

Total mutual visibility in Fibonacci and Lucas cubes

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Abstract

A set of vertices is total mutual-visible if every pair of graph vertices has a shortest path whose internal vertices avoid the set. We prove that the total mutual-visibility numbers of the n -dimensional Fibonacci and Lucas cubes both have order ρ^n/n , where $\rho > 1$ satisfies $\rho^3 = \rho^2 + 1$. The eligible Fibonacci words are exactly those avoiding 11 for which zero padding avoids 0000. Their induced graph decomposes into fibres of recognizing subsets of maximal independent sets of a path, and each fibre is a Cartesian product of smaller Fibonacci cubes. Total visibility becomes open packing in these fibres. An elementary coloring and counting argument proves the order of growth. Typical degree and free-dimension estimates, together with a classical binary-code asymptotic, sharpen the normalized bounds to constants $4.032058\dots$ and $6.346255\dots$. We give the cyclic analogue, an exact component inventory, and reproducible exact values through dimension 20 for Fibonacci cubes and 16 for Lucas cubes. Neither a leading asymptotic constant nor a closed formula in every dimension is established.

Keywords. Total mutual visibility; Fibonacci cube; Lucas cube; open packing; recognizing subset; constrained code.

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1 Introduction and main results

All graphs are finite and simple. A set $X \subseteq V(G)$ is *total mutual-visible* if, for every $u, v \in V(G)$, some shortest u - v path has no internal vertex in X . The endpoints may belong to X . Write $\mu_t(G)$ for the largest possible cardinality. This differs from ordinary mutual visibility, which requires the condition only for pairs in X .

The Fibonacci cube Γ_n has the binary words of length n avoiding 11 as vertices, with adjacency at Hamming distance one. Equivalently, it is the graph of independent sets of the coordinate path P_n , with one-vertex insertions and deletions as edges. We take Γ_0 to be a single vertex. For $n \geq 3$, the Lucas cube Λ_n is the analogous graph of independent sets of the simple cycle C_n . We do not need conventions for cycles of orders one and two.

Cicerone et al. [4, Section 7] explicitly leave total mutual visibility in Fibonacci cubes open, after studying ordinary visibility in them. Piselli's subsequent thesis [9, Section 3.4 and p. 148] treats ordinary visibility and repeats this total-visibility question. The two parameters must not be confused when comparing small values.

Let $\rho > 1$ be the positive root of $\rho^3 = \rho^2 + 1$, and put

$$a = \rho^{-1}, \quad L = 2a^2 + 3a^3 + 4a^4, \quad \kappa = \frac{a}{L}, \quad \eta = \frac{2a^4}{L}. \quad (1)$$

Thus $\rho = 1.465571231876\dots$, $\kappa = 0.248012299967\dots$, and $\eta = 0.157573224016\dots$

Theorem 1.1. *Let b_n count words w of length n avoiding 11 such that $0w0$ avoids 0000. For $n \geq 3$, let ℓ_n count cyclic words of length n avoiding 11 and cyclic 0000. Then*

$$\mu_t(\Gamma_n) = \Theta(\rho^n/n), \quad \mu_t(\Lambda_n) = \Theta(\rho^n/n).$$

More precisely, for either $(G_n, e_n) = (\Gamma_n, b_n)$ or (Λ_n, ℓ_n) ,

$$\frac{1}{\kappa} \leq \liminf_{n \rightarrow \infty} \frac{n\mu_t(G_n)}{e_n} \leq \limsup_{n \rightarrow \infty} \frac{n\mu_t(G_n)}{e_n} \leq \frac{1}{\eta}, \quad (2)$$

where $1/\kappa = 4.032058087974\dots$ and $1/\eta = 6.346255883519\dots$

The language counts satisfy

$$b_n \sim C\rho^n, \quad C = \frac{1 + 2a + a^2 + a^3}{L} = 1.144197371893\dots, \quad \ell_n \sim \rho^n. \quad (3)$$

The inequalities do not assert that the two normalized limits coincide. In particular, no leading constant is determined.

The structural result underlying these estimates is an exact open-packing reduction. For a graph K , let $D_2(K)$ join pairs at graph distance exactly two. An *open packing* is a set whose open neighborhoods are pairwise disjoint; its maximum size is $\text{op}(K)$. For the bipartite graphs used here, $\text{op}(K) = \alpha(D_2(K))$. Adjacent members of an open packing are permitted.

Theorem 1.2. *The graph induced by the eligible vertices of Γ_n has one connected component K_M for each maximal independent set M of P_n . There are nonnegative integers $r_1(M), \dots, r_k(M)$ such that*

$$K_M \cong \Gamma_{r_1(M)} \square \dots \square \Gamma_{r_k(M)}, \quad \mu_t(\Gamma_n) = \sum_M \text{op}(K_M). \quad (4)$$

The factors are determined explicitly by the recognizing-subset constraints of M . An empty product is one vertex.

