nonpositive victim-weight loss gives precisely its last upper-bound term. The derivation uses complementary marginals, never sister independence.

For clarity the full daughter expectations in (B3) have short exact formulas. The identity

$$h_5(k) = \frac{1}{24} \int_0^1 t^k (1-t)^4 dt$$

follows by integrating the degree-four polynomial, or the beta integral. Averaging t^K gives $((1+t)/2)^a$; substitution $x = (1+t)/2$ and subtraction of the integral over $[0, 1/2]$ yield

$$J(a) := \mathbb{E}h_5(K) = 32h_5(a) - \frac{4}{3} \sum_{j=0}^4 \frac{(-1)^j \binom{4}{j}}{(a+j+1)2^{a+j+1}}.$$

Hence, writing $D = 1860480$,

$$\begin{aligned} E_A &= DJ(a) + 2^{-a} \sum_{k=0}^{\min(a, 12)} \binom{a}{k} [A(k) - Dh_5(k)], \\ E_g &= 70E_A + 2^{-a} \sum_{k=0}^{\min(a, 11)} \binom{a}{k} [g(k) - 70A(k)], \quad E_1 = E_A, \quad E_0 = E_A + E_g/\beta. \end{aligned} \tag{B4}$$

The finite corrections are exact because the geometric prefix ends at 12 and $g(k) = 70A(k)$ from 12 onward. Each residual in (B3) is affine in (U_f, v) , so verifying it at the four corners of their

rectangle covers every possible load from the other two types, including loads correlated with the tested count along the path.

For $\beta = 4,000,000$ the certificate enumerates all $2049 \cdot 2 \cdot 4 = 16392$ seed-on rows of $\mathcal{S}_6W + W/2$ and all $2033 \cdot 2 \cdot 4 = 16264$ live seed-free rows. The latter check the stronger $\mathcal{S}_0W + W/2 \leq b_0$; dropping the negative term gives the postwithdrawal inequality in (B2). Removing seeds increases the drift of this decreasing potential, so the seed-on pass alone cannot establish the seed-free result. The row lists and their checks are retained in S1.

The shape properties of (B1) also have an elementary all-count proof. The ratio $A(a+1)/A(a)$ is $3/4$ through the prefix and $(a+1)/(a+6)$ from 12 onward. The unfueled 0-to-1 ratio at beta one million is $6000036/7999996 < 1$; the varying prefix obeys $3(M+6)/(4M) < 1$ for $M \geq 1000000$, after which its multiplier is constant. Both branches are positive and decreasing, $A(15) = 1$, and $C_0 < 80$. Direct rational substitution at 512 gives the prepared bound. For larger beta the unfueled branch is a convex combination of its beta-one-million value and A , so these bounds persist. These shape arguments do not assume survival or acquisition.

Stop the two-old-species potential at its first complete floor failure. Its initial value is at most $8(6/10^8)$, its drift before exit is at most $8b_0$ across both phases, and its stopped value is at least one on a failure. Dynkin's formula gives

$$\mathbb{P}(\text{old floor failure by } 60) \leq 48/10^8 + 480b_0. \quad (\text{B5})$$

No post-exit drift premise is used. For the candidate, $\Phi_j(0) \leq 320$, so the seed-on inequality and an integrating factor yield

$$\mathbb{E}\Phi_j(30) \leq 320e^{-15} + 8b_0(1 - e^{-15}).$$

From the actual time 30 distribution, stop at candidate floor failure. If a count is already below 16 at 30, the stopping time is 30 itself and the barrier already applies. Another 30 time units cost at most $120b_0$. Thus acquisition plus subsequent candidate-floor failure has probability at most $320e^{-15} + 128b_0$. Adding (B5) gives

$$48/10^8 + 320e^{-15} + 608b_0. \quad (\text{B6})$$

This is why no assumption that the candidate is Ready_{512} at time 30 is needed.

B.2 A service inequality on the full shared-fuel law

Fix a site/species pair. Let Q count its ordinary conversions since an exposed block start, without being overwritten at division. Let D be the first complete mark violating the current required count floors. For $0 < z \leq \rho \leq 1$, use the killed exponential weight

$$Y = z^Q \rho^R.$$

Set $H_s = 800 \cdot 3N \cdot 9/16 + 3\beta + 8$. When $R = 0$, recharge contributes $-\beta(1 - \rho)z^Q$. Incoming division either keeps fuel empty or supplies fuel, preserving or decreasing this weight. Seeds and passive deaths leave it unchanged.

When $R = 1$, the selected ordinary job contributes $\beta(a/N)(z - \rho)z^Q$. Discharge and the two other ordinary jobs cost at most $3\beta(1 - \rho)z^Q$. Each active birth/loss pair has rate $800Nu(3/2 - u) \leq 800N(9/16)$; three such pairs give the first part of H_s . Own division intensity is at most 4. Incoming victim replacement has intensity at most 4, so all touching divisions cost at most $8(1 - \rho)z^Q$. This bound includes the full donor/sister fuel mark. Non-touching events preserve the function.

If the required floor is K , then $a \geq K$ before D , and $z - \rho \leq 0$. Therefore the two scalar conditions

$$\theta \leq \beta(1 - \rho), \quad \beta(K/N)(z - \rho) + H_s(1 - \rho) + \theta\rho \leq 0 \quad (\text{B7})$$

imply $L^D Y \leq -\theta Y$. Killing replaces a nonnegative postmark weight by zero and cannot increase this generator. The resulting supermartingale estimate is

$$\mathbb{E}[1_{\{D > w\}} Y_w] \leq e^{-\theta w} Y_0 \leq e^{-\theta w}.$$

On $Q_w < m$, the conservative lower bound $Y_w \geq \rho z^m$ holds. Hence

$$\mathbb{P}(D > w, Q_w < m \mid \text{exposed past}) \leq \rho^{-1} z^{-m} e^{-\theta w}. \quad (\text{B8})$$

This is an intersection bound, not a probability conditional on future floor survival. Division-coupled conversion credits can only help the quota. Seed withdrawal contributes zero to the generator of this count-independent weight, so the argument is unchanged in the two phases.

At $z = 3/4, \rho = 79967/80000, \theta = 1650, K = 16$, recharge gives equality in the first condition, and the second residual is $-598067/10000$. A rational lower Taylor polynomial for e^{165} gives

$$\frac{12}{\rho} (4/3)^{500} e^{-165} < 10^{-8}. \quad (\text{B9})$$

This charges at most twelve coordinate quotas per block. Applying (B8) with past-measurable live-prefix indicators and then event inclusion charges all 600 original blocks by 600×10^{-8} , without changing any block during recovery.

