

Counterexamples to the weakly Gorenstein symmetry conjecture

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Abstract

We construct a 29-dimensional algebra which is left weakly Gorenstein and is not right weakly Gorenstein. We also construct a 145-dimensional gendo-symmetric algebra with the same property. Both constructions work over every field containing an element of infinite multiplicative order. These examples disprove the weakly Gorenstein symmetry conjecture.

1 Introduction

The study of Gorenstein-projective modules has its origins in Auslander and Bridger's theory of G-dimension. In that theory, modules of G-dimension zero are described by reflexivity and Ext vanishing for a module and its algebra dual; see [3, Introduction]. Jorgensen and Şega constructed modules over commutative Artinian local rings showing that reflexivity together with Ext vanishing for the module does not force Ext vanishing for its dual [3, Theorem 1.7]. This raises the question of which algebras allow Gorenstein-projectivity to be detected by Ext vanishing for the module alone.

Let A be a finite-dimensional algebra over a field k . All modules in this paper are finite-dimensional. A left A -module X is *semi-Gorenstein-projective* if $\text{Ext}_A^n(X, A) = 0$ for every integer $n > 0$. It is *Gorenstein-projective* if it is a cycle in a doubly infinite exact complex of finite projective modules which remains exact after applying $\text{Hom}_A(-, A)$. Following Ringel and Zhang [5], the algebra A is *left weakly Gorenstein* if every semi-Gorenstein-projective left module is Gorenstein-projective. The right-hand versions use right modules, equivalently left modules over the opposite algebra A^{op} , whose multiplication reverses that of A .

Ringel and Zhang proved that A is left weakly Gorenstein whenever there are only finitely many isomorphism classes of indecomposable left modules which are both semi-Gorenstein-projective and submodules of projective modules. They also constructed a six-dimensional algebra with a semi-Gorenstein-projective module which does not embed in a projective module [5, Theorems 1.3 and 1.6]. Marczinzik constructed gendo-symmetric algebras which are not weakly Gorenstein [4, Theorem 5.5] and asked whether left weak Gorensteinness implies right weak Gorensteinness [4, §6, Question (3)]. Ringel and Zhang showed that an affirmative answer would imply the Gorenstein symmetry conjecture [5, §9.3]. Huang formulates the left-right question as the *weakly Gorenstein symmetry conjecture* [1, Introduction]: a finite-dimensional algebra is left weakly Gorenstein if and only if it is right weakly Gorenstein.

We disprove this conjecture by constructing a triangular matrix algebra and a gendo-symmetric algebra which are left weakly Gorenstein and are not right weakly Gorenstein.

A module is torsionless if it embeds in a finite projective module. For a module X , its algebra dual X^* is $\text{Hom}_A(X, A)$ on the left and $\text{Hom}_{A^{\text{op}}}(X, A)$ on the right. It is reflexive if the evaluation map $X \rightarrow X^{**}$, $x \mapsto (f \mapsto f(x))$, is an isomorphism. An algebra is gendo-symmetric if it is the endomorphism algebra of a generator over a finite-dimensional symmetric algebra.

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A generator here is a module whose finite direct sums have the regular module as a direct summand; a symmetric algebra is an algebra isomorphic to its vector-space dual as a bimodule.

Theorem 1.1. *Let k contain an element $\alpha \in k^\times$ of infinite multiplicative order. There is a 29-dimensional k -algebra T_α such that every semi-Gorenstein-projective left T_α -module is projective, whereas a 4-dimensional right T_α -module is semi-Gorenstein-projective and is not torsionless.*

There is also a 145-dimensional gendo-symmetric k -algebra E_α such that every semi-Gorenstein-projective left E_α -module is projective, whereas an 8-dimensional right E_α -module is reflexive and semi-Gorenstein-projective but is not Gorenstein-projective. Both algebras have infinite left and right self-injective dimensions.

A Gorenstein-projective module is torsionless and reflexive: these assertions follow by dualizing its complete projective resolution. Thus the modules in Theorem 1.1 witness failure of right weak Gorensteinness.

The construction uses the commutative 14-dimensional algebra and modules of Jorgensen and Şega [2, §2.1 and Theorem 3.2]. Their modules have asymmetric Ext vanishing. We prove that one of these modules detects eventual Ext vanishing for *every* finite module in the first variable. This property forces every semi-Gorenstein-projective left module over the triangular algebra to be projective. A module with vanishing positive Tor supplies the right-hand counterexample. A second detection argument for complexes gives the gendo-symmetric example. Theorem 1.1 is proved in Sections 5 and 7.

Weak Gorensteinness for triangular matrix algebras is also studied through monic modules in [6, Theorem 1.2]. The finite-projective-dimension hypotheses there give criteria in terms of the diagonal algebras. The detecting module used here has infinite projective dimension over the self-injective diagonal algebra.

The parameter can be chosen as 2 over $\mathbb{Q}$, and as an indeterminate over $\mathbb{F}_p$ for any prime p . We also determine the same triangular family at nontrivial roots of unity: in that case every semi-Gorenstein-projective module on either side is projective (Theorem 6.2).

We use $D = \text{Hom}_k(-, k)$ for vector-space duality. The symbols pd_A and id_A denote projective and injective dimensions of left modules, and subscripts A^{op} denote the right-module versions. The assertion $\text{Ext}_A^n(X, L) = 0$ for $n \gg 0$ means that it holds for every integer n above some bound, allowed to depend on X and L . The n th syzygy $\Omega^n X$ is the kernel at step $n - 1$ of a minimal projective resolution, with $\Omega^0 X = X$.

