

# WARING–GOLDBACH PROBLEM: ONE SQUARE, FOUR FIFTH POWERS AND TWENTY-TWO SEVENTH POWERS

RUNBO LI

ABSTRACT. We prove that for every sufficiently large odd integer  $n$ , the equation

$$n = p_1^2 + p_2^5 + p_3^5 + p_4^5 + p_5^5 + p_6^7 + \cdots + p_{27}^7$$

has a solution with all  $p_j$  prime.

## CONTENTS

|                         |   |
|-------------------------|---|
| 1. Introduction         | 1 |
| 2. Preliminaries        | 1 |
| 3. The major arcs       | 2 |
| 4. The minor arcs       | 5 |
| 5. Proof of Theorem 1.1 | 6 |
| References              | 6 |

## 1. INTRODUCTION

Let  $n$  be a sufficiently large positive integer that satisfies some necessary congruence conditions. Using the Hardy–Littlewood circle method, many mathematicians studied the number of representations of  $n$  as the sum of mixed powers of prime numbers. Examples of recent works on this topic include [1], [2], [3], [4], [5], [10] and [12]. In this paper, we prove the following result.

**Theorem 1.1.** *For every sufficiently large odd integer  $n$ , the equation*

$$n = p_1^2 + p_2^5 + p_3^5 + p_4^5 + p_5^5 + p_6^7 + \cdots + p_{27}^7 \quad (1)$$

*has a solution with all  $p_j$  prime.*

## 2. PRELIMINARIES

Let  $n$  be a sufficiently large positive integer,  $P_k = n^{\frac{1}{k}}$  and  $L = \log n$ . Let

$$\begin{aligned} \lambda_j &= \left(\frac{193}{224}\right)^{j-1} \quad \text{for } 1 \leq j \leq 12, & \lambda_{13} &= \frac{8037507997288250129453953306362022519521}{48010323730121098619967367861370284408832}, \\ \lambda_{14} &= \frac{216678498509465522283961262046235598847}{1500322616566284331873980245667821387776}, & \lambda_{15} &= \frac{5848665956963041958482200115320342801}{46885081767696385371061882677119418368}, \\ \lambda_{16} &= \frac{79052567057751589089425005400628381}{732579402620256021422841916829990912}, & \lambda_{17} &= \frac{34273598181843435759730601640169785}{366289701310128010711420958414995456}, \\ \lambda_{18} &= \frac{59695167605853906780155489012271975}{732579402620256021422841916829990912}, & \lambda_{19} &= \frac{26139610423872867716681836920745725}{366289701310128010711420958414995456}, \\ \lambda_{20} &= \frac{23192117260350111119548573410394575}{366289701310128010711420958414995456}, & \lambda_{21} &= \lambda_{22} = \frac{5352027060080794873741978479321825}{91572425327532002677855239603748864}, \end{aligned}$$

2020 Mathematics Subject Classification. 11P32, 11P55.

Key words and phrases. Waring–Goldbach problem, circle method.

and write

$$\Lambda = \sum_{j=1}^{22} \lambda_j = \frac{10735022333353534120309774318126527527533695}{1536330359363875155838955771563849101082624},$$

$$\Theta = \frac{3}{10} + \frac{\Lambda}{7} = \frac{69806580440088359737857907192053053199036027}{53771562577735630454363452004734718537891840} \approx 1.29821.$$

We define

$$f_k(\alpha) = \sum_{\frac{1}{2}P_k < p \leq P_k} e(\alpha p^k) \log p$$

and, for  $1 \leq j \leq 22$ ,

$$f_{k,j}(\alpha) = \sum_{\frac{1}{2}P_k^{\lambda_j} < p \leq P_k^{\lambda_j}} e(\alpha p^k) \log p.$$

We also define

$$F(\alpha) = f_2(\alpha) f_5(\alpha)^4 \prod_{j=1}^{22} f_{7,j}(\alpha).$$

Then by the orthogonality of additive characters, the integral

$$\int_0^1 F(\alpha) e(-\alpha n) d\alpha \quad (2)$$

counts the number of solutions of (1) with prime variables lying in certain ranges weighted by a log factor. Our goal is to show the lower bound

$$\int_0^1 F(\alpha) e(-\alpha n) d\alpha \gg n^\Theta \quad (3)$$

holds for all odd  $n$ .

