

Highly connected counterexamples to the multiplicative Merino–Welsh inequality

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Abstract

For every integer $k \geq 3$, we construct k -connected matroids representable over every infinite field, with girth k , arbitrarily large cogirth, and Merino–Welsh quotient $\Phi(M) = T_M(1, 1)^2 / (T_M(2, 0)T_M(0, 2))$ tending to infinity. In particular, this disproves *Conjecture 7.2 of Csikvári (2026)*, which asserts the multiplicative Merino–Welsh inequality for simple and cosimple matroids, and therefore also the stronger *Conjecture 7.3* therein, as observed by its author. Our examples are rank truncations of direct sums of k -element circuits. We give a 3-connected counterexample on 1011 elements of rank 433, certified by exact integer arithmetic, and a 6048-element counterexample admitting a proof without computation. Taking the direct sum of a counterexample with its dual also gives a negative answer to *Problem 4.3 of Beke, Csáji, Csikvári, and Pituk (2024)*: the original maximum and additive Merino–Welsh inequalities can fail for simple and cosimple matroids. The same construction disproves *Conjecture 2 of Knauer, Martínez-Sandoval, and Merino (2026)*: for every fixed $g \geq 3$, Φ is unbounded even among matroids having both girth and cogirth exactly g . The dual direct-sum examples need not be connected; these constructions do not disprove the graphical or binary versions.

1 Introduction

For a loopless and coloopless matroid M , write

$$\rho(M) = \frac{T_M(2, 0)T_M(0, 2)}{T_M(1, 1)^2}, \quad \Phi(M) = \frac{1}{\rho(M)}.$$

The multiplicative Merino–Welsh inequality asserts $\rho(M) \geq 1$. Beke, Csáji, Csikvári, and Pituk disproved the unrestricted matroid version using parallel thickenings of uniform matroids [1]. Their published Problem 4.3 asks about the simultaneous simple and cosimple restriction. Csikvári explicitly states the multiplicative version for this class as Conjecture 7.2 in [2].

We resolve three named questions in the negative. First, our highly connected examples disprove [2, Conjecture 7.2] directly, and thereby also disprove [2, Conjecture 7.3], which its author notes implies Conjecture 7.2. The latter is an indirect consequence: this manuscript does not construct an explicit bipartite graph violating the permutation Tutte inequality. Second, taking a direct sum with the dual also disproves the maximum and additive inequalities for simple and cosimple matroids, answering [1, Problem 4.3]. Third, the same dual direct sum has equal, fixed girth and cogirth while its Merino–Welsh quotient is unbounded, contradicting [3, Conjecture 2]. These latter direct sums are not asserted to remain connected.

Our construction replaces the two-element circuits underlying the uniform-thickening examples by larger circuits, then truncates the rank of their direct sum. Although a single k -element circuit has $2^k - 1$ independent sets, its $(2, 0)$ Tutte evaluation is $2^k - 2$. The identity

$$(2^k - 1)^2 = 2^k(2^k - 2) + 1$$

allows the exponential number of bases in a suitable rank layer to outgrow the product of the two axial evaluations.

Theorem 1.1. *For every integer $k \geq 3$, there is a sequence of k -connected matroids M_n , representable over every infinite field, such that*

$$g(M_n) = k, \quad g^*(M_n) \longrightarrow \infty, \quad \rho(M_n) \longrightarrow 0.$$

Consequently the multiplicative Merino–Welsh inequality fails even for simple and cosimple, 3-connected, rationally representable matroids.

All mathematical arguments needed for the counterexamples are given below. The numerical search only supplies a smaller concrete parameter choice; it is not used to prove the existence theorem. The literature references identify the questions being answered and do not serve as unproved inputs to the construction.

2 Definitions and exact evaluations

Let C_k denote the circuit matroid on k elements: its independent sets are exactly the proper subsets of its ground set, and its rank is $k - 1$. This convention avoids any ambiguity in the ordering of the two indices of a uniform matroid.