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2 A criterion for eventual Ext vanishing

Lemma 2.1. *Let A be a finite-dimensional local k -algebra with $A/J \cong k$, where $J = \text{rad } A$. Let L be a nonzero left module with $J^3 L = 0$. Its socle is $\text{soc } L = \{\ell \in L : J\ell = 0\}$, which contains $J^2 L$. Put*

$$a = \dim_k L/JL, \quad b = \dim_k JL/J^2 L, \quad c = \dim_k J^2 L, \quad s = \dim_k (\text{soc } L/J^2 L).$$

Suppose $s > 0$ and $bs > a(s + c)$. Then every left module X satisfies

$$\text{Ext}_A^n(X, L) = 0 \ (n \gg 0) \implies \text{pd}_A X < \infty.$$

For a module U over this local algebra, let $\beta_n(U)$ denote the rank of the free module in degree $n \geq 0$ of a minimal projective resolution. If a right module U satisfies $\text{Tor}_n^A(U, L) = 0$ for

$n \gg 0$, its minimal Betti numbers satisfy

$$\beta_{n+1}(U) \geq \frac{b - ac/s}{a} \beta_n(U) \quad (n \gg 0). \quad (1)$$

Proof. Finite projective modules over A are free. Minimal differentials have entries in J . Any resulting k -linear map $L^p \rightarrow L^q$ has image in $(JL)^q$, and its image modulo $(J^2L)^q$ factors through $(L/JL)^p$. Hence

$$\text{rank}_k(L^p \rightarrow L^q) \leq ap + cq. \quad (2)$$

Apply $\text{Hom}_A(-, L)$ to a minimal resolution of X . In its exact tail, the two adjacent ranks at L^{β_n} sum to $(a + b + c)\beta_n$. Equation (2) therefore gives

$$b\beta_n \leq a\beta_{n-1} + c\beta_{n+1}. \quad (3)$$

The differential kills $(\text{soc } L)^{\beta_n}$, so exactness puts this space in the preceding image. Modulo $(J^2L)^{\beta_n}$ the preceding image has dimension at most $a\beta_{n-1}$. Consequently

$$s\beta_n \leq a\beta_{n-1}, \quad s\beta_{n+1} \leq a\beta_n.$$

Substituting the second inequality in (3) gives

$$0 \leq \beta_n \leq \frac{a}{b - ac/s} \beta_{n-1}.$$

The fraction lies strictly between 0 and 1. Iteration forces a nonnegative integer β_n to be zero; the minimal resolution terminates.

For U , tensor its minimal right resolution with L . Exactness and (2), now with the arrows reversed, give

$$b\beta_n \leq a\beta_{n+1} + c\beta_{n-1}, \quad s\beta_{n-1} \leq a\beta_n.$$

Eliminating β_{n-1} proves (1). $\square$

3 The algebra and the detecting module

Fix $\alpha \in k \setminus \{0, 1\}$. The graded vector space $R = R_\alpha$ has the following basis, grouped by degree:

$$1; \quad t, u, v, x, y, z; \quad tv, uv, vx, vy, vz, tx; \quad \omega. \quad (4)$$

The degrees are 0, 1, 2, 3, respectively; R_d denotes the span of the basis elements of degree d , and $R_d = 0$ for all other integers d . Define a commutative bilinear multiplication with unit 1 by the following upper triangular table, and set all products of total degree at least 4 equal to zero:

	t	u	v	x	y	z
t	$-(\alpha + 1)uv + vy$	0	tv	tx	$vx - tv$	$tv - (\alpha + 1)vx$
u		0	uv	0	0	$tx + \alpha uv$
v			0	vx	vy	vz
x				$tx + tv$	0	$-\alpha vx$
y					$tx + (\alpha - 1)uv$	$-vy$
z						0

(5)

The nonzero products of degrees 1 and 2 are

$$t(vx) = u(vz) = v(tx) = x(tv) = x(vx) = y(vy) = z(uv) = \omega, \quad z(tx) = -\alpha\omega. \quad (6)$$

In particular, the degree-two basis labels in (4) are their indicated products.

Lemma 3.1. *These tables define an associative local symmetric algebra of dimension 14. Its radical is the positive-degree part, and its socle is $k\omega$.*

Proof. Only triples of degree-one basis elements require an associativity check. For $r, s, q \in \{t, u, v, x, y, z\}$, compute the coefficient of ω in $(rs)q$ using (5) and (6). The complete list of nonzero values, up to permutation of the three entries, is

$\{r, s, q\}$	tty	ttz	tvx	txz	uvz	vxx	vyg	xxx	xxz	yyz
coefficient	1	$-(\alpha + 1)$	1	$-\alpha$	1	1	1	1	$-\alpha$	-1 .

Each value is unchanged by permuting the entries; all unlisted values are zero. Thus $(rs)q = r(sq)$. Triples involving 1 satisfy associativity by the unit rule, and every other triple has total degree at least 4, so both bracketings vanish.

Let $\varepsilon : R \rightarrow k$ send an element to its coefficient of ω in (4). The matrix of $(r, s) \mapsto \varepsilon(rs)$ between degrees 1 and 2, in the displayed orders, is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\alpha \end{pmatrix}. \quad (7)$$

Its determinant is 1 or -1 . Degrees 0 and 3 pair by $\varepsilon(\omega) = 1$. The full pairing is nondegenerate and associative, so $R \rightarrow DR$, $r \mapsto (s \mapsto \varepsilon(rs))$, is a bimodule isomorphism. Since DR is injective on both sides, this isomorphism also makes R self-injective. The positive-degree ideal J is nilpotent and $R/J = k$, hence it is the radical. Nondegeneracy between degrees 1 and 2 excludes socle elements in those degrees, giving $\text{soc } R = k\omega$. $\square$

These are the multiplication tables of the algebra in [2, §2.1, Proposition 2.2 and Appendix A]. The table description will also be used when α has finite order.

