

Counterexamples to the Two-bases Conjecture for the Tutte Polynomial

21 October 2026

Abstract

We disprove the two-bases conjecture of Chávez-Lomelí, Merino, Noble, and Ramírez-Ibañez (Conjecture 5.8 in *Some inequalities for the Tutte polynomial*), which asserts that $\max\{T_M(2, 0), T_M(0, 2)\} \geq T_M(1, 1)$ whenever a matroid contains two disjoint bases or its ground set is the union of two bases. We construct a connected, self-dual, rationally representable matroid M of rank 32 on 64 elements whose ground set is partitioned into two bases, but for which

$$T_M(2, 0) = T_M(0, 2) = 263656235940864 < 275621102802945 = T_M(1, 1).$$

The construction extends to an explicit family M_m of rank $2m$ on $4m$ elements satisfying $\max\{T_{M_m}(2, 0), T_{M_m}(0, 2)\}/T_{M_m}(1, 1) \rightarrow 0$. Thus the associated additive and multiplicative Merino–Welsh inequalities also fail under the two-bases hypothesis. We additionally construct a simple and cosimple matroid on 60 elements with complementary bases and $T_M(2, 0) < T_M(1, 1)$, disproving the simple-cover formulation of Conjecture 4.2 of Merino, Ibañez, and Rodríguez; its dual disproves the cosimple two-disjoint-bases formulation. All computations admit exact integer certificates, while the infinite-family argument is elementary.

2020 Mathematics Subject Classification. Primary 05B35; Secondary 05A20.

Keywords. Tutte polynomial; two-bases conjecture; Merino–Welsh conjecture; matroids; counterexamples; self-dual matroids.

1 The conjectures and the main result

Throughout, $U_{n,r}$ denotes the uniform matroid on n elements of rank r . For a matroid M with ground set E and rank function r_M , write

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{r_M(E) - r_M(A)} (y - 1)^{|A| - r_M(A)}. \quad (1)$$

In particular, $T_M(1, 1)$ is the number of bases of M .

Chávez-Lomelí, Merino, Noble, and Ramírez-Ibañez [1, Conjecture 5.8] conjectured

$$\max\{T_M(2, 0), T_M(0, 2)\} \geq T_M(1, 1) \quad (2)$$

whenever M has two disjoint bases or E is the union of two bases. These are *alternative* hypotheses. Our counterexamples satisfy the stronger, simultaneous requirement that E is a *disjoint* union of two bases.

Theorem 1. *For each integer $m \geq 1$ there is a connected matroid M_m of rank $2m$ on $4m$ elements with the following properties.*

- (i) *Its ground set is the disjoint union of two bases.*
- (ii) *It is representable over $\mathbb{Q}$ and $M_m \cong M_m^*$.*

(iii) If $a_m = T_{M_m}(2, 0) = T_{M_m}(0, 2)$ and $b_m = T_{M_m}(1, 1)$, then

$$b_m = \sum_{k=0}^m \binom{m}{k}^2 4^k, \quad (3)$$

$$a_m = 2 \sum_{j=0}^{m-1} [z^j] (1+z)^{m-1} (3+z)^m. \quad (4)$$

(iv) In particular,

$$a_{16} = 263656235940864 < 275621102802945 = b_{16}.$$

(v) The ratio satisfies

$$\frac{a_m}{b_m} \leq (m+1) \left(\frac{8}{9}\right)^m \rightarrow 0. \quad (5)$$

Hence the maximum, additive, and multiplicative Merino–Welsh inequalities all fail even for connected matroids whose ground sets are partitioned into two bases.

The unrestricted matroidal forms were already disproved by Beke, Csáji, Csikvári, and Pituk [3]. The point here is that the counterexample is *connected*, satisfies both two-bases hypotheses, and belongs to an explicit family with ratio tending to zero. We make no claim that 64 is the global minimum number of elements.

2 Construction, rank function, and complementary bases

Begin with $U_{2m,m}$ and partition its ground set into two sets of size m , denoted P and S . Double each element of P in parallel, and double each element of S in series. Denote the resulting pairs by

$$P_i = \{p_i^0, p_i^1\}, \quad S_i = \{s_i^0, s_i^1\} \quad (1 \leq i \leq m).$$

The resulting matroid is M_m . We give an explicit rational matrix representation to remove any ambiguity from the construction.