Several ingredients are established results. We apply the distance-two visibility criterion of Bujtás, Klavžar and Tian [3, Theorem 5.2], and the recognizing-subset criterion of Hambardzumyan et al. [5, Lemma 2.1 and Corollary 2.2]. Coding constructions and incidence bounds already give exponential-over-dimension estimates for hypercube visibility [1, 7]. The eligible language is a boundary variant of the classical $(1, 3)$ runlength-limited language; its growth root and typical gap statistics are not new coding constants or methods [8, Lemma 3, Table I and Remark 4]. Kovačević also obtains analogous orders for shift-correcting codes, but in a different, weight-preserving metric. Our conflicts include two insertions or deletions and permit adjacent swaps, so those coding theorems

do not directly settle this invariant. The results here concern the specific visibility constraints, their fibre structure, and the resulting asymptotic and finite conclusions. The bounded literature comparison has not located these conclusions in the inspected sources; we make no categorical priority claim.

Sections 2–3 give the structural reduction and a short elementary proof of the Fibonacci order. Section 4 proves the sharper bounds; only its improved lower constant uses the external asymptotic coding theorem of Kabatiansky and Panchenko [6]. Section 5 treats cycles. Section 6 gives finite checks and the exact coding boundary. The component inventory is recorded in Appendix A.

2 Eligibility and recognizing-subset fibres

For a graph B , let $\mathcal{I}(B)$ be its independent-set graph. Its vertices are the independent subsets of $V(B)$, and two are adjacent if their symmetric difference has size one. This graph is connected and isometric in the Boolean cube: remove the members of $S \setminus T$, then add the members of $T \setminus S$. The resulting path has length $|S \Delta T|$.

We call a vertex *eligible* if its singleton is total mutual-visible. For an independent set S of B , write

$$U(S) = V(B) \setminus N_B[S]$$

for its addable vertices.

Lemma 2.1 (Bujtás–Klavžar–Tian [3]). *For a connected graph G , a set X is total mutual-visible if and only if*

$$N_G(u) \cap N_G(v) \not\subseteq X \quad \text{whenever } \text{dist}_G(u, v) = 2.$$

Proof. Necessity follows by applying the definition to a distance-two pair. For sufficiency, induct on $d = \text{dist}_G(u, v)$. For $d \leq 1$ there is no internal vertex. If $d \geq 2$, take a geodesic beginning u, x_1, x_2 and choose a common neighbor $y \notin X$ of u, x_2 . Then $\text{dist}_G(y, v) = d - 1$: the upper bound follows via x_2 , and a shorter path would contradict the value of d . Apply induction to y, v and prefix the edge uy . $\square$

Proposition 2.2. *Let K be the graph induced by the eligible vertices of $\mathcal{I}(B)$.*

1. *An independent set S is eligible if and only if $U(S)$ is independent.*
2. *For eligible S, T , the visibility conflict is exactly*

$$|S \Delta T| = 2 \quad \text{and the two vertices in } S \Delta T \text{ are nonadjacent in } B.$$

The conflict graph is $D_2(K)$, and $\mu_t(\mathcal{I}(B)) = \text{op}(K)$.

3. *The components of K are the fibres of the unique maximal independent extension $M(S) = S \cup U(S)$.*

Proof. A distance-two pair in $\mathcal{I}(B)$ has one or two common neighbors, corresponding to the two possible orders of its coordinate changes. Two changes at a vertex fail to commute only when they insert adjacent addable vertices. Thus S is the unique common neighbor of a distance-two pair exactly when $U(S)$ contains an edge. Lemma 2.1 proves the first assertion.

Two independent sets at Hamming distance two are opposite corners of a full square unless they exchange adjacent vertices of B . In that exception the corner containing both vertices is absent.

Once all ineligible vertices have been removed, the only remaining local obstructions are therefore the pairs in the second assertion.

Adding a valid vertex preserves eligibility, because it only shrinks the addable set. Hence every conflicting eligible pair has an eligible common neighbor: use their union for a swap, and an intermediate superset of the smaller set for two insertions or deletions. Conversely, a two-edge path inside K cannot realize an adjacent exchange, whose sole common neighbor is ineligible. This proves that the conflict graph is exactly $D_2(K)$. Since K is bipartite, two distinct vertices share an open neighbor exactly when their distance is two, proving the open-packing identity.

If $U(S)$ is independent, then $S \cup U(S)$ is its unique maximal independent extension. If $U(S)$ contains adjacent vertices, extending S after adding either endpoint gives different maximal extensions. Along an eligible insertion the unique extension cannot change, and the same holds for a deletion. Conversely, every eligible S reaches $M(S)$ by successive additions. The fibres are therefore exactly the components. $\square$

A subset S of a maximal independent set M is called *recognizing* if M is its unique maximal independent extension. The following known criterion makes the fibres explicit.

Lemma 2.3 (Hambarzumyan et al. [5]). *A subset $S \subseteq M$ is recognizing if and only if S dominates $V(B) \setminus M$. Equivalently, writing $T = M \setminus S$,*

$$N_B(v) \cap M \not\subseteq T \quad (v \notin M). \quad (5)$$

In particular, a unique neighbor in M of a vertex outside M belongs to every recognizing subset.