For $\lambda \in k$, define an R -linear map on column vectors by

$$\delta_\lambda : R^2 \longrightarrow R^2, \quad \delta_\lambda = \begin{pmatrix} v & y \\ \lambda x & z \end{pmatrix}.$$

Write $\delta_\lambda^\top$ for its 2×2 matrix transpose. Define

$$M = M_\alpha = \text{coker } \delta_\alpha^\top. \quad (8)$$

The quotient inherits this grading; write M_d for its degree- d component. The images e_1, e_2 of the standard basis of R^2 have degree zero and satisfy

$$ve_1 + ye_2 = 0, \quad \alpha xe_1 + ze_2 = 0. \quad (9)$$

Lemma 3.2. *The module M has radical layers of dimensions $(2, 10, 2)$, and*

$$\text{soc } M = k(te_1 + te_2 - xe_2) \oplus k(uv)e_2 \oplus k(vx)e_2. \quad (10)$$

For every left R -module L ,

$$\text{Ext}_R^n(L, M) = 0 \ (n \gg 0) \implies L \text{ is projective}. \quad (11)$$

For every right R -module U with $\text{Tor}_n^R(U, M) = 0$ for $n \gg 0$,

$$\beta_{n+1}(U) \geq 3\beta_n(U) \quad (n \gg 0). \quad (12)$$

Proof. In degree one the two relations in (9) are independent; a basis of the quotient is

$$te_1, ue_1, ye_1, ze_1, te_2, ue_2, ve_2, xe_2, ye_2, ze_2. \quad (13)$$

Multiplying the relations by t, u, v, x, y, z gives the following degree-two reductions:

$$\begin{array}{c|cccccc} & (tv)e_i & (uv)e_i & (vx)e_i & (vy)e_i & (vz)e_i & (tx)e_i \\ \hline i = 1 & 0 & 0 & 0 & (uv)e_2 & 0 & (vx)e_2 \\ i = 2 & (vx)e_2 & (uv)e_2 & (vx)e_2 & 0 & 0 & -\alpha(uv)e_2. \end{array} \quad (14)$$

Here are the eliminations. The u -multiple of the first relation gives $(uv)e_1 = 0$; its v - and x -multiples give $(vy)e_2 = 0$ and $(vx)e_1 = 0$. The v -multiple of the second then gives $(vz)e_2 = 0$. Its x -multiple yields $(tx)e_1 + (tv)e_1 = (vx)e_2$. The t -multiple of the first gives $(tv)e_1 + (vx)e_2 - (tv)e_2 = 0$. The t -multiple of the second, combined with these two equalities, gives $(\alpha - 1)((vx)e_2 - (tv)e_2) = 0$. Thus $(tv)e_2 = (vx)e_2$, $(tv)e_1 = 0$, and $(tx)e_1 = (vx)e_2$. The z -multiple of the first gives $(vz)e_1 = 0$. The u -multiple of the second gives $(tx)e_2 = -\alpha(uv)e_2$. Finally the y -multiple of the first gives $(vy)e_1 = (uv)e_2$.

The linear map from R_2^2 to k^2 prescribed by (14), sending $(uv)e_2$ and $(vx)e_2$ to the standard basis, kills all twelve degree-two relations. Hence these two elements are independent. In degree three the relations give

$$(tx)(v, y) = (\omega, 0), \quad (uv)(\alpha x, z) = (0, \omega),$$

so $M_3 = 0$. This proves the stated radical layers and $J^3 M = 0$.

Write a degree-one element in the order (13) with coefficients $p_1, \dots, p_{10}$. Multiplication by the six generators has the following coordinates in $((uv)e_2, (vx)e_2)$:

$$\begin{array}{c|cc} & (uv)e_2 & (vx)e_2 \\ \hline t & p_1 - (\alpha + 1)p_5 - \alpha p_8 & p_7 - \alpha p_{10} \\ u & p_7 & p_4 \\ v & p_3 + p_6 & p_5 + p_8 \\ x & -\alpha(p_5 + p_8) & p_1 + p_7 + p_8 - \alpha p_{10} \\ y & -p_4 - p_9 & p_3 \\ z & -p_3 & p_2 - \alpha(p_5 + p_8). \end{array} \quad (15)$$

Setting these entries to zero first gives $p_7 = p_4 = p_3 = p_9 = p_6 = 0$ and $p_5 = -p_8$. The z row gives $p_2 = 0$, the t row gives $p_{10} = 0$ and $p_1 = p_5$, and the remaining equation is then satisfied. There is no degree-zero socle element since te_1, te_2 are independent. All of M_2 lies in the socle, proving (10).

Lemma 2.1 applies with $(a, b, c, s) = (2, 10, 2, 1)$, since $10 > 6$. It gives finite projective dimension in (11) and the factor 3 in (12). Over a self-injective algebra a finite projective resolution splits backwards from its last term, so a module of finite projective dimension is projective. $\square$

4 The parameter complex

Lemma 4.1. *For every $\lambda \in k$,*

$$\delta_{\alpha\lambda}\delta_\lambda = 0. \quad (16)$$

If $\lambda \neq 0, 1$, then $\text{rank}_k \delta_\lambda = 14$; if $\lambda = 1$, its rank is 13. The same ranks hold for $\delta_\lambda^\top$.

