

# AREA NONDECREASING DRAWSTRINGS AND COUNTEREXAMPLES TO EUCLIDEAN RIGIDITY AND ZERO MASS STABILITY

JIANGCHENG YOU AND HENG ZHANG

ABSTRACT. Building on the drawstring construction of Kazaras–Xu, we develop a localized refinement that contracts prescribed curves while preserving positive scalar curvature and without decreasing two-dimensional area. As two applications, we disprove part (b) of Gromov’s Euclidean  $C^0$  rigidity conjecture and Sormani’s zero mass stability conjecture.

## CONTENTS

|                                           |    |
|-------------------------------------------|----|
| 1. Introduction                           | 1  |
| 2. The area nondecreasing construction    | 7  |
| 3. A counterexample to Gromov’s question  | 10 |
| 4. A counterexample to Sormani’s question | 14 |
| References                                | 24 |

## 1. INTRODUCTION

This paper studies two questions about the relation between scalar curvature and distance. Gromov asks whether a metric on  $\mathbb{R}^3$  with nonnegative scalar curvature must be flat if it is uniformly comparable to the Euclidean metric and its distances differ from Euclidean distances by at most a fixed constant [14, Section 3.11, p. 156]. Sormani asks whether, as the ADM mass tends to zero, suitable isoperimetric regions must converge to Euclidean balls under additional geometric assumptions [26, Conjecture 1.5]. Motivated by the small mass examples and distance approximation methods of Kazaras–Xu [16], we construct counterexamples to both questions in dimension three. Our main tool is the drawstring construction of Kazaras–Xu [16], which allows prescribed curves to be shortened while controlling scalar curvature. We modify this construction to prevent area decrease and combine it with global distance and isoperimetric arguments to obtain the two counterexamples.

**1.1. The questions of Gromov and Sormani.** Scalar curvature depends on second derivatives of the metric, yet its lower bounds exhibit a remarkable  $C^0$  stability. Gromov proved that a fixed scalar curvature lower bound is preserved under local  $C^0$  convergence of smooth metric tensors to a smooth limit [13], and Bamler subsequently gave a Ricci flow proof [3]. Related developments include Ricci flow definitions of lower scalar curvature bounds for continuous metrics [9], quantitative estimates under  $C^0$  convergence [19, 23], and preservation of scalar curvature lower bounds under suitable weak convergence [24]. Related geometric rigidity results include the polyhedron theorems of Li and Brendle [20, 21, 8].

In the setting of Euclidean rigidity, the classical positive mass theorem establishes flatness from the vanishing of ADM mass, whose definition involves first derivatives of the metric. This raises the question of whether, under nonnegative scalar curvature, rigidity can instead be deduced from  $C^0$  geometric information, without derivative control at infinity. Such information may be expressed either through pointwise control of the metric tensor or through control of its induced distance function. These two forms of control, however, are geometrically distinct.

---

*Date:* October 2026.

*2020 Mathematics Subject Classification.* 53C21, 53C23, 53C24, 83C40.

*Key words and phrases.* Scalar curvature, drawstring construction, Euclidean rigidity, ADM mass, intrinsic flat convergence, isoperimetric regions.

Indeed, distances depend on the lengths of minimizing paths and can therefore be affected by deformations concentrated in small regions. For example, shortening a curve may alter distances between points far from the deformation, even when the metric remains unchanged near those points. Thus control of the metric away from a small set need not imply global control of distances. Against this background, Gromov proposed a Euclidean  $C^0$  rigidity conjecture involving either asymptotic control of the metric tensor or uniform bi-Lipschitz control together with a bounded additive distance error [14]. Let  $g_E$  denote the Euclidean metric. His conjecture takes the following form.

**Conjecture 1.1** ([14, Section 3.11, p. 156]). Let  $n \geq 3$  and let  $g$  be a smooth metric on  $\mathbb{R}^n$  with  $\text{Scal}_g \geq 0$ . Suppose that either

- (a)  $|g - g_E|_{g_E}(x) = o(|x|^{2-n})$  as  $|x| \rightarrow \infty$ ; or
- (b) for some  $L \geq 1$  and  $E_0 < \infty$ ,

$$L^{-2}g_E \leq g \leq L^2g_E, \quad \sup_{x,y \in \mathbb{R}^n} |d_g(x,y) - |x-y|| \leq E_0. \quad (1)$$

Then  $g$  is flat.

Alternative (a) has been established in dimension three by the authors [29] and Mazurowski–Yao [22], respectively. In higher dimensions  $n \geq 4$ , it is proved by Chu–Lee–Wan [11]. In this paper, our concern is alternative (b).

The nonflat scalar flat metrics constructed in [11, Appendix A] already imply the failure of (b) for  $n \geq 4$ . A direct example with positive scalar curvature is

$$U_a(x) = 1 + a(1 + |x|^2)^{-(n-2)/2}, \quad g_{n,a} = U_a^{4/(n-2)}g_E, \quad a > 0.$$

The remaining case of (b) is dimension three.

We fix conventions before stating the second question. For a smooth bounded domain  $\Omega \subset (M, g)$ , let  $d_{g,\Omega}$  be the intrinsic distance on  $\overline{\Omega}$ , with paths constrained to  $\overline{\Omega}$ , and set

$$X_g(\Omega) = (\overline{\Omega}, d_{g,\Omega}, [\![\Omega]\!]_g), \quad \mathcal{E}_R = (\overline{B}_R, |\cdot - \cdot|, [\![B_R]\!]_{g_E}).$$

Here  $B_R$  is the Euclidean coordinate ball centered at the origin,  $[\![\Omega]\!]_g$  is the oriented integration current, and  $d_{\mathcal{F}}$  denotes the intrinsic flat distance of Sormani–Wenger [27]. We abbreviate  $X_g(B_R)$  to  $X_g(R)$  and write  $P_g$  for perimeter. Boundary current spaces carry the restriction of the distance on the region; in particular,  $\partial\mathcal{E}_R$  carries Euclidean chord distance. The following conjecture is posed by Sormani.

**Conjecture 1.2** ([26, Conjecture 1.5]). Let  $(M_j, g_j)$  be complete asymptotically flat three-manifolds without boundary, with nonnegative scalar curvature, no closed minimal surfaces, and  $m_{\text{ADM}}(M_j, g_j) \rightarrow 0$ . Fix  $R > 0$ , and let  $\Omega_j(R)$  be smooth bounded regions maximizing volume among regions with boundary area  $4\pi R^2$ . Suppose that

$$\text{diam}(\overline{\Omega_j(R)}, d_{g_j, \Omega_j(R)}) \leq D_R$$

and that there exist exterior diffeomorphisms  $F_j : M_j \setminus \Omega_j(R) \rightarrow \mathbb{R}^3 \setminus B_R$  such that

$$\|(F_j)_*g_j - g_E\|_{C^2(\{r \geq R\})} \rightarrow 0.$$

Then the regions  $\Omega_j(R)$  converge in the volume preserving intrinsic flat sense to the Euclidean ball.

The second question lies within the framework of stability for the rigidity statement of the positive mass theorem. In the time symmetric setting, the second fundamental form and momentum density vanish. With zero cosmological constant and in units  $G = c = 1$ , the Hamiltonian constraint reduces to  $\text{Scal}_g = 16\pi\mu$ , where  $\mu$  is the energy density measured by the normal observer. Thus the dominant energy condition is equivalent to  $\text{Scal}_g \geq 0$  [4]. For asymptotically flat initial data, the ADM mass represents the total energy of an isolated gravitational system [2]. The positive mass theorem [25, 28] states that a complete asymptotically flat three-manifold without boundary, with nonnegative scalar curvature and zero ADM mass, must be Euclidean. This rigidity result raises a natural physical question: if the total energy of an isolated system is very small, must its spatial geometry be nearly Euclidean, or can substantial geometric distortions persist even as the ADM mass tends to zero? Since the ADM mass is a global quantity, its smallness does not directly control the geometry of individual regions. The stability problem therefore depends crucially on the notion of geometric convergence and the choice of regions to compare.

Lee–Sormani [18] established intrinsic flat stability in the rotationally symmetric setting, where deep, thin wells illustrate why Gromov–Hausdorff convergence need not hold. Such wells can retain nonvanishing depth while their volumes tend to zero, so they may remain visible in Gromov–Hausdorff limits while disappearing in intrinsic flat limits. This motivates intrinsic flat convergence as a possible stability notion, but does not address shortcuts that alter distances across regions of positive volume.

Kazaras–Xu [16, Theorem 1.7 and Corollary 1.8] constructed asymptotically flat metrics on  $\mathbb{R}^3$  with nonnegative scalar curvature, no closed embedded minimal surfaces, and ADM mass tending to zero, whose pointed Gromov–Hausdorff and intrinsic flat limit is obtained from Euclidean space by identifying a circle to a point. As noted in [16, Remark 1.9], these examples already contradict earlier formulations of intrinsic flat stability.

**1.2. Area nondecreasing drawstrings.** Our method for proving the two conjectures is based on the drawstring construction. Earlier shortcut limits arise from the sewing construction of Basilio–Dodziuk–Sormani [5], which uses increasingly many thin tunnels to obtain positive scalar curvature manifolds converging to a space in which a curve is identified to a point. Kazaras–Xu’s original drawstring examples [17] realize pulled string limits on a fixed three-torus with scalar curvature bounded below by  $-1/j$ .

Kazaras–Xu’s codimension-two drawstring theorem [16, Theorem 2.4] realizes prescribed tangential contractions with arbitrarily small scalar curvature loss, without changing the underlying manifold. In dimension three, its possible two-dimensional area decrease can be made arbitrarily small. We remove this decrease by a conformal correction supported in a fixed tube with positive scalar curvature. The prescribed final contraction is retained, and the volume excess is nonnegative and arbitrarily small. Exact area dominance also appears in the spherical examples of [16, Theorem 1.11]; here we require the correction to preserve the metric outside the prescribed tube.

We provide a precise clarification of the two-dimensional area nondecrease. For metrics  $b$  and  $g$  on a three-manifold, write  $g \succeq_2 b$  if

$$|v \wedge w|_g \geq |v \wedge w|_b \quad \text{for all tangent vectors } v, w.$$

Let  $(M, b)$  be an oriented three-manifold and  $\Gamma \subset M$  a smooth embedded closed curve. Put  $t = d_b(\cdot, \Gamma)$  and  $N_b(\Gamma, r) = \{t < r\}$ . Choose  $r_* > 0$  so that  $\overline{N_b(\Gamma, 2r_*)}$  is a regular compact tubular neighborhood, every level  $\{t = s\}$  with  $0 < s \leq 2r_*$  is strictly mean convex, and  $\text{Scal}_b > 0$  on  $\overline{N_b(\Gamma, r_*)}$ . Fix a smooth cutoff  $0 \leq \chi(t) \leq 1$ , equal to one for  $t \leq r_*/2$  and zero for  $t \geq r_*$ . For  $q \in C^\infty(\Gamma)$  with  $0 < q \leq 1$ , set

$$q_- = \min_\Gamma q, \quad A_\varepsilon = -\frac{1}{2} \log(1 - \varepsilon).$$

Our main construction is

**Theorem A** (Area nondecreasing construction). *For every sufficiently small  $\varepsilon > 0$ , there is a smooth metric  $g_\varepsilon$  satisfying*

$$\begin{aligned} g_\varepsilon &= b \quad \text{for } t \geq r_*, & g_\varepsilon|_{T\Gamma} &= q^2 b|_{T\Gamma}, \\ g_\varepsilon &\succeq_2 b, & \text{Scal}_{g_\varepsilon} &\geq \tfrac{1}{2} \text{Scal}_b > 0 \quad \text{on } N_b(\Gamma, r_*). \end{aligned} \tag{2}$$

*All levels  $0 < t \leq 2r_*$  remain strictly mean convex. The measure  $\nu_\varepsilon = dV_{g_\varepsilon} - dV_b$  is nonnegative, supported in  $\overline{N_b(\Gamma, r_*)}$ , and satisfies*

$$\nu_\varepsilon(M) \leq C\varepsilon, \tag{3}$$

$$\min\{q_-^2, 1 - \varepsilon\} b \leq g_\varepsilon \leq e^{4A_\varepsilon} q_-^{-2} b. \tag{4}$$

*The constant  $C$  depends only on the tube and cutoff, not on  $q$ . There is  $0 < r_1 < \min\{\varepsilon, r_*/2\}$  such that  $g_\varepsilon = e^{2A_\varepsilon \chi} b$  for  $t \geq r_1$ . Every  $b$ -normal radial segment from  $\Gamma$  to a point with  $t \leq r_1$  has  $g_\varepsilon$ -length at most  $3e^{A_\varepsilon} r_1$ .*

The correction is  $g_\varepsilon = e^{2A_\varepsilon \chi} \widehat{b}$ , where  $\widehat{b}$  is the Kazaras–Xu metric constructed with tangential factor  $e^{-A_\varepsilon} q$  in [16]. On  $N_b(\Gamma, r_1) \setminus \Gamma$ , the length factors of  $\widehat{b}$  relative to a  $b$ -orthonormal frame adapted to the normal radial, normal angular, and tangential directions are  $e^{-u}$ ,  $e^{-u}h$ , and  $e^u$ , respectively, where  $u \leq 0$  and

$1 - r_1 \leq h \leq 1$ . Its smallest two-dimensional area factor is therefore  $h$ . On the core, where  $\chi = 1$ , the conformal correction gives

$$\frac{|v \wedge w|_{g_\varepsilon}}{|v \wedge w|_b} \geq e^{2A_\varepsilon} h \geq \frac{1 - r_1}{1 - \varepsilon} \geq 1 \quad (v \wedge w \neq 0), \quad \frac{dV_{g_\varepsilon}}{dV_b} = e^{3A_\varepsilon - u} h \geq 1.$$

The initial extra contraction exactly cancels the conformal enlargement along  $\Gamma$ , leaving the prescribed tangential factor  $q$ , i.e.  $g_\varepsilon|_{T\Gamma} = q^2 b|_{T\Gamma}$ . On the fixed transition annulus  $\{r_*/2 < t < r_*\}$ , we have  $\widehat{b} = b$ . Since  $A_\varepsilon = O(\varepsilon)$  and the cutoff  $\chi$  is independent of  $\varepsilon$ , the conformal correction changes the scalar curvature there by at most  $C\varepsilon$ . The positive lower bound for the scalar curvature therefore gives the stated scalar curvature estimate (2) for sufficiently small  $\varepsilon$ . The Kazaras–Xu volume estimate gives  $\text{Vol}_{g_\varepsilon}(N_b(\Gamma, r_1)) = O(r_1^2)$ , while  $g_\varepsilon = e^{2A_\varepsilon \chi} b$  outside the core yields a volume increase of  $O(\varepsilon)$ . These estimates imply (3). Modifications supported in a locally finite family of tubes with pairwise disjoint closures assemble into a smooth metric (see, Corollary 2.1).