Proof. If S dominates the complement of M , no independent extension can add a vertex outside M , so its only maximal extension is M . Otherwise an undominated vertex outside M can be added to S and extended to a different maximal independent set. Formula (5) and the last assertion follow immediately. $\square$

Corollary 2.4. *For Γ_n , the eligible set is*

$$\mathcal{E}_n = \{w \in \{0, 1\}^n : w \text{ avoids } 11, 0w0 \text{ avoids } 0000\}.$$

Two words in $\mathcal{E}_n$ conflict precisely when they differ in two nonadjacent coordinates.

Proof. Two adjacent zero coordinates are both addable precisely when they and their immediate exterior coordinates are zero, with zero padding at the path ends. This is an occurrence of 0000 in 0w0. Apply Proposition 2.2. $\square$

The padding includes all small cases: $\mathcal{E}_0$ contains the empty word, $\mathcal{E}_1 = \{0, 1\}$, and $\mathcal{E}_2 = \{01, 10\}$. In particular, the adjacent exchange $10 \leftrightarrow 01$ is allowed, not a conflict.

Proof of Theorem 1.2. Fix a maximal independent set M of P_n . In (5), every constraint has size one or two. Delete from the variable coordinates every member forced by a singleton constraint, and call the remaining set $R(M)$. A size-two constraint involving a forced coordinate is automatically satisfied. The remaining constraints join pairs of positions at distance two in P_n . They form a disjoint union of paths, say of orders $r_1, \dots, r_k$. The allowed omitted sets T are precisely the independent sets of this residual graph, giving the stated Cartesian product.

The fibres are the components by Proposition 2.2, and the visibility conflict graph is their disjoint union of exact-distance-two graphs. Additivity over components proves (4). Each fibre is isometric

in the cube on its free coordinates: the family of allowed omitted sets is hereditary, so one can pass from T to T' through $T \cap T'$ in exactly $|T \Delta T'|$ steps. $\square$

3 An elementary proof of the asymptotic order

3.1 Counting eligible words

A nonzero word in $\mathcal{E}_n$ has zero-prefix and zero-suffix lengths in $\{0, 1, 2\}$ and internal zero gaps in $\{1, 2, 3\}$. The only all-zero eligible words have lengths zero and one. Thus

$$B(z) := \sum_{n \geq 0} b_n z^n = 1 + z + \frac{z(1 + z + z^2)^2}{1 - z^2 - z^3 - z^4} \quad (6)$$

$$= \frac{1 + 2z + z^2 + z^3}{1 - z^2 - z^3 - z^4}. \quad (7)$$

Consequently $b_0 = 1, b_1 = 2, b_2 = 2, b_3 = 4$, and

$$b_n = b_{n-2} + b_{n-3} + b_{n-4} \quad (n \geq 4). \quad (8)$$

The positive root of $x^4 = x^2 + x + 1$ is ρ , since the polynomial factors as $(x + 1)(x^3 - x^2 - 1)$. Comparing (8) with ρ^n gives, for example,

$$\frac{1}{2}\rho^n \leq b_n \leq 2\rho^n. \quad (9)$$

The smallest-modulus zero of the denominator in (7) is the simple positive root $a = 1/\rho$. It is the only zero of that modulus: equality in the triangle inequality would require the powers of orders two and three to have the same positive phase. Taking its residue gives $b_n \sim C\rho^n$ with C from (3).

3.2 A constructive lower bound

Proposition 3.1. *For $n \geq 1$, put $q_n = 2^{\lceil \log_2 \lceil n/2 \rceil \rceil}$. Then*

$$\mu_t(\Gamma_n) \geq \lceil b_n/q_n \rceil \geq b_n/n.$$

Proof. Assign distinct vectors of $\mathbb{F}_2^{\log_2 q_n}$ to the pairs of coordinates $(1, 2), (3, 4), \dots$, allowing a final singleton and the zero vector as a label. Give each position the label of its pair, and color w by the sum of the labels at its ones. A conflict changes two nonadjacent positions, which lie in distinct pairs and have distinct labels; their sum is nonzero. Thus every color class of $\mathcal{E}_n$ is total mutual-visible. The largest has at least $\lceil b_n/q_n \rceil$ words. Finally $q_n \leq n$, including $n = 1, 2$. $\square$

This is a specialization of standard syndrome coloring, not a claim of a new coding construction. It deliberately need not exploit every allowed adjacent swap.

3.3 The upper bound

Write $K_n = \Gamma_n[\mathcal{E}_n]$. If X is an open packing, then

$$\sum_{w \in X} \deg_{K_n}(w) \leq b_n, \quad (10)$$

because its open neighborhoods are disjoint. This is the familiar packing-incidence principle; related counting bounds in visibility appear in [1].