Fix  $B \geq 1$ . Put

$$\mathfrak{M}(q, a) = \left\{ \alpha \in [0, 1] : \left| \alpha - \frac{a}{q} \right| \leq \frac{L^B}{n} \right\}, \quad \mathfrak{M} = \bigcup_{1 \leq q \leq L^B} \bigcup_{\substack{0 \leq a \leq q \\ (a, q) = 1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = [0, 1] \setminus \mathfrak{M}.$$

Then (3) can be obtained if we get

$$\int_{\mathfrak{M}} F(\alpha) e(-\alpha n) d\alpha \gg n^\Theta \quad (4)$$

and

$$\int_{\mathfrak{m}} F(\alpha) e(-\alpha n) d\alpha \ll n^\Theta L^{-1}. \quad (5)$$

### 3. THE MAJOR ARCS

Define

$$S_k(q, a) = \sum_{\substack{x=1 \\ (x, q)=1}}^q e\left(\frac{ax^k}{q}\right), \quad v_{k,r}(\beta) = \frac{1}{k} \sum_{2^{-k}n^r < m \leq n^r} m^{1/k-1} e(\beta m), \quad v_k(\beta) = v_{k,1}(\beta),$$

$$U(q, a) = S_2(q, a) S_5(q, a)^4 S_7(q, a)^{22}, \quad w(\beta) = v_2(\beta) v_5(\beta)^4 \prod_{j=1}^{22} v_{7,\lambda_j}(\beta).$$

**Lemma 3.1.** *Let  $q, a$  be integers such as  $1 \leq q \leq L^B$ ,  $0 \leq a \leq q$  and  $(a, q) = 1$ . Then for  $\alpha \in \mathfrak{M}(q, a)$ , we have*

$$\sum_{\frac{1}{2}P_k^r < p \leq P_k^r} e(\alpha p^k) \log p = \frac{1}{\varphi(q)} S_k(q, a) v_{k,r}(\alpha - a/q) + O\left(P_k^r \exp(-c\sqrt{L})\right)$$

for some constant  $c > 0$ .

*Proof.* This lemma follows from [[6], Lemma 6] and partial summation. □

We shall use this lemma to give an approximation of  $F(\alpha)$  using functions  $S_k(q, a)$  and  $v_{k,r}$  defined above. By Lemma 3.1, we have

$$F(\alpha) = \frac{1}{\varphi(q)^{27}} U(q, a) w(\alpha - a/q) + O(n^{1+\Theta} L^{-4B}). \quad (6)$$

An integration over  $\mathfrak{M}$  gives

$$\int_{\mathfrak{M}} F(\alpha) e(-\alpha n) d\alpha = \sum_{q \leq L^B} \frac{A_n(q)}{\varphi(q)^{27}} \int_{-L^B/n}^{L^B/n} w(\beta) e(-\beta n) d\beta + O(n^\Theta L^{-B}), \quad (7)$$

where

$$A_n(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^q U(q, a) e(-an/q).$$