For an entirely rational version of (B9), put $P_{200}(x) = \sum_{j=0}^{200} x^j/j!$. The finite comparison is

$$\frac{12}{\rho} \frac{(4/3)^{500}}{P_{200}(165)} < \frac{1948303823}{2500000000000000000} < 10^{-8}.$$

Since $P_{200}(165) \leq e^{165}$, this implies the stated probability bound.

B.3 Closing the base operating event

Each site's actual initiation count dominates its rate-one basal count on $[30, 60]$. For a Poisson mean 30 count, exponential Markov with factor 6 gives at most $6^4 e^{-25}$ for fewer than five firings. Four sites cost at most $5184e^{-25}$, without independence from floors or service.

Adding this to (B6) and (B9), then using $e^{15} > 2900000$ and $e^{25} > 10^{10}$, proves the direct base-event bound. Conservatively replace $608b_0$ by $720b_0$ to retain the accepted allocation

$$a_* = 48/10^8 + 320/2900000 + 600\{(6/5)b_0 + 10^{-8}\} + 5184/10^{10} = 9545923/9062500000. \quad (\text{B10})$$

This event includes candidate establishment, every required floor and grid quota, and each site's five initiating divisions. The spare $112b_0$ is a conservative allocation, not a separate unproved hazard.

C Recovery, resource clocks and finite certificates

C.1 A general potential-and-clock bound

Consider a finite physical marked process stopped on leaving a live set. Reset marks may have null physical updates and are still counted. Write Q_0 for the live generator with reset gains removed and resets treated as killing, and B for the positive live-poststate reset gain. Suppose

$$U \geq 1, \quad U(x) \leq M \text{ at admissible starts,} \quad Q_0 U \leq -\gamma U, \quad BU \leq CU. \quad (\text{C1})$$

Let ν be the reset intensity. A bounded physical statistic F , retained at the terminal stopped mark, is assumed to satisfy

$$0 \leq F \leq F_{\max}, \quad \nu - a \leq -\kappa LF, \quad |\Delta F| \leq b, \quad \sum_e r_e(x)(\Delta_e F)^2 \leq V. \quad (\text{C2})$$

For $T > 0, K \geq 1, \delta > \kappa F_{\max}$ and $0 < \theta < 3/b$, the surviving probability is at most

$$Me^{-\gamma T} \sum_{j=0}^{K-1} \frac{(CT)^j}{j!} + \mathbb{P}\{\text{Poisson}(aT + \delta) \geq K\} + \exp \left[-\theta(\delta/\kappa - F_{\max}) + \frac{VT\theta^2}{2(1 - b\theta/3)} \right]. \quad (\text{C3})$$

Proof. Let $P_j(t)$ be the unnormalized live law with exactly j resets. Its equations are $P'_0 = P_0 Q_0$ and $P'_j = P_j Q_0 + P_{j-1} B$. Apply (C1), an integrating factor and induction to obtain $P_j(t)1 \leq P_j(t)U \leq Me^{-\gamma t}(Ct)^j/j!$. This bounds histories with fewer than K resets.

The compensated resource $M_t^F = F_t - F_0 - \int_0^t LF_s ds$ gives

$$\Lambda_T := \int_0^T \nu_s ds \leq aT + \kappa(F_{\max} + M_T^F).$$

For bounded jumps, the exponential integral remainder and positive Taylor series give $e^{\theta x} - 1 - \theta x \leq x^2 \theta^2 / [2(1 - b\theta/3)]$ for $|x| \leq b$. The compensated-jump exponential supermartingale therefore bounds the probability $\Lambda_T > aT + \delta$ by the third term of (C3).

Finally realize the reset count as $N_T = Y(\Lambda_T)$ for a unit-rate Poisson primitive Y . Although Y and its clock may be dependent,

$$\{N_T \geq K, \Lambda_T \leq aT + \delta\} \subseteq \{Y(aT + \delta) \geq K\}$$

holds pathwise because every Poisson path is nondecreasing. This proves the middle term without assuming clock/count independence. Sum the three events. The restart argument of A.2 gives the same bound conditionally at every admissible completed stopping-time start. $\square$

C.2 The actual fuel clock

For the present model take $F = \sum_i R_i$, the number of fueled sites, and $Z = \sum_i R_i \sum_j a_{ij}/N$. The total division intensity is $\nu = 4 + Z$, with $Z \leq 3F$. Counting the complete marks gives

$$LF = \beta(4 - 2F - Z) - A_{\text{active}} - D_{\text{fuel}}, \quad A_{\text{active}} \geq 400NZ.$$

Consequently, for $\kappa = 3/(5\beta + 1200N)$ and $a = 4 + 4\beta\kappa$,

$$-\kappa LF \geq \kappa\{(\beta + 400N)Z + 2\beta F - 4\beta\} \geq Z - 4\beta\kappa = \nu - a.$$

Fuel lies between zero and four. A complete division can remove at most two fuel tokens, so the jump bound is 2. Recharge, discharge and ordinary jobs contribute at most 20β to the squared-jump rate; active birth/loss contributes at most $5400N$; divisions contribute at most $16 \times 4 = 64$. Thus $V = 20\beta + 5400N + 64$. These are statewise bounds, valid with arbitrary nuisance histories. Seeds change no fuel and do not alter them. In words, every triggered division consumes a fuel token, so a burst of divisions must be paid for by fuel that the recharge channel supplies at a bounded rate; this is what lets the compensated fuel count control the integrated division intensity Λ_T in (C3).