For $w \in \mathcal{E}_n$, let $g(w)$ count internal zero gaps of length three. Each block 10001 permits a distinct eligible insertion $10001 \mapsto 10101$, so $\deg_{K_n}(w) \geq g(w)$. Marking these gaps yields

$$\sum_{n \geq 0} \sum_{w \in \mathcal{E}_n} z^n y^{g(w)} = 1 + z + \frac{z(1 + z + z^2)^2}{1 - z^2 - z^3 - yz^4}. \quad (11)$$

Fix $0 < y < 1$. Its coefficient of z^n is $O(\beta^n)$, where $\beta > 1$ solves $\beta^4 = \beta^2 + \beta + y$; this follows either from the positive recurrence or the simple dominant pole. Since $\beta < \rho$, choose $\delta > 0$ such that $\gamma = \beta y^{-\delta} < \rho$. Then

$$\#\{w \in \mathcal{E}_n : g(w) < \delta n\} \leq y^{-\delta n} \sum_{w \in \mathcal{E}_n} y^{g(w)} = O(\gamma^n).$$

All other members of an open packing have degree at least δn , and (10) gives

$$\mu_t(\Gamma_n) \leq \frac{b_n}{\delta n} + O(\gamma^n) = O(\rho^n/n).$$

Together with Proposition 3.1, this proves the Fibonacci order in Theorem 1.1. The argument does not use the component inventory, computational data, or the asymptotically optimal binary codes used below. The gap-typicality principle has precedent in runlength-limited coding [8]; the weighted-series argument above is included to make this particular bound self-contained.

4 Typical degree and sharper constants

For an eligible word w , write $D(w) = \deg_{K_n}(w)$ and let $m(w) = |R(M(w))|$ be the free dimension of its fibre. Thus the fibre embeds isometrically in $Q_{m(w)}$. We first give a concentration argument for both statistics.

4.1 Gap statistics

List the distances between consecutive ones as $J_1, \dots, J_r \in \{2, 3, 4\}$. Let N_4 count entries equal to four, N_{22} count adjacent pairs $(2, 2)$, and N_{AA} count adjacent pairs with both entries in $A = \{2, 4\}$.

Lemma 4.1. *Uniformly over eligible words,*

$$D(w) = N_4 + N_{22} + O(1), \quad m(w) = N_4 + N_{AA} + O(1), \quad (12)$$

with absolute errors at most four.

Proof. An internal insertion is possible exactly in a distance-four gap. Deleting an interior one preserves eligibility exactly when the gaps on its two sides are both two: the merged gap must be at most four. The remaining possible insertions and deletions occur at the ends, where there are at most two insertion positions and two deletion positions in total. This proves the degree identity.

To form $M(w)$, split every distance-four gap into two distance-two gaps by adding its central vertex; gaps of length two or three stay unchanged. A free interior coordinate of this maximal independent set has a distance-two gap on each side. In the expanded gap list, an adjacent pair of twos occurs once inside each former four-gap, and once across an original pair from $A \times A$. Endpoint completion and endpoint free coordinates change this count by at most four. The two all-zero eligible words are covered by the same bounded-error statement. $\square$

Lemma 4.2. *For every $\varepsilon > 0$ there are $C, c > 0$ such that, for a uniformly chosen $W_n \in \mathcal{E}_n$,*

$$\mathbb{P}(|D(W_n) - \eta n| > \varepsilon n) \leq Ce^{-cn}, \quad (13)$$

$$\mathbb{P}(|m(W_n) - \kappa n| > \varepsilon n) \leq Ce^{-cn}. \quad (14)$$

The constants may depend on ε .

Proof. The identity $a^2 + a^3 + a^4 = 1$ makes $p_j = a^j$, $j \in \{2, 3, 4\}$, a probability distribution with mean L . Take independent random parts J_i with this distribution and put $S_r = J_1 + \dots + J_r$. For a fixed N , each composition of N into parts two, three and four has probability a^N . Conditional on the event H_N that the renewal process hits N , the resulting composition is therefore uniform. Its hitting probabilities have generating function

$$\sum_{N \geq 0} \mathbb{P}(H_N) z^N = \frac{1}{1 - p_2 z^2 - p_3 z^3 - p_4 z^4}.$$

The positive pole at one is simple, with coefficient $1/L$, and is the unique pole on the unit circle because the permitted lengths have greatest common divisor one. Hence $\mathbb{P}(H_N) \rightarrow 1/L > 0$.

Here is an elementary exponential bound for the statistics needed. Given bounded functions $f(j)$ and $h(j, j')$, set

$$T_r = \sum_{i=1}^r f(J_i) + \sum_{i=1}^{r-1} h(J_i, J_{i+1}), \quad \theta = \frac{\mathbb{E}f(J_1) + \mathbb{E}h(J_1, J_2)}{L}.$$

The centered quantity $T_r - \theta S_r$ is a sum of centered single-part terms, centered odd-indexed pair terms, and centered even-indexed pair terms, plus a bounded constant. Within each of the three groups the variables are independent and uniformly bounded. For a bounded centered variable Z , the elementary exponential-moment estimate $\mathbb{E}e^{tZ} \leq e^{C_0 t^2}$ for sufficiently small fixed t follows from its Taylor series. Applying this estimate to each group and then the exponential Markov inequality gives

$$\mathbb{P}(|T_r - \theta S_r| > \varepsilon N) \leq C_1 e^{-c_1 N} \quad \text{for } N/4 \leq r \leq N/2.$$

There are only $O(N)$ possible values of r on H_N . A union bound, followed by division by the hitting probability bounded away from zero, preserves an exponential estimate. Thus uniform compositions satisfy $T_r = \theta N$ with exponentially small probability of a linear deviation.

