

Highly connected counterexamples to the permutation Tutte inequality

21 October 2026

Abstract

We disprove Conjecture 7.3 of Csikvári, *Around the Merino–Welsh conjecture: improving Jackson’s inequality*, which asserts that $\tilde{T}_H(2, 0)\tilde{T}_H(0, 2) \geq 1$ for every bipartite graph H of minimum degree at least two. For each integer $k \geq 2$, we construct an explicit sequence of finite simple balanced bipartite graphs whose minimum degree and vertex connectivity are both exactly k . The graphs admit automorphisms interchanging their two vertex classes, and both $\tilde{T}_H(2, 0)$ and $\tilde{T}_H(0, 2)$ tend to zero along the sequence. Consequently, no fixed minimum-degree or connectivity threshold ensures the inequality, and even the corresponding maximum and additive inequalities fail in these classes. The construction consists of a complete bipartite core with additional vertices of degree k attached to disjoint k -element subsets on each side. A one-variable integral bound gives explicit exponential decay for each fixed k and a sufficient size threshold. We also give a 1000-vertex, 2-connected example whose two axis values are each less than $7/10$, certified by exact rational arithmetic. These are statements about the permutation Tutte polynomial; a failure of a local permutation inequality alone does not give a counterexample to a graph or matroid Merino–Welsh inequality.

2020 Mathematics Subject Classification. Primary 05C31; Secondary 05A20, 05C07, 05C40.

Keywords. Permutation Tutte polynomial; bipartite graph; minimum degree; vertex connectivity; combinatorial inequality.

1 Introduction

Let $H = (L, R; E)$ be a finite bipartite graph with a specified bipartition. For a bijection $\pi : V(H) \rightarrow \{1, \dots, |V(H)|\}$, a vertex is *active* if its label is greater than the labels of all its neighbors. Let $a_L(\pi)$ and $a_R(\pi)$ count the active vertices in the two parts. Beke, Csáji, Csikvári, and Pituk [1] defined the permutation Tutte polynomial by

$$\tilde{T}_H(x, y) = \frac{1}{|V(H)|!} \sum_{\pi} x^{a_L(\pi)} y^{a_R(\pi)}. \quad (1)$$

We use $0^0 = 1$. In particular, $\tilde{T}_H(1, 1) = 1$.

Csikvári [2, Conjecture 7.3 in arXiv v2] conjectured that

$$\tilde{T}_H(2, 0)\tilde{T}_H(0, 2) \geq 1 \quad \text{whenever } \delta(H) \geq 2. \quad (2)$$

The same conjecture appeared as Conjecture 7.1 in the first arXiv version of that paper. The version distinction matters: Conjecture 7.1 in v2 concerns a different, endpoint inequality for matroids.

We give direct counterexamples to (2), with a stronger failure under prescribed minimum degree and vertex connectivity.

Theorem 1.1. *For every integer $k \geq 2$, there is an explicit sequence $(H_{k,n})_{n \geq 1}$ of finite simple balanced bipartite graphs such that*

$$\delta(H_{k,n}) = \kappa(H_{k,n}) = k, \quad \tilde{T}_{H_{k,n}}(2, 0) = \tilde{T}_{H_{k,n}}(0, 2) \longrightarrow 0.$$

The graphs have automorphisms interchanging their bipartition classes, and are pairwise nonisomorphic as n varies. Every

$$n \geq 4(k+1)^2(2k^2 + k + 4)$$

gives a strict counterexample to (2).

Here κ denotes vertex connectivity. By the usual inequalities $\kappa \leq \lambda \leq \delta$, the edge connectivity of these graphs is also exactly k .

The proof conditions on the largest label in one side of a complete bipartite core. The resulting upper bound is a one-variable integral whose exponential factor is uniformly smaller than one. This argument also supplies a finite example, without any claim of minimality.