C.3 The recovery potential and its numerical use

Let $H^+[a][r]$ be the rational table supplied in Supplement S1, with column $r = 0$ for an unfueled site and $r = 1$ for a fueled one. For $16 \leq a < 512$, define

$$A_{\text{rec}}(a) = H^+[a][0], \quad \Delta(a) = H^+[a][0] - H^+[a][1].$$

Set $A_{\text{rec}}(a) = \Delta(a) = 0$ below 16. At and above 512 use the exact formulas

$$f(a) = \frac{(980 - a)(981 - a)(982 - a)}{(a - 10)(a - 9)(a - 8)} \quad (a \leq 768), \quad f(a) = f(768)(10/11)^{a-768} \quad (a > 768),$$

$$A_{\text{rec}}(a) = f(a)/f(512), \quad \Delta(a) = A_{\text{rec}}(a)\{258 + k(a)\}/8000000.$$

For $512 \leq a < 768$, $k(a) = 6(1 - a/N)[f(a + 1)/f(a) - 1] + a[f(a - 1)/f(a) - 1]$; for $a \geq 768$, $k(a) = 5647a/56320 - 6/11$. Set both entries zero below 16 and define

$$h(a, 0) = A_{\text{rec}}(a), \quad h(a, 1) = A_{\text{rec}}(a) - (11/8)\Delta(a), \quad U = \sum_{i,j} h(a_{ij}, R_i). \quad (\text{C4})$$

Heuristically, U measures how far the current population is from Ready_{512} . Each coordinate's contribution h decreases with the count and is normalized by $A_{\text{rec}}(512) = 1$, so a deficient coordinate

contributes at least about 1 and a replenished one much less. Between divisions the chemistry pulls U down exponentially at rate about 154, so a deficit is repaired on a time scale of about one hundredth of a time unit. A division can increase U , by a factor at most 478 as shown below, but divisions arrive at a bounded rate, and the fuel clock of C.2 bounds how many can occur in a window of length $1/2$. The general bound (C3) combines these three facts.

Finite rational comparisons establish the barrier, positivity and fuel order; the seed-free chemical inequalities are $(G_0 + 150)h \leq 0$ below 512 and $(G_0 + 150)h \leq 1/4700$ above it. They include counts 16, 511, 512, 768 and 2048. The positive division operator is unchanged by withdrawing chemical seeds. The direct fixed-data proof below gives $BU \leq 478U$. Since $U \geq 1$ on a live non-Ready₅₁₂ state and removal of resets contributes at least 4 killing,

$$Q_0U \leq -(154 - 12/4700)U, \quad U \leq M = 338971453059/125000 \quad \text{at all admissible starts.} \quad (\text{C5})$$

The recovery live domain is the set where all 12 current counts are at least 16 and at least one is at most 511. Stop on the first floor failure or simultaneous Ready₅₁₂; the terminal complete mark and its fuel bits define that stopping event. The inequality $U \geq 1$ is needed only on this live, non-Ready₅₁₂ domain, not on all count states.

Here is the link from the checked chemical rows to the full population generator. Let $s(a) = 6(1 - a/N)$, $m(a) = a$, $b(a) = 800a(1 - a/N)$, $d(a) = 400a$. With $\sigma = 0$ after withdrawal, the reference two-fuel chemical operator is

$$\begin{aligned} G_\sigma h(a, 0) &= \sigma(1 - a/N)[h(a + 1, 0) - h(a, 0)] + m(a)[h(a - 1, 0) - h(a, 0)] + \beta[h(a, 1) - h(a, 0)], \\ G_\sigma h(a, 1) &= \sigma(1 - a/N)[h(a + 1, 1) - h(a, 1)] + m(a)[h(a - 1, 1) - h(a, 1)] \\ &\quad + b(a)[h(a + 1, 0) - h(a, 1)] + d(a)[h(a - 1, 0) - h(a, 1)] \\ &\quad + [\beta(3 + a/N) + 800N][h(a, 0) - h(a, 1)]. \end{aligned} \quad (\text{C6})$$

The selected coordinate's own ordinary conversion and discharge contribute $\beta(1 + a/N)$. Each foreign coordinate's ordinary/active fuel consumption is bounded by $T(z) = 800z(1 - z/N) + 400z + \beta z/N \leq T(N) = 400N + \beta$, since $\beta \geq 400N$ makes T increasing on $[0, N]$. This proves the final reference coefficient in (C6). The actual fuel-off rate is no larger; multiplying their difference by $h(a, 0) - h(a, 1) \geq 0$ shows that the actual chemical generator is bounded above by the reference generator on each coordinate. Its sum is valid with shared fuel and correlated foreign histories. No independent-coordinate process replaces the population.

C.3.1 A direct certificate for the recovery profile

The reset gain is controlled directly by a fixed rational envelope. Write $A = A_{\text{rec}}$, $D = \Delta$, $t_0 = A$, $t_1 = A - D$, and set $\delta = 10^{-18}$, $\ell = 9997/10000$, $c = 11/8$. Use the supplied two-column table $P(a, r) = H^+[a][r]$ and put

$$X(a, r) = \max\{0, P(a, r) - \delta\}, \quad P(a, r) - \delta \leq t_r(a) \leq P(a, r) \quad (512 \leq a \leq 2048, r \in \{0, 1\}). \quad (\text{C7})$$

Below 512 the table entries themselves define the witness. At and above 512 the witness is the exact formula (C4), and the table is an outward envelope of it, used only through the comparisons (C7).

Supplement S1 [9] supplies the complete table and a standalone exact-arithmetic verifier for these comparisons and for the integrality of every $10^{18}P(a, r)$.

The first finite obligation is

$$A(a) > 0, \quad 0 \leq D(a) \leq A(a)/1251 \quad (16 \leq a < 512). \quad (\text{C8})$$

Below 512 these are finite checks on the tabulated entries themselves.

The corresponding tail comparison has an elementary proof. For $512 \leq a < 768$, the positive rational f decreases and

$$\frac{f(a-1)}{f(a)} = \frac{983-a}{980-a} \frac{a-8}{a-11}.$$

The seed term in $k(a)$ belongs to $[-6, 0]$. The passive term is nonnegative and is less than $768[(203/200)(503/500) - 1] = 1619712/100000 < 17$, using $980 - a > 200$ and $a - 11 > 500$. Thus $-6 \leq k < 17$. For $768 \leq a \leq 2048$, the printed linear formula and $1/10 < 5647/56320 < 1/9$ give $0 < k < 228$. Consequently

$$0 < D(a)/A(a) < 1/10000 \quad (512 \leq a \leq 2048).$$

Together with (C8), this proves positivity, fuel-zero domination and

$$A(a) \leq P(a, 0), \quad \ell X(a, r) \leq h(a, r), \quad 0 < h(a, 1) \leq h(a, 0) \quad (\text{C9})$$

on every live coordinate row. In the tail, use (C7) and $X \leq t \leq P$; below 512, use $P = t$, $X \leq t$ and (C8).