For degree take $f = \mathbf{1}_{\{4\}}$ and $h = \mathbf{1}_{\{(2,2)\}}$; then

$$\theta = \frac{p_4 + p_2^2}{L} = \frac{2a^4}{L} = \eta.$$

For free dimension take the same f and $h = \mathbf{1}_{A \times A}$; then

$$\theta = \frac{p_4 + (p_2 + p_4)^2}{L} = \frac{a}{L} = \kappa,$$

using $a^2 + a^4 = a$.

Finally fix the leading and trailing zero lengths $b_L, b_R \in \{0, 1, 2\}$. The internal gaps compose $N = n - 1 - b_L - b_R$. Conditional on these boundary lengths, every such composition corresponds to exactly one eligible word. The bounded endpoint errors in Lemma 4.1, followed by the finite mixture over the nine boundary choices, give (13) and (14). $\square$

This proof makes the negligible exception quantitatively strong enough: its size is $O(b_n e^{-cn})$, not merely $o(b_n)$. The same typical-run methodology is established in [8]; the pair statistics above identify the degree and free dimension for this visibility problem.

4.2 Averaging classical codes over fibres

Let $A(s, 3)$ be the maximum size of a binary length- s code of minimum Hamming distance at least three, with $A(0, 3) = 1$.

Lemma 4.3. *If K is an eligible fibre of free dimension $m \geq 1$, then*

$$\text{op}(K) \geq |V(K)| \frac{A(m-1, 3)}{2^{m-1}}. \quad (15)$$

Proof. Append a parity bit to a maximum length- $(m-1)$ distance-three code. The resulting even-weight code $E \subseteq Q_m$ has minimum distance at least four. The union $D = E \cup (E + e_1)$ has size $2A(m-1, 3)$ and avoids Hamming distance two: each parity class has minimum distance four, and cross-parity distances are odd. Average $|(D+u) \cap V(K)|$ over all $u \in Q_m$. Its mean is $|V(K)||D|/2^m$, so some translate attains at least this value. Since K is isometric in Q_m , its intersection is an open packing. $\square$

The following is an external coding theorem, not a consequence of the familiar perfect-code subsequence. Kabatiansky and Panchenko [6, p. 550, equation (3)] prove that the optimal radius-one packing density tends to one for each fixed prime-power alphabet. For the binary alphabet this says

$$A(s, 3) \sim \frac{2^s}{s+1} \quad (s \rightarrow \infty) \quad (16)$$

over all integer lengths. Their original freely available three-page paper states an all-length density estimate; this scope is essential to the liminf below. Translate averaging itself is also a standard constrained-coding method; see the discussion and earlier references in [10].

Proof of the Fibonacci bounds in (2). Fix small $\delta, \varepsilon > 0$ with $\delta < \kappa$. Lemma 4.2 shows that the fibres having

$$(\kappa - \delta)n \leq m \leq (\kappa + \delta)n$$

contain $(1 - O(e^{-cn}))b_n$ eligible words. Formula (16) holds uniformly for all sufficiently large such m , so Lemma 4.3 gives

$$\mu_t(\Gamma_n) \geq \frac{(1 - \varepsilon)(1 - O(e^{-cn}))b_n}{(\kappa + \delta)n}.$$

Fibres of dimension zero are singletons and can be ignored for this lower bound. Let $n \rightarrow \infty$, then $\varepsilon, \delta \downarrow 0$.

For the upper bound, discard the $O(b_n e^{-cn})$ eligible words of degree below $(\eta - \varepsilon)n$. Applying (10) to the other members of any open packing gives

$$\mu_t(\Gamma_n) \leq \frac{b_n}{(\eta - \varepsilon)n} + O(b_n e^{-cn}).$$

Multiply by n/b_n , let $n \rightarrow \infty$, and then let $\varepsilon \downarrow 0$. $\square$

The upper constant can also be written $1/\eta = \rho^2 + \frac{3}{2}\rho + 2$. With normalization ρ^n/n , the Fibonacci lower and upper constants are 4.613470267581... and 7.261369303286..., respectively. Neither normalization makes the interval collapse.

5 Lucas cubes

Proposition 2.2 applies to the independent-set graph of C_n . Thus its eligible vertices are precisely the cyclic binary words avoiding 11 and cyclic 0000. A conflict changes two coordinates that are nonadjacent on the cycle; in particular the wraparound adjacent exchange is allowed.