Proof. The four entries of the product in (16) are

$$v^2 + \lambda yx, \quad vy + yz, \quad \alpha\lambda xv + \lambda zx, \quad \alpha\lambda xy + z^2,$$

which vanish by (5). For $\lambda \neq 0$, the degree-one image has basis $(v, \lambda x), (y, z)$. The degree-two image is spanned by the following ten columns:

$$\begin{aligned} & (tv, \lambda tx), (uv, 0), (0, \lambda vx), (vx, \lambda(tx + tv)), (vy, 0), (vz, -\alpha \lambda vx), \\ & (vx - tv, tv - (\alpha + 1)vx), (0, tx + \alpha uv), (vy, vz), (tx + (\alpha - 1)uv, -vy). \end{aligned} \quad (17)$$

They are the images of $(t, 0), (u, 0), (v, 0), (x, 0), (y, 0), (z, 0), (0, t), (0, u), (0, v), (0, y)$. The two omitted degree-one source vectors satisfy

$$\delta_\lambda(0, x) = -\frac{\alpha}{\lambda} \delta_\lambda(v, 0), \quad \delta_\lambda(0, z) = -\delta_\lambda(y, 0).$$

If a linear combination of the ten columns is zero with coefficients $q_1, \dots, q_{10}$, the first coordinates give

$$q_1 = q_7, \quad q_4 = -q_7, \quad q_2 = q_6 = q_{10} = 0, \quad q_5 = -q_9.$$

In the second coordinates, the coefficients of tx, vz, tv, vx then give, respectively,

$$q_8 = 0, \quad q_9 = 0, \quad (1 - \lambda)q_7 = 0, \quad \lambda q_3 - (\alpha + 1)q_7 = 0.$$

For $\lambda \neq 1$ the ten columns are independent. For $\lambda = 1$ these equations leave exactly one free coefficient, so their rank is 9. The degree-three image is all of R_3^2 , since the images of $(vx, 0)$ and $(tx, 0)$ include $(0, \lambda\omega)$ and $(\omega, 0)$. The total ranks are consequently $2 + 10 + 2$ and $2 + 9 + 2$.

On R^2 use the pairing $((r_1, r_2), (s_1, s_2)) \mapsto \varepsilon(r_1 s_1 + r_2 s_2)$. The matrix transpose over the commutative algebra R is the adjoint for this nondegenerate pairing, so it has the same k -linear rank. $\square$

Assume now that α has infinite multiplicative order. By Lemma 4.1, consecutive maps in

$$\dots \xrightarrow{\delta_{\alpha^3}^\top} R^2 \xrightarrow{\delta_{\alpha^2}^\top} R^2 \xrightarrow{\delta_\alpha^\top} R^2 \longrightarrow M \longrightarrow 0 \quad (18)$$

have zero composite and rank 14. Their images therefore equal their kernels, since $\dim_k R^2 = 28$. All entries lie in J , so this is a minimal free resolution.

Define the ideal and quotient

$$K = (t, u, v - x, y - x, z - x) \subset R, \quad N = R/K. \quad (19)$$

The relations identify v, x, y, z with a single element ξ , while t, u vanish. All degree-two products vanish in the quotient: $xy = 0$ gives $\xi^2 = 0$. Conversely the assignment $v, x, y, z \mapsto \xi$, $t, u \mapsto 0$ respects the tables. Thus $N \cong k[\xi]/(\xi^2)$.

Lemma 4.2. *If α has infinite order, then*

$$\mathrm{Tor}_n^R(N, M) = 0 \quad (n > 0), \quad N \otimes_R M \cong k^2,$$

where $1 \otimes e_1, 1 \otimes e_2$ give the displayed basis and J acts trivially.

Proof. Tensor (18) with N . The differential in degree $n \geq 1$ becomes

$$\xi \begin{pmatrix} 1 & \alpha^n \\ 1 & 1 \end{pmatrix} : N^2 \longrightarrow N^2.$$

Its coefficient matrix has determinant $1 - \alpha^n \neq 0$. The image and kernel are both ξN^2 . Hence the positive homology vanishes, and the cokernel in degree zero is $N^2/\xi N^2$ with the asserted basis. $\square$

5 The triangular counterexample

For $\alpha \neq 0, 1$ define

$$T = T_\alpha = \begin{pmatrix} R & M \\ 0 & k \end{pmatrix}, \quad \begin{pmatrix} r & m \\ 0 & c \end{pmatrix} \begin{pmatrix} r' & m' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} rr' & rm' + mc' \\ 0 & cc' \end{pmatrix}. \quad (20)$$

Its dimension is $14 + 14 + 1 = 29$. A left T -module is a triple (Y, V, φ) with Y a left R -module, V a k -space and $\varphi : M \otimes_k V \rightarrow Y$ an R -linear map; the action is

$$\begin{pmatrix} r & m \\ 0 & c \end{pmatrix} (y, v) = (ry + \varphi(m \otimes v), cv).$$

For an R -module Y and a k -space V , write

$$iY = (Y, 0, 0), \quad jV = (M \otimes_k V, V, 1).$$

Here the third entry 1 denotes the identity of $M \otimes_k V$. The left projectives are the direct sums of iP and jV , with P projective over R .

Proposition 5.1. *Every semi-Gorenstein-projective left T -module is projective. More generally, $\text{Ext}_T^n(S, T) = 0$ for $n \gg 0$ implies $\text{pd}_T S \leq 1$.*

Proof. For $S = (Y, V, \varphi)$ choose a finite free surjection $P \twoheadrightarrow Y$. Combining it with φ gives a surjection $iP \oplus jV \twoheadrightarrow S$ whose second component is 1_V . Its kernel has second component zero, so for some R -module L there is an exact sequence

$$0 \longrightarrow iL \longrightarrow iP \oplus jV \longrightarrow S \longrightarrow 0. \quad (21)$$

The functor i is exact and preserves projectives. Termwise Hom on an R -projective resolution gives

$$\text{Ext}_T^n(iL, T) \cong \text{Ext}_R^n(L, R \oplus M) \quad (n \geq 0). \quad (22)$$

Eventual vanishing for S gives eventual vanishing for iL by dimension shifting. Lemma 3.2 makes L projective, proving $\text{pd}_T S \leq 1$. If S is semi-Gorenstein-projective, then $\text{Ext}_T^1(S, iL) = 0$ because iL is a summand of a finite sum of copies of T . Thus (21) splits and S is projective. $\square$

A right T -module is a triple (U, W, ψ) with U a right R -module, W a k -space, $\psi : U \otimes_R M \rightarrow W$ a k -linear map, and action

$$(u, w) \begin{pmatrix} r & m \\ 0 & c \end{pmatrix} = (ur, \psi(u \otimes m) + wc).$$

Put $\mathcal{J}(U) = (U, U \otimes_R M, 1)$, with identity structure map. The right projectives are direct sums of $\mathcal{J}(P)$ and $(0, W, 0)$ with P projective over R .