For $a < 32$ put $M^+(a) = 0$. Otherwise define the complete paired-floor moment and daughter residual by

$$\begin{aligned} M^+(a) &= 2^{-a} \sum_{k=16}^{a-16} \binom{a}{k} P(k, 0), \\ p(a, r) &= 1 + r(2 + a/N), \\ Y(a, r) &= 2p(a, r)M^+(a), \\ d_* &= 93867565097/200000000. \end{aligned} \quad (\text{C10})$$

The remaining reset certificate is the following two finite all-row comparisons:

$$\begin{aligned} \max_{16 \leq a \leq 511, r \in \{0,1\}} [Y(a, r) - d_* X(a, r)] &\leq b_L = -147104211175367700201/500000000000000000, \\ \max_{16 \leq a \leq 2048, r \in \{0,1\}} [Y(a, r) - d_* X(a, r)] &\leq b_A = 3343277526704540851/125000000000000000. \end{aligned} \quad (\text{C11})$$

These are outward upper bounds, not equalities identifying the true maxima. Their signed combination is exactly

$$b_L + 11b_A = -367902757/5000000000000000 < 0. \quad (\text{C12})$$

For independent verification, read the fixed rational P , check its grid integrality and (C7)-(C8), and evaluate (C10) for every $a = 32, \dots, 2048$. Start a summation at $\binom{a}{16}$, and update consecutive coefficients by $\binom{a}{k+1} = \binom{a}{k}(a-k)/(k+1)$; include every k through $a-16$. Compare every one of the 4066 count/fuel residuals with b_A , and the 992 deficient rows with b_L . There are 2,035,153 summand iterations. No search, linear-system solve or population-state enumeration is involved.

Here is the soundness argument: why these finite comparisons bound the reset gain of the full population process. A fixed retained site has three other possible parents. Each has rate at most four and avoids that site as victim with probability $2/3$, so the total retained contribution is at most $8U$. For a chosen parental coordinate (a, r) , the parent's rate is at most $p(a, r)$. On a surviving split, the sisters have counts $K, a-K$, with $16 \leq K \leq a-16$. Fuel-zero domination and (C9), followed by complementary-binomial symmetry, bound their combined expected potential by $2M^+(a)$. This uses the two marginals of one complementary pair, without assuming independent sisters or independent coordinate dynamics. Dropping other coordinates' floor tests and simultaneous-Ready killing only increases the nonnegative gain. Therefore

$$BU \leq 8U + \sum_{i,j} Y(a_{ij}, R_i).$$

Every live nonReady prestate has a deficient coordinate. Apply the first bound in (C11) there and the second at the other eleven coordinates. By (C12) and (C9), $\sum_{i,j} Y(a_{ij}, R_i) \leq d_* \sum_{i,j} X(a_{ij}, R_i) \leq (d_*/\ell)U$. It follows that

$$\begin{aligned} BU &\leq C_{\text{direct}}U < 478U, \\ C_{\text{direct}} &= 8 + d_*/\ell = 95467085097/199940000, \\ 478 - C_{\text{direct}} &= 104234903/199940000 > 0. \end{aligned} \quad (\text{C13})$$

The argument retains the complete physical marks, shared fuel and state-null divisions, and withdrawing chemical seeds changes neither the reset law nor this proof. The mission bounds use the conservative constant 478.

C.3.2 Exact finite verification

The supplied exact-arithmetic verifier checks grid integrality, the high-tail enclosures in (C7), all 496 low-count pairs in (C8), and all 4,066 count/fuel rows in (C11), including 992 deficient rows. It evaluates exactly 2,035,153 binomial summand iterations and obtains the outward endpoints printed in (C11). The result file and verification instructions are in Supplement S1. These fixed-data comparisons are time-independent lemmas.

The table and verifier are self-contained: the verifier reconstructs the analytic high tail from (C4) and evaluates the paired moment with integer binomial recurrences, and no search or auxiliary linear solve is needed. Running it reproduces the finite arithmetic certificate; the soundness argument above is what connects that certificate to the physical reset gain.

C.3.3 Operating comparisons and recovery pricing

The continuously seeded $\beta = 8,000,000$ comparison point of Section 6 uses its own profile, chemical and mission inequalities; its proof record is retained in S1 [9].

Use (C3) with $T = 1/2, K = 14, \delta = 1/100, \theta = 1/640, C = 478$. At $\beta = 4,000,000$, the fuel-clock exponent is $28228801/460320$ and the comparison mean is $1080309/350900$. The conservative exponent $9141301/460320$ and mean $321/100$ suffice. In words, the fuel clock gives the division count in the half-unit window a Poisson comparison mean of about 3.1, so fourteen or more divisions are very unlikely, while the contraction $e^{-\gamma/2} \approx e^{-77}$ overwhelms the reset factor 478 raised to at most thirteen. Positive lower Taylor polynomials and upper geometric remainders yield

$$q_{\text{off}}(1/2) \leq 8622092430140/10^{18} < 1/100000. \quad (\text{C14})$$

This covers histories with 0–13 resets and the event of at least 14 resets. It bounds $\mathbb{P}(H > 1/2, D > 1/2)$; floor exits during recovery are charged separately through G . The $\beta = 8,000,000$ comparison point instead uses $T = 1/5, K = 9$ and obtains a bound below $1/50000$.

The scalar numerical claim in (C14) can be checked using rational expressions alone. Put $P_-(x) = \sum_{j=0}^{200} x^j/j!$, and, for $0 \leq x < 202$, put

$$P_+(x) = P_-(x) + \frac{x^{201}}{201!} \frac{1}{1 - x/202}.$$

The remaining positive exponential tail has successive-term ratio at most $x/202$, so $P_-(x) \leq e^x \leq P_+(x)$. With $\mu = 321/100$ and $E = 9141301/460320$, an explicit rational upper bound in (C14) is

$$\frac{M \sum_{j=0}^{13} (478/2)^j/j!}{P_-(\gamma/2)} + 1 - \frac{\sum_{j=0}^{13} \mu^j/j!}{P_+(\mu)} + \frac{1}{P_-(E)}.$$

This sum is compared with $8622092430140/10^{18}$ in exact arithmetic before the final outward rounding.

C.4 Summary of the finite certificate obligations

Table 6 summarizes the finite checks on which the proof of Theorem 1 relies. Each is a finite exact-arithmetic proof, not an assumption.

The paired-reset moment uses the two marginals of one complementary split and cannot be replaced by two independent daughter probabilities. All of these checks are reproduced by the verifier and records in S1.

C.5 Feed, seeds and divisions

At an occupied fuel site, the combined consuming rate is at most

$$C_* = 4\beta + 1200N + 3 = 18457603.$$

Table 6: Finite certificate obligations in the proof of Theorem 1.