We first show that

$$\int_{-L^B/n}^{L^B/n} w(\beta) e(-\beta n) d\beta \gg n^\Theta. \quad (8)$$

**Lemma 3.2.** ([11], Lemma 6.2]). Suppose that  $|\beta| \leq \frac{1}{2}$ . Then,

$$v_k(\beta) \ll n^{\frac{1}{k}} (1 + n|\beta|)^{-1}.$$

By Lemma 3.2 we know that

$$w(\beta) \ll n^{1+\Theta} (1 + n|\beta|)^{-2}. \quad (9)$$

By (9), we get

$$\begin{aligned} & \int_{-L^B/n}^{L^B/n} w(\beta) e(-\beta n) d\beta \\ &= \int_{-1/2}^{1/2} w(\beta) e(-\beta n) d\beta + O\left(\int_{L^B/n}^{1/2} |w(\beta)| d\beta\right) \\ &= \int_{-1/2}^{1/2} w(\beta) e(-\beta n) d\beta + O(n^\Theta L^{-B}) \\ &= \frac{1}{2 \cdot 5^4 \cdot 7^{22}} \sum_{\substack{m_1 + \dots + m_{27} = n \\ 2^{-2}n < m_1 \leq n \\ 2^{-5}n < m_2, m_3, m_4, m_5 \leq n \\ 2^{-7}n^{\lambda_j-5} < m_j \leq n^{\lambda_j-5} (6 \leq j \leq 27)}} m_1^{-\frac{1}{2}} (m_2 m_3 m_4 m_5)^{-\frac{4}{5}} \prod_{j=6}^{27} m_j^{-\frac{6}{7}} + O(n^\Theta L^{-B}) \\ &\gg \sum_{\substack{m_1 + \dots + m_{27} = n \\ \frac{1}{4}n < m_1 \leq \frac{1}{2}n \\ \frac{1}{32}n < m_2, m_3, m_4 \leq \frac{3}{64}n \\ \frac{1}{128}n^{\lambda_j-5} < m_j \leq \frac{3}{256}n^{\lambda_j-5} (6 \leq j \leq 27)}} m_1^{-\frac{1}{2}} (m_2 m_3 m_4 m_5)^{-\frac{4}{5}} \prod_{j=6}^{27} m_j^{-\frac{6}{7}} \gg n^\Theta. \end{aligned} \quad (10)$$

Next, we want to deal with the sum

$$\sum_{q \leq L^B} \frac{A_n(q)}{\varphi(q)^{27}}. \quad (11)$$

Define the series

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \frac{A_n(q)}{\varphi(q)^{27}}.$$

**Lemma 3.3.** ([6], Lemma 5]). For  $k \geq 2$ , we have

$$S_k(q, a) \ll q^{\frac{1}{2}+\varepsilon}.$$

**Lemma 3.4.** ([11], Lemma 4.3]. For  $(a, p) = 1$ , we have

$$|S_k(p, a)| \leq ((k, p-1) - 1)\sqrt{p} + 1.$$

**Lemma 3.5.** ([6], Lemma 4]). Let  $p^n \mid k$ ,  $p^{n+1} \nmid k$  and

$$l = \begin{cases} n+2, & p=2, \\ n+1, & p \neq 2. \end{cases}$$

If  $t > l$ , then

$$S_k(p^t, a) = 0.$$

By Lemma 3.3, we know that  $U(q, a) \ll q^{\frac{27}{2}+\varepsilon}$  for  $(a, q) = 1$ , and hence  $A_n(q) \ll q^{\frac{29}{2}+\varepsilon}$  uniformly in  $n$ . Thus,  $\mathfrak{S}(n)$  converges absolutely, and

$$\sum_{q \leq L^B} \frac{A_n(q)}{\varphi(q)^{27}} = \mathfrak{S}(n) + O(L^{-B}). \quad (12)$$

Since the complete exponential sums are multiplicative, for  $(q_1, q_2) = 1$  we have

$$A_n(q_1 q_2) = A_n(q_1) A_n(q_2).$$

Moreover, By Lemma 3.5 and the fact that  $A_n(4) = S_5(4, a) = 0$ , we have  $A_n(q) = 0$  unless  $q$  is square-free. Hence

$$\mathfrak{S}(n) = \prod_p \mathfrak{S}_p(n) = \prod_p \left( 1 + \frac{A_n(p)}{(p-1)^{27}} \right). \quad (13)$$

By the orthogonality of additive characters modulo  $p$ , we have

$$\mathfrak{S}_p(n) = \frac{pM_n(p)}{(p-1)^{27}}, \quad (14)$$

where  $M_n(p)$  denote the number of solutions  $(x_1, \dots, x_{27}) \in (\mathbb{F}_p^\times)^{27}$  of

$$x_1^2 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{j=6}^{27} x_j^7 \equiv n \pmod{p}. \quad (15)$$

When  $p = 2$ , every unit is 1, and

$$x_1^2 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{j=6}^{27} x_j^7 \equiv 1 \pmod{2}.$$

Hence  $n \equiv 1 \pmod{2}$  is necessary.