The area comparison in (2) has a useful global consequence. Suppose that a smooth relatively compact region  $\Omega$  uniquely maximizes  $b$ -volume among finite volume sets with perimeter at most  $P_b(\Omega)$ , and that  $g \succeq_2 b$  with  $g = b$  outside a compact subset of  $\Omega$ . Writing  $\nu = dV_g - dV_b \geq 0$ , every set  $E$  of finite  $g$ -volume and finite perimeter satisfying  $P_g(E) \leq P_g(\Omega) = P_b(\Omega)$  obeys

$$P_b(E) \leq P_b(\Omega), \quad \text{Vol}_g(\Omega) - \text{Vol}_g(E) = \text{Vol}_b(\Omega) - \text{Vol}_b(E) + \nu(\Omega \setminus E) \geq 0.$$

The equality case preserves uniqueness. This is Lemma 2.2, and it is the mechanism that retains exact isoperimetry in the small mass construction.

**1.3. Uniform additive distance control.** Both applications use the metric family

$$\beta_a = \varphi_a^4 g_E, \quad \varphi_a(x) = 1 + \frac{a}{\sqrt{1 + |x|^2}}, \quad a > 0. \quad (5)$$

Writing  $r = |x|$ , these metrics are complete and asymptotically flat, with

$$\text{Scal}_{\beta_a}(r) = \frac{24a}{(\sqrt{1 + r^2} + a)^5} > 0, \quad m_{\text{ADM}}(\beta_a) = 2a.$$

For the first application,  $a > 0$  is fixed and the coordinates are isotropic coordinates. The background itself does not satisfy a bounded additive distance estimate (1) in dimension three. For example, its radial distance obeys

$$d_{\beta_a}(0, re_1) - r = 2a \operatorname{arsinh} r + a^2 \arctan r,$$

which diverges as  $r \rightarrow \infty$ . If a metric agrees with  $\beta_a$  outside  $B_{R_0}$ , every path from the origin to  $re_1$ ,  $r > R_0$ , has length at least  $\int_{R_0}^r \varphi_a(s)^2 ds$ . Thus compactly supported modifications cannot provide the global bound sought here; the drawstrings in the first construction extend to infinity.

**Theorem B** (Counterexample to Gromov’s Conjecture 1.1). *For every  $\lambda > 1$ ,  $E > 0$ , and  $V > 0$ , there are  $a > 0$  and a smooth complete metric  $g$  on  $\mathbb{R}^3$  such that*

$$\begin{aligned} \text{Scal}_g &> 0, \quad g_E \leq g \leq \lambda^2 g_E, \\ 0 &\leq d_g(x, y) - |x - y| \leq E \quad (x, y \in \mathbb{R}^3). \end{aligned} \quad (6)$$

Moreover,  $g \succeq_2 \beta_a$ ,

$$0 \leq \int_{\mathbb{R}^3} (dV_g - dV_{\beta_a}) \leq V,$$

and, as  $|x| \rightarrow \infty$ ,

$$|g - g_E|_{g_E} = O(|x|^{-1}), \quad |g - g_E|_{g_E} \neq o(|x|^{-1}), \quad \inf_{\mathbb{R}^3} \text{Scal}_g = 0. \quad (7)$$

Since  $\text{Scal}_g > 0$ , this disproves alternative (b) of Conjecture 1.1 in dimension three, even with arbitrarily small additive error and a bi-Lipschitz constant arbitrarily close to one. The critical tensor decay in (7) is consistent with the rigidity results for alternative (a).

*Idea of construction.* Our construction here adapts the finite net drawstring method of Kazaras–Xu [16, Theorem 1.6 and Section 5.2], who use drawstrings along finitely many disjoint closed curves to approximate the distances of a smaller conformal metric on a closed three-manifold. We extend this approach to  $\mathbb{R}^3$  by constructing short paths in each unit cube and choosing summable local distance errors to obtain a global additive distance bound.

We tile  $\mathbb{R}^3$  by unit cubes  $Q_k$ . In each cube, choose finitely many pairwise disjoint segments whose endpoint pairs approximate every pair in  $Q_k \times Q_k$  to a prescribed accuracy, and complete them to disjoint embedded closed curves. For small  $\eta_k > 0$ , prescribe along each such curve  $\Gamma_i$

$$q_i = \frac{1 + \eta_k}{\varphi_a^2|_{\Gamma_i}}, \quad g|_{T\Gamma_i} = (1 + \eta_k)^2 g_E|_{T\Gamma_i}.$$

Thus the selected segments have  $g$ -lengths close to their Euclidean lengths, and sufficiently thin tubes ensure  $g \geq g_E$ . Given  $x, y \in Q_k$ , choose a selected segment  $[p_i, z_i]$  whose endpoints are close to  $x$  and  $y$ , respectively. Joining  $x$  to  $p_i$  and  $z_i$  to  $y$  by two short connectors gives a path of  $g$ -length at most  $|x - y| + \tau_k$ , provided the endpoint approximation and  $\eta_k$  are sufficiently small; see Figure 1.

Choose the local errors so that  $\sum_k \tau_k \leq E$ . To obtain the global estimate (6), divide an arbitrary Euclidean segment  $[x, y] \subset \mathbb{R}^3$  into portions lying in successive cubes, and replace each portion by a corresponding short path. Concatenating these paths gives a path from  $x$  to  $y$  whose length exceeds  $|x - y|$  by at most the sum of the local errors. Consequently,

$$0 \leq d_g(x, y) - |x - y| \leq E.$$

Since all modifications are supported strictly inside the cubes, the resulting local metrics agree with the same smooth background near cube boundaries and assemble into a smooth global metric. Positive scalar curvature and area nondecrease are preserved by Theorem A.

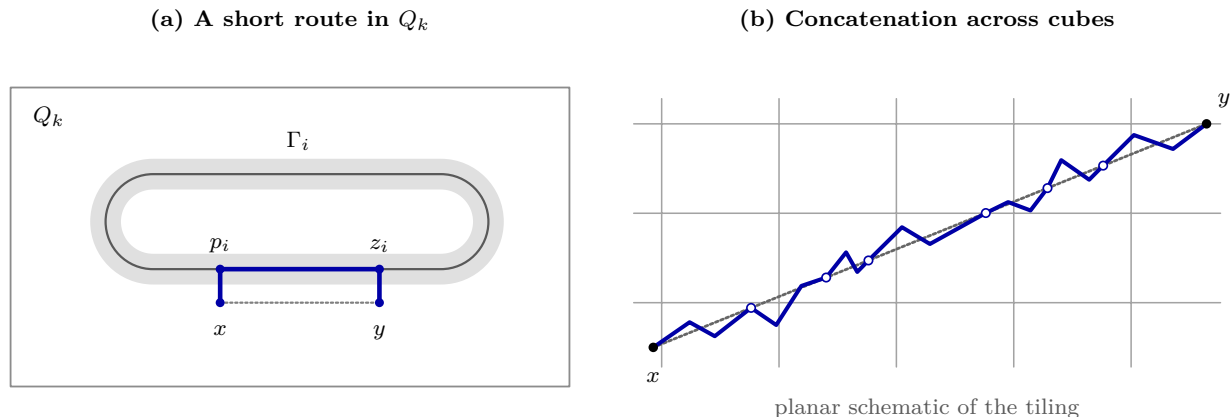


FIGURE 1. The construction for Gromov's question. (a) A selected drawstring segment in  $Q_k$ , together with short connectors, provides a route from  $x$  to  $y$  with small additive length error. (b) Concatenating these local routes across cubes yields a uniform global distance bound by summability of the local errors.

**1.4. Small mass with exact isoperimetry.** For the second application, let  $a \downarrow 0$  and write  $\beta_a$  in area-radius coordinates  $\rho = r\varphi_a(r)^2$  as  $b_a$ . In these coordinates,

$$b_a = \frac{d\rho^2}{W_a(\rho)} + \rho^2 g_{\mathbb{S}^2}, \quad W_a(\rho(r)) = \left(1 + \frac{2r\varphi'_a(r)}{\varphi_a(r)}\right)^2.$$

For small  $a$ , one has  $\frac{1}{2} \leq W_a < 1$  for  $\rho > 0$ , and the scalar curvature is strictly decreasing with  $\rho$ . For all sufficiently small  $a > 0$ , every centered ball is uniquely isoperimetric in  $(\mathbb{R}^3, b_a)$ , and  $P_{b_a}(B_R) = 4\pi R^2$ . Section 4.1 proves this using Brendle's classification of constant mean curvature surfaces [7] and isoperimetric existence [10]. All coordinates in the remainder of this subsection are area-radius coordinates.

Fix the circle

$$\Gamma = \{x \in \mathbb{R}^3 : x_3 = 0, x_1^2 + x_2^2 = 1\}. \quad (8)$$

For  $q \in (0, 1)$ , let  $D_q$  be the length metric obtained by charging Euclidean length away from  $\Gamma$  and  $q$  times Euclidean length along  $\Gamma$ . Explicitly, for Lipschitz paths  $\gamma : [0, 1] \rightarrow \mathbb{R}^3$  joining  $x$  to  $y$ , set

$$D_q(x, y) = \inf_{\gamma} \int_0^1 \theta_q(\gamma(s)) |\gamma'(s)|_{g_E} ds, \quad \theta_q(z) = \begin{cases} q, & z \in \Gamma, \\ 1, & z \notin \Gamma. \end{cases}$$

In particular,  $q|x - y| \leq D_q(x, y) \leq |x - y|$ . For  $R > 1$  and  $x, y \in \overline{B}_R$ , the distance  $D_q(x, y)$  can be computed using only paths in  $\overline{B}_R$ , since radial projection onto the ball fixes the endpoints and does not increase the weighted path length determined by  $D_q$ ; see Section 4.2. For  $R > 1$ , define

$$X_q(R) = (\overline{B}_R, D_q, T_{q,R}), \quad T_{q,R} = \text{id}_{\#} \llbracket B_R \rrbracket_{g_E},$$

where  $\text{id} : (\overline{B}_R, |\cdot - \cdot|) \rightarrow (\overline{B}_R, D_q)$ .

**Theorem C** (counterexample to Sormani's Conjecture 1.2). *For every  $\lambda > 1$  and  $q \in (\lambda^{-1}, 1)$ , there are smooth complete asymptotically flat metrics  $g_j$  on  $\mathbb{R}^3$ , with no closed embedded minimal surfaces, such that*

$$\text{Scal}_{g_j} > 0, \quad m_{\text{ADM}}(g_j) = m_j \downarrow 0, \quad \lambda^{-2}\delta \leq g_j \leq \lambda^2\delta. \quad (9)$$

*For every  $R > 2$ ,  $B_R$  uniquely maximizes volume among finite volume sets with perimeter at most  $4\pi R^2$ , and  $P_{g_j}(B_R) = 4\pi R^2$ . For each fixed  $R > 2$  and all sufficiently large  $j$ ,*

$$0 \leq \text{Vol}_{g_j}(B_R) - \frac{4\pi R^3}{3} \leq C_R m_j, \quad (10)$$

$$\text{diam}(\overline{B}_R, d_{g_j, B_R}) \leq 2R + C_R m_j, \quad (11)$$

$$\sup_{p, z \in \overline{B}_R} |d_{g_j, B_R}(p, z) - D_q(p, z)| \leq C_R m_j, \quad (12)$$

$$d_{\mathcal{F}}(X_{g_j}(R), X_q(R)) \leq C_R m_j, \quad (13)$$

$$d_{\mathcal{F}}(X_{g_j}(R), \mathcal{E}_R) \geq c_{R,q} - C_R m_j, \quad (14)$$

where  $c_{R,q}$  is a positive constant. Moreover, the metrics converge to the Euclidean metric in  $C^2$  outside  $B_R$  at rate  $O(m_j)$ , the mass of  $T_{q,R}$  equals the Euclidean volume of  $B_R$ , and the intrinsic and ambient distances agree on  $\overline{B}_R$ , i.e.

$$d_{g_j, B_R} = d_{g_j}|_{\overline{B}_R \times \overline{B}_R}. \quad (15)$$

Thus  $\Omega_j(R) = B_R$ , with  $R > 2$  fixed independently of  $j$ , satisfies the hypotheses of Conjecture 1.2, with identity exterior maps, but has a positive intrinsic flat separation from  $\mathcal{E}_R$  in the sense of (14). Uniqueness rules out obtaining the conjectured limit by selecting different maximizers at the same area. The contraction factor  $q$  is fixed, so the limiting length of  $\Gamma$  is  $2\pi q > 0$ . Partial drawstring limits were studied in [16, Theorem 1.5]; here we identify the intrinsic flat current limit while imposing exact global isoperimetry and a prescribed uniform bi-Lipschitz bound.

The result is compatible with positive stability theorems that impose additional geometric control or change the domains being compared. Under their uniform asymptotic flatness and topological assumptions, Kazaras–Khuri–Lee [15] prove pointed Gromov–Hausdorff stability with a uniform Ricci curvature lower bound. Dong–Song [12] obtain convergence after removing regions whose boundary areas tend to zero, using the intrinsic length distances of the retained regions. Such paths must avoid the removed set, so that their conclusion does not require the interior shortcuts in our full isoperimetric regions to disappear.

*Idea of construction.* The construction applies Theorem A near  $\Gamma$  in  $b_{a_j}$ , with the modification supported in a fixed compact subset of  $B_{3/2}$ . Hence the ADM mass remains  $2a_j$ , and Lemma 2.2 preserves every centered volume maximizer  $B_R$ ,  $R > 2$ . Shrinking the drawstring cores and letting the conformal corrections tend to zero gives uniform distance convergence to  $D_q$ ; the current comparison in Lemma 4.8 then gives intrinsic flat convergence. The boundary metric space  $(\partial B_R, D_q)$  is not isometric to the Euclidean chord sphere (Lemma 4.7). Since taking boundaries does not increase intrinsic flat distance, this yields the positive gap in (14), despite convergence of the volumes. This obstruction concerns the distance inherited from the region. The induced Riemannian metric on each  $\partial B_R$  remains exactly round, but paths through the interior change

the distances between boundary points. The restrictions of  $D_q$  and the Euclidean distance  $|\cdot - \cdot|$  to  $\partial B_R$  induce the same standard spherical length distance. Hence any isometry between these two boundary metric spaces would also preserve Euclidean chord distances, contradicting  $D_q(p, z) < |p - z|$  for some  $p, z \in \partial B_R$ , as illustrated in Figure 2. More quantitatively, Lemma 4.7 gives, for every  $R > 1$ ,

$$\sup_{p, z \in \partial B_R} (|p - z| - D_q(p, z)) = 2(\sqrt{1 - q^2} - q \arccos q) > 0.$$

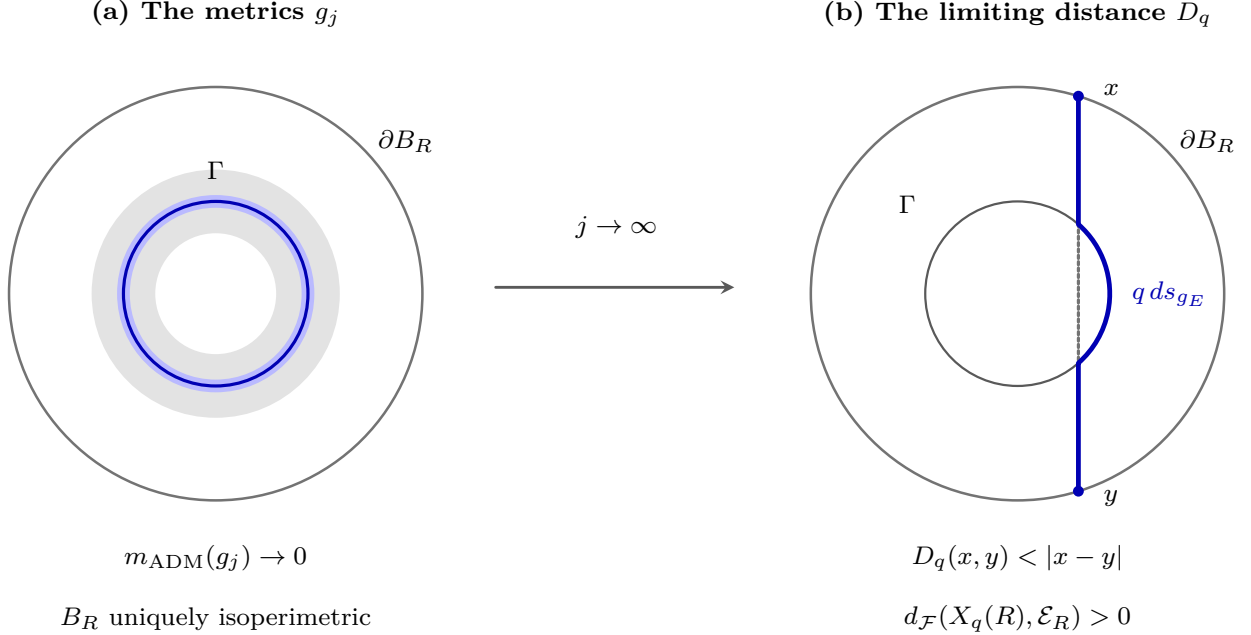


FIGURE 2. The construction for Sormani's question. (a) The drawstring construction contracts the fixed circle  $\Gamma$  without changing the ADM mass or the unique volume maximizing property of  $B_R$ . (b) The persistent shortcuts along  $\Gamma$  produce a non-Euclidean intrinsic flat limit  $X_q(R)$ .