Obligation	Concrete objects and finite checks
Branching and retention	The explicit function (B1); full binomial daughter moments; all 2049 counts, two fuels and four load corners for the seeded bound; live counts 16–2048 for the seed-free bound
Literal recovery input	Exact rational profile values below 512; positive tail (C4); barriers, maximum and fuel order; 4066 physical count/fuel rows for seed-free chemistry
Paired reset	For $K \sim \text{Binomial}(a, 1/2)$, exact truncated moment of the profile over $16 \leq K \leq a - 16$; retained-victim coefficient 8; deficient-coordinate max inequality over all count/fuel rows
Scalar bounds	Taylor-polynomial/rational inequalities for service, recovery, waits and stock caps; outward rounding only after summing where specified

Indeed each of three species contributes $800Nu(3/2 - u) + (\beta + 1)u$, increasing through $u = 1$ at this beta; add discharge β . Basal victim-fuel loss has intensity F ; triggered victim loss is at most $3F$.

Let A_T count accepted recharges. For $t \geq s \geq 0$, tilting only these marks by e^t and using $v = e^{-sF}$ gives

$$L_t v / v \leq \beta(4 - F)(e^{t-s} - 1) + F(C_* + 1 + 3e^s)(e^s - 1) \leq 4k(t, s),$$

where k is the maximum of the two coefficients. In a triggered division with occupied victim fuel, split $e^{2s} - 1 = (e^s - 1) + e^s(e^s - 1)$: the parent token is charged in C_* , the extra victim token in the last term. Thus

$$\mathbb{P}(A_{60} > B) \leq \exp\{s(4 - F_0) + 240k(t, s) - tB\}. \quad (\text{C15})$$

Choose $t = 1/10000, s = 400/22457607, B = 800000000$. The elementary $e^x \leq (1 - x)^{-1}$ for $0 \leq x < 1$ bounds the two coefficients by explicit rationals. Cross multiplication shows that the first dominates:

$$k(t, s) \leq \max \left\{ \frac{\beta(t - s)}{1 - t + s}, \left(C_* + 1 + \frac{3}{1 - s} \right) \frac{s}{1 - s} \right\} = \frac{\beta(t - s)}{1 - t + s}.$$

Substitution in (C15) therefore bounds the exponent by

$$-5508991479342664251200/5043026607980323551 < -1092 < -20.$$

Seeds preserve A and F , so their withdrawal does not alter this proof. Seed intensity is at most 72 before 30 and zero afterward. With exponential parameter $1/10$ and $e^{1/10} < 10/9$, the 5000-seed tail is below e^{-260} . Actual division intensity is at most 16 throughout; parameter $1/5$ and $e^{1/5} - 1 < 31/140$ give the 1200-division tail at most $e^{-192/7} < e^{-20}$. State-dependent accepted counts need not be independent Poisson variables for these exponential moment bounds.

Finally complete-mark conservation gives

$$F_0 + A = F + B_{\text{cat}} + H + Q + U_{\text{waste}} + W_R, \quad B_{\text{cat}} + H + Q + U_{\text{waste}} \leq A + 4,$$

$$B_{\text{cat}} + E \leq A + 4 + E, \quad W_R \leq D, \quad W_S \leq 3ND. \quad (\text{C16})$$

These identities and inequalities hold at every prefix, including terminal jumps. Use A.4 to transfer the unlimited-stock mission to finite stocks on the simultaneous cap event. Substituting (B10) and (C14) into (A2) and charging three global tails proves (7), with a rational outward bound (4).

D Majority restoration with continual errors

We give the complete conventional proof of Theorem 2: the exact count law, inequalities uniform over the parameter box, return potentials, the noisy-parent copying estimate and the finite witness. The proof record is retained in S1 [10].

D.1 The exact law and operated cycle

A compartment contains n molecules in two conformations. Fix either label, let y be the number wrong relative to it and put $k = n - y$. Fuel and waste are chemostatted, with reverse factor $r = e^{-A}$; A is the affinity in units of $k_B T$. The concentration is fixed as n varies. With intensive restoration rate $a = 1$, the wrong-count birth and death rates are

$$u_n(y) = \frac{ky(y-1)}{n^2} + r \frac{k(k-1)(k-2)}{n^2} + \kappa k, \quad d_n(y) = \frac{yk(k-1)}{n^2} + r \frac{y(y-1)(y-2)}{n^2} + \kappa y. \quad (\text{D1})$$

The three terms in each direction are respectively forward cooperative conversion consuming fuel, reverse conversion releasing fuel, and spontaneous error. Falling factorials are exact, including at the boundary; a zero factor makes its channel zero. In particular $u_n(0) > 0$. Purity is not absorbing, and the argument never conditions on absence of errors. Exchanging the labels gives $u_n(n-y) = d_n(y)$.

For copying, write the wrong counts as (p, z) for parent and new compartment. Parent jumps have rates $\kappa(n-p), \kappa p$. New-compartment jumps have spontaneous rates $\kappa(n-z), \kappa z$, together with copying rates

$$\begin{array}{c|cc} & \text{forward} & \text{reverse} \\ z \mapsto z+1 & \alpha(n-z)p/n & r\alpha(n-z)(n-p)/n \\ z \mapsto z-1 & \alpha z(n-p)/n & r\alpha zp/n. \end{array} \quad (\text{D2})$$

Only parent molecules catalyze the copying reactions. Cooperative restoration is off during contact; spontaneous errors continue in both compartments. No operation reads the label or freezes the parent.

Take $n = 100m$, integer $m \geq 10$, and fixed parameters

$$\alpha = a = 1, \quad \frac{9}{200} \leq r \leq \frac{11}{200}, \quad \frac{9}{1000} \leq \kappa \leq \frac{11}{1000}. \quad (\text{D3})$$

Start from any parent with $y_0 \leq 15m$ and a new compartment whose wrong count is $\text{Binomial}(n, 1/2)$. Run (D2) for $T_c = 3$, then separate the pair and restore each under (D1) for $T_r = 20$. Each output is ready when its wrong count is at most $15m$. Subsequent restoration laws are independent conditional on the pair at separation; the copying counts themselves need not be independent. The proof below uses only their marginal capture bounds and a union bound.

For either finite count chain use $Lf(x) = \sum_c w_c(x)[f(\tau_c x) - f(x)]$. The finite law and time-dependent expectation identity follow by A.2, or directly by differentiating the finite matrix exponential. All forthcoming probability estimates are at specified fixed times.