Proposition 1.2. *There is a simple balanced bipartite graph H with 1000 vertices, 160400 edges, and $\delta(H) = \kappa(H) = 2$, for which*

$$\tilde{T}_H(2, 0) = \tilde{T}_H(0, 2) < \frac{7}{10}, \quad \tilde{T}_H(2, 0)\tilde{T}_H(0, 2) < \frac{49}{100}.$$

The general existence theorem is proved by inequalities valid for all k ; it does not rely on a search or numerical integration. The rational calculation in Section 5 gives a smaller concrete witness.

2 The graph construction

Definition 2.1. For integers $k \geq 2$, $q \geq 0$, and $n \geq 1$, put $m = (k+q)n$. Start with a complete bipartite core $K_{m,m}$, with parts A and B . Choose pairwise disjoint sets $A_1, \dots, A_n \subseteq A$ and $B_1, \dots, B_n \subseteq B$, each of size k . For each i , add a vertex α_i adjacent precisely to A_i , and a vertex β_i adjacent precisely to B_i . There are no other vertices or edges. Denote the resulting graph by $H(k, q, n)$. Its bipartition is

$$L = A \dot{\cup} \{\beta_1, \dots, \beta_n\}, \quad R = B \dot{\cup} \{\alpha_1, \dots, \alpha_n\}.$$

The qn unused core vertices on each side are called bare core vertices.

Lemma 2.2. *The graph $H = H(k, q, n)$ is simple and balanced, with*

$$|V(H)| = 2(k+q+1)n, \quad |E(H)| = (k+q)^2n^2 + 2kn, \quad \delta(H) = \kappa(H) = k.$$

There is an automorphism interchanging L and R . Consequently $\tilde{T}_H(x, y) = \tilde{T}_H(y, x)$.

Proof. The new vertices have degree k . The core vertices in the specified blocks have degree $m+1$, and any bare core vertices have degree m . Since $m \geq k$, the minimum degree is k . The order and size follow directly from the construction.

Delete any set of fewer than k vertices. At least one vertex remains on each side of the core, so the remaining core is connected. Every remaining new vertex retains a core neighbor. The whole remaining graph is therefore connected, proving $\kappa(H) \geq k$. Conversely, deleting the k neighbors of a new vertex isolates that vertex, with other vertices remaining, so $\kappa(H) \leq k$.

Match A_i with B_i for each i , match the two bare core sets, and interchange α_i and β_i . These correspondences give an automorphism swapping the two parts. Applying it to the labelings in (1) proves the polynomial symmetry. $\square$

Write

$$X_{k,q,n} = \tilde{T}_{H(k,q,n)}(2, 0) = \tilde{T}_{H(k,q,n)}(0, 2). \quad (3)$$

3 A one-variable integral bound

Assign independent uniform labels in $[0, 1]$ to every vertex. Almost surely these labels are distinct, and all strict total orders are equally likely. Thus (1) is equivalently

$$\tilde{T}_H(x, y) = \mathbb{E}[x^{a_L} y^{a_R}], \quad \tilde{T}_H(2, 0) = \mathbb{E}[2^{a_L} \mathbf{1}_{\{a_R=0\}}].$$

For fixed k, q , define on $[0, 1]$

$$F_k(t) = \frac{(2 + kt)(2 - t)^k - 2}{k + 1}, \quad (4)$$

$$U(t) = (2 - t)^q F_k(t), \quad (5)$$

$$V(t) = t^{k+q} \left(2 - \frac{kt}{k+1} \right), \quad (6)$$

$$Q(t) = U(t)V(t). \quad (7)$$

Lemma 3.1. *For every $k \geq 2$, $q \geq 0$, and $n \geq 1$,*

$$0 < X_{k,q,n} \leq I_{k,q,n} := \int_0^1 U(t)^n d(V(t)^n) = n \int_0^1 Q(t)^{n-1} U(t) V'(t) dt. \quad (8)$$

In particular, $I_{k,q,n}$ is an explicitly computable rational number.