Finally, mean convex barriers and the Penrose inequality rule out closed minimal surfaces for sufficiently small ADM mass. This follows the comparison strategy in [16, Section 5.3].

**1.5. Organization of the paper.** Section 2 establishes the local drawstring construction and its isoperimetric consequence. Section 3 assembles these local constructions to obtain a counterexample to Gromov's conjecture. Section 4 applies the drawstring construction to produce asymptotically flat metrics with vanishing ADM mass but a non-Euclidean intrinsic flat limit, yielding a counterexample to Sormani's conjecture.

**On the use of AI.** ChatGPT 6 was employed to discuss potential ideas. Kimi K3 was used for grammatical refinement, spelling correction, and suggestions on the phrasing of section and lemma headings. Neither ChatGPT 6 nor Kimi K3 was used to produce any manuscript text, arguments, proofs, or computations. The authors take full responsibility for this work.

## 2. THE AREA NONDECREASING CONSTRUCTION

We first prove Theorem A. We then assemble the local modifications and show how area nondecrease preserves a fixed perimeter volume maximizer.

**2.1. The local modification.** Let  $b, \Gamma, t, r_*$ , and  $\chi$  be as in the introduction, and let  $\pi$  denote the normal tubular projection onto  $\Gamma$ . On the punctured tube, define  $\mathcal{V}_x = \ker d\pi_x$  and  $\mathcal{H}_x = \mathcal{V}_x^\perp$ , thus  $\mathcal{V}_x$  is tangent to the two-dimensional normal fiber through  $x$ , while  $\mathcal{H}_x$  is its  $b$ -orthogonal complement. Let  $e_\theta$  be a  $b$ -unit vector tangent to the normal circle in each fiber, and let  $\omega = b(e_\theta, \cdot)$  be its metric dual. Denote by  $P_{\mathcal{H}}$  the  $b$ -orthogonal projection onto  $\mathcal{H}$ , and define

$$b_{\mathcal{H}}(v, w) = b(P_{\mathcal{H}}v, P_{\mathcal{H}}w).$$

Since  $\nabla^b t$  is the unit radial vector and is orthogonal to  $e_\theta$ , the metric admits the orthogonal decomposition

$$b = dt^2 + \omega^2 + b_{\mathcal{H}}. \quad (16)$$

In general,  $b_{\mathcal{H}}$  need not coincide with  $\pi^*(b|_{T\Gamma})$ .

We use the following consequences of Kazaras–Xu [16, Theorem 2.4]. For every smooth function  $v \leq 0$  on  $\Gamma$  and every  $\varepsilon > 0$ , there exist  $r_1 < \varepsilon$  and a smooth metric

$$\widehat{b} = e^{-2u} dt^2 + e^{-2u} h^2 \omega^2 + e^{2u} b_{\mathcal{H}}, \quad u = (v \circ \pi)w(t),$$

where  $h = h(t)$  and  $w = w(t)$  satisfy

$$0 \leq w \leq 1, \quad w(0) = 1, \quad 1 - r_1 \leq h \leq 1.$$

Moreover,

$$\begin{aligned} \widehat{b} &= b \quad \text{outside } N_b(\Gamma, r_1), \\ \widehat{b}|_{T\Gamma} &= e^{2v} b|_{T\Gamma}, \\ \text{Scal}_{\widehat{b}} &\geq \text{Scal}_b - \varepsilon, \\ \text{Vol}_{\widehat{b}}(N_b(\Gamma, r_1)) &\leq 12\pi \text{Length}_b(\Gamma) r_1^2. \end{aligned} \quad (17)$$

The level surfaces  $\{t = s\}$  satisfy

$$H_b(\{t = s\}) \geq \frac{1}{4s \log(1/s)} > 0, \quad 0 < s \leq r_1. \quad (18)$$

Furthermore, the estimates in [16, (4.51)–(4.52)] show that every normal radial segment in  $N_b(\Gamma, r_1)$  has  $\widehat{b}$ -length at most  $3r_1$ .

After decreasing  $r_*$  if necessary, the tubular requirements of [16, General Setup 2.1] are satisfied. Indeed, the normal exponential map is regular on  $N_b(\Gamma, 2r_*)$ , and

$$\begin{aligned} H_b(\{t = s\}) &= s^{-1} + \mathcal{O}(1), \\ \text{Area}_b(\{t = s\}) &= 2\pi s \text{Length}_b(\Gamma) + \mathcal{O}(s^2) \end{aligned}$$

as  $s \downarrow 0$ , which imply the required mean-curvature and area bounds. Moreover, [16, Remark 4.2] gives  $h = 1$  and  $w = 0$  for  $t \geq r_1$ , hence  $\widehat{b} = b$  outside  $N_b(\Gamma, r_1)$ .

*Proof of Theorem A.* Write  $A = A_\varepsilon = -\frac{1}{2} \log(1 - \varepsilon)$  and apply the drawstring construction of Kazaras–Xu [16, Theorem 2.4] with  $v = \log q - A$ . Choose  $\varepsilon < r_*/2$ , so that  $r_1 < r_*/2$ , and define  $g_\varepsilon = e^{2A\chi} \widehat{b}$ . Since  $\widehat{b}$  is smooth and  $\chi = 1$  near  $\Gamma$ , the metric  $g_\varepsilon$  is smooth across  $\Gamma$ . Moreover,

$$g_\varepsilon = b \quad \text{for } t \geq r_*; \quad g_\varepsilon|_{T\Gamma} = e^{2A} e^{2(\log q - A)} b|_{T\Gamma} = q^2 b|_{T\Gamma}.$$

We first prove  $g_\varepsilon \succeq b$ . On  $0 < t < r_1$ , where  $\chi = 1$ , the orthogonal splitting (16), together with  $u \leq 0$  and  $0 < h \leq 1$ , gives

$$|\xi|_{g_\varepsilon}^2 \geq e^{4A} h^2 |\xi|_b^2 \geq \left( \frac{1 - r_1}{1 - \varepsilon} \right)^2 |\xi|_b^2 \geq |\xi|_b^2$$

for every  $\xi \in \Lambda^2 T_x M$ , since  $e^{2A} = (1 - \varepsilon)^{-1}$  and  $r_1 < \varepsilon$ . Similarly,

$$\frac{dV_{g_\varepsilon}}{dV_b} = e^{3A - u} h \geq \frac{1 - r_1}{(1 - \varepsilon)^{3/2}} \geq 1.$$

For  $t \geq r_1$ , we have  $g_\varepsilon = e^{2A\chi} b$  with  $A\chi \geq 0$ , so both inequalities remain valid. They extend across  $\Gamma$  by continuity. Thus  $g_\varepsilon \succeq b$  and  $dV_{g_\varepsilon} \geq dV_b$  everywhere.

We next establish the two sided metric estimate (4). On  $0 < t < r_1$ , we have

$$\log q_- \leq A + u = (1 - w)A + w \log q(\pi(x)) \leq A.$$

Using (16),  $u \leq 0$ , and  $1 - r_1 \leq h \leq 1$ , we obtain

$$\begin{aligned} g_\varepsilon &= e^{2(A-u)}(dt^2 + h^2\omega^2) + e^{2(A+u)}b_{\mathcal{H}} \\ &\geq (1 - \varepsilon)(dt^2 + \omega^2) + q_-^2 b_{\mathcal{H}} \geq \min\{q_-^2, 1 - \varepsilon\} b, \end{aligned}$$

on the other hand,

$$g_\varepsilon \leq e^{2(A-u)}b \leq e^{4A}q_-^2 b.$$

For  $t \geq r_1$ , the same bounds follow from

$$b \leq g_\varepsilon = e^{2A\chi}b \leq e^{2A}b.$$

By continuity, the estimates extend across  $\Gamma$ , proving (4).

We now verify the scalar curvature bound (2). Set  $\kappa = \min_{N_b(\Gamma, r_*)} \text{Scal}_b > 0$ . For a conformal change in dimension three,

$$\text{Scal}_{e^{2f}\widehat{b}} = e^{-2f}(\text{Scal}_{\widehat{b}} - 4\Delta_{\widehat{b}}f - 2|df|_{\widehat{b}}^2).$$

On  $t \leq r_*/2$ , the scalar curvature estimate in (17) of Kazaras–Xu therefore yields

$$\text{Scal}_{g_\varepsilon} = e^{-2A} \text{Scal}_{\widehat{b}} \geq (1 - \varepsilon)(\text{Scal}_b - \varepsilon) \geq \frac{9}{16} \text{Scal}_b > \frac{1}{2} \text{Scal}_b,$$

provides  $\varepsilon \leq \min\{1/4, \kappa/4\}$ .

On  $r_*/2 < t < r_*$ , we have  $\widehat{b} = b$ . Let  $M_1 = \sup |d\chi|_b$  and  $M_2 = \sup |\Delta_b \chi|$  on this fixed annulus. Then

$$\text{Scal}_{g_\varepsilon} \geq e^{-2A\chi}(\text{Scal}_b - 4AM_2 - 2A^2M_1^2) \geq \frac{9}{16} \text{Scal}_b > \frac{1}{2} \text{Scal}_b,$$

provides we further decrease  $\varepsilon$  so that

$$4AM_2 + 2A^2M_1^2 \leq \frac{\kappa}{4}.$$

This proves the claimed scalar curvature bound (2) throughout  $N_b(\Gamma, r_*)$ .

We next check strict mean convexity. On  $0 < t \leq r_*/2$ , we have  $\chi = 1$ , so  $H_{g_\varepsilon} = e^{-A}H_{\widehat{b}} > 0$ , by (18) for  $t \leq r_1$  and by  $\widehat{b} = b$  and  $H_b > 0$  for  $r_1 \leq t \leq r_*/2$ . On  $r_*/2 < t < r_*$ , there holds  $\widehat{b} = b$ , and the conformal mean curvature formula gives

$$H_{g_\varepsilon} = e^{-A\chi}(H_b + 2A\chi').$$

Since  $H_b$  has a positive minimum on this annulus,  $\chi'$  is bounded, and  $A \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we have  $H_{g_\varepsilon} > 0$  for sufficiently small  $\varepsilon$ . Finally, on  $r_* \leq t \leq 2r_*$ , the fact  $g_\varepsilon = b$  implies  $H_{g_\varepsilon} = H_b > 0$ .

It remains to estimate the volume change (3). Since  $dV_{g_\varepsilon} \geq dV_b$  and  $g_\varepsilon = b$  for  $t \geq r_*$ , the measure  $\nu_\varepsilon = dV_{g_\varepsilon} - dV_b$  is nonnegative and supported in  $N_b(\Gamma, r_*)$ . By the volume estimate in (17),

$$\nu_\varepsilon(N_b(\Gamma, r_1)) \leq 12\pi e^{3A} \text{Length}_b(\Gamma)r_1^2.$$

On the remaining part of  $N_b(\Gamma, r_*)$ ,  $g_\varepsilon = e^{2A\chi}b$ , so

$$\nu_\varepsilon(N_b(\Gamma, r_*) \setminus N_b(\Gamma, r_1)) \leq (e^{3A} - 1) \text{Vol}_b(N_b(\Gamma, r_*)).$$

Consequently,

$$\begin{aligned} \nu_\varepsilon(M) &\leq 12\pi e^{3A} \text{Length}_b(\Gamma)r_1^2 \\ &\quad + (e^{3A} - 1) \text{Vol}_b(N_b(\Gamma, r_*)) \\ &\leq C(r_1^2 + A) \leq C\varepsilon, \end{aligned}$$

where  $C$  depends only on the fixed background tube and cutoff, independently of  $q$  and  $r_1$ . This proves (3).

Finally, for  $x \in N_b(\Gamma, r_1)$ , let  $\gamma_x$  be the  $b$ -normal radial segment joining  $\pi(x)$  to  $x$ . Since  $\chi = 1$  along  $\gamma_x$ , the radial integral estimate in [16, (4.51)–(4.52)] gives

$$L_{g_\varepsilon}(\gamma_x) = e^A L_{\widehat{b}}(\gamma_x) \leq 3e^A r_1.$$

Moreover,  $g_\varepsilon = e^{2A\chi}b$  for  $t \geq r_1$ . The proof is complete.  $\square$

**2.2. Assembly and isoperimetric comparison.** Let  $U_i = N_b(\Gamma_i, r_{*,i})$  be tubes with pairwise disjoint closures, each satisfying the hypotheses of Theorem A for a background metric  $b$ . Assume that the family is locally finite, meaning that every compact subset of  $M$  meets only finitely many  $U_i$ . On each  $\Gamma_i$ , prescribe a smooth function  $q_i : \Gamma_i \rightarrow (0, 1]$ , and let  $v_i > 0$ .

Apply Theorem A in each  $U_i$ , choosing  $\varepsilon_i$  so that the increase in volume in  $U_i$  is bounded by  $v_i$ . Since the tubes have pairwise disjoint closures and form a locally finite family, the resulting metrics combine to give a smooth global metric. All local conclusions remain valid, and the total increase in volume is bounded by  $\sum_i v_i$ . Completeness follows from the uniform lower metric bound when the background is complete.

