

AN EXPLICIT UPPER-BOUND CERTIFICATE FOR THE ESSENTIAL MINIMUM OF THE ZHANG–ZAGIER HEIGHT

RICARDO PINHO

ABSTRACT. We give an exact certificate proving $C_{82} < 0.2534449$ for the essential minimum of $h(\alpha) + h(1 - \alpha)$, where h is the absolute logarithmic Weil height. We use Doche’s limit-point method, building on Grinsztajn’s factor library and, in particular, Épi’s unequal-multiplicity construction. We retain Épi’s denominator factors, add Doche’s P_7 , and add 25 numerator factors found by LP-dual-weighted evaluation lattices with LLL. Joint linear-program re-optimisation is followed by exact rational rounding. Two separately written ball-arithmetic checkers certify the bound; a further adversarial computational audit accepts the certificate.

1. INTRODUCTION

For a primitive irreducible polynomial $f \in \mathbb{Z}[z]$ of degree d and a root α , put

$$M(f) = |\mathrm{lc}(f)| \prod_{f(\beta)=0} \max(1, |\beta|), \quad h(\alpha) = \frac{\log M(f)}{d}, \quad h_{\mathrm{ZZ}}(\alpha) = h(\alpha) + h(1 - \alpha).$$

All logarithms are natural. The essential minimum considered here is

$$(1) \quad C_{82} = \sup\{H \in \mathbb{R} : \{\alpha \in \overline{\mathbb{Q}} : h_{\mathrm{ZZ}}(\alpha) \leq H\} \text{ is finite}\}.$$

The notation follows constant 82a of Tao and contributors’ optimization constants table [7]. Unlike the minimum of the nonzero values of h_{ZZ} , this constant concerns infinitely many algebraic numbers of unbounded degree.

The problem belongs to the arithmetic theory of small points and essential minima developed by Zhang [9]. Zagier [8] proved

$$h_{\mathrm{ZZ}}(\alpha) \geq \tfrac{1}{2} \log((1 + \sqrt{5})/2)$$

away from the roots of $z(z - 1)(z^2 - z + 1)$. Doche developed the limit-point construction used here [2, 3]. The table lists Grinsztajn’s 2026 bound 0.2536331090204145 [6]. More recently, Épi (BakeLab, September 2026) reported 0.2535744749068227 [4]; an independent re-enclosure of its unchanged construction gives approximately 0.2535689302702101. These earlier constructions are central to the present one. The table lists the lower bound 0.24874 due to Flammang [5]. Burgos Gil, Menares, Qu and Sombra [1] prove strong duality for upper- and lower-bound methods and theoretical computability of essential minima. Their general results are not needed for the argument below.

Theorem 1. *The essential minimum of the Zhang–Zagier height satisfies*

$$C_{82} \leq B < 0.2534449.$$

Here B is the explicit functional in (3) evaluated on the exact certificate specified below.

A read-only check of the live table, its open pull requests, and the Épi repository on 16 October 2026 found no smaller bound. This note reports a verified upper bound, without a claim of optimality.

Date: 16 October 2026.

Key words and phrases. Zhang–Zagier height, essential minimum, Mahler measure, rigorous computation.

2. CONSTRUCTION AND METHOD

Write $T(z) = z(1 - z)$ and

$$Q(X) = \prod_j R_j(X)^{m_j}, \quad \log |A(X)| = \sum_i q_i \log |S_i(X)|, \quad D = \deg Q.$$

The product A is formal; no branches of rational powers are chosen. The labels follow Doche, Grinsztajn and Épi; the exact integer coefficient arrays (descending order) in `certificate.json`, rather than the labels, define the polynomials. Table 1 gives all active factors and exact weights.

Épi introduced unequal denominator multiplicities and the extra factors E_{d5} (degree 28) and E_{n6} (degree 16). Our construction keeps Épi's denominator factors, adds Doche's P_7 , and adds 25 numerator factors found by LP-dual-weighted evaluation lattices with LLL reduction. The factor E_{n6} was included in the starting library but is inactive in the final certificate. The perturbative factors R_0, R_2 come from Grinsztajn's public construction. Thus unequal multiplicities and the Épi factors are inherited, rather than introduced here.