Proof. In the expectation defining $X_{k,q,n}$, retain the condition that all α_i are inactive, and omit the condition that all vertices of B are inactive. Omitting this condition can only increase the nonnegative expectation. Condition on the labels of B , and let t be their maximum.

A bare vertex of A has activity weight 2 when its label exceeds t , and weight 1 otherwise. Its integrated weight is $2 - t$.

Consider a block A_i and its neighbor α_i , whose label is z . The inactivity of α_i requires that some label in A_i exceed z . A vertex in A_i is active precisely when its label exceeds both t and z . If $z \leq t$, the integral over the k labels in A_i is $(2 - t)^k - z^k$; if $z > t$, it is $(2 - z)^k - z^k$. Integration over z therefore gives

$$\begin{aligned} \int_0^t ((2 - t)^k - z^k) dz + \int_t^1 ((2 - z)^k - z^k) dz \\ = \frac{(2 + kt)(2 - t)^k - 2}{k + 1} = F_k(t). \end{aligned}$$

The disjoint blocks and bare vertices of A hence contribute $U(t)^n$.

For a block B_i with maximum label s , the expected activity weight of β_i is $s + 2(1 - s) = 2 - s$. Its integrated weight over labels in $[0, t]^k$ is

$$\int_0^t k s^{k-1} (2 - s) ds = t^k \left(2 - \frac{kt}{k+1} \right).$$

Thus the weighted measure of all labels of B being at most t is

$$t^{qn} \left[t^k \left(2 - \frac{kt}{k+1} \right) \right]^n = V(t)^n.$$

Integrating $U(t)^n$ against this weighted maximum-label distribution proves the first upper bound. Notice that this is a finite positive measure, not necessarily a probability measure; no normalization is omitted. Differentiating V^n gives the last expression in (8). Positivity of $X_{k,q,n}$ follows, for example, from the positive-probability event on which every label in L exceeds every label in R . Finally, the integrand is a polynomial with rational coefficients. $\square$

Lemma 3.2. *Let $\rho = \max_{0 \leq t \leq 1} Q(t)$. Then*

$$X_{k,q,n} \leq nC_{k,q}\rho^{n-1}, \quad C_{k,q} = 2^q \frac{(2^{k+1} - 2)(k+2)}{(k+1)^2} < 2^{q+k+2}. \quad (9)$$

Proof. We have $F'_k(t) = -kt(2-t)^{k-1} \leq 0$, so U is nonnegative and nonincreasing. Also $V(0) = 0$ and $V' \geq 0$, as is clear either from its integral interpretation or from (6). Consequently

$$\int_0^1 U(t)V'(t) dt \leq U(0)V(1) = 2^q \frac{(2^{k+1} - 2)(k+2)}{(k+1)^2}.$$

Combine this with (8). The final rough bound follows from $2^{k+1} - 2 < 2^{k+1}$ and $(k+2)/(k+1)^2 < 2$. $\square$

4 Uniform decay for every prescribed minimum degree

Lemma 4.1. *Fix $k \geq 2$ and take $q = 2k^2 + 1$. For every $t \in [0, 1]$,*

$$Q(t) \leq \exp\left(-\frac{1}{2(k+1)^2}\right) < 1. \quad (10)$$

Proof. Put $s = 1 - t$ and $D = (k+1)^2$. Directly from (4)–(7),

$$Q(t) = \frac{(1-s^2)^q}{D} \left([(k+2)^2 - k^2s^2](1-s^2)^k - 2(k+2+ks)(1-s)^k \right).$$

The coefficient $(k+2)^2 - k^2s^2$ is positive. Apply $(1-s^2)^k \leq 1$ and Bernoulli's inequality $(1-s)^k \geq 1 - ks$ to obtain

$$Q(t) \leq (1-s^2)^q \left(1 - \frac{1}{D} + \frac{2k}{k+1}s + \frac{k^2}{D}s^2 \right). \quad (11)$$

