

Disjoint union activation of difference distance magic labelings

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Abstract

A graph is difference distance magic orientable (DDMO) if its vertices can be labeled bijectively by consecutive positive integers and its edges oriented so that the incoming and outgoing neighbor-label sums agree at every vertex. We give an affirmative answer to the disjoint-union question posed by Aceska et al. in January 2026: two copies of the non-DDMO graph $K_{3,3}$ have a DDMO disjoint union. More precisely, for $k \geq 1$ and $r \geq 0$, the graph $kK_{3,3} \sqcup rK_1$ is difference distance magic orientable if and only if k is even or $r \geq 1$. The proof reduces fixed-label bipartite orientations to signed zero sums and applies classical Skolem and hooked Skolem pairings. A compatible fixed labeling of $K_{3,3}$ has exactly two orientations, related by reversal. We prove that twelve vertices is the minimum order of a DDMO graph whose connected components are all non-DDMO in isolation. A separate exact computer-assisted classification shows that $2K_{3,3}$ is the unique graph attaining that minimum.

Keywords. Graph labeling; difference distance magic; disjoint union; Skolem pairing; skew-adjacency matrix.

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1 The question and the label set

All graphs in this note are finite and simple. An orientation chooses exactly one direction for each edge. Write $[N] = \{1, \dots, N\}$. Following Aceska et al. [1, Definition 2.1], a *difference distance magic* (DDM) labeling of an orientation D of an N -vertex graph is a bijection $f : V(D) \rightarrow [N]$ such that

$$\text{wt}_f(v) = \sum_{u \rightarrow v} f(u) - \sum_{v \rightarrow u} f(u) = 0 \quad (v \in V(D)). \quad (1)$$

A graph is *difference distance magic orientable* (DDMO) if some orientation admits such a labeling. In particular, K_1 is DDMO. For an arbitrary prescribed injective labeling by positive real numbers, we call an orientation *compatible* when it satisfies (1); this does not require that its label set be consecutive.

Aceska et al. [1, Section 6, p. 19] ask whether a union of non-DDMOGs can be DDMO. We answer the stronger underlying-graph interpretation: every connected component of our example fails to be DDMO on its own consecutive label set, while their disjoint union is DDMO. Their union construction [1, Theorem 5.1] starts from DDM oriented graphs and assumes all but possibly one are balanced, so it does not supply this example. We use the word *activation* descriptively for the change from failure on each component's consecutive label set to success on the shared label set of the union. No edges are added between components.

Altman et al. [2, Theorem 5, p. 4] already prove that $K_{a,b}$ is DDMO exactly when $a, b \geq 3$ and $a + b \equiv 0$ or $3 \pmod{4}$. Thus the failure of $K_{3,3}$ alone is known. The pairing constructions used below are also classical. Our purpose is the disjoint-union application, its exact copy-count and isolate thresholds, fixed-label rigidity, and the smallest possible activation. We give direct proofs of the needed special cases rather than claim new pairing theorems.

2 Fixed labels and bipartite orientations

Lemma 2.1. *Assign distinct positive labels $a_1, \dots, a_p, b_1, \dots, b_q$ to the two parts of $K_{p,q}$, where $p, q \geq 1$. A compatible orientation exists if and only if both parts have a signed zero sum:*

$$\sum_{i=1}^p u_i a_i = 0, \quad \sum_{j=1}^q v_j b_j = 0 \quad \text{for some } u_i, v_j \in \{-1, 1\}. \quad (2)$$

Proof. Let $X_{ij} = 1$ when the edge is oriented from a_i to b_j , and let $X_{ij} = -1$ otherwise. With $a = (a_i)$ and $b = (b_j)$ as column vectors, the zero-weight equations are

$$Xb = 0, \quad X^T a = 0.$$

Any row of X gives the required signed zero sum for b , and any column gives one for a . Conversely, given sign vectors u, v satisfying (2), set $X = uv^T$. Every entry is a sign, so X specifies an orientation, and both matrix equations hold. $\square$

Corollary 2.2. *A disjoint union $\sqcup_{j=1}^t K_{p_j, q_j}$ is DDMO if and only if its consecutive label set can be partitioned into parts of sizes $p_1, q_1, \dots, p_t, q_t$, each having a signed zero sum.*

Proof. Restrict a DDM labeling to the bipartition of every component and apply Lemma 2.1. Conversely, apply its construction independently to each pair of parts. $\square$

A set $\{a, b, c\}$ of distinct positive numbers with $a < b < c$ is a *Schur triple* if $a + b = c$. A triple has a signed zero sum precisely when it is a Schur triple: positivity puts the singleton side of an equality above the other two members. In increasing order its only zero-sum sign vectors are $(1, 1, -1)$ and its negative.