Proposition 5.2. *If α has infinite order, the right T -module $S_0 = \mathcal{J}(N)$ has dimension 4, satisfies $\text{Ext}_{T^{\text{op}}}^n(S_0, T) = 0$ for every $n > 0$, and its evaluation map has a one-dimensional kernel.*

Proof. Lemma 4.2 gives the dimension. Apply $\mathcal{J}(-)$ to a projective resolution of the right R -module N . Exactness in its first component is the original resolution, and exactness in its second component follows from the Tor vanishing in Lemma 4.2. Thus it is a projective resolution of S_0 . For every projective P ,

$$\text{Hom}_{T^{\text{op}}}(\mathcal{J}(P), T) \cong \text{Hom}_R(P, R).$$

Indeed, a morphism to $(0, k, 0)$ is zero by compatibility with the structure map, and a morphism to $\mathcal{J}(R)$ is uniquely determined by $P \rightarrow R$. Since R is injective, this proves the Ext vanishing. As a left module $S_0^* = i(N^*)$, where $N^* = \text{Hom}_R(N, R) = \text{ann}_R K$ and $\text{ann}_R K = \{r \in R : Kr = 0\}$.

To compute evaluation, consider the degree-two element

$$h = tv + tx + (\alpha + 1)uv + vy.$$

The tables give

$$th = uh = 0, \quad vh = xh = yh = zh = \omega.$$

Thus $Kh = K\omega = 0$. Frobenius duality gives $\dim_k \text{ann}_R K = \dim_k DN = 2$; the independent elements h, ω therefore form a basis of this annihilator. The degree-two reductions give

$$he_1 = he_2 = (uv)e_2 + (vx)e_2 \neq 0, \quad \omega M = 0. \quad (23)$$

The first component of $S_0 \rightarrow S_0^{**}$ is $N \rightarrow N^{**}$ over the symmetric algebra R , and is an isomorphism. Its second component is

$$N \otimes_R M \longrightarrow \text{Hom}_R(N^*, M), \quad n \otimes m \longmapsto (f \longmapsto f(n)m).$$

Identify $f \in N^*$ with $f(1) \in kh \oplus k\omega$. Equation (23) shows that the two basis elements $1 \otimes e_1$ and $1 \otimes e_2$ have the same nonzero image. The kernel is exactly

$$k(1 \otimes (e_1 - e_2)).$$

If a module embeds in a finite projective, its regular-module-valued functions separate elements, so its evaluation is injective. Therefore S_0 is not torsionless and is not Gorenstein-projective. $\square$

Proposition 5.3. *For every $\alpha \neq 0, 1$, both self-injective dimensions of T_α are infinite.*

Proof. The module M has dimension 14 and top dimension 2. A projective module over the local algebra R is free and has dimension 14 times its top dimension. Thus M is not projective. Since R is self-injective, finite projective or injective dimension would force M to be projective. Hence both dimensions of M are infinite. Equation (22), for arbitrary L , implies $\text{id}_T T = \infty$.

Let $s = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in T$. The dual of the right projective $sT = (0, k, 0)$ is the left module $(0, k, 0)$. There is an exact sequence

$$0 \longrightarrow iM \longrightarrow jk \longrightarrow D(sT) \longrightarrow 0.$$

Applying i to a minimal R -resolution of M gives a minimal T -resolution of iM , so $\text{pd}_T iM = \infty$. Therefore $\text{pd}_T D(sT) = \infty$. It is a summand of DT , and vector-space duality gives $\text{id}_{T^{\text{op}}} T = \text{pd}_T DT = \infty$. $\square$

Propositions 5.1–5.3 prove the assertion about T_α in Theorem 1.1.

6 Finite-order parameters

Lemma 6.1. *Let R be a finite-dimensional self-injective local algebra with residue field k , and let $T = \begin{pmatrix} R & M \\ 0 & k \end{pmatrix}$. Every semi-Gorenstein-projective right T -module has the form*

$$\mathcal{J}(U) \oplus (0, W, 0), \quad \text{Tor}_n^R(U, M) = 0 \quad (n > 0).$$

Proof. Write a minimal right projective resolution as $Q_n = \mathcal{J}(P_n) \oplus (0, W_n, 0)$. A morphism $\mathcal{J}(P_n) \rightarrow (0, W_{n-1}, 0)$ is zero. The diagonal map $(0, W_n, 0) \rightarrow (0, W_{n-1}, 0)$ is zero by minimality, since its coefficients lie in the radical of k , which is zero. Applying $\text{Hom}_{T^{\text{op}}}(-, (0, k, 0))$ therefore gives a complex with zero differentials and term DW_n . Ext vanishing implies $W_n = 0$ for $n > 0$.

Consequently the resolution has only $\mathcal{J}(P_n)$ in positive degrees, and its degree-one differential lands in $\mathcal{J}(P_0)$. On taking the first component it becomes a projective R -resolution of $U = \text{coker}(P_1 \rightarrow P_0)$. On taking the second component, positive exactness gives $\text{Tor}_n^R(U, M) = 0$ for $n > 0$. The cokernel in degree zero is $\mathcal{J}(U) \oplus (0, W_0, 0)$, as asserted. $\square$

Theorem 6.2. *If $\alpha \in k^\times$ has finite order $m \geq 2$, every semi-Gorenstein-projective module on either side of T_α is projective.*

Proof. The left side is Proposition 5.1. For the right side it suffices by Lemma 6.1 to prove that eventual vanishing of $\text{Tor}_n^R(U, M)$ forces U projective.