For $1 \leq i \leq 2m$, let

$$v_i = (1, i, i^2, \dots, i^{m-1})^\top \in \mathbb{Q}^m,$$

and let $e_1, \dots, e_m$ be the standard basis of $\mathbb{Q}^m$. Represent the elements by columns in $\mathbb{Q}^{2m}$ as follows:

$$p_i^0, p_i^1 : \begin{pmatrix} v_i \\ 0 \end{pmatrix}, \begin{pmatrix} v_i \\ 0 \end{pmatrix}; \quad s_i^0, s_i^1 : \begin{pmatrix} v_{m+i} \\ e_i \end{pmatrix}, \begin{pmatrix} 0 \\ e_i \end{pmatrix}. \quad (6)$$

The Vandermonde determinant shows that any at most m of the distinct core vectors $v_1, \dots, v_{2m}$ are independent.

For $A \subseteq E(M_m)$ define

$$\begin{aligned} u(A) &= |\{i : A \cap P_i \neq \emptyset\}|, \\ t(A) &= |\{i : A \cap S_i \neq \emptyset\}|, \\ v(A) &= |\{i : S_i \subseteq A\}|. \end{aligned}$$

The lower coordinates of the matrix in (6) contribute one independent direction for each nonempty S_i . If both columns in S_i are present, their difference supplies the core direction v_{m+i} ; the nonempty P_i supply the remaining core directions. It follows that

$$r_{M_m}(A) = t(A) + \min\{m, u(A) + v(A)\}. \quad (7)$$

In particular, $r(M_m) = 2m$.

Consider the disjoint sets

$$B_0 = \{p_i^0, s_i^0 : 1 \leq i \leq m\}, \quad B_1 = \{p_i^1, s_i^1 : 1 \leq i \leq m\}.$$

For either set $u = t = m$ and $v = 0$, so (7) gives rank $2m$. Thus B_0 and B_1 are bases, and

$$E(M_m) = B_0 \dot{\cup} B_1. \quad (8)$$

More generally, every transversal choosing one member of each of the $2m$ pairs is a basis.

Lemma 1. *The matroid M_m is connected and isomorphic to its dual.*

Proof. Let $d(A) = |\{i : P_i \subseteq A\}|$ be the number of full parallel pairs. Then $|A| = u(A) + d(A) + t(A) + v(A)$. By the dual-rank identity $r_{M_m^*}(A) = |A| - 2m + r_{M_m}(E \setminus A)$ and (7),

$$r_{M_m^*}(A) = u(A) + \min\{m, d(A) + t(A)\}.$$

This is precisely $r_{M_m}(\phi(A))$, where the permutation ϕ interchanges the parallel pair P_i with the series pair S_i , preserving the superscript for each i . Hence $M_m \cong M_m^*$.

For connectedness, every $(m+1)$ -element circuit of the uniform core lifts to a circuit of M_m by choosing one representative of each parallel pair and both elements of each series pair appearing in the core circuit. Indeed, (7) gives nullity one for the lifted set and shows that deleting any element produces an independent set. Any two elements from distinct pairs can be placed in the same such lifted circuit. The two elements of a parallel pair form a circuit; the two elements of a series pair occur together in a lifted circuit. Thus every two elements belong to a common circuit and M_m is connected. $\square$

In particular, M_m has neither loops nor coloops. Self-duality is an isomorphism, not an assertion that each basis has a basis as its literal complement.

3 Exact Tutte evaluations

3.1 Counting bases

In a basis, each parallel pair contributes at most one element. By (7), every series pair must contribute at least one. Suppose exactly k parallel pairs contribute one element each. The basis must contain both elements from precisely $m - k$ series pairs, and one element from each of the remaining k series pairs. There are

$$\binom{m}{k} 2^k \binom{m}{m-k} 2^k = \binom{m}{k}^2 4^k$$

choices. Summing proves (3); equivalently,

$$b_m = [z^m](1+2z)^m(2+z)^m = [z^m](2+5z+2z^2)^m. \quad (9)$$

3.2 The two endpoints

Set

$$F_m(z) = (1+z)^m(3+z)^m = \sum_{j=0}^{2m} f_{m,j} z^j.$$

At $(2,0)$, the rank expansion (1) is the signed sum $\sum_{A \subseteq E} (-1)^{|A|-r(A)}$. Using $d(A)$ as in Lemma 1, we have