**Corollary 2.1** (Locally finite assembly). *There is a smooth metric  $g$  with*

$$g = b \quad \text{off} \quad \bigcup_i U_i, \quad g|_{T\Gamma_i} = q_i^2 b|_{T\Gamma_i}, \quad g \succeq_2 b,$$

$$\text{Scal}_g \geq \frac{1}{2} \text{Scal}_b > 0 \quad \text{on} \quad \bigcup_i U_i, \quad 0 \leq \int_M (dV_g - dV_b) \leq \sum_i v_i.$$

For each  $i$ , let  $\varepsilon_i$  be the parameter used in the local construction and put  $q_{i,-} = \min_{\Gamma_i} q_i$ . Then

$$\min\{q_{i,-}^2, 1 - \varepsilon_i\} b \leq g \leq e^{4A\varepsilon_i} q_{i,-}^{-2} b \quad \text{on } U_i.$$

Moreover, for  $0 < s \leq r_{*,i}$ , every level surface  $\{x \in M : d_b(x, \Gamma_i) = s\}$  is strictly mean convex with respect to  $g$ . If  $b$  is complete and  $g \geq cb$  for a constant  $c > 0$ , then  $g$  is complete.

We next show that an area nondecreasing metric modification supported inside a volume maximizing region preserves its maximizing property.

**Lemma 2.2** (Preservation of volume maximizers). *Let  $\Omega \Subset M$  be a smooth domain that uniquely maximizes  $b$ -volume among finite volume sets  $E$  of finite perimeter satisfying  $P_b(E) \leq P_b(\Omega)$ . Suppose  $g$  is a smooth metric such that  $g \succeq_2 b$  and  $g = b$  outside a compact subset of  $\Omega$ . Then  $P_g(\Omega) = P_b(\Omega)$ , and  $\Omega$  uniquely maximizes  $g$ -volume among finite volume sets  $E$  of finite perimeter satisfying  $P_g(E) \leq P_g(\Omega)$ .*

*Proof.* Choose a compact set  $K \Subset \Omega$  such that  $g = b$  on  $M \setminus K$ . We first show that  $dV_g \geq dV_b$ . At each point, choose a  $b$ -orthonormal basis in which  $g$  has eigenvalues  $\lambda_1, \lambda_2, \lambda_3 > 0$ . Since  $g \succeq_2 b$ , the area of every two-plane is nondecreasing. Hence there holds  $\lambda_1 \lambda_2 \lambda_3 \geq 1$ , and then  $dV_g/dV_b \geq 1$ , which means  $\nu = dV_g - dV_b$  is a nonnegative measure supported in  $K$ .

Let  $A = P_b(\Omega)$ . Since  $g = b$  near  $\partial\Omega$ , we have  $P_g(\Omega) = P_b(\Omega) = A$ .

Now let  $E$  be a finite volume set of finite perimeter satisfying  $P_g(E) \leq A$ . Since  $g \succeq_2 b$ , integration of the area inequality over the reduced boundary  $\partial^*E$  gives

$$P_b(E) \leq P_g(E) \leq P_g(\Omega) = P_b(\Omega) = A.$$

Moreover,  $dV_g \geq dV_b$  implies that  $\text{Vol}_b(E) \leq \text{Vol}_g(E) < \infty$ . Thus  $E$  is an admissible competitor for the  $b$ -volume maximization problem, and the hypothesis on  $\Omega$  gives

$$\text{Vol}_b(E) \leq \text{Vol}_b(\Omega),$$

with equality only when  $E = \Omega$  up to null sets.

Finally, since  $\nu$  is nonnegative and supported in  $K \subset \Omega$ , we obtain

$$\text{Vol}_g(\Omega) - \text{Vol}_g(E) = \text{Vol}_b(\Omega) - \text{Vol}_b(E) + \nu(\Omega \setminus E) \geq 0.$$

If equality holds, both terms in the last expression must vanish. In particular,  $\text{Vol}_b(E) = \text{Vol}_b(\Omega)$ , so the uniqueness hypothesis implies  $E = \Omega$  up to null sets. This proves the claim.  $\square$

### 3. A COUNTEREXAMPLE TO GROMOV'S QUESTION

Fix  $\lambda > 1$  and choose  $a > 0$  so small that  $\Lambda := (1 + a)^6 < \lambda$ . Throughout this section,  $b = \beta_a = \varphi_a^4 g_E$  in isotropic coordinates. We first construct short routes in a single cube while preserving  $g \geq g_E$ . We then assign summable errors to the cubes of a Euclidean tiling.

**3.1. Construction in a cube.** We use a finite family of disjoint segments to approximate all pairs of points in a cube.

**Lemma 3.1.** *Let  $Q \subset \mathbb{R}^3$  be a closed cube and let  $\rho > 0$ . There is a finite family of pairwise disjoint nondegenerate segments  $[p_i, z_i] \subset \text{int } Q$  such that, for every  $(x, y) \in Q \times Q$ , there exists some  $i$  satisfies  $|x - p_i| < \rho$  and  $|y - z_i| < \rho$ . Each segment is contained in a smooth embedded closed curve  $\Gamma_i \subset \text{int } Q$ , and these curves may be chosen pairwise disjoint.*

*Proof.* Choose finitely many ordered pairs  $(a_i, b_i) \in \text{int } Q \times \text{int } Q$ ,  $1 \leq i \leq N$ , such that every  $(x, y) \in Q \times Q$  satisfies

$$|x - a_i| < \frac{\rho}{2}, \quad |y - b_i| < \frac{\rho}{2}$$

for some  $i$ .

We construct the segments  $[p_i, z_i]$  inductively. Suppose that  $[p_j, z_j]$ ,  $j < i$ , have been chosen and are pairwise disjoint and nondegenerate. Let  $L_j$  be the line through  $p_j, z_j$ . Choose  $p_i \in B_{\rho/4}(a_i) \cap \text{int } Q$  outside the finitely many lines  $L_j$ . For each  $j < i$ , let  $H_j$  be the plane containing  $p_i$  and  $L_j$ . Choose  $z_i \in B_{\rho/4}(b_i) \cap \text{int } Q$  outside  $\{p_i\}$  and the finitely many planes  $H_j$ . These choices are possible since finite unions of lines or planes have empty interior in  $\mathbb{R}^3$ .

Then  $p_i \neq z_i$ , and the four endpoints of  $[p_i, z_i]$  and  $[p_j, z_j]$  are not coplanar for every  $j < i$ . Hence the segments are pairwise disjoint. Since  $Q$  is convex, they are contained in  $\text{int } Q$ . Moreover, whenever  $|x - a_i| < \rho/2$  and  $|y - b_i| < \rho/2$ ,

$$|x - p_i| < \frac{3\rho}{4} < \rho, \quad |y - z_i| < \frac{3\rho}{4} < \rho.$$

Finally, choose pairwise disjoint tubular neighborhoods of the segments with compact closures in  $\text{int } Q$ . Within each neighborhood, extend the segment slightly at both ends and join the extensions by a return arc disjoint from the original segment. Smoothing the joins away from the original segment produces a smooth embedded closed curve  $\Gamma_i$  containing  $[p_i, z_i]$ . The curves are pairwise disjoint.  $\square$

Applying the local drawstring construction along the closed curves provided by Lemma 3.1, we obtain a metric whose intrinsic distance on  $Q$  are arbitrarily close to Euclidean distance, while preserving the required curvature and area bounds. The resulting metric also has arbitrarily small volume excess.

**Lemma 3.2.** *For every closed unit cube  $Q$ ,  $\tau > 0$ , and  $v > 0$ , there is a smooth metric  $G_Q$  on  $\mathbb{R}^3$ , equal to  $b$  outside a compact subset of  $\text{int } Q$ , such that*

$$\begin{aligned} \text{Scal}_{G_Q} &\geq \frac{1}{2} \text{Scal}_b > 0, \quad G_Q \succeq_2 b, \quad g_E \leq G_Q \leq \Lambda^2 g_E, \\ 0 &\leq \int_{\mathbb{R}^3} (dV_{G_Q} - dV_b) < v, \\ d_{G_Q, Q}(x, y) &\leq |x - y| + \tau \quad (x, y \in Q). \end{aligned} \tag{19}$$

Moreover, for  $x \in Q$ ,

$$G_Q(x) \leq \varphi_a(x)^4 \left( \max_Q \varphi_a \right)^8 g_E, \tag{20}$$

See (5) for the definition of  $\varphi_a$ .

*Proof.* Put  $\mu = \min_Q \varphi_a > 1$  and  $M = \max_Q \varphi_a$ . Choose  $\eta, \rho > 0$  such that

$$1 + \eta < \mu^2, \quad 2(\Lambda + 1)\rho + \sqrt{3}\eta < \tau. \tag{21}$$

Apply Lemma 3.1 to obtain finitely many segments  $[p_i, z_i]$  and pairwise disjoint curves  $\Gamma_i$ ,  $i = 1, \dots, N$ . Prescribe

$$q_i = \frac{1 + \eta}{\varphi_a^2|_{\Gamma_i}} \in (0, 1). \tag{22}$$

Choose radii  $r_{*,i} > 0$  so that the tubes  $N_b(\Gamma_i, 2r_{*,i})$  have pairwise disjoint compact closures contained in  $\text{int } Q$  and the hypotheses of Theorem A hold. Write  $U_i = N_b(\Gamma_i, r_{*,i})$ ,  $t_i = d_b(\cdot, \Gamma_i)$ , and let  $\pi_i$  be the normal tubular projection onto  $\Gamma_i$ . Decrease the radii further so that

$$\frac{\varphi_a(x)^2}{\varphi_a(\pi_i(x))^2} \geq \frac{1}{1+\eta} \quad (x \in U_i), \quad \mu^2(1 - r_{*,i}) > 1. \quad (23)$$

Indeed, with  $L_Q = \sup_Q |d \log \varphi_a|_b$ , integration along the normal radial segment gives

$$|\log \varphi_a(x) - \log \varphi_a(\pi_i(x))| \leq L_Q t_i(x) \leq L_Q r_{*,i}.$$

Thus it suffices to require  $2L_Q r_{*,i} \leq \log(1+\eta)$  and  $r_{*,i} < 1 - \mu^{-2}$ .

Choose  $v_i > 0$  with  $\sum_{i=1}^N v_i < v$ . In each  $U_i$ , use the metric constructed in the proof of Theorem A, choosing  $\varepsilon_i > 0$  sufficiently small that the volume excess is less than  $v_i$  and

$$\varepsilon_i < \frac{r_{*,i}}{2}, \quad A_i = -\frac{1}{2} \log(1 - \varepsilon_i) \leq \log M.$$

Then  $r_{1,i} < \varepsilon_i < r_{*,i}/2$ . Assemble these metrics using Corollary 2.1. The resulting smooth metric  $G_Q$  equals  $b$  outside the compact set  $\bigcup_{i=1}^N \overline{U_i} \subset \text{int } Q$ . The scalar curvature, area, and volume conclusions follow from Theorem A, and disjointness of the tubes.

We verify the lower tensor bound  $g_E \leq G_Q$ . On  $0 < t_i < r_{1,i}$ , we have

$$A_i + u_i = A_i + (\log q_i(\pi_i(x)) - A_i)w_i \geq \log q_i(\pi_i(x)).$$

Using  $b = dt_i^2 + \omega_i^2 + b_{\mathcal{H}} = \varphi_a^4 g_E$ , we obtain

$$\begin{aligned} G_Q &= e^{2(A_i - u_i)}(dt_i^2 + h_i^2 \omega_i^2) + e^{2(A_i + u_i)} b_{\mathcal{H}} \\ &\geq \min\{(1 - r_{1,i})^2, q_i(\pi_i(x))^2\} b \\ &= \min\left\{\varphi_a(x)^2(1 - r_{1,i}), (1+\eta) \frac{\varphi_a(x)^2}{\varphi_a(\pi_i(x))^2}\right\}^2 g_E \\ &\geq g_E, \end{aligned}$$

where the last inequality follows from  $\mu^2(1 - r_{1,i}) > 1$  and (23). By continuity, the same bound holds on the central curves. On  $U_i \cap \{t_i \geq r_{1,i}\}$ ,  $G_Q = e^{2A_i \chi_i} b \geq b \geq g_E$ , while outside all the tubes  $G_Q = b \geq g_E$ .

For the upper bound  $G_Q \leq \Lambda^2 g_E$  and (20), put  $q_{i,-} = \min_{\Gamma_i} q_i$ . By (22),  $q_{i,-} \geq (1+\eta)M^{-2}$ . The estimate (4) gives, throughout  $U_i$ ,

$$G_Q \leq e^{4A_i} q_{i,-}^{-2} b \leq \frac{M^8}{(1+\eta)^2} b \leq M^8 b,$$

where  $A_i$  satisfies  $A_i \leq \log M$  by construction. The same bound holds on  $Q \setminus \bigcup_i U_i$ , where  $G_Q = b$ . This proves (20). Since  $\varphi_a \leq 1+a$  and  $M \leq 1+a$ , it follows that  $G_Q \leq (1+a)^{12} g_E = \Lambda^2 g_E$  on  $Q$ ; outside  $Q$ , this bound follows from  $G_Q = b = \varphi_a^4 g_E$ .

Finally, we prove (19) to conclude the entire proof. On each central curve,

$$G_Q|_{T\Gamma_i} = q_i^2 b|_{T\Gamma_i} = (1+\eta)^2 g_E|_{T\Gamma_i}.$$

Given  $x, y \in Q$ , choose  $i$  as in Lemma 3.1 and set  $D = |x - p_i| + |y - z_i| < 2\rho$ . Join  $x$  to  $y$  by the segments  $[x, p_i]$ ,  $[p_i, z_i]$ , and  $[z_i, y]$ , all contained in  $Q$  by convexity. The segment  $[p_i, z_i]$  has  $G_Q$ -length  $(1+\eta)|p_i - z_i|$ , since it lies on  $\Gamma_i$ . The global bound  $G_Q \leq \Lambda^2 g_E$  shows that the other two segments have total  $G_Q$ -length at most  $\Lambda D$ . Since  $|p_i - z_i|$  is bounded by  $|x - y| + D$  and  $|p_i - z_i|$  is bounded by  $\sqrt{3}$ , we obtain

$$\begin{aligned} d_{G_Q, Q}(x, y) &\leq \Lambda D + (1+\eta)|p_i - z_i| \\ &\leq |x - y| + (\Lambda + 1)D + \sqrt{3}\eta \\ &< |x - y| + \tau, \end{aligned}$$

by our choice of  $\rho$  and  $\eta$  in (21). □

**3.2. The global distance bound.** We now proceed to prove Theorem B.

*Proof of Theorem B.* Tile  $\mathbb{R}^3$  by  $Q_k = k + [0, 1]^3$ ,  $k := (k_1, k_2, k_3) \in \mathbb{Z}^3$ , and assign

$$\tau_k = \frac{E}{27} 2^{-(|k_1|+|k_2|+|k_3|)}, \quad v_k = \frac{V}{27} 2^{-(|k_1|+|k_2|+|k_3|)}.$$

We have  $\sum_k \tau_k = E$  and  $\sum_k v_k = V$ . Apply Lemma 3.2 in each cube and set

$$D_k = G_{Q_k} - b, \quad g = b + \sum_{k \in \mathbb{Z}^3} D_k.$$

Since  $\text{supp } D_k \subseteq \text{int } Q_k$  and the cubical tiling is locally finite,  $g$  is smooth and agrees with  $b$  near all cube boundaries. Moreover,  $g = G_{Q_k}$  on  $Q_k$ , so Lemma 3.2 gives

$$\text{Scal}_g \geq \frac{1}{2} \text{Scal}_b > 0, \quad g_E \leq g \leq \Lambda^2 g_E, \quad g \succeq_2 b, \quad 0 \leq \int_{\mathbb{R}^3} (dV_g - dV_b) \leq V.$$

Moreover,  $g$  is complete.