Fix pairwise disjoint sets $E_1, \dots, E_n$, each of size k , and put $E = \bigcup_i E_i$. Define

$$N_{k,n} = C_k^{\oplus n}, \quad s = (k - 1)n.$$

For $1 \leq r \leq s$, let $M_{k,n,r} = \text{Tr}_r(N_{k,n})$, where Tr_r means truncation to rank r . Explicitly,

$$\text{rk}_M(A) = \min \left\{ r, \sum_{i=1}^n \min\{|A \cap E_i|, k - 1\} \right\}. \quad (1)$$

Equivalently, the independent sets satisfy both $|A| \leq r$ and $|A \cap E_i| \leq k - 1$ for every i . These are the usual independent sets of a truncated partition matroid; in particular they define a matroid. One can also verify augmentation directly: if I, J obey the constraints and $|I| < |J|$, some block has $|I \cap E_i| < |J \cap E_i|$, so an element of $(J \setminus I) \cap E_i$ can be added to I .

We use the subset expansion

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{r - \text{rk}_M(A)} (y - 1)^{|A| - \text{rk}_M(A)}. \quad (2)$$

Introduce three polynomials for one block:

$$F_k(z) = (1 + z)^k - z^k, \quad (3)$$

$$W_k(z) = (1 + z)^k - z^k - z^{k-1}, \quad (4)$$

$$D_k(z) = (1 + z)^k - z^k + z^{k-1}. \quad (5)$$

Write $f_j = [z^j]F_k(z)^n$, $w_j = [z^j]W_k(z)^n$, and $d_j = [z^j]D_k(z)^n$. Each of these coefficient arrays is nonnegative.

Lemma 2.1. *For $M = M_{k,n,r}$,*

$$T_M(1, 1) = f_r, \quad (6)$$

$$T_M(2, 0) = 2 \sum_{\substack{0 \leq j < r \\ j \equiv r-1 \pmod{2}}} w_j, \quad (7)$$

$$T_M(0, 2) = 2^{kn} - 2 \sum_{\substack{0 \leq j < r \\ j \equiv r-1 \pmod{2}}} d_j. \quad (8)$$

In particular,

$$T_M(2, 0) \leq (2^k - 2)^n, \quad T_M(0, 2) \leq 2^{kn}. \quad (9)$$

Proof. The independent-set generating polynomial of one circuit is F_k . A basis of M is precisely an independent r -subset of $N_{k,n}$. This proves (6).

The signed rank generating polynomial of one block is W_k : for ranks below $k - 1$ its coefficients are $\binom{k}{j}$, whereas at rank $k - 1$ the k subsets of size $k - 1$ contribute positively and the full block contributes negatively. Thus

$$W_k(z)^n = \sum_{A \subseteq E} (-1)^{|A| - \text{rk}_N(A)} z^{\text{rk}_N(A)}.$$

Applying (1) in (2) gives

$$T_M(2, 0) = \sum_{j < r} w_j + \sum_{j \geq r} (-1)^{j-r} w_j.$$

Since

$$W_k(1) = 2^k - 2, \quad W_k(-1) = 0,$$

the second sum can be replaced by $-\sum_{j < r} (-1)^{j-r} w_j$, proving (7). The total sums of the even and odd coefficients of $W_k(z)^n$ both equal $(2^k - 2)^n/2$. Nonnegativity now gives the first bound in (9).