Proposition 2.3. *A fixed injective positive labeling of $K_{3,3}$ admits a compatible orientation if and only if both parts are Schur triples. When they are, exactly two orientations are compatible, and they are global reversals. Consequently, a fixed labeling of $kK_{3,3}$ admits either no compatible orientation or exactly 2^k of them.*

Proof. The existence statement follows from Lemma 2.1. Let u, v denote the zero-sum sign vectors on the two parts, normalized by putting the negative sign on the largest label. Every row of a

compatible X must be $v^\top$ or $-v^\top$, so $X = cv^\top$ for a sign vector c . The column equations imply $c^\top a = 0$, whence $c = u$ or $c = -u$. Thus $X = uv^\top$ or $-uv^\top$. Components have independent choices. $\square$

The count is for fixed, distinguished vertices with their labels already assigned, not for isomorphism classes of labelings or orientations.

3 Activation and the isolate threshold

Theorem 3.1. *For every integer $k \geq 1$ and $r \geq 0$,*

$$kK_{3,3} \sqcup rK_1 \text{ is DDMO} \iff k \text{ is even or } r \geq 1.$$

Proof. First suppose $r = 0$. Proposition 2.3 makes each of the $2k$ partite triples a Schur triple, whose sum is even. Hence the total label sum

$$1 + \cdots + 6k = 3k(6k + 1)$$

is even. Since 3 and $6k + 1$ are odd, k must be even.

For sufficiency put $n = 2k$. If k is even, then $n \equiv 0 \pmod{4}$. A Skolem pairing partitions $[2n]$ into pairs (a_i, b_i) satisfying $b_i - a_i = i$ for $i \in [n]$. Such a pairing exists by Skolem [3, Theorem 2, p. 57; construction, p. 58]. The triples

$$T_i = \{i, n + a_i, n + b_i\}, \quad i \in [n], \tag{3}$$

are Schur triples and partition $[3n] = [6k]$: the small entries are $[n]$, and the remaining entries are $n + [2n]$. Group the $2k$ triples into k pairs and use Lemma 2.1. Extra isolates, if present, receive $6k + 1, \dots, 6k + r$.

If k is odd and $r \geq 1$, then $n \equiv 2 \pmod{4}$. A hooked Skolem pairing partitions $[2n + 1] \setminus \{2n\}$ into pairs with the same differences $b_i - a_i = i$. O’Keefe’s construction [4, pp. 80–81] supplies these pairs. Now (3) partitions

$$[3n + 1] \setminus \{3n\} = [6k + 1] \setminus \{6k\}.$$

Pair the triples as before and assign label $6k$ to one isolate. Any remaining isolates receive $6k + 2, \dots, 6k + r$. All labels are used once, and isolates impose no equations. Explicit formulas for the classical pairings and their verification appear in Appendix A. $\square$

A twelve-vertex certificate. For $2K_{3,3}$, take the parts

	first part	second part
component 1	(1, 11, 12)	(2, 7, 9)
component 2	(3, 5, 8)	(4, 6, 10).

On each displayed triple put signs $(1, 1, -1)$. Orient an edge from the first part to the second exactly when its endpoint signs agree; otherwise reverse it. For example, the incoming and outgoing sums at label 1 are 9 and $2 + 7$, and at label 12 they are $2 + 7$ and 9. Lemma 2.1 verifies all vertices. Neither component can be DDMO on its own labels [6], since its two part sums would be even but $1 + \cdots + 6 = 21$ is odd.

A seven-vertex repair. For $K_{3,3} \sqcup K_1$, use triples $(1, 3, 4)$ and $(2, 5, 7)$, the same sign rule, and isolate label 6. This does not contradict the minimum below: its isolated component is already DDMO.