For $\lambda \neq 0$ define $C_\lambda = \text{coker } \delta_\lambda^\top$. Its presentation $R^2 \twoheadrightarrow C_\lambda$ is a projective cover, since its relations lie in the radical. If λ and $\alpha\lambda$ are both different from 1, Lemma 4.1 and (16) imply

$$\Omega C_\lambda \cong C_{\alpha\lambda}.$$

At $\lambda = \alpha^{-1}$, the kernel of $\delta_{\alpha^{-1}}^\top$ has dimension 14 and contains the 13-dimensional image of $\delta_1^\top$. At $\lambda = 1$, its kernel has dimension 15 and contains the 14-dimensional image of $\delta_\alpha^\top$. The induced maps on cokernels therefore give

$$\begin{aligned} 0 \longrightarrow k \longrightarrow C_1 \longrightarrow \Omega^{m-1}M \longrightarrow 0, \\ 0 \longrightarrow k \longrightarrow M \longrightarrow \Omega C_1 \longrightarrow 0. \end{aligned} \tag{24}$$

The kernels are one-dimensional and hence are the unique simple module k . For the first sequence we used $C_{\alpha^{-1}} \cong \Omega^{m-2}M$; when $m = 2$ this is $C_\alpha = M$.

Suppose $\text{Tor}_n^R(U, M) = 0$ for $n \gg 0$. The long exact sequences of (24), followed by dimension shifting, give

$$\text{Tor}_n^R(U, k) \cong \text{Tor}_n^R(U, C_1), \quad \text{Tor}_{n+2}^R(U, C_1) \cong \text{Tor}_n^R(U, k) \quad (n \gg 0).$$

Thus $\beta_{n+2}(U) = \beta_n(U)$ eventually. Equation (12) then yields

$$\beta_n = \beta_{n+2} \geq 3\beta_{n+1} \geq 9\beta_n.$$

It follows that $\beta_n = 0$. The resolution terminates, and self-injectivity of R makes U projective. Both summands in Lemma 6.1 are now projective. $\square$

7 The gendo-symmetric counterexample

We first prove a detection result for a general triangular algebra. In this section R is any finite-dimensional self-injective k -algebra, and M is a nonzero finite left R -module such that

$$\text{for every finite left } L, \quad \text{Ext}_R^n(L, M) = 0 \ (n \gg 0) \implies L \text{ is projective.} \tag{25}$$

Put $A = \begin{pmatrix} R & M \\ 0 & k \end{pmatrix}$, with diagonal idempotents r, s corresponding to R, k . The functors i, j on left modules retain the definitions of Section 5 with A in place of T . The proof of Proposition 5.1 uses only self-injectivity and (25), and gives

$$\text{Ext}_A^n(U, A) = 0 \ (n \gg 0) \implies \text{pd}_A U \leq 1. \tag{26}$$

We use cohomological complexes, with differential $d_Q^n : Q^n \rightarrow Q^{n+1}$, cohomology $H^n(Q) = \ker d_Q^n / \operatorname{im} d_Q^{n-1}$, and projective resolutions in nonpositive degrees. For an idempotent r , the left component rX is the image of multiplication by r on X ; components of complexes are taken termwise. The category $D^-(A\text{-mod})$ is the derived category of bounded-above complexes of finite left modules. A complex is *perfect*, written $V \in \operatorname{per} A$, if it is represented by a bounded complex of finite projectives. Derived tensor and derived Hom are computed by bounded-above projective resolutions in their resolved variables. The shift $Q[1]$ has $Q[1]^n = Q^{n+1}$ and differential $d_{Q[1]}^n = -d_Q^{n+1}$.

Lemma 7.1. *Let $V \in D^-(A\text{-mod})$. If*

$$rH^n(V) = 0 \quad (n \ll 0), \quad H^n \mathbf{R} \operatorname{Hom}_A(V, A) = 0 \quad (n \gg 0),$$

then V is perfect.

Proof. Take a bounded-above minimal projective representative $Q^\bullet$ of V . Such a representative is obtained from a bounded-above projective resolution by splitting off invertible differential blocks, starting at the highest degree; in each degree only finitely many finite projective summands occur. Its terms have the form $Q^n = iP^n \oplus jW^n$, where P^n is projective over R and W^n is a finite k -space. We have

$$\operatorname{Hom}_A(jW, iP') = 0, \quad \operatorname{Hom}_A(jW, jW') \cong \operatorname{Hom}_k(W, W').$$

Thus minimality makes the j -diagonal differential zero. In this decomposition,

$$d_Q^n = \begin{pmatrix} d_P^n & 0 \\ h^n & 0 \end{pmatrix}, \quad h^n : iP^n \longrightarrow jW^{n+1}. \quad (27)$$

The complex $P^\bullet$ is minimal over R , and $(d_P)^2 = 0$.

Let $Q^* = \operatorname{Hom}_A(Q^\bullet, A)$, where $(Q^*)^n = \operatorname{Hom}_A(Q^{-n}, A)$ and $d_{Q^*}^n(g) = (-1)^{n+1} g \circ d_Q^{-n-1}$. Its right r -component is

$$Q^*r = \operatorname{Hom}_A(Q^\bullet, Ar) \cong \operatorname{Hom}_R(P^\bullet, R).$$

It is exact in sufficiently large positive degrees by hypothesis. The functor $\operatorname{Hom}_{R^{\operatorname{op}}}(-, R)$ is exact, since R is right self-injective, and finite projectives identify with their double duals. Therefore $P^\bullet$ is exact in sufficiently negative degrees.

Taking left r -components in (27) gives the short exact sequence of complexes

$$0 \longrightarrow (M \otimes_k W^\bullet, 0) \longrightarrow rQ^\bullet \longrightarrow P^\bullet \longrightarrow 0. \quad (28)$$

The differential of the first complex is zero. In sufficiently negative degrees both $H^{n-1}(P)$ and $H^n(rQ)$ vanish, so the long exact sequence forces $M \otimes_k W^n = 0$. As $M \neq 0$, this gives $W^n = 0$.

