

AN EXPLICIT UPPER-BOUND CERTIFICATE FOR THE SCHUR–SIEGEL–SMYTH TRACE CONSTANT

RICARDO PINHO

ABSTRACT. We document an explicit certificate proving $C_{86} < 1.8117251$ for the smallest limiting trace-to-degree ratio of totally positive algebraic integers. The construction uses the primal problem of Orloski and Talebizadeh Sardari, with their four-polynomial set enlarged to eleven. A discretised solution is rounded to a rational step density, then normalised to a probability measure. Exact rational calculations and rigorous ball-arithmetic inequalities verify Smith’s sufficient criterion for an arithmetic measure. An independently written agent checker accepts the certificate and rejects four corrupted controls. This note records the proof, data provenance, reproduction results and limitations of the previously released certificate; it makes no optimality claim.

1. STATEMENT

An algebraic integer α of degree d is *totally positive* if all its conjugates $\alpha_1, \dots, \alpha_d$ are strictly positive real numbers. Write $\overline{\text{tr}}(\alpha) = d^{-1} \sum_j \alpha_j$ and define

$$(1) \quad C_{86} = \sup\{t \in \mathbb{R} : \#\{\alpha \text{ totally positive} : \overline{\text{tr}}(\alpha) < t\} < \infty\}.$$

This is the constant denoted λ_{SSS} in the literature and constant 86a in Tao and contributors’ table [9]. It concerns a limiting ratio, allowing finitely many exceptional integers.

Theorem 1. *The Schur–Siegel–Smyth trace constant satisfies*

$$C_{86} \leq \frac{144938007612277893}{79999999999994000} = 1.811725095153609 \dots < 1.8117251.$$

The exact rational number, rather than its decimal expansion, is the bound proved here. The certificate was already released in [7] and incorporated into the table through PR 227, merged on 18 October 2026 [6]. This paper supplies its mathematical and computational account. Table inclusion is not journal peer review.

2. BACKGROUND

The classical Schur–Siegel–Smyth problem asked whether the smallest limiting ratio was 2. Schur and Siegel established successively stronger lower bounds, and Smyth developed the auxiliary-polynomial method; Smith [8], whose introduction reviews this history, disproved the conjectured value 2 by proving $C_{86} < 1.89831$. His converse to Fekete’s theorem characterises which probability measures on suitable real compact sets occur as limits of conjugate distributions.

We use the positive logarithmic potential convention

$$U_\mu(z) = \int \log |z - y| d\mu(y).$$

Smith uses its negative. A probability measure ν is called *arithmetic* here if $\int \log |Q| d\nu \geq 0$ for every nonzero $Q \in \mathbb{Z}[x]$. Smith’s Theorem 1.5 in arXiv v2 states that, on a compact real set Σ with at most countably many components and logarithmic capacity greater than 1, this condition is

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equivalent to being the weak limit of root-counting probability measures of infinitely many distinct monic irreducible integer polynomials with roots in Σ .

Orloski and Talebizadeh Sardari [4] give a finite-polynomial primal problem that produces such measures. Their four-polynomial choice

$$(2) \quad A_4 = \{x, 1 - x, x^2 - 3x + 1, x^3 - 5x^2 + 6x - 1\}$$

gives $C_{86} \leq 1.8216$. In the checked v2 full text this is Corollary 1.12: $|\Lambda_{A_4} - 1.8215998| \leq 10^{-7}$. The present construction uses their framework and seven further small-trace integer polynomials; it introduces no new converse theorem.

Serre's obstruction, discussed in [4, §1.2] and [8, §5.2], puts the ceiling $1.898302\dots$ on the classical auxiliary-function lower-bound method. Smith's theorem turns the associated measure into an upper bound. This ceiling on lower-bound certificates does not prevent smaller upper bounds obtained from other arithmetic measures.