5.1 Counting and the fibre structure

The four-state zero-run transfer matrix is

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \ell_n = \text{tr}(T^n) \quad (n \geq 3). \quad (17)$$

The states record the number of zeros since the preceding one. A cyclic eligible word contains a one and hence determines a unique closed state walk, proving the trace identity. The characteristic polynomial is $x^4 - x^2 - x - 1$. Its dominant root is the simple root ρ , and all other roots have smaller modulus, so $\ell_n \sim \rho^n$. The initial trace values are 0, 2, 3, 6 for exponents one through four; the first two are matrix values, not graph conventions for Λ_1 and Λ_2 .

Proposition 5.1. *The eligible components of Λ_n , $n \geq 3$, are indexed by maximal independent sets M of C_n . If the cyclic support gaps of M include a three, the corresponding fibre is a Cartesian product of Fibonacci cubes. If all gaps are two, then n is even and M is one of the two alternating sets. For even $n \geq 6$ its fibre is $\Lambda_{n/2}$. At $n = 4$ each of these two fibres is $\Gamma_2 = P_3$, and at $n = 3$ there are three singleton fibres.*

Proof. A maximal independent set of a cycle has support distances two or three. Each gap three forces its two endpoint coordinates by Lemma 2.3. If such a gap exists, deleting all forced coordinates breaks the remaining pair-constraint graph into paths. If every gap is two, all coordinates are free and the pair constraints form the cycle on $n/2$ vertices. When $n = 4$, the repeated pair constraint gives a single edge, not a two-cycle; its independent-set graph is P_3 . The $n = 3$ assertion follows directly or from the same recognition constraints. $\square$

5.2 Cyclic concentration and bounds

For cyclic eligible words, let the gap list include the final gap back to the initial one. Mark one of the ones as a root. Ordered compositions of n into parts 2, 3, 4, together with one of the n possible labelled root coordinates, are in bijection with such rooted words. Every unrooted word has between $n/4$ and $n/2$ roots. Passing between uniform rooted and unrooted words therefore changes probabilities by at most a fixed factor.

The degree and free-dimension identities in (12) hold with cyclic adjacent-pair counts; deleting the pair across the artificial root changes them by $O(1)$. Lemma 4.2's composition proof consequently gives the same exponentially concentrated densities η and κ . This also handles the alternating fibres in Proposition 5.1: every fibre is a subset of a cube on its free coordinates, so Lemma 4.3 does not require Fibonacci rather than Lucas factors.

Disjoint-neighborhood counting and translate averaging now prove (2) with ℓ_n in place of b_n . The Lucas order also has a proof independent of (16): distinct binary coordinate labels give an

eligible conflict coloring with $2^{\lceil \log_2 n \rceil} \leq 2n$ colors, while the same positive typical degree yields an $O(\ell_n/n)$ upper bound. Together with (17), this completes Theorem 1.1.

6 Exact values and the remaining coding problem

6.1 Finite computation and validation

The conflict characterization gives an exact component-wise maximum-independent-set computation. We used the identity

$$\alpha(H) = \max\{\alpha(H - v), 1 + \alpha(H - N_H[v])\} \quad (18)$$

with memoization and additive decomposition over connected components. The implementation uses integer bitsets, retains witnesses, and evaluates both branches. Thus the reported optimum is an exhaustive exact computation, not an optimization heuristic. Table 1 gives the Fibonacci values.

Table 1: Eligible counts and exact total mutual-visibility numbers for Fibonacci cubes.

n	b_n	$\mu_t(\Gamma_n)$	n	b_n	$\mu_t(\Gamma_n)$
1	2	2	11	77	41
2	2	2	12	112	62
3	4	3	13	165	73
4	5	5	14	241	111
5	8	5	15	354	156
6	11	9	16	518	201
7	17	12	17	760	303
8	24	14	18	1113	405
9	36	24	19	1632	549
10	52	28	20	2391	812

Three distinct checks support the implementation. First, a separate breadth-first-search test directly uses the original all-pairs geodesic definition, allowing arrival at a selected vertex but not continuation through it. It checks every subset of the eligible set through $n = 7$. Singleton exclusion and downward closure justify restricting the subset search to eligible vertices; it does not enumerate every subset of $V(\Gamma_7)$. Second, direct common-neighbor constraints agree with the simplified characterization through $n = 12$, and component keys, product orders, and free-coordinate counts are checked through $n = 17$. Third, a separate numerical mixed-integer model built from the original common-neighbor constraints matches the table through $n = 20$. Its optimal status and zero reported gaps are numerical cross-checks, not rational certificates; (18) is the exact-arithmetic underpinning.

The component-type generating function of Appendix A gives a further reconstruction: only 31 distinct product types are needed across dimensions zero through twenty. This independently generates the graphs being optimized, but reuses the same maximum-independent-set kernel and is not an independent solver. At dimension twenty there are 265 eligible fibres of 21 types, comprising 2391 eligible vertices. The largest fibre has 89 vertices; the largest component of its exact-distance-two conflict graph has 45. These two notions of component should not be interchanged.