$$|A| - r(A) = d(A) + \max\{0, u(A) + v(A) - m\}.$$

For each fixed nonempty parallel pair, the two singleton choices contribute weight 2 and the full-pair choice weight -1 , for a total weight 1. For each series pair not selected in full, the empty choice and the two singleton choices contribute total weight 3. Grouping by $j = u(A) + v(A)$ gives

$$a_m = \sum_{j \leq m} f_{m,j} + \sum_{j > m} (-1)^{j-m} f_{m,j}. \quad (10)$$

Since $F_m(-1) = 0$, this also equals

$$\begin{aligned} a_m &= 2 \sum_{\substack{0 \leq j < m \\ j \equiv m-1 \pmod{2}}} f_{m,j} = 2[z^{m-1}] \frac{F_m(z)}{1-z^2} \\ &= 2[z^{m-1}] \frac{(1+z)^{m-1}(3+z)^m}{1-z} = 2 \sum_{j=0}^{m-1} [z^j] (1+z)^{m-1} (3+z)^m. \end{aligned} \quad (11)$$

This is (4). Tutte duality and Lemma 1 give $T_{M_m}(0, 2) = a_m$. For an additional independent check, direct signed-rank enumeration yields

$$T_{M_m}(0, 2) = \sum_{j=0}^{2m} (-1)^{m-\min(m,j)} [z^j] (1+4z+3z^2)^m. \quad (12)$$

3.3 A 64-element counterexample

For $m = 16$, the integer evaluations from (3) and (4) are

$$\begin{aligned} b_{16} &= \sum_{k=0}^{16} \binom{16}{k}^2 4^k = 275\,621\,102\,802\,945, \\ a_{16} &= 2 \sum_{j=0}^{15} [z^j] (1+z)^{15} (3+z)^{16} = 263\,656\,235\,940\,864. \end{aligned}$$

The exact positive gap is

$$b_{16} - a_{16} = 11\,964\,866\,862\,081 > 0. \quad (13)$$

Since the two endpoints coincide, putting $M = M_{16}$ gives

$$\begin{aligned} \max\{T_M(2, 0), T_M(0, 2)\} &< T_M(1, 1), \\ T_M(2, 0) + T_M(0, 2) &< 2T_M(1, 1), \\ T_M(2, 0)T_M(0, 2) &< T_M(1, 1)^2. \end{aligned}$$

The normalized maximum ratio is approximately 0.95659; the multiplicative ratio is approximately 0.91506. Within this one-parameter family, exact evaluation shows that $m = 16$ is the first counterexample. This is not a global minimality assertion.

4 Infinite families and quantitative failure

By (11), all terms are nonnegative and

$$0 < a_m < 2(1+1)^{m-1}(3+1)^m = 8^m.$$

Applying Cauchy–Schwarz to $c_k = \binom{m}{k} 2^k$, we get

$$b_m = \sum_{k=0}^m c_k^2 \geq \frac{(\sum_{k=0}^m c_k)^2}{m+1} = \frac{9^m}{m+1}.$$

Consequently,

$$\frac{\max\{T_{M_m}(2, 0), T_{M_m}(0, 2)\}}{T_{M_m}(1, 1)} = \frac{a_m}{b_m} \leq (m+1) \left(\frac{8}{9}\right)^m \xrightarrow{m \rightarrow \infty} 0. \quad (14)$$

In particular, the bound itself certifies a counterexample for every $m \geq 29$: one checks $9^{29} > 30 \cdot 8^{29}$, and the quantity $9^m / ((m+1)8^m)$ increases for $m \geq 8$. The separate exact calculation above establishes the smaller counterexample $m = 16$.

5 The earlier one-sided conjecture

Merino, Ibañez, and Rodríguez [2, Conjecture 4.2] conjectured that a *cosimple* matroid with two disjoint bases satisfies $T_M(0, 2) \geq T_M(1, 1)$; dually, a *simple* matroid whose ground set is the union of two bases should satisfy $T_M(2, 0) \geq T_M(1, 1)$. Our M_m has both parallel and series pairs, so we construct a separate example for these stronger simplicity hypotheses.

Proposition 1. *Let $C_3 = U_{3,2}$ and define*

$$L = \text{Tr}_{30}(C_3^{\oplus 20}),$$

where Tr_{30} means truncation to rank 30. Then L is a 60-element simple and cosimple matroid with its ground set partitioned into two bases, and

$$T_L(2, 0) = 3\,612\,141\,626\,425\,344 < 4\,143\,252\,234\,043\,404 = T_L(1, 1). \quad (15)$$

Thus L disproves the simple-cover formulation and L^* disproves the cosimple-packing formulation of [2, Conjecture 4.2].