In the other direction, Orloski, Talebizadeh Sardari and Smith [3] prove the published lower bound 1.80203 ; their article appeared online in 2024 and in volume 94 (2025) of *Mathematics of Computation*. Niedbala Giraudin's September 2026 preprint [1] reports 1.80220 . We checked these source statements, without independently reverifying their certificates. Both lower values are below the upper bound in Theorem 1.

3. THE CONSTRUCTION AND ITS IMPLICATION

For a finite set $A \subset \mathbb{Z}[x]$, the primal problem of [4, §1.3.1] minimises $\int x d\mu$ over nonnegative, compactly supported measures on $[0, \infty)$ and nonnegative weights b_Q , subject to

$$(3) \quad U_\mu(z) \geq \sum_{Q \in A} b_Q \log |Q(z)| \quad (z \in \mathbb{C}), \quad \int \log |Q| d\mu \geq 0 \quad (Q \in A), \quad \mu(\mathbb{R}) \geq 1.$$

The rounded measure below has mass slightly less than 1. Thus it is the pair $(\mu/m, (b_Q/m)_{Q \in A})$, where $m = \mu(\mathbb{R})$, that is feasible. All potential and logarithmic-integral inequalities are homogeneous under this simultaneous scaling.

The data are in `cert_n1200_A11.json` [7]. The grid is $e_i = i/200$, $0 \leq i \leq 1200$, and

$$(4) \quad d\mu(x) = \sum_{i=0}^{1199} d_i \mathbf{1}_{[e_i, e_{i+1}]}(x) dx.$$

All d_i and b_Q are nonnegative rationals with denominator dividing 10^{12} . There are exactly 629 positive-density cells; their union has 25 connected components and lies in $[1/20, 1037/200]$. The complete density array is specified by the hashed certificate, rather than transcribed here. Table 1 gives every polynomial and weight in its order in that file; the decimal weights are exact.

The construction code solves a sampled linear program on this grid, adds points with deficient potential slack, and uses target slack $\delta = 10^{-4}$. Cell densities are rounded to the nearest multiple of 10^{-12} and weights down to multiples of 10^{-12} . These search steps only produce a candidate: feasibility is established afterwards on entire intervals. Exact summation gives

$$(5) \quad m = \frac{3999999999999}{4000000000000}, \quad T := \int x d\mu = \frac{144938007612277893}{8000000000000000},$$

$$\sum_{Q \in A} b_Q \deg Q = \frac{6638236823}{8000000000} < m.$$

For clarity, we give the reduction from finite checks to the assertion about algebraic integers, rather than assuming the candidate is an optimiser of the primal problem.

TABLE 1. The eleven polynomials and the unnormalised rational weights b_Q . The first four are (2).

$Q(x)$	b_Q
x	0.038759530583
$1 - x$	0.079868388122
$x^2 - 3x + 1$	0.046317667230
$x^3 - 5x^2 + 6x - 1$	0.053369420893
$x^4 - 7x^3 + 14x^2 - 8x + 1$	0.019753937664
$x^4 - 7x^3 + 13x^2 - 7x + 1$	0.036545507753
$x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1$	0.016049367094
$x^5 - 9x^4 + 26x^3 - 29x^2 + 11x - 1$	0.009731525491
$x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1$	0.007106935573
$x^3 - 6x^2 + 9x - 1$	0.018100957401
$x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1$	0.002893658474

Proposition 2. Suppose μ is a nonzero bounded-density measure on a compact subset $K \subset (0, \infty)$, and A consists of pairwise nonassociate primitive irreducible integer polynomials. Suppose $b_Q > 0$, $\sum_Q b_Q \deg Q < m = \mu(\mathbb{R})$, $\int \log |Q| d\mu \geq 0$ for each $Q \in A$, and

$$(6) \quad f(x) := U_\mu(x) - \sum_{Q \in A} b_Q \log |Q(x)| \geq 0 \quad (x \in K).$$

If $K \subset \Sigma \subset (0, \infty)$ for a real compact interval Σ of capacity greater than 1, then $C_{86} \leq T/m$.