Similarly d_j counts all subsets of $N_{k,n}$ having rank j : the full block contributes one additional subset of rank $k - 1$, which explains D_k . Hence

$$T_M(0, 2) = \sum_{j < r} (-1)^{r-j} d_j + \sum_{j \geq r} d_j.$$

The total number of subsets is $\sum_j d_j = 2^{kn}$; subtracting twice the terms with negative sign proves (8) and the second bound. $\square$

3 Unbounded quotients and connectivity

For each n , choose r_n to be an index of a largest coefficient of $F_k(z)^n$, breaking ties arbitrarily. There are $s + 1$ coefficients, whose sum is $(2^k - 1)^n$. Therefore

$$T_{M_{k,n,r_n}}(1, 1) \geq \frac{(2^k - 1)^n}{s + 1}. \quad (10)$$

Lemma 2.1 implies

$$\rho(M_{k,n,r_n}) \leq (s + 1)^2 \left(\frac{2^k(2^k - 2)}{(2^k - 1)^2} \right)^n = (s + 1)^2 \left(1 - \frac{1}{(2^k - 1)^2} \right)^n. \quad (11)$$

For fixed k the last expression tends to zero.

We next check that this rank choice has all the required structural properties; no assumption about an approximate location of a mode is needed.

Lemma 3.1. *For fixed $k \geq 3$, the maximizing indices above satisfy $r_n \rightarrow \infty$ and $s - r_n \rightarrow \infty$.*

Proof. Fix a nonnegative integer L . For $j \leq L$,

$$[z^j]F_k(z)^n \leq \binom{kn}{j} \leq (kn)^L.$$

There are k^n bases of $N_{k,n}$. Every independent set of size $s - d$, where $d \leq L$, extends to one of these bases. Consequently

$$[z^{s-d}]F_k(z)^n \leq k^n \binom{s}{d} \leq k^n s^L.$$

Both bounds are eventually smaller than $(2^k - 1)^n / (s + 1)$, because $2^k - 1 > k$. A largest coefficient cannot occur at either end within distance L . As L was arbitrary, the conclusion follows. $\square$

We use the standard connectivity function $\lambda_M(A) = \text{rk}_M(A) + \text{rk}_M(E \setminus A) - r$. A j -separation is a partition with both parts of size at least j and $\lambda_M(A) \leq j - 1$. A matroid is k -connected if it has no j -separation for $1 \leq j < k$.

Lemma 3.2. *If $r \geq k - 1$ and $D := s - r \geq k - 1$, then $M_{k,n,r}$ is k -connected, has girth k , and has cogirth*

$$g^*(M) = kn - (r - 1) - \left\lfloor \frac{r - 1}{k - 1} \right\rfloor = D + 1 + \left\lceil \frac{D + 1}{k - 1} \right\rceil. \quad (12)$$

Proof. Every set of fewer than k elements is independent in $N_{k,n}$ and remains independent after truncation. Each full block E_i remains a circuit. Thus $g(M) = k$.

Fix $1 \leq j < k$ and a partition $E = A \sqcup B$ with $|A|, |B| \geq j$. Put $a = \text{rk}_N(A)$ and $b = \text{rk}_N(B)$. Since the girth of N is k , we have $a, b \geq j$. Also $a + b \geq s$. If $a, b < r$, then

$$\lambda_M(A) = a + b - r \geq s - r = D \geq k - 1 \geq j.$$

If, say, $a \geq r$, then $\lambda_M(A) = \min(b, r) \geq j$. This excludes every such separation, proving k -connectivity.

For the cogirth formula, consider a subset H with $\text{rk}_M(H) \leq r - 1$. Its rank in N is then at most $r - 1$. A block contributes at most its rank plus one to $|H|$, and the extra element is available only if the block is full, which costs $k - 1$ units of rank. Therefore

$$|H| \leq (r - 1) + \left\lfloor \frac{r - 1}{k - 1} \right\rfloor.$$

This bound is attained by taking as many full blocks as possible at rank budget $r - 1$, followed by the required proper subset of one further block. The resulting set is a hyperplane of the truncation. The complements of hyperplanes are exactly the cocircuits, giving the first formula in (12). The second follows from $s = (k - 1)n$ and $D = s - r$. $\square$

Completion of the proof of Theorem 1.1. For sufficiently large n , Lemma 3.1 ensures the hypotheses of Lemma 3.2. Formula (12) also shows $g^*(M_n) \rightarrow \infty$, and (11) shows $\rho(M_n) \rightarrow 0$.