For $0 < t \leq 1$, both $Q(t)$ and the parenthesis are positive. Using $\log(1-s^2) \leq -s^2$ and $\log z \leq z - 1$ yields

$$\log Q(t) \leq -\frac{1}{D} + \frac{2k}{k+1}s - \left(q - \frac{k^2}{D} \right) s^2.$$

Since $q - k^2/D > 2k^2$, completing the square gives

$$\log Q(t) \leq -\frac{1}{D} + \frac{k^2}{D(q - k^2/D)} < -\frac{1}{2D}.$$

At $t = 0$ we have $Q(0) = 0$, so (10) holds on the closed interval. $\square$

Proof of Theorem 1.1. Take $q = 2k^2 + 1$ and $H_{k,n} = H(k, q, n)$. Lemma 2.2 proves all structural assertions. Lemmas 3.2 and 4.1 give

$$0 < X_{k,q,n} \leq nC_{k,q} \exp\left(-\frac{n-1}{2(k+1)^2}\right) \rightarrow 0. \quad (12)$$

The vertex count is strictly increasing in n , proving that the graphs are pairwise nonisomorphic.

For the stated sufficient threshold, set $D = (k+1)^2$, $L = q + k + 3$, and $n_0 = 4DL$. Then $D \leq L$ and $L \geq 14$. The elementary inequality $4L^2 \leq 2^L$ holds for integers $L \geq 8$: it is an equality at $L = 8$, and the ratio $2^L/(4L^2)$ increases thereafter. Hence

$$n_0 \leq 4L^2 \leq 2^L, \quad C_{k,q} < 2^{L-1}.$$

As $(n_0 - 1)/(2D) = 2L - 1/(2D) > 2L - 1$, the upper bound in (12) at $n = n_0$ is strictly smaller than

$$2^{2L-1}e^{-(2L-1)} < 1.$$

The function $ne^{-(n-1)/(2D)}$ is nonincreasing for $n \geq 2D$. Thus every integer $n \geq n_0$ has $X_{k,q,n} < 1$, as required. $\square$

Corollary 4.2. *For every fixed integer $k \geq 2$,*

$$\inf_H \tilde{T}_H(2, 0) \tilde{T}_H(0, 2) = 0,$$

where the infimum is over finite simple balanced bipartite graphs with $\delta(H) = \kappa(H) = k$. The same infimum is zero for $\max\{\tilde{T}_H(2, 0), \tilde{T}_H(0, 2)\}$ and for $(\tilde{T}_H(2, 0) + \tilde{T}_H(0, 2))/2$. These statements remain true when the graphs are required to have an automorphism interchanging their parts.

Proof. All three quantities are nonnegative, and the sequence in Theorem 1.1 makes each tend to zero. $\square$

5 A finite exact certificate

Take $H = H(2, 2, 100)$. Its core is $K_{400,400}$; on each side, partition 200 core vertices into 100 pairs and attach one vertex on the opposite side to each pair. The remaining 200 core vertices on each side are bare. Lemma 2.2 gives

$$|V(H)| = 1000, \quad |E(H)| = 160400, \quad \delta(H) = \kappa(H) = 2.$$

The choice $q = 2$ is used only for this finite witness; the uniform choice in Theorem 1.1 is $q = 2k^2 + 1$.