First suppose that the straight segment from  $x$  to  $y$  crosses cube faces transversely and misses all grid edges. It meets finitely many cube interiors, each in a single segment because cubes are convex. For the portion in  $Q_k$ , use (19). Concatenating these paths gives

$$d_g(x, y) \leq \sum_{Q_k \text{ crossed}} (|x_k - y_k| + \tau_k) \leq |x - y| + E. \quad (24)$$

Such pairs  $(x, y)$  are dense in  $\mathbb{R}^3 \times \mathbb{R}^3$ . Moreover, the upper tensor bound  $g \leq \Lambda^2 g_E$  implies

$$|d_g(x, y) - d_g(x', y')| \leq \Lambda (|x - x'| + |y - y'|)$$

for any pairs  $(x, y), (x', y') \in \mathbb{R}^3 \times \mathbb{R}^3$ . Thus, approximating arbitrary endpoints  $x, y \in \mathbb{R}^3$  by pairs satisfy (24), and passing to the limit gives (6).

If  $x \in Q_k$  and  $|x| \rightarrow \infty$ , the bounded diameter of the cubes implies  $\max_{Q_k} \varphi_a = 1 + \mathcal{O}_a(|x|^{-1})$ . Combining (20) with  $g \geq g_E$  yields  $g - g_E = \mathcal{O}_a(|x|^{-1})$ . At the vertices  $x_j = (j, 0, 0)$ , the constructed metric  $g$  equals  $b$  in a neighborhood, so

$$g(x_j) - g_E = \left( \left( 1 + \frac{a}{\sqrt{1+j^2}} \right)^4 - 1 \right) g_E = \left( \frac{4a}{j} + \mathcal{O}_a(j^{-2}) \right) g_E.$$

Thus the error is not  $o(|x|^{-1})$ . At the same points,  $\text{Scal}_g(x_j) = \text{Scal}_b(x_j) \rightarrow 0$ , proving  $\inf \text{Scal}_g = 0$ . These conclude (7).  $\square$

*Remark 3.3* (Constant choices of  $q_i$ ). In Lemma 3.2, one may replace  $q_i = (1+\eta)/\varphi_a^2|_{\Gamma_i}$  by constants. Indeed, subdivide  $Q$  into finitely many sufficiently small subcubes  $P$  and choose

$$q_i \equiv q_P := \frac{1+\eta}{(\min_P \varphi_a)^2} \quad (\Gamma_i \subset \text{int } P).$$

Since  $\varphi_a$  varies arbitrarily little on each  $P$ , we may arrange

$$1 + \eta \leq q_P \varphi_a^2 \leq 1 + 2\eta.$$

Applying Lemma 3.1 and Theorem A in each  $P$  gives the same curvature, area, tensor, and volume bounds. Choosing  $\eta$  and the local distance errors sufficiently small, and summing along straight segments, recovers the distance estimate of Lemma 3.2.

Theorem B also gives higher-dimensional examples by taking products. For  $n > 3$ , let

$$g_n = g \oplus (g_E)_{n-3},$$

where  $g$  is supplied by Theorem B and  $(g_E)_m$  denotes the Euclidean metric on  $\mathbb{R}^m$ . The product metric is complete and satisfies

$$\text{Scal}_{g_n}(x, z) = \text{Scal}_g(x) > 0, \quad (g_E)_n \leq g_n \leq \lambda^2 (g_E)_n.$$

For  $x, y \in \mathbb{R}^3$  and  $z, w \in \mathbb{R}^{n-3}$ , the product distance formula gives

$$\begin{aligned} 0 &\leq d_{g_n}((x, z), (y, w)) - |(x, z) - (y, w)| \\ &= \sqrt{d_g(x, y)^2 + |z - w|^2} - \sqrt{|x - y|^2 + |z - w|^2} \\ &\leq d_g(x, y) - |x - y| \leq E. \end{aligned}$$

Here we used (6) and the fact that  $s \mapsto \sqrt{s^2 + t^2}$  is 1-Lipschitz on  $[0, \infty)$  for each fixed  $t \geq 0$ .

Thus the scalar curvature, tensor, and additive distance conclusions extend to every dimension  $n > 3$ . Thus Theorem B also yields counterexamples to Gromov's Conjecture 1.1 (b) in every dimension  $n > 3$ .

#### 4. A COUNTEREXAMPLE TO SORMANI'S QUESTION

We now apply the drawstring construction to Sormani's zero mass stability question. By contracting a fixed circle while the background mass tends to zero, we obtain a counterexample to the proposed intrinsic flat convergence.

**4.1. Radial backgrounds and exact isoperimetry.** For  $\beta_a$  in (5), introduce area radius  $\rho = r\varphi_a(r)^2$  and the radial map  $F_a(y) = \varphi_a(|y|)^2 y$ . Set

$$L_a(r) = 1 + \frac{2r\varphi'_a(r)}{\varphi_a(r)} = 1 - \frac{2ar^2}{(1+r^2)(\sqrt{1+r^2}+a)}, \quad \frac{d\rho}{dr} = \varphi_a^2 L_a.$$

For small  $a > 0$ , the map  $F_a$  is a smooth diffeomorphism of  $\mathbb{R}^3$ . In the coordinates  $x = F_a(y)$ , with  $\rho = |x|$ , the background is

$$b_a = (F_a^{-1})^* \beta_a = \frac{d\rho^2}{W_a(\rho)} + \rho^2 g_{\mathbb{S}^2}, \quad W_a(\rho(r)) = L_a(r)^2. \quad (25)$$

Thus  $\frac{1}{2} \leq W_a < 1$  for  $\rho > 0$ , after  $a$  is decreased, and  $W_a(0) = 1$ . Its Hawking mass function is

$$\mathbf{m}_a(\rho(r)) := \frac{\rho(r)}{2}(1 - W_a(\rho(r))) = \frac{2ar^3}{(1+r^2)^{3/2}} + \frac{2a^2r^3}{(1+r^2)^3}.$$

In particular,  $\mathbf{m}_a(0) = 0$  and  $\lim_{\rho \rightarrow \infty} \mathbf{m}_a(\rho) = 2a = m_{\text{ADM}}(b_a)$ , and we have

$$\mathbf{m}'_a(\rho) = \frac{\rho^2 \text{Scal}_{b_a}(\rho)}{4}, \quad \text{Scal}'_{b_a}(\rho) < 0 \quad (\rho > 0).$$

then there holds

$$3\mathbf{m}_a(\rho) - \rho \mathbf{m}'_a(\rho) = \frac{3}{4} \int_0^\rho s^2 (\text{Scal}_{b_a}(s) - \text{Scal}_{b_a}(\rho)) ds > 0 \quad (\rho > 0).$$

The explicit formulas also give, uniformly for small  $a$ ,

$$g_E \leq b_a \leq (1 + Ca)g_E, \quad \|b_a - g_E\|_{C^2(\{|x| \geq R\})} \leq C_R a \quad (R > 0). \quad (26)$$

Smoothness at the origin follows from the smooth radial diffeomorphism  $F_a$ , and  $b_a \rightarrow g_E$  smoothly on compact sets.

We next establish a general criterion for the unique isoperimetry of centered balls  $B_R$  in rotationally symmetric manifolds. Consider a smooth complete metric on  $\mathbb{R}^3$  of the form

$$b = \frac{d\rho^2}{W(\rho)} + \rho^2 g_{\mathbb{S}^2}, \quad W(\rho) = 1 - \frac{2\mathbf{m}(\rho)}{\rho},$$

where  $\mathbf{m}$  is the Hawking mass function. Assume that  $b$  is asymptotically flat, with  $\partial^\alpha(b - g_E) = \mathcal{O}(\rho^{-1-|\alpha|})$  for  $|\alpha| \leq 2$ , and that

$$\frac{1}{2} \leq W < 1, \quad \mathbf{m}' \geq 0, \quad \text{Scal}_b(\rho) = \frac{4\mathbf{m}'}{\rho^2} \text{ is nonincreasing,} \quad 3\mathbf{m} - \rho \mathbf{m}' > 0$$

for  $\rho > 0$ .

**Proposition 4.1.** *For every  $R > 0$ ,  $B_R$  is uniquely isoperimetric in  $(\mathbb{R}^3, b)$ . Equivalently,  $B_R$  uniquely maximizes volume among finite volume sets of perimeter at most  $4\pi R^2$ .*

*Proof.* Set

$$s(\rho) = \int_0^\rho W(t)^{-1/2} dt, \quad h(s) = \rho(s).$$

Then  $b = ds^2 + h(s)^2 g_{\mathbb{S}^2}$ , with  $h' = \sqrt{W} > 0$ , and

$$\begin{aligned} \frac{2h''}{h} - \frac{1 - (h')^2}{h^2} &= -\frac{\text{Scal}_b(\rho)}{2}, \\ \frac{h''}{h} + \frac{1 - (h')^2}{h^2} &= \frac{3\mathfrak{m} - \rho\mathfrak{m}'}{\rho^3} > 0. \end{aligned}$$

Thus the hypotheses of Brendle's theorem [7, Theorem 1.4] are satisfied, so every closed embedded constant mean curvature surface is a centered sphere.

Since  $\text{Scal}_b = 4\mathfrak{m}'/\rho^2 \geq 0$  and

$$H_{\partial B_R} = \frac{2\sqrt{W(R)}}{R} > 0,$$

the maximum principle excludes closed minimal surfaces. By Carlotto–Chodosh–Eichmair [10, Proposition K.1], isoperimetric regions exist for every prescribed volume. Their boundaries have constant mean curvature, and Brendle's theorem [7, Theorem 1.4] implies that each boundary component is a centered sphere. There can be only one component, since consecutive boundary spheres have opposite outward mean curvature. Thus every isoperimetric region is a centered ball, uniquely determined by its volume. Then the equivalent volume maximizing formulation follows.  $\square$

For  $b_a$  we consider (defined as (25)), with all the hypotheses of Proposition 4.1 have already been checked and directly computation, we obtain the following consequence,

**Corollary 4.2.** *For all sufficiently small  $a > 0$ , every centered ball  $B_R$  is uniquely isoperimetric in  $(\mathbb{R}^3, b_a)$ . For every fixed  $R > 0$ ,*

$$P_{b_a}(B_R) = 4\pi R^2, \quad 0 \leq \text{Vol}_{b_a}(B_R) - \frac{4\pi R^3}{3} \leq C_R a.$$

**4.2. Distances with contraction along a circle.** We describe contraction along a circle by assigning a reduced length cost to paths traveling on the circle. Let  $K \subset \mathbb{R}^3$  be a closed Euclidean ball, let  $b$  be a smooth Riemannian metric on a neighborhood of  $K$ , and let  $\Gamma \subset \text{int } K$  be a smooth embedded closed curve. Fix  $q \in (0, 1)$ . For a Lipschitz path  $\gamma : [0, 1] \rightarrow K$ , define

$$L_{q,b}(\gamma) := \int_0^1 \theta_q(\gamma(s)) |\gamma'(s)|_b ds, \quad \theta_q(x) = \begin{cases} q, & x \in \Gamma, \\ 1, & x \notin \Gamma, \end{cases} \quad (27)$$

and

$$D_{q,b}^K(x, y) = \inf_{\substack{\gamma(0)=x, \gamma(1)=y \\ \gamma([0,1]) \subset K}} L_{q,b}(\gamma).$$

Here and below, the infimum is taken over Lipschitz paths. Since  $b$  and the Euclidean metric are uniformly equivalent on  $K$ , either metric may be used to define Lipschitz regularity. In particular,

$$q d_{b,K} \leq D_{q,b}^K \leq d_{b,K},$$

where  $d_{b,K}$  denotes the intrinsic distance induced by  $b$  on  $K$ .

For  $t > 0$ , consider the closed Euclidean neighborhood  $N_t$  defined as  $\{x \in K : \text{dist}_{g_E}(x, \Gamma) \leq t\}$ , the corresponding weighted distance is

$$D_{q,b,t}^K(x, y) = \inf_{\substack{\gamma(0)=x, \gamma(1)=y \\ \gamma([0,1]) \subset K}} L_{q,b,t}(\gamma), \quad (28)$$

where

$$L_{q,b,t}(\gamma) := \int_0^1 \theta_{q,t}(\gamma(s)) |\gamma'(s)|_b ds, \quad \theta_{q,t}(x) = \begin{cases} q, & x \in N_t, \\ 1, & x \in K \setminus N_t, \end{cases}$$

and the infimum in (28) again runs over Lipschitz paths in  $K$ . By construction, for  $0 < t' < t$ ,

$$q d_{b,K} \leq D_{q,b,t}^K \leq D_{q,b,t'}^K \leq D_{q,b}^K.$$

Moreover,

**Lemma 4.3.** *As  $t \downarrow 0$ , the distances  $D_{q,b,t}^K$  increase uniformly on  $K \times K$  to  $D_{q,b}^K$  with respect to the Euclidean distance.*

*Proof.* Fix  $x, y \in K$  and  $t_j \downarrow 0$ , and write

$$\ell = \lim_{j \rightarrow \infty} D_{q,b,t_j}^K(x, y) \leq D_{q,b}^K(x, y). \quad (29)$$

Choose Lipschitz paths  $\gamma_j : [0, 1] \rightarrow K$  from  $x$  to  $y$  such that

$$D_{q,b,t_j}^K(x, y) \leq L_{q,b,t_j}(\gamma_j) \leq D_{q,b,t_j}^K(x, y) + j^{-1}.$$

Since

$$q \text{Length}_b(\gamma_j) \leq L_{q,b,t_j}(\gamma_j) \leq d_{b,K}(x, y) + j^{-1},$$

their Euclidean lengths  $\text{Length}_{g_E}(\gamma_j)$  are uniformly bounded. After reparametrization by normalized Euclidean arclength and passage to a subsequence,  $\gamma_j$  converge uniformly to a Lipschitz path  $\gamma : [0, 1] \rightarrow K$  from  $x$  to  $y$  with respect to the Euclidean distance.