Let $\mathbb{F}$ be any infinite field. Each C_k has a representation using a basis $e_1, \dots, e_{k-1}$ together with $e_1 + \dots + e_{k-1}$. The direct sum gives a representation of N . A sufficiently generic linear map $\mathbb{F}^s \rightarrow \mathbb{F}^r$ preserves independence of every originally independent set of size at most r . Such a map exists: for each of finitely many independent sets, some minor of its projected columns is a nonzero polynomial in the entries of the map; the product of these polynomials is nonzero and cannot vanish everywhere over an infinite field. For sets of larger rank, an independent r -subset survives, while no rank can exceed r . Thus the projected rank is exactly (1). This proves representability over $\mathbb{F}$. $\square$

Remark 3.3. These examples are also laminar matroids: their independent sets are specified by capacities on the disjoint blocks E_i and on their union E , a laminar family. This assertion follows directly from the displayed independent-set description.

4 A concrete counterexample and exact verification

Take $k = 3$, $n = 337$, and $r = 433$:

$$M = \text{Tr}_{433}(C_3^{\oplus 337}).$$

The ground set has 1011 elements, the rank is 433, and the corank is 578. Since $s - r = 674 - 433 = 241 \geq 2$, the matroid is 3-connected. Its girth is 3 and (12) gives $g^*(M) = 363$. In particular, it is simple and cosimple.

For $k = 3$, the three one-block polynomials are

$$F_3(z) = 1 + 3z + 3z^2, \quad W_3(z) = 1 + 3z + 2z^2, \quad D_3(z) = 1 + 3z + 4z^2.$$

Applying Lemma 2.1 with exact integer arithmetic gives

$$\rho(M) = 0.987478184929103952874888526897 \dots < 1.$$

A shorter certificate uses only $b = [z^{433}](1 + 3z + 3z^2)^{337}$:

$$b^2 > 48^{337} \geq T_M(2, 0)T_M(0, 2). \quad (13)$$

The following self-contained Python code verifies the strict integer comparison in (13).

```
c = [1]
for _ in range(337):
    d = [0] * (len(c) + 2)
    for j, value in enumerate(c):
        d[j] += value
        d[j + 1] += 3 * value
        d[j + 2] += 3 * value
    c = d
assert c[433] ** 2 > 48 ** 337
```

A separate verification script `verify_simple_cosimple_mw.py` computes all three Tutte evaluations and writes their full integer values to `certificate_n337_r433.json`. The source archive for this manuscript is self-contained and does not require the script. All inequality decisions use integers. Decimal ratios are displayed only for readability.

Two verification routes were used. First, independent coefficient implementations agreed on the 1011-element example. Second, for $1 \leq n \leq 4$ and every $1 \leq r \leq 2n$ (20 parameter pairs), direct enumeration of every subset using (1) agreed with all three coefficient formulas. An optional exhaustive search within this family first finds a failure at $n = 337$, at ranks 433, 434, and 435, after checking 113232 parameter pairs. This is a family-specific search result, not a claim of minimality among all simple and cosimple matroids.

5 Consequences and scope

The permutation Tutte formulation

Csikvári [2, Conjecture 7.3] conjectures a multiplicative permutation Tutte inequality for bipartite graphs of minimum degree at least two, and observes that it implies his Conjecture 7.2. Our counterexample to Conjecture 7.2 therefore also implies that Conjecture 7.3 is false. This is a logical consequence, rather than an explicit bipartite-graph construction.

The maximum and additive versions

Let M be any one of the simple and cosimple multiplicative counterexamples, and put $Q = M \oplus M^*$. Direct sum and duality give

$$T_Q(2, 0) = T_Q(0, 2) = T_M(2, 0)T_M(0, 2) < T_M(1, 1)^2 = T_Q(1, 1).$$

Thus Q is simple and cosimple and violates both the maximum and additive versions as well. This answers the simple and cosimple question in [1, Problem 4.3], alongside the explicit multiplicative formulation in [2, Conjecture 7.2]. The direct sum Q is disconnected; we do not claim these latter two versions fail under arbitrarily high connectivity.