The negative tail of Q is therefore iP , and the corresponding positive tail of Q^* is $\operatorname{Hom}_R(P^\bullet, M)$. It is exact by hypothesis. Choose an integer a far enough into this tail, and put $L = \operatorname{coker}(P^{a-1} \rightarrow P^a)$. The exact negative tail is a minimal projective resolution of L , and its Hom into M is eventually exact. Equation (25) makes L projective. Its minimal resolution terminates at P^a , so $P^n = 0$ for $n < a$. Hence Q is bounded. $\square$

Define the trivial extension $B = A \ltimes DA$ by

$$(a, \lambda)(a', \lambda') = (aa', a\lambda' + \lambda a'), \quad (a\lambda a')(x) = \lambda(a'xa).$$

The functional $(a, \lambda) \mapsto \lambda(1)$ gives a symmetric nondegenerate associative pairing on B ; hence B is symmetric. Regard A as a right B -module via the projection $\pi : B \rightarrow A$, $(a, \lambda) \mapsto a$. With ordinary composition of endomorphisms, set

$$E = \text{End}_B(B \oplus A) = \begin{pmatrix} B & DA \\ A & A \end{pmatrix}. \quad (29)$$

The diagonal actions on the off-diagonal blocks use π on B . The product of an upper-right element λ with a lower-left element v is $(0, \lambda v) \in B$; the reverse product is zero. These statements follow by identifying a right B -map $B \rightarrow A$ with its value at 1, and a right B -map $A \rightarrow B$ with an element of $\text{ann}_B(DA) = \{b \in B : b(0, \lambda) = 0 \text{ for every } \lambda \in DA\} = DA$. Let $e, f \in E$ be the diagonal idempotents for B, A , respectively. They are distinct from $r, s \in A$. The algebra E is gendo-symmetric: by definition it is the endomorphism algebra of a generator over a symmetric algebra, and $B \oplus A$ contains B .

Lemma 7.2. *The map*

$$p : E \longrightarrow A, \quad \begin{pmatrix} b & \lambda \\ v & a \end{pmatrix} \longmapsto \pi(b)$$

is a surjective algebra map. Multiplication identifies $Ef \otimes_A fE$ with EfE , and there is an exact sequence of left E -modules

$$0 \longrightarrow Ef \longrightarrow Ee \longrightarrow {}_pA \longrightarrow 0. \quad (30)$$

Here ${}_pA$ is the left E -module with action $z \cdot a = p(z)a$ for $z \in E$, $a \in A$, and Ee is injective.

Proof. The products through f have blocks DA, DA, A, A , and the natural maps from the four tensor blocks are isomorphisms. These are precisely the kernel of p . Also ${}_AfE = A \oplus A$ is free. In column coordinates $Ef = (DA, A)$ and $Ee = (B, A)$; the injection in (30) is $(\lambda, a) \mapsto ((0, \lambda), a)$ and its quotient is $B \rightarrow A$.

Write $\tau(b) = \lambda(1)$ for $b = (a, \lambda) \in B$. The pairing between Ee and eE given by $(u, v) \mapsto \tau(vu)$ is E -balanced. In coordinates it pairs B with B by the symmetric form on B , and A with DA by evaluation $\lambda(a)$. It is nondegenerate. Thus $Ee \cong D(eE)$ as left E -modules, and is injective because eE is right projective. $\square$

Theorem 7.3. *Under (25), every finite left E -module X with $\text{Ext}_E^n(X, E) = 0$ for $n \gg 0$ has finite projective dimension. Every semi-Gorenstein-projective left E -module is projective, and every Gorenstein-projective module on either side of E is projective.*

Proof. Let A_p be A as a left A -right E bimodule via p on the right, and put $V = A_p \otimes_E^{\mathbf{L}} X$. The short exact sequence $0 \rightarrow EfE \rightarrow E \rightarrow A_p \rightarrow 0$ and Lemma 7.2 give a triangle

$$Ef \otimes_A^{\mathbf{L}} fX \longrightarrow X \longrightarrow {}_pV \longrightarrow (Ef \otimes_A^{\mathbf{L}} fX)[1] \quad (31)$$

in the derived category of left E -modules. Indeed, fE is free on the left over A and projective on the right over E , so its derived tensor with X is fX . Each term defining V from a resolution of X by finite projective modules is a finite left A -projective, since $A_p \otimes_E Ee = A$ and $A_p \otimes_E Ef = 0$. Thus $V \in D^-(A\text{-mod})$.

The long exact sequence of (30) gives eventual vanishing of $\text{Ext}_E^n(X, {}_pA)$. Derived tensor-Hom adjunction identifies this with

$$H^n \mathbf{R} \text{Hom}_A(V, A) = 0 \quad (n \gg 0). \quad (32)$$

One can compute the adjunction termwise using the same projective resolution of X ; its tensor terms are the projectives just described.

Take the e -component of (31) and restrict from B to its subalgebra A . This gives

$$DA \otimes_A^L fX \longrightarrow eX \longrightarrow V \longrightarrow (DA \otimes_A^L fX)[1]. \quad (33)$$

The right A -module $rDA = D(Ar)$ is $(DR, 0, 0)$. Since R is self-injective, DR is right projective, and

$$0 \longrightarrow (0, DR \otimes_R M, 0) \longrightarrow (DR, DR \otimes_R M, 1) \longrightarrow (DR, 0, 0) \longrightarrow 0$$

is a projective resolution of length at most one. Taking left r -components in (33) therefore yields $rH^n(V) = 0$ for $n < -2$. Lemma 7.1, with (32), proves that V is perfect.