The sampled Doche functional is optimised by a linear program jointly in denominator multiplicities and numerator exponents. Its dual weights select evaluation points and external fields for integer evaluation lattices. LLL reduction supplies candidate numerator polynomials; exact factorisation and numerical screening select columns for further LP re-optimisation. The retained weights are then rounded to positive integer denominator multiplicities and nonnegative rational numerator exponents. This is a heuristic search, not an exhaustive optimisation. Only the rounded exact data enter the proof and the rigorous checkers.

The formal degree is $\deg A := \sum_i q_i \deg S_i$. Every denominator factor is monic, all numerator–denominator pairs are coprime, and exact rational arithmetic verifies

$$(2) \quad D = 999999992, \quad \deg A < 944662259 < D.$$

Nonmonic numerator factors are permitted; their leading-coefficient contributions are retained in every logarithmic evaluation.

Set $u(t) = e^{2\pi it}$ and, away from zeros, define

$$g(t) = \sum_i q_i \log |S_i(T(u(t)))| - \log |Q(T(u(t)))|.$$

The bound to be certified is

$$(3) \quad B = \frac{1}{D} \left(\log M(Q \circ T) + \int_0^1 (g(t))_+ dt \right), \quad x_+ = \max(0, x).$$

At numerator zeros the positive part is zero; isolated denominator zeros are treated as integrable logarithmic singularities. Formula (3) is the logarithmic form of Doche's formula [3, equation (9)]. Notice that the Mahler term is for $Q \circ T$, not for Q .

3. WHY THE CERTIFICATE PROVES THE THEOREM

We give the infinite-sequence argument explicitly, including the step that handles reducible polynomials.

Proposition 2. *Suppose $Q \in \mathbb{Z}[X]$ is monic up to sign with $D = \deg Q > 0$, $S_i \in \mathbb{Z}[X] \setminus \{0\}$, and $q_i \in \mathbb{Q}_{\geq 0}$ satisfy $\sum_i q_i \deg S_i < D$. Suppose also that each S_i with $q_i > 0$ is coprime to Q . Then $C_{82} \leq B$, with B defined by (3). In fact there are distinct algebraic integers α_k with $\limsup_k h_{ZZ}(\alpha_k) \leq B$.*

Proof. Choose a positive integer b clearing all the denominators and put

$$U(z) = \prod_i S_i(T(z))^{bq_i}, \quad W(z) = Q(T(z))^b, \quad N = \deg W = 2bD.$$

TABLE 1. Active factors of the exact certificate. Numerator labels **select** and **merged** denote the 25 lattice-derived factors.

			Denominator	Degree	Multiplicity
			Q1	28	9297152
			Q2	28	5575669
			R0	28	3376055
			R2	28	854210
			P7	12	4177785
			P9	16	14295157
			Ed5	28	6652059

Numerator	Degree	Exponent q_i	Numerator	Degree	Exponent q_i
P1	1	309690889482955/4194304	select5_0	39	4617249463684/4001225
P2	1	630788667754879/8773059	select5_7	34	4710839667389/9906458
P3	3	21260800937839/8055645	select5_9	32	1079755879201/2164874
P4	4	52745169098993/7487927	select5_10	35	6714480457493/8818907
P6	8	29007076082772/3169231	select7_0	42	9317325406677/3194929
P8	12	3410042187423/8533421	select8_0	22	10953669139763/9925694
select2_1	34	1992723892935/9974618	select9_1	24	3449087714362/9861087
select2_11	24	702728210855/1985031	select9_2	37	4659890228461/8743052
select2_15	27	450900060781/9270228	select9_3	42	6289381318901/9096035
select2_22	28	6599318260539/9993640	select9_5	42	651575276147/7044436
select2_25	32	562420442071/2264313	select9_6	36	11838421919974/8950455
select2_36	18	3362145432269/9532037	select9_7	39	127103111402/1055161
select3_7	28	4352002110254/4887323	select10_0	41	4884194025059/4010746
select4_0	24	3578264620002/7819705	select11_0	42	6519687014701/5871539
select4_1	42	13531153728682/6884769	merged_0	30	3775427748764/6478995
select4_3	38	5381514591569/5240791			