Proof. Off K , U_μ is harmonic and each $-b_Q \log |Q|$ is superharmonic, with value $+\infty$ at its roots. The bounded density makes U_μ continuous on $\mathbb{C}$. At infinity,

$$f(z) = \left(m - \sum_Q b_Q \deg Q \right) \log |z| - \sum_Q b_Q \log |\text{lc}(Q)| + o(1) \longrightarrow +\infty.$$

The minimum principle on a sufficiently large disk with K removed therefore extends (6) to all $z \in \mathbb{C}$.

Put $\nu = \mu/m$ and $a_Q = b_Q/m$. For any primitive irreducible $P \in \mathbb{Z}[x]$ not associated to an element of A , write $P(x) = c \prod_{j=1}^n (x - \alpha_j)$. The nonzero integer resultants give

$$\begin{aligned} \int \log |P| d\nu &= \log |c| + \sum_j U_\nu(\alpha_j) \\ &\geq \left(1 - \sum_Q a_Q \deg Q \right) \log |c| + \sum_Q a_Q \log |\text{Res}(P, Q)| \geq 0. \end{aligned}$$

For the elements of A the direct integral constraints give this inequality; integer constants and products then give it for every nonzero integer polynomial. This is Smith's sufficiency argument [8, Proposition 5.15, v2].

Apply Smith's Theorem 1.5 to ν on Σ . Its distinct monic irreducible polynomials define distinct totally positive algebraic integers. Weak convergence on the fixed compact interval preserves the mean, so their trace-to-degree ratios tend to $\int x d\nu = T/m$. Definition (1) gives the assertion. $\square$

For our data take $K = \text{supp } \mu$ and $\Sigma = [1/20, 1037/200]$. Its capacity is its length divided by four, namely $1027/800 > 1$. The verification below establishes the remaining hypotheses. Equation (5) then proves Theorem 1.

4. VERIFICATION

The transcendental quantities are evaluated from closed forms, without numerical quadrature. Put $G(u) = u \log |u| - u$, with $G(0) = 0$. A cell's potential is

$$H_i(x) = G(x - e_i) - G(x - e_{i+1}).$$

It decreases to its cell midpoint and increases thereafter. On any real interval $[s, t]$, its minimum is attained at an endpoint or at that midpoint, if present. For a real root r , $\log |x - r|$ is bounded above by the logarithm of the larger endpoint distance. Consequently,

$$(7) \quad f(x) \geq \sum_i d_i \min_{[s,t]} H_i - \sum_Q b_Q \sum_{Q(r)=0} \log \max\{|s - r|, |t - r|\} \quad (s \leq x \leq t),$$

since all leading coefficients in Table 1 have modulus 1. Adaptive subdivision certifies intervals until their lower bounds are positive. This proves a continuum inequality, including cell edges.

The original `verify_cert.py` rederives positivity, mass, moments and the condition at infinity using exact fractions, computes the eleven logarithmic integrals with 50-digit `mpmath`, and checks (7) in floating point with an allowance of 10^{-8} . It is a diagnostic cross-check, not by itself a rigorous transcendental verifier. The companion `verify_arb.py` uses floating point only to choose a partition, then recomputes its lower bounds in 128-bit Arb ball arithmetic. Endpoint balls of radius 10^{-15} cover the rounded dyadic subdivisions of the rational grid cells. The fresh run returned the following excerpts:

```
|A|=11 cells=629 (2) min=0.0001
(3) 155719 subintervals, depth 10
exact = 144938007612277893/79999999999994000
arb check: 78174 subintervals, failures 0,
min LB lower endpoint 1.133e-08
STEP3_RIGOROUS_OK
```

The first count includes unsuccessful parent intervals; the Arb count is the accepted partition. The rigorous direct-integral and algebraic checks are supplied by the independent checker described next.