The Lucas values appear in Table 2. Direct common-neighbor and complete fibre-reconstruction checks run through $n = 16$, and the full geodesic-definition test checks every eligible subset for

Table 2: Cyclic eligible counts and exact total mutual-visibility numbers, with $\Lambda_n = \mathcal{I}(C_n)$ for $n \geq 3$.

n	ℓ_n	$\mu_t(\Lambda_n)$	n	ℓ_n	$\mu_t(\Lambda_n)$
3	3	3	10	47	31
4	6	4	11	66	33
5	5	5	12	99	61
6	11	7	13	143	78
7	14	14	14	212	96
8	22	14	15	308	158
9	30	21	16	454	190

$3 \leq n \leq 7$. The degree/free-dimension boundary identities are additionally checked for all eligible Fibonacci words through $n = 28$. The code and saved results accompany the manuscript in a separate reproducibility supplement. These finite validations support, but do not replace, the all-dimension proofs; no proof-assistant or SAT/LRAT certificate is claimed.

6.2 An explicit classical coding subproblem

The exact reduction (4) is not a closed formula because its open-packing summands remain genuine extremal problems. The following makes this boundary concrete.

Proposition 6.1. *For every $k \geq 1$, an eligible fibre of Γ_{7k-6} is isomorphic to Q_k . Moreover,*

$$\text{op}(Q_k) = 2A(k-1, 3). \quad (19)$$

Proof. Put $n = 7k - 6$ and define

$$R = \{1 + 7j : 0 \leq j \leq k-1\}, \quad M = R \cup \{7j + 3, 7j + 6 : 0 \leq j \leq k-2\}.$$

For $k = 1$ this is the unique nonempty maximal independent set of P_1 . In general M is maximal: its support begins and ends at the path endpoints, with consecutive gaps of lengths two or three. Every gap of length three forces its two endpoints, and exactly the members of R remain free. They are seven apart and have no pair constraints between them. All their subsets may be deleted independently, so the fibre is Q_k .

Identity (19) is a known open-packing identity [2, Theorem 2.2]; a short coding proof is included. A subset of Q_k avoiding distance two splits into even- and odd-weight codes, each of minimum distance at least four. Puncturing one coordinate gives a distance-three code of length $k-1$, so each part has at most $A(k-1, 3)$ members. Conversely, parity extension of an optimal such code and an odd translate give two parity parts with total size $2A(k-1, 3)$. $\square$

Thus a universal formula for all product contributions would include exact finite binary one-error-correcting code sizes. This is not a proof that a simpler formula for the aggregate sequence cannot exist, and it is not a hardness claim about the one-parameter family. Nor does it make the classical asymptotic (16) an open problem. The unresolved issue here is the packing efficiency within the constrained fibres: stronger constructions or converses are needed to close the interval in (2). Natural questions are whether the normalized sequence converges and which products of Fibonacci cubes admit exact open-packing values. Open packing is not multiplicative: $\text{op}(\Gamma_1) = 2$, whereas $\text{op}(\Gamma_1 \square \Gamma_1) = 2$, not four.

A Exact component inventory

The product decomposition admits a compact formal inventory. Give a fibre of type $(r_1, \dots, r_k)$ the monomial $\prod_i y_{r_i}$, with $y_0 = 1$. Let $\mathcal{F}(z; \mathbf{y})$ sum these monomials over all maximal independent sets of all paths, weighted by z^n . Define

$$Y(q) = \sum_{r \geq 0} y_r q^r, \quad I(z) = 1 + z^2 Y(z^2), \quad J(z) = z + (1 + z^3) Y(z^2).$$

Proposition A.1. *The exact component-type inventory is*

$$\mathcal{F}(z; \mathbf{y}) = 1 + \frac{Y(z^2) - 1}{z} + 2z^2 Y(z^2) + z^3 I(z) + \frac{z^4 J(z)^2}{1 - z^3 I(z)}. \quad (20)$$

Proof. A nonempty maximal independent set has boundary zero lengths $b_L, b_R \in \{0, 1\}$ and support gaps two or three. Regard a gap three as a barrier, because both its endpoints are forced. A run of h gaps two between barriers contributes $\max(h - 1, 0)$ free coordinates, giving

$$\sum_{h \geq 0} z^{2h} y_{\max(h-1, 0)} = I(z).$$

At a boundary with zero length b , the corresponding number is $\max(h - b, 0)$; summing over $b = 0, 1$ gives $J(z)$. An initial support vertex contributes z , the first barrier contributes z^3 , and each further barrier together with its preceding run contributes $z^3 I(z)$. This gives the last term in (20).