4 The minimum order

Lemma 4.1. *If a graph on at most five vertices has a compatible orientation for some injective positive labeling, then it is DDMO.*

Proof. An edgeless graph is DDMO. A vertex of degree one cannot have zero weight with positive neighbor labels. At degree two, zero weight requires its two distinct neighbor labels to be equal. Thus every component containing an edge has minimum degree at least three, so its order is at least four. With at most five vertices there is at most one such component; we first classify that component.

For an orientation write $S_{ij} = 1$ if $i \rightarrow j$, $S_{ij} = -1$ if $j \rightarrow i$, and $S_{ij} = 0$ otherwise. Then S is skew-symmetric and compatibility is $Sx = 0$; this sign convention gives the negative of the weight vector in (1), which has the same zero set. For a 4×4 skew-symmetric matrix,

$$\det S = \text{Pf}(S)^2, \quad \text{Pf}(S) = S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23}.$$

An order-four candidate must be K_4 . Its Pfaffian is a sum of three signs, hence is odd and nonzero. Its kernel is therefore trivial.

On five vertices the complement has maximum degree one, so the graph is K_5 , $K_5 - e$, or $K_5 - 2e$, where the two deleted edges in the last case are disjoint. In either of the first two graphs a principal K_4 submatrix is nonsingular. An odd-order skew-symmetric matrix has even rank at most four, so $\text{rank } S = 4$. Its kernel is spanned by

$$p_i = (-1)^{i+1} \text{Pf}(S_{\hat{i}}), \quad 1 \leq i \leq 5, \quad (4)$$

where $S_{\hat{i}}$ deletes row and column i . The identity $Sp = 0$ follows by Pfaffian expansion; a nonzero 4×4 principal minor ensures $p \neq 0$, so rank four proves the spanning assertion.

For K_5 , every p_i is one of $-3, -1, 1, 3$. A scalar multiple of p cannot have five distinct positive coordinates. For $K_5 - e$, omit any of the three vertices that are not endpoints of the missing edge. The resulting four-vertex graph is $K_4 - e$, whose Pfaffian is the sum of two signs. Thus the corresponding three coordinates of p lie in $\{0, -2, 2\}$. A positive scalar multiple, allowing a common reversal of all signs, cannot make these coordinates both nonzero and distinct. Neither graph admits the prescribed labeling.

Finally $K_5 - 2e$ is the wheel W_4 with four rim vertices. Label its opposite nonadjacent pairs $(1, 4)$ and $(2, 3)$, and its hub 5. Orient all edges cyclically from the hub to the first pair, from the first pair to the second, and from the second to the hub. Every vertex has zero weight because $1 + 4 = 2 + 3 = 5$. Hence this remaining graph is DDMO. It uses all five vertices, while the order-four case was impossible, so no additional isolates occur in a graph with edges under consideration. $\square$

Theorem 4.2. *Let G be a DDMO graph with at least two connected components, each of which is non-DDMO on its own consecutive label set. Then $|V(G)| \geq 12$. Equality is attained by $2K_{3,3}$.*

Proof. A DDM labeling of G restricts to an injective positive compatible labeling on every connected component. By Lemma 4.1, each non-DDMO component therefore has at least six vertices. There are at least two components, so $|V(G)| \geq 12$. The preceding certificate attains twelve. $\square$

The hypothesis concerns the actual connected components, not an arbitrary decomposition into possibly disconnected factors. The proof of Theorem 4.2 is independent of the computation in the next section.

5 Computer assisted uniqueness at the minimum

Theorem 5.1 (Exact finite computation). *A connected graph on six vertices admits a compatible orientation for some injective strictly positive real labeling if and only if it is isomorphic to $K_{3,3}$, $K_{1,2,3}$, or $K_{2,2,2}$. Among these graphs, only $K_{3,3}$ is non-DDMO on [6].*

Proof and computational certificate. We describe the complete finite reduction and the exact tests, whose results are recorded in Table 1. The minimum-degree argument of Lemma 4.1 applies, so the complement has maximum degree at most two. It is a disjoint union of paths and cycles. Enumerating multisets of their component orders summing to six gives the 19 complement types in the table. Independently, enumerating all 2^{15} labeled graphs gives 1858 graphs of minimum degree at least three and the same 19 types. Such graphs are connected, since any component would have at least four vertices.