D.2 An unstopped two-potential return bound

Let $V, W \geq 0$, $\beta_1, \beta, \delta \geq 0$, and $\mu, v_S, v_H, w_R > 0$. Suppose

$$LV \leq \beta_1, \quad LW \leq -\mu W + \beta + \delta \mathbf{1}_S, \quad S \subseteq \{V \geq v_S\}, \quad \text{Bad} \subseteq \{V \geq v_H\} \cup \{W \geq w_R\}.$$

For every initial state x_0 and $T \geq 0$,

$$\mathbb{P}_{x_0}(X_T \in \text{Bad}) \leq \frac{V(x_0) + \beta_1 T}{v_H} + \frac{e^{-\mu T} W(x_0) + [\beta + \delta(V(x_0) + \beta_1 T)/v_S]/\mu}{w_R}. \quad (\text{D4})$$

Indeed $\mathbb{E}V(X_t) \leq V(x_0) + \beta_1 t \leq V(x_0) + \beta_1 T$. Markov's inequality therefore bounds $\mathbb{P}(X_t \in S)$ by this last expression divided by v_S . Insert that bound into the expectation inequality for W , multiply by $e^{\mu t}$, and integrate. Dropping the factor $1 - e^{-\mu T} \leq 1$ gives the numerator in (D4). Apply Markov's inequality to V, W on the displayed covering of Bad . The chain has not been stopped at an excursion above a barrier.

For integers $0 \leq L < R < I < H \leq n$, $L + 1 < H$, and slopes $q, q_1 > 1$, set

$$V(y) = q_1^{(\min(y, H) - L)^+}, \quad W(y) = \begin{cases} q^{(y-L)^+}, & y < H, \\ 0, & y \geq H. \end{cases} \quad (\text{D5})$$

Suppose on $L < y < H$ that $u_n \leq d_n/q_1$ and $(q-1)(u_n - d_n/q) \leq -\lambda$, and suppose $u_n(y) \leq U$ for $y \leq L$, $d_n(H) \leq U$. Then

$$LV \leq (q_1 - 1)U, \quad LW \leq -\lambda W + (q-1)U + \lambda + Uq^{H-1-L} \mathbf{1}_{\{y=H\}}. \quad (\text{D6})$$

For V , the generator is zero below L and above H . At L it is $u_n(L)(q_1 - 1)$; between the endpoints it is $q_1^{y-1-L}(q_1 - 1)(q_1 u_n - d_n) \leq 0$; at H its downward contribution is nonpositive. For W , the generator is zero below L and above H , is $u_n(L)(q-1)$ at L , and equals $W(q-1)(u_n - d_n/q)$ on the interior up to $H-2$. At $H-1$, replacing the upward target value by zero only decreases this generator. At H , the sole contribution is $d_n(H)q^{H-1-L}$. The added constant $(q-1)U + \lambda$ covers the plateau where $W = 1$. This proves every boundary case in (D6).

The exceptional re-entry set $\{H\}$ is contained in $\{V \geq q_1^{H-L}\}$, and $\{y > R\} \subseteq \{V \geq q_1^{H-L}\} \cup \{W \geq q^{R+1-L}\}$. Hence from every $y_0 \leq I$, (D4) gives

$$\mathbb{P}_{y_0}(Y_T > R) \leq \pi + \frac{e^{-\lambda T} q^{I-L} + [(q-1)U + \lambda + Uq^{H-1-L}\pi]/\lambda}{q^{R+1-L}}, \quad \pi = \frac{q_1^{I-L} + (q_1 - 1)UT}{q_1^{H-L}}. \quad (\text{D7})$$

The two potentials have different jobs. The potential W with the gentler slope q certifies the exponential return toward L , but it is cut off at the barrier H , and since the chain is not stopped there the proof must pay for excursions above H that re-enter. The steeper potential V makes such excursions rare, at the price π . With a single slope the re-entry term would grow like q^{I-R} ; with the thresholds and slopes chosen below it carries the factor $(q^{23}q_1^{-10})^m < 1$ instead.

D.3 Exact inequalities on the entire parameter box

Choose $(L, R, I, H) = (10m, 15m, 28m, 38m)$, that is 10%, 15%, 28% and 38% wrong molecules, with $q = 11/10$, $q_1 = 27/20$, $\lambda = n/1000$, $U = 33n/100$. The thresholds have a direct reading: an output is ready below R ; the copying estimate of D.4 delivers both outputs below I ; the restoring drift is certified downward on the whole band from L to H ; and H is placed far enough above I that excursions past it are rare. We give the algebraic certificate for (D6) rather than sampled drift values.

Write $N = n$, $Y = y$, $K = N - Y$. For either slope $q_* \in \{11/10, 27/20\}$, the quantity $N^2(d_n/q_* - u_n)$ is affine in r, κ . Its coefficients are $Y(Y-1)(Y-2)/q_* - K(K-1)(K-2)$ and $(Y/q_* - K)N^2$. Both are negative on $N/10 \leq Y \leq 19N/50$, $N \geq 1000$: here $100 \leq Y < K$, so the nonnegative falling factorial is strictly smaller for Y than for K , and $q_* > 1$. The minimum is therefore at $r_1 = 11/200$, $\kappa_1 = 11/1000$. At that vertex, expand

$$N^2(d_n/q_* - u_n) = H_{q_*} + A_{q_*} + B_{q_*},$$

$$H_{q_*} = \frac{YK^2 + r_1Y^3 + \kappa_1N^2Y}{q_*} - (Y^2K + r_1K^3 + \kappa_1N^2K),$$

$$A_{q_*} = YK(1 - 1/q_*) + 3r_1(K^2 - Y^2/q_*) \geq 0, \quad B_{q_*} = 2r_1(Y/q_* - K) \geq -2r_1N. \quad (\text{D8})$$

Put $s = Y - N/10$, $t = 19N/50 - Y$. Then $s, t \geq 0$ and $N = 25(s + t)/7$. The following table gives the exact coefficients of $H_{q_*} - (c + 10^{-4})N^3$; the identity is checked by substituting for Y, N and collecting the four monomials.

Table 7: Exact coefficients of $H_{q_*} - (c + 10^{-4})N^3$ in the monomials s^3, s^2t, st^2, t^3 .