An independent AI-agent referee wrote `referee_check.py` from scratch, without importing the construction code. Its released version in [7] checks increasing edges, nonnegative densities and weights, support away from zero, exact mass and trace, the exact claimed bound, irreducibility, pairwise coprimality and real-root counts. It isolates roots by exact rational intervals of width at most 10^{-35} and encloses the eleven direct integrals using Arb. All lie in $[0.0000999999986, 0.0001000000008]$, hence are strictly positive.

For the potential inequality it uses a different enclosure: rewrite U_μ as a sum of G terms at density jumps, bound nearby singular terms directly, and use a midpoint value and an interval derivative for the remaining terms. The fresh run reports

```
E4 ok
E5: cells=629, subintervals_proved=6218,
min_certified_lower_bound=7.359790864206603e-08,
max_depth=5, n_failures=0
ALL_OK True
```

The four corrupted-certificate controls were also rerun. Multiplying densities by 0.98 produces a certified negative value of f ; multiplying weights by 1.3 fails the condition at infinity; adding $1/10$ to the first positive cell density fails a direct logarithmic integral and produces negative potential differences; subtracting 10^{-12} from the claimed bound fails exact equality with T/m . All four were rejected. This agent audit is not external peer review.

5. REPRODUCIBILITY

The public repository is <https://github.com/rpinho/c86-trace-certificate>, at commit 116a58ff99681226ce6299d4a02eeff1e7d3c5c7.

The SHA-256 of its `cert_n1200_A11.json` is
 722a287b74a9f84f8c6c2f505e7e6bb987ca
 634710b031287a5b01619bffe53d.

The line break is typographical. A fresh clone was checked on 19 October 2026 with both original scripts copied unchanged from the construction archive, and with the public referee script. The original and public JSON files parse to identical objects; their byte hashes differ because of formatting. The hash above identifies the public file. The public repository contains the certificate and referee checker; the original pair and complete replay logs are retained with this note’s local reproduction report.

From a fresh checkout with its requirements installed, the public verification command is

```
python -I referee_check.py cert_n1200_A11.json 2
```

The final argument limits the pool to two workers. The archived original commands are

```
python verify_cert.py CERT.json  
python verify_arb.py CERT.json 2
```

Limit BLAS and OpenMP to one thread per process. Neither the floating-point optimiser nor its search history is needed to check the released certificate.

6. LIMITATIONS AND PRIOR-ART CHECK

This is an upper bound only. We prove neither optimality for the eleven-polynomial problem nor a matching lower bound for C_{86} . We do not explicitly list a sequence of minimal polynomials achieving the certified limiting distribution. The mathematical implication uses Smith’s published theorem and the minimum-principle argument above; the software checks are not a formal proof in a proof assistant.

The live 86a source and open issues/PRs were checked on 19 October 2026. The source lists 1.8117251 with an asterisk for a self-archived, unrefereed certificate, and 1.8216 as the best unasterisked upper bound; the lower-bound update to 1.80220 remains open as PR 223. The checked arXiv records and full texts include [8, 4, 3, 1]; recent arXiv and public-code searches did not locate a smaller upper bound than this certificate. Search coverage does not prove priority.

Orloski’s seminar abstract at Hebrew University, dated 2 January 2025, announces 1.81618 [2]. Penn State’s official Ph.D. list confirms his 2025 dissertation, *Distributions of Conjugates of Algebraic Integers and the Schur–Siegel–Smyth Trace Problem* [5]. Its library search record is indexed as open access, but direct access was blocked during this audit; the thesis’s numerical contents remain unchecked. Thus we do not assert exhaustive priority over the thesis. This limitation does not affect the bound proved by the certificate.

AI ASSISTANCE AND AUTHOR REVIEW

Analysis and proof drafting by AI agents under my direction; I reviewed the results at summary level. The construction, software verification and independent agent audit also used AI assistance. Independence here means separate code and reasoning, not an unaffiliated human referee.

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INDEPENDENT RESEARCHER