When  $p = 3$ , the square is 1, and every fifth and seventh power is  $\pm 1$ . It is easy to see that the sum of 26 fifth and seventh powers contains all residue classes modulo 3, and (15) has a solution.

When  $p = 5$ , we have  $x^5 = x$ . Since  $\mathbb{F}_5^\times + \mathbb{F}_5^\times = \mathbb{F}_5$ , the sum of two fifth powers contains all residue classes modulo 5, and (15) has a solution.

When  $p = 7$ , we have  $x^7 = x$ . Since  $\mathbb{F}_7^\times + \mathbb{F}_7^\times = \mathbb{F}_7$ , the sum of two seventh powers contains all residue classes modulo 7, and (15) has a solution.

When  $p \geq 11$ , we define  $\mathcal{A}_k = \mathcal{A}_k(p) = \{x^k : x \in \mathbb{F}_p\}$ . It is known that  $|\mathcal{A}_k(p)| = \frac{p-1}{(k, p-1)}$ . Applying the Cauchy–Davenport theorem repeatedly, we have

$$\begin{aligned} |\mathcal{A}_2 + 4\mathcal{A}_5 + 22\mathcal{A}_7| &\geq \min(p, |\mathcal{A}_2| + 4|\mathcal{A}_5| + 22|\mathcal{A}_7| - 26) \\ &\geq \min\left(p, \frac{p-1}{2} + \frac{4(p-1)}{5} + \frac{22(p-1)}{7} - 26\right) \\ &\geq \min\left(p, \frac{311p-2130}{70}\right). \end{aligned}$$

Since  $p \geq 11$ , we have  $\frac{311p-2130}{70} > p$ , and thus  $\mathcal{A}_2 + 4\mathcal{A}_5 + 22\mathcal{A}_7 = \mathbb{F}_p$ . Now (15) has a solution for every prime  $p \geq 11$ .

Combining above cases, we know that  $M_n(p) \geq 1$  if  $n$  is odd, and also  $\mathfrak{S}(n) > 0$  by (13). Now, by Lemma 3.4 and trivial estimates, we have

$$\begin{aligned} \left| \frac{A_n(p)}{(p-1)^{27}} \right| &\leq \frac{p(\sqrt{p}+1)(4\sqrt{p}+1)^4(6\sqrt{p}+1)^{22}}{(p-1)^{27}} \\ &= \frac{p^{\frac{29}{2}}}{(p-1)^{27}} \left(1 + \frac{1}{\sqrt{p}}\right) \left(4 + \frac{1}{\sqrt{p}}\right)^4 \left(6 + \frac{1}{\sqrt{p}}\right)^{22} \\ &\leq \frac{p^{15}}{(p/2)^{27}} \left(1 + \frac{1}{\sqrt{p}}\right) \left(4 + \frac{1}{\sqrt{p}}\right)^4 \left(6 + \frac{1}{\sqrt{p}}\right)^{22} \\ &\leq \frac{2^{27} \cdot 2 \cdot 5^4 \cdot 7^{22}}{p^{12}} \end{aligned}$$

for  $p > 10^{10}$ . Thus, we get

$$\prod_{p > 10^{10}} \left(1 + \frac{A_n(p)}{(p-1)^{27}}\right) \geq \prod_{p > 10^{10}} \left(1 - \frac{2^{28} \cdot 5^4 \cdot 7^{22}}{p^{12}}\right) > c \quad (16)$$

with some  $c > 0$ . By (13), (14), (16) and  $M_n(p) \geq 1$ , we have

$$\mathfrak{S}(n) > c \prod_{p \leq 10^{10}} \frac{pM_n(p)}{(p-1)^{27}} > c \prod_{p \leq 10^{10}} p^{-26} \gg 1. \quad (17)$$

Now, by (7), (10), (12) and (17), the proof of (4) is completed.