(q_*, c)	s^3	s^2t	st^2	t^3
$(11/10, 1/100)$	5439579/6036800	7877943/1207360	1905857/241472	61505/241472
$(27/20, 0)$	2684653/14817600	12605801/2963520	3549799/592704	15845/197568

Every entry is positive. Since $10^{-4}N^3 \geq 2r_1N$ for $N \geq 1000$, (D8) proves $d_n/q_* - u_n \geq cN$. For $q_* = 11/10$, multiplication by $q_* - 1 = 1/10$ gives the required contraction $N/1000$; for $q_* = 27/20$ it proves the escape inequality. Finally each direction's forward rate is at most $n/4$, its reverse rate at most rn , and its spontaneous rate at most κn . Thus both rates are at most $(1/4 + r + \kappa)n \leq 0.316n < U$, proving the caps. These are exact polynomial inequalities over the whole box.

With $T = 20$, (D7) consequently becomes

$$\pi_m = \frac{q_1^{18m} + 231m}{q_1^{28m}}, \quad \rho_m = \pi_m + \frac{e^{-2m}q^{18m} + 34 + 330q^{28m-1}\pi_m}{q^{5m+1}}. \quad (\text{D9})$$

Every captured state $y_0 \leq 28m$ has restoration failure at most ρ_m , uniformly in (D3).

D.4 Copying while the parent fluctuates

For the copying pair let $v = z - n/2$, $w = p - n/2$, $\Lambda = \alpha(1 + r) + 2\kappa$, $\varrho = \alpha(1 - r)$. The daughter up/down rates from (D2), including spontaneous errors, are $(n/2 - v)(\Lambda/2 + \varrho w/n)$ and $(n/2 + v)(\Lambda/2 - \varrho w/n)$. Thus the exact mean drifts are $Lw = -2\kappa w$, $Lv = -\Lambda v + \varrho w$.

For $0 < \theta \leq 1$, set $G_s = \exp\{\theta[A(s)v + B(s)w] + \varphi(s)\}$, with $A' = \Lambda A$, $B' = 2\kappa B - \varrho A$. Put $P(x) = e^x - 1 - x$, $P_-(x) = e^{-x} - 1 + x$. For nonnegative x , $0 \leq P_-(x) \leq P(x)$. Expanding the jump exponentials, their linear terms cancel the time derivative exactly. The total daughter rate is at most $n(\alpha + \kappa)$, because the copying sum is at most αn for $0 < r < 1$; the parent rate is κn . Consequently

$$\frac{(\partial_s + L)G_s}{G_s} \leq \varphi'(s) + n[(\alpha + \kappa)P(\theta A) + \kappa P(\theta B)]. \quad (\text{D10})$$

For $0 \leq x \leq \theta \leq 1$, the power series gives $P(x) \leq x^2[1/2 + \theta \sum_{j \geq 3} 1/j!] \leq c_\theta x^2$, where $c_\theta = 1/2 + 2\theta/9$. To justify the last rational bound, $j! \geq 6 \cdot 4^{j-3}$ for $j \geq 3$, so the sum is at most $(1/6)/(1 - 1/4) = 2/9$. If $0 \leq A \leq 1$, $0 \leq B \leq B_{\max} \leq 1$, choose $\varphi' = -n\theta^2 c_\theta[(\alpha + \kappa)A^2 + \kappa B_{\max}^2]$. Equation (D10) and the finite expectation identity give $\mathbb{E}G_T \leq \mathbb{E}G_0$.

For a daughter tail at time T , choose

$$A(s) = e^{-\Lambda(T-s)}, \quad B(s) = B_0[e^{-2\kappa(T-s)} - e^{-\Lambda(T-s)}], \quad B_0 = \frac{1-r}{1+r}.$$

These solve the backward equations, obey $A(T) = 1, B(T) = 0$, $0 \leq B \leq B_0 < 1$, and $B(0) = h(T) = B_0(e^{-2\kappa T} - e^{-\Lambda T})$. Initially $p = y_0$ is deterministic and z is binomial. Its exponential moment is $\cosh(\theta A(0)/2)^n \leq \exp(n\theta^2 A(0)^2/8)$; the inequality follows by integrating $\tanh x \leq x$. Since $\int_0^T A(s)^2 ds \leq 1/(2\Lambda)$, exponential Markov's inequality proves

$$\mathbb{P}(z_T > I) \leq \exp \left\{ -\theta(I + 1 - n/2) + \theta h(T)(y_0 - n/2) + n\theta^2 \left[c_\theta \left(\frac{\alpha + \kappa}{2\Lambda} + \kappa B_0^2 T \right) + \frac{e^{-2\Lambda T}}{8} \right] \right\}. \quad (\text{D11})$$

The integer threshold is $I + 1$. For the parent choose $A = 0$, $B(s) = e^{-2\kappa(T-s)}$, $B_{\max} = 1$ instead. The same argument yields

$$\mathbb{P}(p_T > I) \leq \exp\{-\theta(I + 1 - n/2) + \theta e^{-2\kappa T}(y_0 - n/2) + n\theta^2 c_\theta \kappa T\}. \quad (\text{D12})$$

The parent remains a random part of the joint chain throughout this calculation. In particular the κB^2 charge is not removed by treating it as a deterministic source; no independent-daughter-molecule law conditioned on a fluctuating parent is assumed.

Here are sufficient rational specializations of these bounds at $T = 3$. Over (D3), $e^{-6\kappa} \geq 117/125$, $e^{-3\Lambda} \leq 413/10000 < 1/4$, $B_0 \geq 189/211$. Hence $h(3) \geq (189/211)(117/125 - 413/10000) \geq 4/5$. For completeness the first exponential bound follows from $(1 - 33/32000)^{64} \geq 117/125$ and $e^{-x} \geq 1 - x$ at $x = 33/32000$. For the second, $3\Lambda \geq 3189/1000$ and $e^{3189/1000} \geq$

$(27182818283/10^{10})^3[1 + 189/2000 + (189/2000)^2/2]^2 \geq 2425/100$. The lower bound on e follows from $\sum_{j=0}^{13} 1/j! > 27182818283/10^{10}$; all remaining comparisons here are rational cross multiplications.