Set  $O = \{s \in [0, 1] : \gamma(s) \notin \Gamma\}$ . For any finite collection of pairwise disjoint compact intervals  $I_1, \dots, I_N \subset O$ , the sets  $\gamma(I_k)$  have positive Euclidean distance from  $\Gamma$ . Thus, for all sufficiently large  $j$ , the paths  $\gamma_j$  lie outside  $N_{t_j}$  on these intervals, and

$$L_{q,b,t_j}(\gamma_j) \geq q \text{Length}_b(\gamma_j) + (1 - q) \sum_{k=1}^N \text{Length}_b(\gamma_j|_{I_k}).$$

Lower semicontinuity of  $b$ -length gives

$$\ell \geq q \text{Length}_b(\gamma) + (1 - q) \sum_{k=1}^N \text{Length}_b(\gamma|_{I_k}).$$

Since  $O$  is relatively open in  $[0, 1]$ , taking the supremum over these interval collections yields

$$\begin{aligned} \ell &\geq q \text{Length}_b(\gamma) + (1 - q) \int_O |\gamma'|_b ds \\ &= L_{q,b}(\gamma) \geq D_{q,b}^K(x, y). \end{aligned} \quad (30)$$

Combining (29) and (30), we obtain the pointwise convergence.

Choose  $C < \infty$  such that  $b \leq C^2 g_E$  on  $K$ , then the triangle inequality gives

$$|D_{q,b,t}^K(x, y) - D_{q,b,t}^K(x', y')| \leq C(|x - x'| + |y - y'|).$$

The same estimate holds for  $D_{q,b}^K$ . Compactness of  $K \times K$  now upgrades pointwise convergence to uniform convergence.  $\square$

Suppose now that  $g_i$  are smooth metrics on a neighborhood of  $K$  satisfying, for a fixed  $1 \leq L < \infty$ ,

$$q^2 b \leq g_i \leq L^2 b, \quad g_i|_{T\Gamma} = q^2 b|_{T\Gamma}, \quad (31)$$

on  $K$ , and that there are  $t_i, \alpha_i \downarrow 0$  such that

$$b \leq g_i \leq (1 + \alpha_i)^2 b \quad \text{on } K \setminus N_{t_i}, \quad (32)$$

thus  $g_i \geq \theta_{q,t_i}^2 b$  on  $K$ , and for every Lipschitz path  $\gamma$  in  $K$ , there holds  $\text{Length}_{g_i}(\gamma) \geq L_{q,b,t_i}(\gamma)$ , taking infimum gives  $d_{g_i,K} \geq D_{q,b,t_i}^K$ . Lemma 4.3 then yields

$$\liminf_{i \rightarrow \infty} d_{g_i,K}(x, y) \geq D_{q,b}^K(x, y).$$

For the reverse inequality, we have

**Proposition 4.4.**  *$d_{g_i,K} \rightarrow D_{q,b}^K$  uniformly on  $K \times K$  with respect to the Euclidean distance.*

*Proof.* It suffices to show that

$$\limsup_{i \rightarrow \infty} d_{g_i, K}(x, y) \leq L_{q, b}(\gamma), \quad (33)$$

Since the compactness of  $K \times K$ , together with the uniform Lipschitz estimate, upgrades pointwise convergence to uniform convergence.

Fix a Lipschitz path  $\gamma : [0, 1] \rightarrow K$  from  $x$  to  $y$ . If  $\gamma(s) \notin \Gamma$ , then  $\gamma(s) \notin N_{t_i}$  for all sufficiently large  $i$ , so  $|\gamma'(s)|_{g_i} \rightarrow |\gamma'(s)|_b$  wherever  $\gamma'(s)$  exists. For almost every  $s$  with  $\gamma(s) \in \Gamma$ , the derivative  $\gamma'(s)$  is tangent to  $\Gamma$ , and (31) gives  $|\gamma'(s)|_{g_i} = q|\gamma'(s)|_b$ . Since  $|\gamma'|_{g_i} \leq L|\gamma'|_b$  and  $|\gamma'|_b \in L^1([0, 1])$ , dominated convergence yields  $\text{Length}_{g_i}(\gamma) \rightarrow L_{q, b}(\gamma)$ . Taking the infimum over Lipschitz paths from  $x$  to  $y$  proves (33).  $\square$

Fix a background metric  $b$  and a curve  $\Gamma$  satisfying the hypotheses of Theorem A. Take the function  $q$  in that theorem to be a constant in  $(0, 1)$ . For all sufficiently small  $\varepsilon < 1 - q^2$ , the resulting metrics  $g_\varepsilon$  satisfy (31) and (32). Thus Proposition 4.4 applies. In the small mass construction below, the background metrics  $b_a$  satisfy the hypotheses of Theorem A for every sufficiently small  $a > 0$ . We apply Proposition 4.4 with each  $b_a$  fixed, and then choose the drawstring parameters by a diagonal argument.

We now specialize the weighted distance to the Euclidean background  $g_E$  and the unit circle  $\Gamma$  defined in (8). For  $0 \leq q \leq 1$ , define

$$D_q(x, y) = \inf_{\substack{\gamma : [0, 1] \rightarrow \mathbb{R}^3 \\ \gamma(0)=x, \gamma(1)=y}} \int_0^1 \theta_q(\gamma(s)) |\gamma'(s)|_{g_E} ds,$$

where  $\theta_q$  is defined in (27). For  $R > 1$ , let  $\Pi_R : \mathbb{R}^3 \rightarrow \overline{B}_R$  be the Euclidean radial projection. Since  $\Pi_R$  is 1-Lipschitz and  $\theta_q \circ \Pi_R = \theta_q$ , we have

$$L_{q, g_E}(\Pi_R \circ \gamma) \leq L_{q, g_E}(\gamma)$$

for every Lipschitz path  $\gamma$ . Consequently,

$$D_{q, g_E}^{\overline{B}_R} = D_q|_{\overline{B}_R \times \overline{B}_R}.$$

The definition extends to  $q = 0$ , in which case  $D_0$  is a pseudometric that identifies precisely the points of  $\Gamma$ .

Write  $d_\Gamma(a, b)$  for the shorter-arc distance between  $a, b \in \Gamma$ . The following proposition gives an explicit formula for  $D_q$ .

**Proposition 4.5.** *For  $0 \leq q \leq 1$ , the weighted distance  $D_q(x, y)$  is realized by either a straight segment or a path consisting of two straight segments joined by a shorter arc of  $\Gamma$ . More precisely,*

$$D_q(x, y) = \min \left\{ |x - y|, \min_{a, b \in \Gamma} (|x - a| + q d_\Gamma(a, b) + |b - y|) \right\}. \quad (34)$$

For every  $R > 1$  and  $p, z \in \partial B_R$ ,

$$\frac{R-1}{R} |p - z| \leq D_q(p, z) \leq |p - z|. \quad (35)$$

*Proof.* For  $s \in [0, \pi]$ , set  $f_q(s) = \min\{qs, 2\sin(s/2)\}$ , and define

$$\rho_q(a, b) := f_q(d_\Gamma(a, b)) = \min\{q d_\Gamma(a, b), |a - b|\}$$

for  $a, b \in \Gamma$ . Since  $f_q$  is nondecreasing and concave, with  $f_q(0) = 0$ , we have

$$f_q(s) + f_q(t) \geq f_q(\min\{s + t, \pi\}) \quad (s, t \in [0, \pi]).$$

Together with the triangle inequality for  $d_\Gamma$ , this shows that  $\rho_q$  satisfies the triangle inequality on  $\Gamma$ .

Define an distance on  $\mathbb{R}^3$  by

$$\tilde{D}_q(x, y) = \min \left\{ |x - y|, \min_{a, b \in \Gamma} (|x - a| + \rho_q(a, b) + |b - y|) \right\}. \quad (36)$$

The Euclidean triangle inequality implies that, RHS of (36) is precisely the RHS of (34). The triangle inequality for  $\rho_q$  implies the same inequality for  $\tilde{D}_q$ . Thus, to prove (34), it remains to show that  $\tilde{D}_q = D_q$ .

We have

$$\tilde{D}_q(x, y) \leq |x - y|, \quad \tilde{D}_q(a, b) \leq q d_\Gamma(a, b) \quad (a, b \in \Gamma). \quad (37)$$

Fix a Lipschitz path  $\gamma : [0, 1] \rightarrow \mathbb{R}^3$  joining  $x$  to  $y$ , and set

$$h(s) = \tilde{D}_q(x, \gamma(s)),$$

then  $h$  is absolutely continuous and  $|h'| \leq |\gamma'|_{g_E}$  almost everywhere.

For almost every  $s$  with  $\gamma(s) \in \Gamma$ , the derivative  $\gamma'(s)$  is tangent to  $\Gamma$ . We may therefore choose  $z_\tau \in \Gamma$  such that

$$\begin{aligned} |z_\tau - \gamma(s + \tau)| &= o(|\tau|), \\ d_\Gamma(z_\tau, \gamma(s)) &= |\tau| |\gamma'(s)|_{g_E} + o(|\tau|). \end{aligned}$$

Consequently, (37) reads

$$\begin{aligned} \tilde{D}_q(\gamma(s), \gamma(s + \tau)) &\leq \tilde{D}_q(\gamma(s), z_\tau) + \tilde{D}_q(z_\tau, \gamma(s + \tau)) \\ &\leq q d_\Gamma(\gamma(s), z_\tau) + |z_\tau - \gamma(s + \tau)| \\ &= q|\tau| |\gamma'(s)|_{g_E} + o(|\tau|). \end{aligned}$$

Thus

$$|h'(s)| \leq \theta_q(\gamma(s)) |\gamma'(s)|_{g_E}$$

for almost every  $s \in [0, 1]$ . Integrating and taking the infimum over  $\gamma$  give  $\tilde{D}_q \leq D_q$ .

Conversely, direct segments and segment-arc-segment paths give  $D_q \leq \tilde{D}_q$ . Therefore  $D_q = \tilde{D}_q$ , proving (34).

If  $x, y \in \bar{B}_R$  and  $R > 1$ , all the routes in (34) remain in  $\bar{B}_R$ , by convexity and  $\Gamma \subset B_R$ . Finally, for  $p, z \in \partial B_R$ ,

$$D_q(p, z) \geq \min\{|p - z|, 2(R - 1)\} \geq \frac{R - 1}{R} |p - z|,$$

this proves (35).  $\square$

*Remark 4.6.* In particular,

$$D_0(x, y) = \min\{|x - y|, \text{dist}(x, \Gamma) + \text{dist}(y, \Gamma)\}, \quad (38)$$

$$D_1(x, y) = |x - y|, \quad (39)$$

$$\sup_{x, y \in \mathbb{R}^3} |D_q(x, y) - D_{q'}(x, y)| \leq \pi |q - q'|, \quad 0 \leq q, q' \leq 1. \quad (40)$$

hold.

Away from  $\Gamma$ , the distance  $D_q$  agrees locally with the Euclidean distance from its definition. For  $q < 1$ , however, distances between points on every sphere  $\partial B_R$ ,  $R > 1$ , are affected by the contraction along  $\Gamma$ . The following lemma quantifies this effect. Set

$$\Delta(q) = 2(\sqrt{1 - q^2} - q \arccos q) > 0, \quad 0 \leq q < 1.$$

**Lemma 4.7.** *For  $R > 1$  and  $0 \leq q < 1$ , the metric spaces  $(\partial B_R, D_q)$  and  $(\partial B_R, |\cdot - \cdot|)$  are not isometric, although both metrics induce the same intrinsic length distance on  $\partial B_R$ . Moreover,*

$$\max_{p, z \in \partial B_R} (|p - z| - D_q(p, z)) = \sup_{x, y \in \mathbb{R}^3} (|x - y| - D_q(x, y)) = \Delta(q) > 0. \quad (41)$$

*Proof.* If  $p, z \in \partial B_R$  satisfy  $|p - z| < 2(R - 1)$ , then (34) gives  $D_q(p, z) = |p - z|$ . Thus sufficiently fine partitions give the same length for every continuous curve in either metric. Their associated length distances are consequently both the round distance  $d_{\partial B_R}$ .

Choose

$$a_\pm = (q, \pm \sqrt{1 - q^2}, 0), \quad p_\pm = (q, \pm \sqrt{R^2 - q^2}, 0).$$

Then  $a_{\pm} \in \Gamma$ ,  $p_{\pm} \in \partial B_R$ . Let  $\gamma$  consist of the straight segment from  $p_+$  to  $a_+$ , a shorter arc of  $\Gamma$  from  $a_+$  to  $a_-$ , and the straight segment from  $a_-$  to  $p_-$ ,

$$\begin{aligned} L_{q,g_E}(\gamma) &= |p_+ - a_+| + q d_{\Gamma}(a_+, a_-) + |a_- - p_-| \\ &= 2\sqrt{R^2 - q^2} - 2\sqrt{1 - q^2} + 2q \arccos q \\ &= |p_+ - p_-| - \Delta(q). \end{aligned}$$

Since  $D_q(p_+, p_-) \leq L_{q,g_E}(\gamma)$ , we obtain

$$\begin{aligned} |p_+ - p_-| - D_q(p_+, p_-) &\geq \Delta(q), \\ \implies \max_{p, z \in \partial B_R} (|p - z| - D_q(p, z)) &\geq \Delta(q). \end{aligned} \tag{42}$$

For another direction, fix  $x, y \in \mathbb{R}^3$  and  $a, b \in \Gamma$ , and set  $s = d_{\Gamma}(a, b) \in [0, \pi]$ . The Euclidean triangle inequality gives

$$|x - y| - (|x - a| + qs + |b - y|) \leq |a - b| - qs = 2 \sin(s/2) - qs \leq \Delta(q),$$

where the last inequality follows by maximizing  $2 \sin(s/2) - qs$  over  $s \in [0, \pi]$ . The maximum occurs at  $s = 2 \arccos q$  and equals  $\Delta(q)$ . By (34),

$$\begin{aligned} |x - y| - D_q(x, y) &\leq \Delta(q), \\ \implies \sup_{x, y \in \mathbb{R}^3} (|x - y| - D_q(x, y)) &\leq \Delta(q) \end{aligned} \tag{43}$$

hold for all  $x, y \in \mathbb{R}^3$ . Combining (42) with (43) proves (41).

At last, Suppose there were an isometry  $F : (\partial B_R, D_q) \rightarrow (\partial B_R, |\cdot - \cdot|)$ . Since both metrics induce the standard spherical geodesic distance  $d_{\partial B_R}$ ,  $F$  also preserves  $d_{\partial B_R}$ . By the identity

$$|p - z| = 2R \sin \left( \frac{d_{\partial B_R}(p, z)}{2R} \right),$$

$F$  must preserve Euclidean chord distances. Consequently,

$$D_q(p, z) = |F(p) - F(z)| = |p - z|$$

for all  $p, z \in \partial B_R$ , contradicting (41). □

**4.3. Integral current comparison.** We use integral currents in metric spaces in the sense of Ambrosio–Kirchheim [1]. For two precompact integral current spaces  $X_i = (X_i, d_i, T_i)$ ,  $i = 0, 1$ , of dimension  $n$ , the intrinsic flat distance of Sormani–Wenger [27], as considered in [26], is defined by

$$d_{\mathcal{F}}(X_0, X_1) = \inf \{ \mathbf{M}_Z(A) + \mathbf{M}_Z(B) : \varphi_{1\#}T_1 - \varphi_{0\#}T_0 = A + \partial B \}.$$

Here the infimum is taken over all complete metric spaces  $Z$ , isometric embeddings  $\varphi_i : (\overline{X}_i, d_i) \rightarrow Z$ , and integral currents  $A$  and  $B$  in  $Z$  of dimensions  $n$  and  $n + 1$ , respectively. The notation  $\varphi_{i\#}T_i$  denotes the pushforward of  $T_i$  under  $\varphi_i$ ,  $\mathbf{M}_Z$  denotes the mass of a current in  $Z$ .