Proof. Write $E = \bigcup_{i=1}^{20} E_i$, where each E_i is a three-element circuit of the direct sum. The rank function is

$$r_L(A) = \min \left\{ 30, \sum_{i=1}^{20} \min\{|A \cap E_i|, 2\} \right\}.$$

Choose two elements from each of the first ten blocks and one element from each remaining block. This is an independent 30-element set; its complement has the same property. Hence the two sets are complementary bases.

Every two-element set is independent, so L is simple. Removing any two elements reduces the rank of $C_3^{\oplus 20}$ from 40 to at least 38, and the truncation still has rank 30. Therefore L has no cocircuit of size one or two and is cosimple.

The independent-set enumerator of a single triangle is $1 + 3z + 3z^2$, so

$$T_L(1, 1) = [z^{30}](1 + 3z + 3z^2)^{20} = 4\,143\,252\,234\,043\,404. \quad (16)$$

Set $q_j = [z^j](1 + 3z + 3z^2)^{20}$. For a triangle, the rank-grouped signed-nullity weights are 1, $3z$, and $3z^2$; the final coefficient is $3 - 1$, from the three independent pairs and the dependent triple. The truncation rank formula gives

$$T_L(2, 0) = \sum_{j \leq 30} q_j + \sum_{j > 30} (-1)^{j-30} q_j = 2 \sum_{\substack{0 \leq j < 30 \\ j \text{ odd}}} q_j.$$

The final identity uses $(1 - 3 + 3)^{20} = 0$. It yields the exact value in (15). The weaker bound suffices:

$$T_L(2, 0) \leq \sum_{j=0}^{40} q_j = 6^{20} = 3\,656\,158\,440\,062\,976 < T_L(1, 1).$$

□

It is important not to confuse the two results: L itself is *not* a maximum-form counterexample, since

$$T_L(0, 2) = 243\,284\,737\,468\,137\,472 > T_L(1, 1).$$

The 64-element M_{16} is the maximum-form counterexample.

The one-sided examples can also be made into an infinite family. For $t \geq 1$, put $L_t = \text{Tr}_{30t}(C_3^{\oplus 20t})$. Grouping blocks into t groups of twenty gives $T_{L_t}(1, 1) \geq T_L(1, 1)^t$, while the same coefficient argument gives $T_{L_t}(2, 0) \leq 6^{20t}$. Thus

$$\frac{T_{L_t}(2, 0)}{T_{L_t}(1, 1)} \leq \left(\frac{6^{20}}{4\,143\,252\,234\,043\,404} \right)^t \rightarrow 0.$$

Each L_t remains simple and cosimple and admits complementary bases.

6 Scope and verification

The main family is not graphic. Indeed, deleting one member of each parallel pair and contracting one member of each series pair recovers $U_{2m,m}$. For $m \geq 2$, the latter has a $U_{4,2}$ minor. Therefore M_m is not binary, and the construction does not address the graphic Merino–Welsh conjecture.

The accompanying standalone Python script uses only the standard library and checks all numerical claims by exact integer arithmetic. For $m = 16$, it computes the two endpoints and the number of bases by an independent rank-grouped sum over the six possible states of each parallel/series pair, as well as by (3)–(4). It verifies the $m = 1, 2, 3$ cases using matrix rank over $\mathbb{F}_{101}$, checks the complementary bases for $m = 16$, and independently enumerates rank/size states for the 60-element truncation. These checks are not substitutes for independent review of the arguments given above.

References

- [1] L. E. Chávez-Lomelí, C. Merino, S. D. Noble, and M. Ramírez-Ibañez, *Some inequalities for the Tutte polynomial*, European Journal of Combinatorics **32** (2011), 422–433. doi:10.1016/j.ejc.2010.11.005. Conjecture 5.8 in arXiv:1004.2639.
- [2] C. Merino, M. Ibañez, and M. G. Rodríguez, *A note on some inequalities for the Tutte polynomial of a matroid*, Electronic Notes in Discrete Mathematics **34** (2009), 603–607. doi:10.1016/j.endm.2009.07.102. Conjecture 4.2.
- [3] C. Beke, G. K. Csáji, P. Csikvári, and S. Pituk, *The Merino–Welsh conjecture is false for matroids*, Advances in Mathematics **446** (2024), article 109674. doi:10.1016/j.aim.2024.109674.