For the daughter put $\theta = 1/8$, so $c_\theta = 19/36$. The bounds $(1 + \kappa)/(2\Lambda) \leq 1011/2126 < 12/25$, $\kappa B_0^2 T \leq 33/1000$, and $e^{-2\Delta T}/8 \leq 1/128$ show that the exponent in (D11), for $y_0 \leq 0.15n$ and $I = 0.28n$, is at most

$$-\frac{3n}{400} - \frac{1}{8} + \frac{n}{64} \left[\frac{19}{36} \left(\frac{12}{25} + \frac{33}{1000} \right) + \frac{1}{128} \right] = -\frac{3n}{400} - \frac{1}{8} + \frac{4457n}{1024000} \leq -\frac{3n}{1000}.$$

For the parent put $\theta = 1$, $c_\theta = 13/18$. Its exponent in (D12) is at most

$$\left[\frac{11}{50} - \frac{7}{20} \frac{117}{125} + \frac{13}{18} \frac{33}{1000} \right] n - 1 = -\frac{2513n}{30000} - 1 \leq -\frac{2n}{25}.$$

Thus, for all parameters and all ready parents under consideration,

$$\mathbb{P}(z_3 > 28m) \leq e^{-3m/10}, \quad \mathbb{P}(p_3 > 28m) \leq e^{-8m}. \quad (\text{D13})$$

These deliberately crude algebraic bounds suffice for the capture rates in Theorem 2 without sharper variance estimates.

D.5 Exponential scale, the finite witness and repeated cycles

Apply (D9) after separation conditional on the actual pair. If a count is captured its return failure is at most ρ_m ; otherwise charge its failure to (D13). A union bound proves the one-cycle, two-output estimate

$$\varepsilon_m = e^{-3m/10} + e^{-8m} + 2\rho_m. \quad (\text{D14})$$

No independence between the two copying errors is used. Reflection of (D1), the paired copying channels and the racemic initial law proves the same statement for either inherited label.

To give the decay and the finite witness explicitly, let $u = (20/27)^m$, $v = (10/11)^m$, $y = (11/10)^m$, $E = e^{-2m}$. Expanding (D9) gives

$$\rho_m = u^{10} + 231mu^{28} + \frac{10}{11}Ey^{13} + \frac{340}{11}v^5 + \frac{33000}{121}(y^{23}u^{10} + 231my^{23}u^{28}). \quad (\text{D15})$$

Here the superscripts on u, v, y are ordinary powers. The following exact rational comparisons prove $\rho_m \leq 40(5/8)^m$ for every integer $m \geq 10$. Set $a_0 = 5/8$. Bound $m \leq 2^m$. The six bases in (D15), after absorbing that factor where needed and dividing by a_0 , are bounded as follows:

For example the first normalized base is $(20/27)^{10}/a_0$, the second is $2(20/27)^{28}/a_0$, and the last is $2(11/10)^{23}(20/27)^{28}/a_0$. For the third, use $e^{-2} \leq 13534/100000$, following from the preceding lower bound on e . Each base is at most one, so its m -th power is at most its tenth power; the right column is an outward rational bound for that tenth power. Weighting the column by the coefficients in (D15) gives

Table 8: Normalized bases of the six terms in (D15) and outward bounds on their tenth powers.

Term without coefficient	Upper bound on normalized base	Upper bound on its m -th power for $m \geq 10$
u^{10}	1/10	1/10 ⁸
mu^{28}	1/10	1/10 ⁸
Ey^{13}	3/4	3/50
v^5	1987/2000	937/1000
$y^{23}u^{10}$	357/500	69/2000
$my^{23}u^{28}$	1/10	1/10 ⁸

$$\frac{1}{10^8} + \frac{231}{10^8} + \frac{10}{11} \frac{3}{50} + \frac{340}{11} \frac{937}{1000} + \frac{33000}{121} \left(\frac{69}{2000} + \frac{231}{10^8} \right) < 40. \quad (\text{D16})$$

This supplies the claimed decay using a short list of exact power comparisons, without a fitted asymptotic estimate. For $m \geq 27$ the simpler bound (D17) also follows. Indeed $e^{-3/10} \geq (1 - 3/640)^{64} \geq 74/100$, $e^{-8} \leq 1/15$, so $e^{-8m} \leq (1/1000)e^{-3m/10}$. Also $(5/8)/(74/100) \leq 845/1000$ and $(845/1000)^{27} \leq 107/10000$. Therefore

$$\varepsilon_m \leq \left(1 + \frac{1}{1000} + 80 \frac{107}{10000} \right) e^{-3m/10} < 2e^{-3m/10} = 2e^{-0.003n}. \quad (\text{D17})$$

At $m = 40$ retain (D14) rather than weakening it to (D17):

$$\varepsilon_{40} \leq e^{-12} + e^{-320} + 80(5/8)^{40} \leq \frac{1}{162754} + \frac{1}{10^{10}} + \frac{80}{10^8} < 10^{-5}. \quad (\text{D18})$$

The exponential comparisons can be reduced to finite rational statements: $\sum_{j=0}^{64} 12^j/j! \geq 162754$, $320^5/5! > 10^{10}$, and $10^8 5^{40} \leq 8^{40}$. Thus the quoted hundred-cycle witness has a complete conventional finite proof; the displayed $2e^{-12}$ bound alone would yield only $1 - 200e^{-12}$, about 0.998771.

The bound is uniform over every ready parent. Conditional on each successful cycle and the selected ready output, the next failure probability is at most ε_m . Iterating conditional expectation, or summing first failures, gives selected-lineage success at least $1 - g\varepsilon_m$ over g cycles; selecting either ready output at each cycle satisfies the hypothesis. A full binary family charges $2^g - 1$ cycles instead. In particular (D18) gives at least 0.999 for 100 selected-lineage cycles at $n = 4000$. This is a finite mission, not open-ended evolution.

D.6 What its resource estimate measures

The counting convention is expected gross fuelled reaction events, not a finite-stock high-probability ceiling. During copying the combined forward-and-reverse fuelled rate is $\alpha[(n-z)(p+r(n-p)) + z((n-p)+rp)]/n \leq \alpha n = n$, since each parent-dependent bracket is at most n . This gives at most $3n$ expected gross events. In a restoring compartment the total forward rate is $ky(n-2)/n^2 \leq n/4$; the sum of reverse rates is at most rn . Charging both forward and reverse events therefore costs at most $(1/4 + 11/200)n < 0.31n$ per unit time. Two compartments restored for 20 time units contribute at most $12.4n$. The whole cycle is bounded by $15.4n$ expected gross events, including both directions in both phases. This is not a pathwise fuel inventory. Net fuel consumption is

bounded by the gross count, so the expected chemical work supplied by these chemostats is at most $A 15.4n$. The apparatus, supplied blank compartment, switching and chemostats remain physical assumptions; this is not their total energetic price.

At all times there are two compartments with total spontaneous-flip intensity $2\kappa n$, so the expected number of errors through time 23 is exactly $46\kappa n$. The count compensator identity in A.2 justifies both expectations. No rare-error premise $\kappa nT \ll 1$ is used.