These are coprime integer polynomials, since a common composed zero would map under T to a common original zero. We have $\deg U < N$, and W is monic up to sign. For $F_n = U^n - W^n$, factorization over roots of unity gives

$$(4) \quad \frac{\log M(F_n)}{n} = \frac{1}{n} \sum_{\zeta^n=1} \log M(U - \zeta W) \longrightarrow L := \int_0^1 \log \max(|U(u(t))|, |W(u(t))|) dt.$$

Indeed, $U - \zeta W$ has fixed degree N and leading coefficient of modulus one. Continuity of its roots as a multiset implies continuity of its logarithmic Mahler measure in ζ . Thus the sums converge to its phase mean, which Jensen's formula evaluates as L . The interchange of the circle and phase integrals is justified as follows. The positive part of $\log |U - \zeta W|$ is uniformly bounded on the two circles. Its phase mean is $\log \max(|U|, |W|)$, which is bounded below because U, W have no common zero. Tonelli's theorem then makes the negative part integrable as well. Splitting off $\log |W|$ and applying Jensen in z gives

$$(5) \quad L = b \log M(Q \circ T) + b \int_0^1 (g(t))_+ dt = bDB.$$

For each prime p , consider

$$G_p(z) = \frac{U(z)^p - W(z)^p}{U(z) - W(z)} = \sum_{j=0}^{p-1} U(z)^{p-1-j} W(z)^j.$$

This integer polynomial is monic up to sign, has degree $N(p-1)$, and is invariant under $z \mapsto 1-z$. A root cannot be a zero of W ; its value of U/W is a nontrivial p th root of unity. At $U/W = 1$ the sum is $pW^{p-1} \neq 0$. Consequently G_p and G_r are coprime for distinct primes p, r .

Factor G_p into primitive irreducible integer polynomials, retaining multiplicities. Each factor is monic up to sign, and the height formula shows that the average of $h(\alpha)$ over the roots is $\log M(G_p)/\deg G_p$. Symmetry gives the same average for $h(1 - \alpha)$. Hence

$$\frac{1}{\deg G_p} \sum_{G_p(\alpha)=0} h_{ZZ}(\alpha) = \frac{2 \log M(G_p)}{N(p-1)} \rightarrow \frac{2L}{N} = B,$$

by (4), since $\log M(G_p) = \log M(F_p) - \log M(U - W)$. On each irreducible factor, h_{ZZ} is constant on its Galois orbit. Select one root of a factor with height sum at most this average. Coprimality across primes makes the selected roots distinct, even if an individual G_p has repeated roots. Every threshold strictly greater than B therefore contains infinitely many such roots, which proves the claim by (1). $\square$

The exact checks in (2) verify the hypotheses. The certified upper bounds in the next section then prove Theorem 1. The factor two from $\deg T = 2$ cancels the factor two in the root average; this explains the divisor D in (3).

4. COMPUTER VERIFICATION

The public repository supplies `verify.py` and `independent_check.py`, written separately at the application level. Both compose the factors with T exactly, certify complete root multisets with FLINT/Arb, and compute the Mahler term by Jensen’s identity. They use exact rational weights and certify the integral over the full circle, without assuming that a sample grid locates every positive arc. They share FLINT/Arb and mathematical identities, so their agreement does not constitute independent arithmetic-library verification.

The first checker integrates explicit Taylor upper majorants on exact dyadic panels. Root-distance bounds exclude negative panels; separated root balls bound the second derivative on remaining panels. The second checker uses certified analytic quadrature on positive panels and full interval enclosures on unresolved crossing panels. Both use real coefficients to double the integral over $[0, 1/2]$. The Taylor ball bounds an upper majorant; it is not a two-sided enclosure of B . The analytic checker encloses B itself.