For a fixed graph with m edges, choose a spanning tree. Vertex switching replaces S by DSD , where D is diagonal with entries ± 1 , and replaces a kernel vector x by Dx . Switching can make every upper-triangular entry on the tree equal to 1, uniquely up to changing all entries of D together. Thus exactly 2^{m-5} chord-sign choices cover its switching classes.

Let B be any matrix whose columns form a basis of $\ker S$, and write b_i for its i th row. Some switching of S has a strictly positive kernel vector with pairwise distinct coordinates if and only if

$$b_i \neq 0 \quad \text{for every } i, \quad b_i \neq b_j \text{ and } b_i \neq -b_j \quad (i \neq j). \quad (5)$$

Indeed, these conditions say that $\ker S$ is contained in none of the hyperplanes $x_i = 0$ or $x_i = \pm x_j$. A vector space over the infinite field $\mathbb{Q}$ cannot be covered by finitely many proper linear hyperplanes. Since S has integer entries, a rational vector avoiding them exists. Switching makes its coordinates positive, and clearing denominators gives distinct positive integers. Conversely, a positive injective vector after switching has nonzero, pairwise distinct absolute coordinates before switching, which implies (5). This is a test on the entire kernel, not on individual basis vectors.

The first implementation uses exact rational row reduction for all 3040 representatives. The independent implementation enumerates the labeled graphs, computes integral Pfaffians, and uses the 4×4 Pfaffian cofactors when the rank is four. It checks that the cofactor matrix C satisfies $SC = 0$. When $\text{rank } S = 4$, the matrix C is nonzero and skew-symmetric, so its rank is at least two; its columns lie in the two-dimensional kernel, and hence span it. The same coordinate-hyperplane test therefore applies. Only the five rank-two cases require a separately computed exact nullspace in this implementation. Both implementations give the table: 2510 representatives have rank six, 525 have rank four, and five have rank two; precisely the five rank-two representatives satisfy (5). They occur only for complements $2C_3$, $P_1 \sqcup P_2 \sqcup C_3$, and $3P_2$, respectively the three graphs stated.

For completeness, the last two graphs have simple DDM certificates on [6]. Use the parts (1), (4, 3), (6, 2, 5) for $K_{1,2,3}$, with signs (+), (+, −), (−, +, +). For $K_{2,2,2}$ use (2, 1), (4, 3), (6, 5), each with signs (+, −). Every signed part sum is 1. Order the three parts cyclically. Between consecutive parts, orient an edge forward when its endpoint signs agree and backward otherwise. At each vertex the contributions from the two other parts cancel, so every weight is zero. The parity argument already excludes $K_{3,3}$ on [6]. $\square$

Table 1: Exact enumeration on six vertices. P_j has j vertices and C_j is a j -cycle. Rank columns count spanning-tree-normalized switching representatives. “Pass” counts those satisfying (5).

Complement	Edges	Rank 2	Rank 4	Rank 6	Pass
$2C_3$	9	1	9	6	1
$C_3 \sqcup P_3$	10	0	20	12	0
C_6	9	0	6	10	0
$P_1 \sqcup C_5$	10	0	0	32	0
$2P_1 \sqcup C_4$	11	0	0	64	0
$3P_1 \sqcup C_3$	12	0	80	48	0
$6P_1$	15	0	0	1024	0
$4P_1 \sqcup P_2$	14	0	136	376	0
$3P_1 \sqcup P_3$	13	0	0	256	0
$2P_1 \sqcup 2P_2$	13	0	76	180	0
$2P_1 \sqcup P_4$	12	0	0	128	0
$P_1 \sqcup P_2 \sqcup C_3$	11	2	38	24	2
$P_1 \sqcup P_2 \sqcup P_3$	12	0	40	88	0
$P_1 \sqcup P_5$	11	0	20	44	0
$P_2 \sqcup C_4$	10	0	12	20	0
$3P_2$	12	2	68	58	2
$P_2 \sqcup P_4$	11	0	20	44	0
$2P_3$	11	0	0	64	0
P_6	10	0	0	32	0
Total		5	525	2510	5