The following comparison estimate follows by adapting the metric gluing and homotopy argument of Sormani–Wenger [27, proof of Lemma 5.4]. If two metrics on the same manifold differ by at most  $\eta$ , their isometric copies can be placed in a common metric space with corresponding points at distance  $\eta/2$ . After a Kuratowski embedding, straight line interpolation gives a homotopy with controlled Lipschitz constant. The homotopy formula and the Lipschitz mass estimate then yield the following bound, which does not depend on the inverse Lipschitz constants. We omit the proof.

**Lemma 4.8.** *Let  $(P, h)$  be a compact connected oriented Riemannian  $n$ -manifold, possibly with boundary, with length distance  $d_h$  and integration current  $T$ . Suppose  $d_0, d_1$  are metrics on  $P$  such that the identity maps  $\text{id}_i : (P, d_h) \rightarrow (P, d_i)$  are bi-Lipschitz and satisfy  $\text{Lip}(\text{id}_i) \leq L$ , where  $L \geq 1$ . Set*

$$Y_i = (P, d_i, \text{id}_{i\#}T), \quad \eta = \sup_{p, z \in P} |d_0(p, z) - d_1(p, z)|, \quad C_L = \sqrt{1 + L^2},$$

then

$$d_{\mathcal{F}}(Y_0, Y_1) \leq \frac{\eta}{2} (C_L^{n+1} \text{Vol}_h(P) + C_L^n \text{Area}_h(\partial P)).$$

For a closed surface and  $L = 1$ , this becomes

$$d_{\mathcal{F}}(Y_0, Y_1) \leq \sqrt{2} \eta \operatorname{Area}_h(P). \quad (44)$$

Taking boundaries does not increase the intrinsic flat distance. The following result follows from [27, proof of Theorem 4.6].

**Lemma 4.9.** *For precompact integral current spaces  $X_0, X_1$  of the same positive dimension,*

$$d_{\mathcal{F}}(\partial X_0, \partial X_1) \leq d_{\mathcal{F}}(X_0, X_1).$$

*Here each boundary carries the metric restricted from the completion of its original space.*

This lemma implies that, to obtain an obstruction for the intrinsic flat distance  $d_{\mathcal{F}}$ , we only need to deduce an obstruction on its boundary.

We now describe the current spaces appearing in Theorem C. For  $R > 1$ ,  $\llbracket B_R \rrbracket_{g_E}$  is the Euclidean integration current and  $T_R = \partial \llbracket B_R \rrbracket_{g_E} = \llbracket \partial B_R \rrbracket$  the current of the outward oriented sphere. Recall that

$$\begin{aligned} X_q(R) &= (\overline{B}_R, D_q, T_{q,R}), \quad (0 < q \leq 1), \\ X_{g_j}(R) &= (\overline{B}_R, d_{g_j, B_R}, \llbracket B_R \rrbracket_{g_j}), \quad \mathcal{E}_R = (\overline{B}_R, |\cdot - \cdot|, \llbracket B_R \rrbracket_{g_E}). \end{aligned}$$

Here  $\llbracket B_R \rrbracket_{g_j}$  and  $T_{q,R}$  are the respective identity pushforwards of  $\llbracket B_R \rrbracket_{g_E}$ . Thus these spaces represent the same oriented ball with multiplicity one, equipped with different distances. Moreover,

$$\mathbf{M}(T_{q,R}) = \mathbf{M}(\llbracket B_R \rrbracket_{g_E}) = \frac{4\pi R^3}{3}. \quad (45)$$

This identity reflects the shortcuts along  $\Gamma$  change global distances while preserving the three-dimensional mass.

For  $q \in [0, 1]$ , consider the boundary current spaces

$$Y_q(R) = (\partial B_R, D_q, \operatorname{id}_{\#} T_R), \quad Y_E(R) = (\partial B_R, |\cdot - \cdot|, \operatorname{id}_{\#} T_R).$$

These are integral current spaces, with  $\partial X_q(R) = Y_q(R)$  for  $q > 0$  and  $\partial \mathcal{E}_R = Y_E(R)$ . Their boundary currents both have mass  $4\pi R^2$ . Distances between boundary points are measured using paths in  $\overline{B}_R$ , which may travel along  $\Gamma$ . Consequently, for  $q < 1$ , some boundary point pairs have  $D_q$ -distance strictly smaller than their Euclidean chord distance, even though the boundary current masses agree. In view of (38) and (39), when  $q = 0$ , the circle  $\Gamma$  is contracted to a single point in the ambient metric, whereas  $q = 1$  corresponds to the Euclidean metric, so that  $Y_1(R) = Y_E(R)$ .

For  $0 \leq q < 1$ , Lemma 4.7 and [27, Theorem 3.27] give

$$c_{R,q} := d_{\mathcal{F}}(Y_q(R), Y_E(R)) > 0. \quad (46)$$

Lemma 4.9 therefore implies

$$d_{\mathcal{F}}(X_q(R), \mathcal{E}_R) \geq c_{R,q}.$$

Thus the boundary maintains a positive intrinsic flat separation from the Euclidean ball, even though the volume and boundary area are Euclidean. This is the obstruction underlying (14).

The comparison estimates (41), (40), and (44) also yield

$$c_{R,q} \leq 4\sqrt{2} \pi R^2 \Delta(q), \quad (47)$$

$$d_{\mathcal{F}}(Y_q(R), Y_{q'}(R)) \leq 4\sqrt{2} \pi^2 R^2 |q - q'|, \quad q, q' \in [0, 1].$$

Consequently, the family  $Y_q(R)$ ,  $q \in [0, 1]$ , is Lipschitz continuous in intrinsic flat distance  $d_{\mathcal{F}}$ .

We can also briefly analyze the quantification of the gap  $c_{R,q}$ . For  $R > 1$  and  $0 \leq q < 1$ , the argument for (46) gives no explicit positive lower bound for  $c_{R,q}$ . Since  $\Delta'(q) = -2 \arccos q$  and  $\Delta(1) = 0$ , we have

$$\Delta(q) = 2 \int_q^1 \arccos s \, ds = \frac{4\sqrt{2}}{3} (1-q)^{3/2} + O((1-q)^{5/2}) \quad (q \uparrow 1).$$

Together with (47), this yields  $c_{R,q} = \mathcal{O}_R((1-q)^{3/2})$ . A natural question is to obtain explicit positive lower bounds for  $c_{R,q}$  and determine its sharp asymptotic behavior as  $q \uparrow 1$ . The discrepancy  $\Delta(q)$  is measured under the fixed coordinate identification, whereas intrinsic flat distance involves an infimum over common ambient spaces and isometric embeddings. Thus  $\Delta(q)$  alone does not provide the desired lower bound.

**4.4. Construction of the small mass sequence.** We now in the position of proving Theorem C. Fix  $\lambda > 1$  and  $q \in (\lambda^{-1}, 1)$ . All coordinates in this subsection are area radius coordinates, and  $b_a$  is the metric in (25).

Since  $b_a \rightarrow g_E$  smoothly near the unit circle  $\Gamma$ , we may choose  $r_* > 0$ , independent of all sufficiently small  $a > 0$ , such that  $N_{b_a}(\Gamma, 2r_*) \subset \mathbb{R}^3$  is a regular tubular neighborhood of  $\Gamma$  with closure contained in  $B_{3/2} \setminus \overline{B}_{1/2}$ . In particular, every point in this neighborhood has a unique nearest point on  $\Gamma$ ,  $t_a = d_{b_a}(\cdot, \Gamma)$  is smooth away from  $\Gamma$ , and the distance levels  $\{t_a = t\}$  are smooth for  $0 < t \leq 2r_*$ . Since the Euclidean distance tubes of  $\Gamma$  have mean curvature  $H = 1/t + \mathcal{O}(1)$  as  $t \rightarrow 0$ , after decreasing  $r_*$  if necessary, that there exists  $c_* > 0$ , independent of  $a$ , such that

$$\text{Scal}_{b_a} \geq c_* a \quad \text{on } N_{b_a}(\Gamma, 2r_*), \quad H_{b_a}(\{t_a = t\}) \geq \frac{1}{2t} \quad (0 < t \leq 2r_*).$$

For each sufficiently small  $a > 0$ , choose the radial cutoff  $\chi_a$  as in Theorem A, equal to 1 for  $t_a \leq r_*/2$  and 0 for  $t_a \geq r_*$ . Since  $r_*$  is independent of  $a$  and  $b_a \rightarrow g_E$  smoothly near  $\Gamma$ , these cutoffs may be chosen so that

$$|d\chi_a|_{b_a} + |\Delta_{b_a} \chi_a| \leq C$$

with  $C$  independent of  $a$ .

We now construct the sequence of metrics required in Theorem C. For each sufficiently small  $a > 0$ , apply Theorem A to the background  $b_a$  with the fixed contraction factor  $q$ . For  $0 < \varepsilon < 1 - q^2$  sufficiently small, this gives a smooth metric  $g_{a,\varepsilon}$  satisfying  $g_{a,\varepsilon} \succeq_2 b_a$  and

$$q^2 g_E \leq g_{a,\varepsilon} \leq e^{4A_\varepsilon} q^{-2} (1 + Ca) g_E,$$

by (4) and (26). Moreover,  $g_{a,\varepsilon} = b_a$  outside  $N_{b_a}(\Gamma, r_*) \Subset B_{3/2}$ . Consequently,  $g_{a,\varepsilon}$  is complete and asymptotically flat, with

$$\text{Scal}_{g_{a,\varepsilon}} > 0, \quad m_{\text{ADM}}(g_{a,\varepsilon}) = 2a.$$

Then we choose  $a_j \downarrow 0$  and  $\varepsilon_j \downarrow 0$  so that  $g_j = g_{a_j, \varepsilon_j}$  has ADM mass tending to 0, satisfies the tensor and volume bounds in Theorem C, and has intrinsic distances converging to the prescribed limit  $D_q$ . Together with the geometric properties established below, this sequence will yield a counterexample to Conjecture 1.2.

Since  $q > \lambda^{-1}$ , we choose  $a_j \downarrow 0$  sufficiently small that

$$q^{-2}(1 + Ca_j) < \lambda^2 \quad \text{for every } j.$$

For each  $j$ , fix the background  $b_{a_j}$  and set  $K_j = \overline{B}_{j+3}$ . By Proposition 4.4, the intrinsic distances induced by  $g_{a_j, \varepsilon_j}$  on  $K_j$  satisfy

$$\lim_{\varepsilon \downarrow 0} \sup_{p, z \in K_j} |d_{g_{a_j, \varepsilon_j}, K_j}(p, z) - D_{q, b_{a_j}}^{K_j}(p, z)| = 0.$$

Moreover, the volume estimate (3) in Theorem A gives

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} (dV_{g_{a_j, \varepsilon_j}} - dV_{b_{a_j}}) = 0.$$

We may thus choose  $\varepsilon_j > 0$  sufficiently small that

$$\varepsilon_j \leq a_j^2, \quad \varepsilon_j < 1 - q^2, \quad e^{4A_{\varepsilon_j}} q^{-2} (1 + Ca_j) \leq \lambda^2,$$

and

$$\nu_j(\mathbb{R}^3) = (dV_{g_{a_j, \varepsilon_j}} - dV_{b_{a_j}})(\mathbb{R}^3) \leq a_j^2, \tag{48}$$

$$\sup_{p, z \in K_j} |d_{g_{a_j, \varepsilon_j}, K_j}(p, z) - D_{q, b_{a_j}}^{K_j}(p, z)| \leq a_j. \tag{49}$$

The parameter  $\varepsilon_j$  is also chosen small enough for all the conclusions of Theorem A to hold.

Set

$$g_j = g_{a_j, \varepsilon_j}, \quad m_j = 2a_j.$$

Then the scalar curvature, ADM mass, and tensor bounds in (9) of Theorem C are satisfied. In addition,  $g_j \succeq_2 b_{a_j}$  and the volume excess is bounded by  $a_j^2$  from (48).

To establish the distance estimate (12) in Theorem C, notice that By (26) and  $D_{q,g_E}^{\overline{B}_R} = D_q$  for  $R > 1$ , we have

$$D_q \leq D_{q,b_{a_j}}^{\overline{B}_R} \leq \sqrt{1 + Ca_j} D_q \quad \text{on } \overline{B}_R \times \overline{B}_R.$$

Together with (49), this yields (12). Since  $D_q(p, z) \leq |p - z| \leq 2R$ , the diameter estimate (11) follows.

Finally, we must show that the constructed metrics  $g_j$  contain no closed embedded minimal surfaces. This is established in the next proposition using the mean convex tubes and the smallness of the ADM mass.

**Proposition 4.10.** *For all sufficiently large  $j$ ,  $(\mathbb{R}^3, g_j)$  contains no closed embedded minimal surface.*

*Proof.* Suppose that  $\Sigma$  is such a surface. First, the maximum principle ensures that  $\Sigma \subset \overline{B}_2$  and  $\Sigma$  cannot lie entirely in  $N_{b_{a_j}}(\Gamma, 2r_*)$ . Thus there is  $p \in \Sigma$  with  $d_{b_{a_j}}(p, \Gamma) > 2r_*$ . Choose a fixed  $s_0 \in (0, r_*/2)$ , independent of  $j$ , such that  $B_{b_{a_j}}(x, s_0)$  is a normal geodesic ball for every  $x \in \overline{B}_2$  and all sufficiently large  $j$ . Moreover,  $B_{b_{a_j}}(p, s_0)$  is disjoint from  $N_{b_{a_j}}(\Gamma, r_*)$ , where the metric modification is supported. Consequently,  $g_j = b_{a_j}$  on this ball, and  $\Sigma$  is  $b_{a_j}$ -minimal there.

Write  $\rho = d_{b_{a_j}}(p, \cdot)$  and  $A(s) = \text{Area}_{b_{a_j}}(\Sigma \cap B_{b_{a_j}}(p, s))$ . Since  $b_{a_j} \rightarrow g_E$  smoothly on compact subsets of  $\mathbb{R}^3$ , the standard Hessian expansion in geodesic normal coordinates gives

$$\text{Hess}_{b_{a_j}}(\rho^2/2) = b_{a_j} + \mathcal{O}(\rho^2)$$

for  $\rho < s_0$ , with a constant independent of  $j$  and  $p$ . Applying the divergence theorem on  $\Sigma$  and then the coarea formula yields, at regular radii,

$$(2 - Cs^2)A(s) \leq s \int_{\Sigma \cap \partial B_{b_{a_j}}(p, s)} |\nabla_\Sigma \rho| d\ell \leq sA'(s).$$

Since  $A$  is nondecreasing, the differential inequality also holds in the distributional sense. Integrating from zero, using  $A(s)/(\pi s^2) \rightarrow 1$  at the smooth embedded point  $p$ , gives

$$\text{Area}_{g_j}(\Sigma) \geq A(s_0) \geq \pi s_0^2 e^{-Cs_0^2/2} =: A_* > 0. \quad (50)$$

The constant  $A_*$  is independent of  $j$ .