#### 4. THE MINOR ARCS

We first recall some mean value estimates proved by Andrade [3] and Kumchev [7]. Note that we can remove the logarithmic weights and the restriction to primes in  $f_k(\alpha)$  and  $f_{k,j}(\alpha)$  at the cost of a power of  $L$ , and deal with the classical Weyl sums.

**Lemma 4.1.** *We have*

$$\int_0^1 |f_2(\alpha)|^2 |f_5(\alpha)|^6 d\alpha \ll n^{\frac{6}{5} + \varepsilon}.$$

*Proof.* Taking  $k_1 = k_2 = k_3 = 5$  in [[3], Lemma 4], we get Lemma 4.1. □

**Lemma 4.2.** *We have*

$$\int_0^1 \left| \prod_{j=1}^{22} f_{7,j}(\alpha) \right|^2 d\alpha \ll P_7^{\Lambda + \varepsilon}.$$

*Proof.* Taking  $i = 1$  and  $P = P_7$  in [[7], Lemma 1], we get Lemma 4.2. □

Write  $\tau = 2^{-100}$ . Define

$$\mathcal{E} = \{\alpha \in [0, 1] : |f_5(\alpha)| \leq P_5^{(3\Lambda - 14)/7 - 10\tau}\}.$$

Using Hölder's inequality, together with Lemma 4.1, Lemma 4.2 and the definition of  $\mathcal{E}$ , we get

$$\begin{aligned} \int_{\mathcal{E}} |F(\alpha)| d\alpha &\leq \left( \int_0^1 |f_2(\alpha)|^2 |f_5(\alpha)|^6 d\alpha \right)^{\frac{1}{2}} \left( \int_0^1 \left| \prod_{j=1}^{22} f_{7,j}(\alpha) \right|^2 d\alpha \right)^{\frac{1}{2}} \sup_{\alpha \in \mathcal{E}} |f_5(\alpha)| \\ &\ll n^{\frac{3}{5} + \varepsilon} P_7^{\Lambda/2 + \varepsilon} P_5^{(3\Lambda - 14)/7 - 10\tau} \ll n^{\theta - \tau}, \end{aligned} \quad (18)$$

where

$$\theta = \frac{139593870697983127505152898301285690218027381}{107543125155471260908726904009469437075783680} \approx 1.29803.$$

Note that  $\Theta - \theta > 0.0001$ .

## 5. PROOF OF THEOREM 1.1

**Lemma 5.1.** *Suppose that  $X^{\frac{1}{24}} \leq P \leq X^{\frac{9}{20}}$ . Then either we have*

$$\sum_{X < p \leq 2X} e(\alpha p^5) \ll X^{\frac{47}{48} + \varepsilon},$$

*or there exist integers  $q$  and  $a$  such that  $1 \leq q \leq P$ ,  $(a, q) = 1$  and  $|q\alpha - a| \leq PX^{-5}$ .*

*Proof.* Taking  $k = 5$  in [[9], Lemma 2.2], we get Lemma 5.1. □

Now suppose that  $\alpha \in [0, 1] \setminus \mathcal{E}$ . We have  $|f_5(\alpha)| > P_5^{(3\Lambda-14)/7-10\tau}$  in this case. Taking  $X = \frac{1}{2}P_5$  and  $P = \frac{1}{32}n^{\frac{1}{19}}$  in Lemma 5.1, we find that the first case will never occur since  $\frac{3\Lambda-14}{7} - \frac{47}{48} > 0.01$ . Hence, we must have

$$[0, 1] \setminus \mathcal{E} \subset \mathfrak{R} = \bigcup_{1 \leq q \leq \frac{1}{32}n^{\frac{1}{19}}} \bigcup_{\substack{0 \leq a \leq q \\ (a, q) = 1}} \mathfrak{R}(q, a), \quad (19)$$

where

$$\mathfrak{R}(q, a) = \{\alpha \in [0, 1] : |q\alpha - a| \leq n^{-\frac{118}{119}}\}. \quad (20)$$