The following decimals are rounded upward from the certified endpoints:

Checker	Upper bound for B
<code>verify.py</code>	0.253444687333627
<code>independent_check.py</code>	0.253444687337420
Additional adversarial evaluator	0.253444687331849

The additional adversarial computational referee returned *ACCEPT*. It checked the same certificate hash, the strict degree inequality and all 217 cross-factor gcds, reconstructed the infinite-sequence proof, and used a separate evaluator with Taylor sign isolation and Arb analytic integration. Its Épi control reproduced the unchanged public construction; a doubled- Q_1 numerical mutation failed the target, and malformed-input controls were rejected. This is a computational audit, not journal peer review. The certificate’s stored floating search objective is not used as evidence for the bound.

With Python and the repository’s pinned `python-flint` dependency, the principal checks are reproduced by

```
python verify.py certificate.json --tol 1e-10 --out taylor.json
python independent_check.py certificate.json analytic.json 1e-8
```

These tolerances control subdivision; the computed interval endpoints, not a presumed tolerance error, certify the exact target 0.2534449. No search rerun or expanded product polynomial is required.

DATA AVAILABILITY AND LIMITATIONS

The certificate and both checkers are available at <https://github.com/rpinho/c82-height-certificate>. The SHA-256 digest of `certificate.json` is

b33626f367b6f52922746c0fee7bb0084
49cd0bb18efcb983ab20ff073350fce.

The line break and final punctuation are not part of the digest. The bound does not determine C_{82} or establish optimality within this factor family. Proposition 2 gives infinitely many algebraic integers without expanding their defining polynomials.

AI ASSISTANCE

Analysis and proof drafting were done by AI agents (OpenAI Codex and Claude) under the author’s direction; the author reviewed the results at summary level. AI agents also prepared the certificate, the checking software and this note. This does not claim a line-by-line human verification of the proof or the software; correctness rests on the exact certificate, checked by two separately written ball-arithmetic checkers.

REFERENCES

1. José Burgos Gil, Ricardo Menares, Binggang Qu, and Martín Sombra, *Closing the gap around the essential minimum of height functions with linear programming*, arXiv:2601.18978v2 [math.NT], 2026, <https://arxiv.org/abs/2601.18978v2>.
2. Christophe Doche, *On the spectrum of the Zhang–Zagier height*, Math. Comp. **70** (2001), no. 233, 419–430, <https://doi.org/10.1090/S0025-5718-00-01183-2>.
3. ———, *Zhang–Zagier heights of perturbed polynomials*, J. Théor. Nombres Bordeaux **13** (2001), no. 1, 103–110, https://www.numdam.org/item/JTNB_2001__13_1_103_0/.
4. Épi Team (BakeLab), *Épi Results: Zhang–Zagier essential minimum*, GitHub repository, Bake AI, 2026, https://github.com/BakeLab/Epi-results/tree/main/records/zhang-zagier/zhang_zagier_c82, commit 97fa6a420f417458e11723c8bceb70f1529e447a; accessed 16 October 2026.
5. V. Flammang, *On the Zhang–Zagier measure*, Int. J. Number Theory **14** (2018), no. 10, 2663–2671, <https://doi.org/10.1142/S1793042118501609>.
6. Max Grinsztajn, *Code and certification data for an improved C_{82} Zhang–Zagier bound*, GitHub repository, 2026, <https://github.com/maaxgrin/zhang-zagier-c82-bound>, commit cb88f92434f649a1588b31ae91f67852830c6f77; accessed 16 October 2026.
7. Terence Tao and contributors, *Optimization constants in mathematics: Essential minimum of the Zhang–Zagier height (constant 82a)*, Online table, <https://teorth.github.io/optimizationproblems/constants/82a.html>; accessed 16 October 2026.
8. Don Zagier, *Algebraic numbers close to both 0 and 1*, Math. Comp. **61** (1993), no. 203, 485–491, <https://doi.org/10.2307/2152970>.
9. Shouwu Zhang, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. **8** (1995), no. 1, 187–221, <https://doi.org/10.1090/S0894-0347-1995-1254133-7>.

INDEPENDENT RESEARCHER