Here $U = u/3$ and $V = v/3$, where

$$\begin{aligned} u(t) &= 24 - 24t - 18t^2 + 32t^3 - 14t^4 + 2t^5, \\ v(t) &= 6t^4 - 2t^5. \end{aligned}$$

Therefore the rational upper bound in (8) is

$$I_{2,2,100} = \frac{100}{9^{100}} \int_0^1 (u(t)v(t))^{99} u(t)v'(t) dt < \frac{7}{10}. \quad (13)$$

We describe the exact finite arithmetic certifying this strict comparison. Write

$$(uv)^{99} uv' = \sum_{j=0}^{999} c_j t^j, \quad \Lambda = \text{lcm}(1, 2, \dots, 1000).$$

All coefficients c_j are integers. Set

$$N = 100 \sum_{j=0}^{999} c_j \frac{\Lambda}{j+1}, \quad D_0 = 9^{100} \Lambda. \quad (14)$$

Then $I_{2,2,100} = N/D_0$, and the calculation gives $N > 0$ and $7D_0 - 10N > 0$. To make the arithmetic reproducible without special software, let $a = (24, -24, -18, 32, -14, 2)$ and $b = (0, 0, 0, 0, 6, -2)$. Compute polynomial products by

$$(f * g)_j = \sum_{i=0}^j f_i g_{j-i},$$

starting with the coefficient array (1), multiplying by $a * b$ a total of 99 times, and finally multiplying by a and by $(b_1, 2b_2, \dots, 5b_5)$. Formula (14) then involves only integer arithmetic. The complete positive integer $7D_0 - 10N$ is recorded in Appendix A.

The resulting rational number has reduced numerator and denominator of 519 digits each. For orientation only,

$$I_{2,2,100} = 0.67702186967725368440568057708376844 \dots$$

This decimal is the *upper bound*, not an asserted exact value of $\tilde{T}_H(2, 0)$. A second calculation, expanding u^{100} and differentiating the binomial expansion $v^{100} = 2^{100}t^{400}(3-t)^{100}$, gives the same reduced rational number. Both comparisons with $7/10$ use exact arithmetic.

By (8) and symmetry, both actual axis values are less than $7/10$, proving Proposition 1.2. Exhaustive permutation enumeration on four small graphs from the construction (including its $k = 1$ instance P_4) was also used as a consistency check of the activity convention and the integral upper bound. Neither that check nor the finite calculation is needed for the general existence proof.

6 Scope and provenance

The theorem disproves Csikvári’s minimum-degree-two permutation Tutte conjecture and rules out any repair based solely on imposing a fixed larger minimum degree or vertex connectivity. The maximum and additive failures in Corollary 4.2 are consequences of the same two axis values tending to zero; we do not attribute these variants as separately numbered conjectures in [2].

The graph has core degrees growing with n and additional vertices of fixed degree k . This distinction is compatible with positive results that impose an upper degree bound as well as a lower one, such as [2, Lemma 6.1].

The permutation Tutte polynomial is different from the ordinary Tutte polynomial. The local-to-global transfer principle of [1] gives a sufficient condition, applied to every local basis exchange graph, for a matroid inequality. Failure of that local sufficient condition does not reverse the implication. Thus the present graph construction alone does not disprove the graphical or matroid Merino–Welsh inequalities.

AI assistance and verification. This manuscript was developed through iterative interactions with OpenAI’s ChatGPT under the submitter’s direction. AI assistance contributed to the construction, proof development, literature checks, exact-arithmetic verification, and writing. During preparation of this version, the explicit construction and proof were reconstructed and checked afresh from the available research context. The finite certificate was recomputed with two exact-arithmetic implementations. These checks do not constitute independent human verification, and this statement does not assert that every argument has been independently checked and understood by a human. No formal proof-assistant verification or peer review is claimed.

A The finite integer gap

For N, D_0 as defined in (14), the positive integer $7D_0 - 10N$ is the following decimal integer, with line breaks inserted for readability:

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4350969505057189536482398692866289335462873580972577588807126752109988
9348225219944922442710889169163352011412078616241284105368960944637519
7242409792971467767621007319274205288085888668095720443738761486980092
4368438343186993017750597292518186243613934489522989297368486580624150
0510394461875890392532064768268950304706850172766548316320495415236467
3159351153309323137288352934369711412726030838128522754176012154400029
5010575297025858135495637341370608125136763132389859755509124589056029
85401258881147058848296214608759232000
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