If a closed embedded minimal surface exists, there is a nonempty outermost minimal surface  $\mathcal{H}_j$ , which is smooth and outer minimizing; see [6, Section 2]. Applying the Riemannian Penrose inequality [6, Theorem 19] to the exterior of  $\mathcal{H}_j$  gives

$$\text{Area}_{g_j}(\mathcal{H}_j) \leq 16\pi m_j^2.$$

Every component of  $\mathcal{H}_j$  is a closed embedded minimal surface and therefore has area at least  $A_*$  by (50). This is impossible for large  $j$ , since  $m_j \rightarrow 0$ .  $\square$

*Completion of the proof of Theorem C.* Fix  $R > 2$ . By Corollary 4.2,  $B_R$  uniquely maximizes volume under the perimeter bound  $4\pi R^2$  in  $(\mathbb{R}^3, b_{a_j})$ . Since  $g_j \succeq b_{a_j}$  and  $g_j = b_{a_j}$  outside  $B_{3/2}$ , Lemma 2.2 shows that  $B_R$  has the same volume maximizing property in  $(\mathbb{R}^3, g_j)$ , with  $P_{g_j}(B_R) = 4\pi R^2$ . Moreover,

$$0 \leq \text{Vol}_{g_j}(B_R) - \frac{4\pi R^3}{3} = \text{Vol}_{b_{a_j}}(B_R) - \frac{4\pi R^3}{3} + \nu_j(\mathbb{R}^3) \leq C_R a_j + a_j^2.$$

This proves (10). The exterior  $C^2$  estimate follows from  $g_j = b_{a_j}$  on  $\{|x| \geq R\}$  and (26).

We next prove (15), which reveals that the intrinsic and ambient distances agree on  $\overline{B}_R$ . Define the radial retraction  $\Pi_R(\rho, \theta) = (\min\{\rho, R\}, \theta)$ . For  $\rho > R$  and  $X = c\partial_\rho + Y$ , where  $Y$  is angular,

$$|d\Pi_R(X)|_{g_j}^2 = R^2 |Y|_{g_{S^2}}^2 \leq \frac{c^2}{W_{a_j}(\rho)} + \rho^2 |Y|_{g_{S^2}}^2 = |X|_{g_j}^2.$$

Inside  $B_R$  the retraction is the identity. It is therefore length nonincreasing on rectifiable curves and fixes their endpoints in  $\overline{B}_R$ . Consequently,

$$d_{g_j, B_R}(p, z) = d_{g_j}(p, z) \quad (p, z \in \overline{B}_R),$$

It remains to prove (13) and (14). Set  $d_j = d_{g_j, B_R}$  and  $\eta_j = \sup_{p, z \in \overline{B}_R} |d_j(p, z) - D_q(p, z)|$ . By (12),  $\eta_j \leq C_R m_j$ . It is not hard to show that the identity maps from  $(\overline{B}_R, |\cdot - \cdot|)$  to  $(\overline{B}_R, d_j)$  and  $(\overline{B}_R, D_q)$  are

bi-Lipschitz, with Lipschitz constants at most  $\lambda$ . Applying Lemma 4.8 with  $P = \overline{B}_R$ ,  $h = g_E$ ,  $d_0 = d_j$ ,  $d_1 = D_q$ ,  $n = 3$ , and  $L = \lambda$ , we obtain

$$\begin{aligned} d_{\mathcal{F}}(X_{g_j}(R), X_q(R)) &\leq \frac{\eta_j}{2} \left( (1 + \lambda^2)^2 \text{Vol}_{g_E}(B_R) + (1 + \lambda^2)^{3/2} \text{Area}_{g_E}(\partial B_R) \right) \\ &\leq C_R \eta_j \leq C_R m_j. \end{aligned}$$

This proves (13).

The triangle inequality, Lemma 4.9, and (46) now give

$$\begin{aligned} d_{\mathcal{F}}(X_{g_j}(R), \mathcal{E}_R) &\geq d_{\mathcal{F}}(X_q(R), \mathcal{E}_R) - d_{\mathcal{F}}(X_{g_j}(R), X_q(R)) \\ &\geq d_{\mathcal{F}}(Y_q(R), Y_E(R)) - C_R m_j \\ &= c_{R,q} - C_R m_j. \end{aligned}$$

This proves (14). Together with (45), this completes the proof of Theorem C.  $\square$

For each fixed  $R > 2$ , the regions  $\Omega_j(R) = B_R$  satisfy all hypotheses of Conjecture 1.2, with identity exterior maps. They converge in the volume preserving intrinsic flat sense to  $X_q(R)$ , whereas the positive boundary gap excludes  $\mathcal{E}_R$  as a limit. Moreover,  $g_j \rightarrow g_E$  pointwise on  $\mathbb{R}^3 \setminus \Gamma$ . Since  $\Gamma$  has zero Euclidean volume, the uniform tensor bounds and dominated convergence give  $g_j \rightarrow g_E$  in  $L^p_{\text{loc}}(g_E)$  for every  $1 \leq p < \infty$ , despite the non-Euclidean distance limit  $D_q$ . This is consistent with Dong–Song’s stability after excision [12] and Kazaras–Khuri–Lee’s pointed Gromov–Hausdorff stability under a uniform Ricci curvature lower bound [15], the latter assumption being violated by the present sequence.

**4.5. Variants of the small mass construction.** Theorem C establishes the failure of zero mass stability using a fixed positive contraction factor  $q$ , while keeping the metrics  $g_j$  uniformly comparable to the Euclidean metric. We record two variants that clarify the scope of this construction. The first replaces the radial background metric  $b_a$  by one with an exact Schwarzschild exterior. The second allows the contraction factor  $q$  to tend to 0, so that the limiting distance identifies all points of  $\Gamma$ .

For the first variant, choose a smooth nonnegative function  $\zeta$  supported in  $[0, 2]$ , positive and strictly decreasing on  $(0, 2)$ , such that  $\zeta(r) = \psi(r^2)$  near  $r = 0$  for some smooth function  $\psi$ , and normalized by  $\int_0^2 \zeta(t) t^2 dt = 1$ . Set

$$F(r) = \int_0^r \zeta(t) t^2 dt, \quad \bar{b}_m = \frac{dr^2}{1 - 2mF(r)/r} + r^2 g_{\mathbb{S}^2}.$$

For sufficiently small  $m > 0$ , this metric is smooth and complete, and

$$g_E \leq \bar{b}_m \leq (1 + Cm) g_E, \quad \text{Scal}_{\bar{b}_m} = 4m\zeta(r) \geq 0.$$

Since  $F(r) = 1$  for  $r \geq 2$ , the exterior is exactly Schwarzschild of mass  $m$ . Moreover, every centered ball in  $(\mathbb{R}^3, \bar{b}_m)$  is uniquely isoperimetric. Then applying Theorem A to  $\bar{b}_{m_j}$ , we obtain

**Corollary 4.11.** *Theorem C remains valid with  $\text{Scal}_{g_j} > 0$  replaced by  $\text{Scal}_{g_j} \geq 0$  and with the additional conclusion*

$$g_j = \frac{dr^2}{1 - 2m_j/r} + r^2 g_{\mathbb{S}^2} \quad (r \geq 2).$$

This variant shows that the failure of zero mass stability persists even with an exact Schwarzschild exterior. Thus the obstruction arises from the interior drawstring contraction rather than from the choice of exterior geometry. Compared with Theorem C, the only weakening is that the scalar curvature is nonnegative rather than strictly positive everywhere.

Now let’s turn to the second variant. In Theorem C, the fixed contraction factor  $q \in (0, 1)$  leaves  $\Gamma$  with positive limiting length. We now take  $q_m = m$ , so that

$$\text{Length}_{g_m}(\Gamma) = 2\pi m \rightarrow 0.$$

The resulting limiting distance is

$$D_0(x, y) = \min\{|x - y|, \text{dist}(x, \Gamma) + \text{dist}(y, \Gamma)\},$$

which identifies all points of  $\Gamma$ .

Applying Theorem A to the exact Schwarzschild backgrounds  $\bar{b}_m$  with this contraction factor  $q_m$ , together with the preceding distance and boundary current comparisons, yields the following complete contraction variant.

**Corollary 4.12.** *There exist a sequence  $m_j \downarrow 0$  and smooth complete metrics  $g_j$  on  $\mathbb{R}^3$  with nonnegative scalar curvature, ADM mass  $m_j$ , no closed embedded minimal surfaces, and an exact Schwarzschild exterior for  $r \geq 2$ .*

*For every  $R > 2$ , the ball  $B_R$  uniquely maximizes volume among finite volume sets with perimeter at most  $4\pi R^2$ , and  $P_{g_j}(B_R) = 4\pi R^2$ . Moreover, for each fixed  $R > 2$ , there exists a constant  $C_R > 0$ , independent of  $j$ , such that*

$$0 \leq \text{Vol}_{g_j}(B_R) - \frac{4\pi R^3}{3} \leq C_R m_j,$$

$$\text{diam}(\bar{B}_R, d_{g_j, B_R}) \leq 2R + C_R m_j.$$

Furthermore,

$$\sup_{p, z \in \partial B_R} |d_{g_j, B_R}(p, z) - D_0(p, z)| \leq C_R m_j,$$

$$d_{\mathcal{F}}(X_{g_j}(R), \mathcal{E}_R) \geq c_{R,0} - C_R m_j,$$

where  $c_{R,0} > 0$  is defined in (46).

*Remark 4.13.* Unlike Theorem C, this variant completely collapses  $\Gamma$  in the limiting distance. However, it necessarily loses the uniform bi-Lipschitz bound, because  $\text{Length}_{g_m}(\Gamma) \rightarrow 0$ . Thus the fixed positive contraction in Theorem C yields a stronger counterexample in terms of uniform metric control, whereas the complete contraction variant Corollary 4.12 realizes the limiting case  $q = 0$ , in which all points of  $\Gamma$  are identified.

Finally, let's compare the main results we established. Theorems B and C use the same local drawstring construction but serve different purposes.

In Theorem C, the modifications are confined to a fixed compact region, preserving the desired ADM mass and isoperimetric properties. In the exact Schwarzschild variant of Corollary 4.11, radial distances satisfy, for fixed  $R_0 > 2$ ,

$$\int_{R_0}^L \frac{dr}{\sqrt{1 - 2m/r}} = L + m \log L + \mathcal{O}_{m, R_0}(1) \quad (L \rightarrow \infty).$$

Since the corresponding Euclidean distance is  $L - R_0$ , the additive distance error grows logarithmically and is therefore unbounded for every fixed  $m > 0$ .

By contrast, Theorem B uses a locally finite collection of modifications extending to infinity to achieve global additive distance control, without establishing the corresponding ADM mass and isoperimetric conclusions. Thus the two theorems illustrate distinct global applications of the same local construction.

**Declaration of competing interest.** The authors declare that they have no known competing financial interests in this paper.

**Data availability.** No data were used for the research described in the article.

## REFERENCES

- [1] L. Ambrosio and B. Kirchheim, *Currents in metric spaces*, Acta Math. **185** (2000), 1–80.
- [2] R. Arnowitt, S. Deser, and C. W. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. **122** (1961), 997–1006.
- [3] R. H. Bamler, *A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature*, Math. Res. Lett. **23** (2016), 325–337.
- [4] R. Bartnik and J. Isenberg, *The constraint equations*, in P. T. Chruściel and H. Friedrich (eds.), *The Einstein Equations and the Large Scale Behavior of Gravitational Fields*, Birkhäuser, Basel, 2004, pp. 1–38.
- [5] J. Basilio, J. Dodziuk, and C. Sormani, *Sewing Riemannian manifolds with positive scalar curvature*, J. Geom. Anal. **28** (2018), 3553–3602.
- [6] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), 177–267.

- [7] S. Brendle, *Constant mean curvature surfaces in warped product manifolds*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 247–269.
- [8] S. Brendle, *Scalar curvature rigidity of convex polytopes*, Invent. Math. **235** (2024), 669–708.
- [9] P. Burkhardt-Guim, *Pointwise lower scalar curvature bounds for  $C^0$  metrics via regularizing Ricci flow*, Geom. Funct. Anal. **29** (2019), 1703–1772.
- [10] A. Carlotto, O. Chodosh, and M. Eichmair, *Effective versions of the positive mass theorem*, Invent. Math. **206** (2016), 975–1016.
- [11] J. Chu, M.-C. Lee, and J. Wan, *Rigidity of positive mass theorem with fast metric decay*, [arXiv:2607.17236v2](#) (2026).
- [12] C. Dong and A. Song, *Stability of Euclidean 3-space for the positive mass theorem*, Invent. Math. **239** (2025), 287–319.
- [13] M. Gromov, *Dirac and Plateau billiards in domains with corners*, Cent. Eur. J. Math. **12** (2014), 1109–1156.
- [14] M. Gromov, *Four lectures on scalar curvature*, [arXiv:1908.10612v6](#) (2021).
- [15] D. Kazaras, M. Khuri, and D. A. Lee, *Stability of the positive mass theorem under Ricci curvature lower bounds*, Math. Res. Lett. **31** (2024), 747–794.
- [16] D. Kazaras and K. Xu, *Codimension 2 drawstrings with scalar curvature lower bounds*, [arXiv:2501.09149](#) (2025).
- [17] D. Kazaras and K. Xu, *Drawstrings and flexibility in the Geroch conjecture*, [arXiv:2309.03756v3](#) (2026).
- [18] D. A. Lee and C. Sormani, *Stability of the positive mass theorem for rotationally symmetric Riemannian manifolds*, J. Reine Angew. Math. **686** (2014), 187–220.
- [19] M.-C. Lee, *Quantification of scalar curvature under  $C^0$  convergence using smoothing*, [arXiv:2604.17759](#) (2026).
- [20] C. Li, *A polyhedron comparison theorem for 3-manifolds with positive scalar curvature*, Invent. Math. **219** (2020), 1–37.
- [21] C. Li, *Correction to: A polyhedron comparison theorem for 3-manifolds with positive scalar curvature*, Invent. Math. **228** (2022), 535–538.
- [22] L. Mazurowski and X. Yao, *A positive mass theorem for continuous metrics*, [arXiv:2606.19123](#) (2026).
- [23] L. Mazurowski and X. Yao, *Quantification of  $C^0$  convergence in dimension three*, [arXiv:2604.14087](#) (2026).
- [24] L. Mazurowski and X. Yao, *Scalar curvature under weak limits of manifolds*, [arXiv:2605.03136](#) (2026).
- [25] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), 45–76.
- [26] C. Sormani, *Geometric stability of the Schoen–Yau zero mass rigidity theorem*, Surv. Differential Geom. **29** (2024), 309–353; [arXiv:2604.17599v2](#).
- [27] C. Sormani and S. Wenger, *The intrinsic flat distance between Riemannian manifolds and other integral current spaces*, J. Differential Geom. **87** (2011), 117–199.
- [28] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), 381–402.
- [29] J. You and H. Zhang, *Gromov’s Euclidean endpoint  $C^0$  rigidity for the positive mass theorem*, [arXiv:2606.15372v2](#) (2026).

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, CHINA  
 Email address: yjcmp@mail.ustc.edu.cn

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, CHINA  
 Email address: hengz@mail.ustc.edu.cn