Since eX is a module, (33) now implies $\mathrm{Tor}_n^A(DA, fX) = 0$ for $n \gg 0$. Finite-dimensional duality gives

$$D \mathrm{Tor}_n^A(DA, fX) \cong \mathrm{Ext}_A^n(fX, DDA) \cong \mathrm{Ext}_A^n(fX, A).$$

By (26), $\mathrm{pd}_A fX \leq 1$. Hence the first term of (31) is perfect over E : tensor a finite A -projective resolution with Ef . The third term is also perfect, because V is perfect over A and (30) gives $\mathrm{pd}_{E(pA)} \leq 1$. Closure of bounded projective complexes under cones makes X perfect, or equivalently of finite projective dimension.

If X is semi-Gorenstein-projective and has positive finite projective dimension d , take a shortest projective resolution. Its last differential $P_d \rightarrow P_{d-1}$ is injective with $P_d \neq 0$. Vanishing of $\mathrm{Ext}_E^d(X, E)$ would make $P_{d-1}^* \rightarrow P_d^*$ surjective, hence split because P_d^* is projective. Double duality on projectives would split the last differential, shortening the resolution. This contradiction proves that X is projective.

Finally the algebra dual of a right Gorenstein-projective module is left Gorenstein-projective and hence projective. Dualizing again, using reflexivity of Gorenstein-projective modules, proves that the right module is projective as well. $\square$

Lemma 7.4. *For any finite-dimensional A and its algebra E in (29), define on right modules*

$$F(U) = U \otimes_A fE = (U, U), \quad (x, y) \begin{pmatrix} b & \lambda \\ v & a \end{pmatrix} = (x\pi(b) + yv, ya).$$

The functor F is exact, fully faithful, and preserves and reflects projectivity. Every $F(U)$ is reflexive, and

$$\mathrm{Ext}_{E^{\mathrm{op}}}^n(F(U), E) \cong \mathrm{Ext}_{A^{\mathrm{op}}}^n(U, A) \quad (n > 0) \quad (34)$$

as k -vector spaces.

Proof. The left A -module $fE = A \oplus A$ is free, so F is exact; it sends A to the right projective fE . Adjunction gives

$$\mathrm{Hom}_{E^{\mathrm{op}}}(F(U), F(U')) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(U, F(U')f) = \mathrm{Hom}_{A^{\mathrm{op}}}(U, U').$$

Thus F is fully faithful. If $F(U)$ is projective, applying F to a projective surjection onto U gives a split surjection. Full faithfulness pulls the splitting back, proving U projective. Apply F to a projective resolution of U and use $Ef = DA \oplus A$ as a right A -module. Since DA is right injective, Hom adjunction gives (34).

For reflexivity, put $H = F(U)^*$. The same adjunction gives its left E -module components as

$$eH = \mathrm{Hom}_{A^{\mathrm{op}}}(U, DA) = DU, \quad fH = U^* = \mathrm{Hom}_{A^{\mathrm{op}}}(U, A).$$

The upper-right action is

$$DA \otimes_A U^* \longrightarrow DU, \quad \lambda \otimes g \longmapsto (u \mapsto \lambda(g(u))),$$

and the lower-left action is zero. The B -action on DU factors through π .

A left E -map $H \rightarrow Ee = (B, A)$ has upper component with image in DA , since DA annihilates DU and its annihilator in B is DA . Every left A -map $DU \rightarrow DA$ has the form

$$\varphi \longmapsto (a \longmapsto \varphi(ua))$$

for a unique $u \in U$, by vector-space duality. Write $\gamma : U^* \rightarrow A$ for its lower component. Compatibility with the upper-right action forces $\gamma(g) = g(u)$: evaluation at $a \in A$ gives the equality $\lambda(g(u)a) = \lambda(\gamma(g)a)$ for all $\lambda \in DA$. Conversely these formulas define an E -map. Its image already lies in $Ef \subset Ee$. It follows that both $\text{Hom}_E(H, Ee)$ and $\text{Hom}_E(H, Ef)$ identify with U . Under these identifications evaluation $F(U) \rightarrow H^*$ is the identity on each of its two components. It is therefore an isomorphism of right E -modules. $\square$

Proof of the gendo-symmetric assertion in Theorem 1.1. Take $A = T_\alpha$ with α of infinite order. Lemma 3.2 supplies (25). Set $E_\alpha = \text{End}_{A \ltimes DA}((A \ltimes DA) \oplus A)$. Theorem 7.3 makes every semi-Gorenstein-projective left E_α -module projective. The module S_0 of Proposition 5.2 is nonprojective and semi-Gorenstein-projective. Lemma 7.4 makes $F(S_0)$ reflexive, semi-Gorenstein-projective and nonprojective. Theorem 7.3 makes every right Gorenstein-projective module projective, so $F(S_0)$ is not Gorenstein-projective. Equation (29) gives $\dim_k E_\alpha = 5 \dim_k A = 145$, and $\dim_k F(S_0) = 2 \dim_k S_0 = 8$.

Equation (34) and Proposition 5.3 give $\text{id}_{E^{\text{op}}} E = \infty$. If $\text{id}_E E = d < \infty$, the d th syzygy of every finite left module would be semi-Gorenstein-projective and hence projective by Theorem 7.3. Thus every finite left module would have projective dimension at most d . For every finite right module Y and finite left module Z , duality then gives

$$\text{Tor}_{d+1}^E(Y, Z) = D \text{Ext}_E^{d+1}(Z, DY) = 0.$$

Taking $Z = E/\text{rad } E$ detects the $(d+1)$ st term of a minimal right projective resolution, so every finite right module has projective dimension at most d . In particular $F(S_0)$ has finite projective dimension. The last-differential argument in Theorem 7.3, on the right, makes a semi-Gorenstein-projective module of finite projective dimension projective. This contradicts Lemma 7.4. Hence $\text{id}_E E = \infty$. $\square$

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