Define a function  $\Upsilon : \mathfrak{R} \rightarrow [0, 1]$  by

$$\Upsilon(\alpha) = (q + n|q\alpha - a|)^{-1}.$$

Taking  $k = 2, 5$  in [[8], Theorem 2] and by partial summation, we have

$$f_2(\alpha) \ll P_2 L^C \Upsilon(\alpha)^{\frac{1}{2} - \varepsilon} \quad (21)$$

and

$$f_5(\alpha) \ll P_5 L^C \Upsilon(\alpha)^{\frac{1}{2} - \varepsilon} \quad (22)$$

for some constant  $C > 0$ . By (21), (22) and trivial upper bounds for  $f_{7,j}(\alpha)$  ( $1 \leq j \leq 22$ ), we have

$$F(\alpha) = f_2(\alpha)f_5(\alpha)^4 \prod_{j=1}^{22} f_{7,j}(\alpha) \ll n^{1+\Theta} L^{27C} \Upsilon(\alpha)^{\frac{49}{20}}. \quad (23)$$

Integrating over  $\mathfrak{R} \setminus \mathfrak{M}$  gives

$$\int_{\mathfrak{R} \setminus \mathfrak{M}} |F(\alpha)| d\alpha \ll n^{1+\Theta} L^{27C} \int_{\mathfrak{R} \setminus \mathfrak{M}} |\Upsilon(\alpha)|^{\frac{49}{20}} d\alpha \ll n^{\Theta} L^{27C - \frac{9}{20}B+1}. \quad (24)$$

Choosing  $B = 200C + 100$  in (24) and by the fact that  $\mathfrak{m} \subset \mathcal{E} \cup \mathfrak{R} \setminus \mathfrak{M}$ , the proof of (5) is completed. From (4) and (5) we get (3), and the proof of Theorem 1.1 is completed.

## REFERENCES

- [1] G. M. L. Andrade. On some Waring–Goldbach problems. *Ramanujan J.*, 69(67), 2026.
- [2] G. M. L. Andrade. On sums of two squares, two cubes, and higher powers of primes. *Bull. Braz. Math. Soc., New Series*, 57(35), 2026.
- [3] G. M. L. Andrade. Waring–Goldbach problems for one square and higher powers. *Indag. Math., in press, corrected proof. arXiv e-prints*, page arXiv:2603.05550v1, 2026.
- [4] G. M. L. Andrade and H. Godinho. Representing integers as sums of mixed powers of primes. *Indag. Math., in press, corrected proof. arXiv e-prints*, page arXiv:2512.05154v1, 2025.
- [5] J. Brüdern. Another Waring–Goldbach problem. *Acta. Arith.*, 210:125–132, 2023.
- [6] L.-K. Hua. Some results in the additive prime number theory. *Quart. J. Math. Oxford Ser.*, 9(1):68–80, 1938.
- [7] A. V. Kumchev. On the Waring–Goldbach problem for seventh powers. *Proc. Amer. Math. Soc.*, 133(10):2927–2937, 2005.
- [8] A. V. Kumchev. On Weyl sums over primes and almost primes. *Michigan Math. J.*, 54(2):243–268, 2006.
- [9] A. V. Kumchev and T. D. Wooley. On the Waring–Goldbach problem for seventh and higher powers. *Monatsh. Math.*, 183(2):303–310, 2017.
- [10] L. Long, J. Li, and M. Zhang. On Waring–Goldbach problem for two squares, three biquadratics, one quintic and one septic. *arXiv e-prints*, page arXiv:2610.15142v1, 2026.
- [11] R. C. Vaughan. *The Hardy–Littlewood method, 2nd Edition*, volume 125 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997.
- [12] H. Wang and S. Tian. On Waring–Goldbach problem with mixed powers. *arXiv e-prints*, page arXiv:2608.12397v1, 2026.

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, LONDON SW7 2AZ, UK  
*Email address:* `runbo.li26@imperial.ac.uk`, `runbo.li.carey@gmail.com`