With no barrier, the free dimension is $\max(h + 1 - b_L - b_R, 0)$. The boundary choices $(0, 0)$, $(0, 1)$ or $(1, 0)$, and $(1, 1)$ give respectively $(Y(z^2) - 1)/z$, $2z^2 Y(z^2)$, and $z^3 I(z)$. Finally the empty path contributes one. $\square$

Setting $y_r = 1$ gives $(1 + z + z^2)/(1 - z^2 - z^3)$, the established maximal-independent-set count for paths. Setting $y_r = F_{r+2}$, where $F_0 = 0, F_1 = 1$, gives $B(z)$ from (7). Applying instead the linear rule

$$\prod_i y_{r_i} \mapsto \text{op}(\Gamma_{r_1} \square \cdots \square \Gamma_{r_k})$$

gives the exact total-visibility sequence. This rule is linear on inventory monomials, not multiplicative on factors.

For completeness, the specialization marking free dimension while weighting each component by its order is

$$F(z, t) = \sum_{n \geq 0} \sum_{w \in \mathcal{E}_n} z^n t^{m(w)} = \frac{N(z, t)}{D(z, t)},$$

where substituting $y_r = F_{r+2} t^r$ into (20) gives

$$\begin{aligned} N(z, t) &= 1 + 2tz + (2 - t)z^2 + t^2 z^3 + (1 - t^2)z^4 + (t - t^2)z^6, \\ D(z, t) &= 1 - tz^2 - z^3 - t^2 z^4 + (t - 1)z^5 + (t^2 - t)z^7. \end{aligned}$$

Here $Y(z^2) = (1 + tz^2)/(1 - tz^2 - t^2 z^4)$. At $t = 1$ this reduces to (7). If $a(t)$ is the dominant simple zero of $D(z, t)$ near $t = 1$, implicit differentiation gives

$$-\frac{a'(1)}{a} = \frac{D_t(a, 1)}{a D_z(a, 1)} = \frac{a}{L} = \kappa,$$

an independent algebraic check of the free-dimension density. The proof of concentration in the main text does not require this additional rational-function calculation.

The supplement compares the full component inventory against direct maximal-independent-set enumeration through $n = 18$, checks the scalar counts and free-dimension specializations through $n = 60$, and reconstructs the exact values through $n = 20$. These are finite checks of the formal identity, whose all-dimension proof is Proposition A.1.

Reproducibility and license

The manuscript, verification code, and results are available in the public GitHub repository. The reproducibility supplement contains the original exact solver, witnesses and finite data, independent definition and structural checks, the component-inventory implementation, and the separate numerical optimization cross-check. It specifies dependencies and the precise ranges of each check. It contains no copies of third-party papers. The manuscript and accompanying prose/data are licensed under Creative Commons Attribution 4.0 International (CC BY 4.0). No separate software license is designated here; the supplement identifies that distinction. No formal verification status is implied.

References

- [1] M. Axenovich and D. Liu, *Visibility in hypercubes*, Graphs and Combinatorics **42** (2026), Article 38. doi:10.1007/s00373-026-03025-9. Open full text at the DOI.
- [2] B. Brešar, S. Klavžar and D. F. Rall, *Packings in bipartite prisms and hypercubes*, Discrete Mathematics **347**(4) (2024), 113875. Free author preprint: <https://arxiv.org/abs/2309.04963>.
- [3] C. Bujtás, S. Klavžar and J. Tian, *Visibility polynomials, dual visibility spectrum, and characterization of total mutual-visibility sets*, Aequationes Mathematicae **99** (2025), 1883–1901. doi:10.1007/s00010-025-01197-y. Free preprint: <https://arxiv.org/abs/2412.03066>.
- [4] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra and F. Piselli, *Mutual and total mutual visibility in hypercube-like graphs*, Applied Mathematics and Computation **491** (2025), 129216. doi:10.1016/j.amc.2024.129216. Free institutional full text.
- [5] S. Hambardzumyan, V. V. Mkrtchyan, V. L. Musoyan and H. Sargsyan, *The hardness of the independence and matching clutter of a graph*, Opuscula Mathematica **36**(3) (2016), 375–397. doi:10.7494/OpMath.2016.36.3.375. Free publisher full text.
- [6] G. A. Kabatiansky and V. I. Panchenko, *Packings and coverings of the Hamming space by unit balls*, Doklady Akademii Nauk SSSR **303**(3) (1988), 550–552 (Russian); English translation, Soviet Mathematics Doklady **38**(3) (1989), 564–566. Free original full text.
- [7] D. Korže and A. Vesel, *Variety of mutual-visibility problems in hypercubes*, preprint (2024). <https://arxiv.org/abs/2405.05650>.
- [8] M. Kovačević, *Runlength-limited sequences and shift-correcting codes: asymptotic analysis*, IEEE Transactions on Information Theory **65**(8) (2019), 4804–4814. doi:10.1109/TIT.2019.2907979. Free author preprint: <https://arxiv.org/abs/1803.06117>.
- [9] F. Piselli, *Distributed Tasks For Swarm Robotics Under Weak Assumptions*, doctoral thesis, University of Florence (2026). <https://floren.unifi.it/handle/2158/1465052>. Free institutional full text.
- [10] V. A. Rameshwar and N. Kashyap, *Estimating the sizes of binary error-correcting constrained codes*, preprint (2023). <https://arxiv.org/abs/2301.05098>.