Fixed girth and cogirth

Knauer, Martínez-Sandoval, and Merino [3, Conjecture 2] ask for a finite upper bound on $\Phi(M)$ depending only on a common girth and cogirth parameter $g \geq 3$. Fix $k \geq 3$ and take M_n from Theorem 1.1. Set $Q_n = M_n \oplus M_n^*$. For all sufficiently large n ,

$$g(Q_n) = g^*(Q_n) = k, \quad \Phi(Q_n) = \Phi(M_n)^2 \longrightarrow \infty.$$

This gives a negative answer even when both girth and cogirth are required to be *exactly* the prescribed value.

Relation to graphical and binary matroids

The original direct sum $N_{k,n}$ is graphic, but its truncation need not be. In the 1011-element example, keep one element from each of four blocks. In the other 333 blocks choose an independent set C of size 431, contract C , and delete every element except the four kept elements. The resulting rank function is

$$\text{rk}(X) = \min(2, |X|),$$

so the minor is the four-element rank-two uniform matroid $U_{2,4}$. It is not binary: a two-dimensional binary vector space has only three nonzero vectors. Thus this example is neither binary nor graphic, and it does not settle those separate restrictions.

Appendix: a finite certificate without computation

There is also a completely explicit parameter choice whose failure can be checked by hand:

$$k = 3, \quad n = 2016, \quad r = 2592.$$

This gives 6048 elements. Consider only independent sets with exactly 288 empty blocks, 864 blocks of occupancy one, and 864 blocks of occupancy two. Their total size is 2592, and their number is

$$L = \frac{2016!}{288! 864! 864!} 3^{1728} \leq [z^{2592}](1 + 3z + 3z^2)^{2016}.$$

We claim

$$L \geq \frac{7^{2016}}{\binom{2018}{2}}. \tag{14}$$

Use the multinomial distribution with 2016 trials and probabilities $(1/7, 3/7, 3/7)$. The count vector $(288, 864, 864)$ is its integer mean and is a mode. To verify the latter assertion directly,

write this vector as $(a_i)_{i=0}^2$ and let $(m_i)_{i=0}^2$ be any other count vector with the same sum. The ratio of the probability at (m_i) to the probability at (a_i) is

$$\prod_{i=0}^2 \frac{a_i! a_i^{m_i - a_i}}{m_i!} \leq 1.$$

Each factor is at most one: for an increase it is a product of $a_i/(a_i + j)$, and for a decrease it is a product of $(a_i - j)/a_i$. There are $\binom{2018}{2}$ possible count vectors, so the mode has probability at least its reciprocal. Its probability is exactly $L/7^{2016}$, proving (14).

Finally, the binomial theorem gives

$$\left(\frac{49}{48}\right)^{48} = \left(1 + \frac{1}{48}\right)^{48} > 2, \quad \binom{2018}{2} = 2035153 < 2097152 = 2^{21}.$$

Since $2016 = 48 \cdot 42$, it follows that

$$\left(\frac{49}{48}\right)^{2016} > 2^{42} > \binom{2018}{2}^2.$$

Writing $b = [z^{2592}](1 + 3z + 3z^2)^{2016}$, we conclude

$$b^2 \geq \frac{49^{2016}}{\binom{2018}{2}^2} > 48^{2016} \geq T_M(2, 0)T_M(0, 2).$$

Its rank is at least two and its truncation deficit is $4032 - 2592 = 1440$, so it is 3-connected, simple, and cosimple by Lemma 3.2. This finite proof requires neither a numerical search nor a floating-point estimate.

References

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- [3] K. Knauer, L. Martínez-Sandoval, and C. Merino, *Merino–Welsh inequalities for matroids with controlled lattices of cyclic flats*, [arXiv:2609.38047](https://arxiv.org/abs/2609.38047), version 1, 29 September 2026. Conjecture 2.