Corollary 5.2 (Computer assisted). *Up to graph isomorphism, $2K_{3,3}$ is the unique DDMO graph of order twelve whose connected components are all non-DDMO in isolation.*

Proof. Theorem 4.2 forces exactly two components of order six. Their restricted labels are injective and positive. Theorem 5.1 and its [6] certificates force both components to be $K_{3,3}$. $\square$

6 Extension via a stated conservative labeling theorem

The previous results do not use the external theorem discussed here. Licona and Tey [5, Theorem 4.17, p. 23] state that every snowflake with M edges has a conservative labeling if $M \equiv 0, 3 \pmod{4}$, and a near-conservative labeling otherwise. These are bijective edge labelings by $[M]$ and $[M-1] \cup \{M+1\}$, respectively, with zero signed incident sum at every vertex of degree at least three. A snowflake is obtained by identifying one leaf from each of a collection of disjoint stars [5, Definition 4.1, p. 13].

Proposition 6.1 (Conditional on the stated snowflake theorem). *Assume Theorem 4.17 of [5]. Let $s \geq 1$, $r \geq 0$, and $a_i, b_i \geq 1$, and put $M = \sum_{i=1}^s (a_i + b_i)$. Then*

$$\left(\bigsqcup_{i=1}^s K_{a_i, b_i} \right) \sqcup rK_1 \text{ is DDMO} \iff \begin{cases} a_i, b_i \geq 3 \\ r \geq 1 \text{ or } M \equiv 0, 3 \pmod{4} \end{cases} \text{ for every } i,$$

Here M counts the nonisolated vertices, not the total number of vertices.

Proof. The degree-one and degree-two obstructions require all part sizes to be at least three. If $r = 0$, Lemma 2.1 makes every part sum even, so $M(M + 1)/2$ is even, equivalently $M \equiv 0, 3 \pmod{4}$.

For sufficiency, create one star with d edges for each prescribed part size d . If there are at least three stars, identify one leaf of each to form a snowflake and apply the assumed theorem. Each original star center still has degree $d \geq 3$, so its incident edge labels form a signed-zero block of the required size. Splitting the identified vertex back into leaves leaves these blocks unchanged. Lemma 2.1 now supplies the orientations.

For exactly two stars, if some size $d \geq 6$, replace it temporarily by two sizes 3 and $d - 3$. The preceding construction applies to the resulting three stars. Merge the two corresponding signed-zero blocks to recover size d ; their signed sum remains zero. To avoid depending on a degree-two-center boundary convention for the external theorem, the six remaining cases are supplied directly below. Each row gives two signed-zero blocks; the absolute values cover $[M]$ in the ordinary cases and $[M - 1] \cup \{M + 1\}$ otherwise:

(a, b)	first signed block	second signed block
(3, 3)	(1, 3, -4)	(2, 5, -7)
(3, 4)	(1, 2, -3)	(4, -5, -6, 7)
(3, 5)	(1, 3, -4)	(2, 5, -6, 7, -8)
(4, 4)	(1, -2, -3, 4)	(5, -6, -7, 8)
(4, 5)	(1, -2, -3, 4)	(-5, -6, -7, 8, 10)
(5, 5)	(-1, 2, -3, -4, 6)	(-5, -7, -8, 9, 11).

In the ordinary cases, any isolates receive labels $M + 1, \dots, M + r$. In the near cases, give one isolate label M and any further isolates labels $M + 2, \dots, M + r$. $\square$

Why the dependence is explicit. The printed Lemma 3.17 of [5, p. 13] has a parity inconsistency: $n_1 = 3$, $n_2 = 1$, and $k = t = 0$ satisfy its stated condition on $n = n_1 + n_2$, but would give signed-zero blocks with absolute values partitioning [14]. Their total absolute sum would be even, whereas $1 + \dots + 14 = 105$. We have not fully verified the external proof or established a counterexample to its Theorem 4.17. Accordingly Proposition 6.1 is a conditional reduction, not an independently established general classification in this note.

Licon and Tey's introduction [5, p. 3] attributes the related galaxy result to Goldfeder and Tey (2018), their reference 13. We do not claim the partition principle as new. The freely available 2024 paper is directly cited here; we did not obtain a freely readable full text of that older dependency.

7 Scope and reproducibility

The infinite classification in Theorem 3.1, the rigidity statement, and the twelve-vertex lower bound have mathematical proofs independent of finite testing. Only Theorem 5.1 and Corollary 5.2 depend on exhaustive computation. All enumerations are exact; no floating-point feasibility decision or SAT/ILP solver is used.

Data and code availability. Repository for the manuscript and verification materials: <https://github.com/dapyle11/Disjoint-union-activation-of-difference-distance-magic-labelings>. The accompanying source package includes the LaTeX manuscript, the classical-construction checker, two six-vertex classifiers, machine-readable results, commands, and a checksum

manifest. The principal classifier uses only the Python standard library. The independent Pfaffian checker uses SymPy for its five rank-two nullspaces. The construction checker verifies every pair difference, label and edge coverage, and vertex weight for $1 \leq k \leq 1000$, as well as extra-isolate and fixed-label orientation checks. These tests check the implementations, not the infinite theorems.

Corollary 2.2 is an exact reduction for arbitrary complete-bipartite unions; it should not be confused with a classification of all prescribed part sizes. The independently established classification here is for the stated copies of $K_{3,3}$, with arbitrary isolates; Section 6 identifies the additional external dependency for a broader conclusion. The results are a short answer and refinement of the published union question, rather than a claim to new Skolem constructions or a general theory of graph activation.

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A Explicit classical pairings

We record the particular cases used in Theorem 3.1, with attribution to their original constructions. A pair (a, b) below has difference $b - a$; the list is reindexed by difference when inserted into (3). An index range with upper endpoint below its lower endpoint is empty.

Skolem pairing for $n = 4m$. For $m = 1$, use $(6, 7), (1, 3), (2, 5), (4, 8)$. For $m \geq 2$, Skolem's construction [3, p. 58] is

$$\begin{array}{ll}
 (4m + j, 8m - j) & 0 \leq j \leq 2m - 1, \\
 (2m + 1, 6m), & (2m, 4m - 1), \\
 (j, 4m - 1 - j) & 1 \leq j \leq m - 1, \\
 (m, m + 1), & \\
 (m + 2 + j, 3m - 1 - j) & 0 \leq j \leq m - 3.
 \end{array}$$

The first row gives every even difference $2, 4, \dots, 4m$. The other rows give, respectively, $4m - 1$ and $2m - 1$; the odd differences $2m + 1, 2m + 3, \dots, 4m - 3$; 1; and the odd differences $3, 5, \dots, 2m - 3$. These lists partition $[4m]$. Their endpoints partition $[8m]$: the first row uses $[4m, 6m - 1] \cup [6m + 1, 8m]$, and the remaining rows use each integer in $[1, 4m - 1] \cup \{6m\}$ once, as is seen by their consecutive endpoint ranges.

Hooked pairing for $n = 4m+2$. For $m = 0$, use $(1, 2), (3, 5)$. For $m \geq 1$, O’Keefe’s construction [4, pp. 80–81] is

$$\begin{array}{ll}
(j, 4m+2-j) & 1 \leq j \leq 2m, \\
(2m+1, 6m+2), & (4m+2, 6m+3), \\
(4m+3, 8m+5), & \\
(4m+3+j, 8m+4-j) & 1 \leq j \leq m-1, \\
(5m+2+j, 7m+3-j) & 1 \leq j \leq m-1, \\
(7m+3, 7m+4). &
\end{array}$$

The first row gives all even differences $2, 4, \dots, 4m$. The next rows give $4m+1, 2m+1$, and $4m+2$; then the odd differences from $2m+3$ through $4m-1$ and from 3 through $2m-1$; and finally 1 . Thus every difference in $[4m+2]$ occurs once. The first row uses $[1, 2m] \cup [2m+2, 4m+1]$. The remaining endpoints, in increasing consecutive ranges together with their displayed singletons, fill its complement in $[8m+5] \setminus \{8m+4\}$ exactly. The omitted value is $2n$, as required.

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