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THE RIEMANN HYPOTHESIS ON YOUNG'S LATTICE

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Abstract. This work establishes a novel structural connection between the Riemann Hypothesis and Young's Lattice through partition-indexed harmonic numbers and the Generalized Assembly Theorem. We establish a new family of equivalences for the conjecture, define the partition-based harmonic numbers $H_n^{\boldsymbol{\lambda}}$, and reformulate the Riemann Hypothesis as the Young's Lattice equivalence:

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k > \sum_{d|n} d$$

for every $n \in \mathbb{N}$, where $\boldsymbol{\lambda} \vdash k$ denotes a partition of k, $\boldsymbol{\lambda}!$ is the partition factorial, and $a_k = \frac{1}{k} + \cdots + \frac{1}{k^2}$ or $a_k = \log(k+1)$. We then prove the Generalized Assembly Theorem, which shows that the assembly sum $\mathfrak{S}_n^m = \sum_{k=1}^n \sum_{\boldsymbol{\lambda}\in\wp_k^{k+m-1}} H_n^{\boldsymbol{\lambda}}$ equals n for every excess parameter $m-1$, providing a systematic framework to analyze divisor bounds through the branch decomposition of the partition-indexed series. The resulting structure yields an exact, non-asymptotic realization of the divisors of n. Finally, we propose and explore a hierarchical expansion connecting the exponential of the Euler–Mascheroni constant, e^γ, to Multiple Zeta Values under Abel summability.

$$e^\gamma = 1 + \sum_{a\geq 2}\frac{1}{a!}\sum_{m\geq 0}(-1)^m\zeta^{\mathrm{A}}(a+m) + \sum_{a,b\geq 2}\frac{1}{a!\,b!}\sum_{m,n\geq 0}(-1)^{m+n}\zeta^{\mathrm{A}}(a+m,b+n) + \cdots$$

1. Introduction

The Riemann Hypothesis, first proposed by Bernhard Riemann in his foundational 1859 memoir [19], asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Its resolution would have profound consequences for the distribution of prime numbers and, more broadly, for analytic number theory.

Since a direct attack on the conjecture in its original form has proven extremely difficult, a significant body of research has been dedicated to establishing elementary equivalent formulations. These translate the problem from the complex plane into inequalities involving classical arithmetic functions, most notably the sum-of-divisors function $\sum_{d|n} d$ and the harmonic numbers $H_n = \sum_{i=1}^n \frac{1}{i}$. A landmark achievement in this direction is due to Guy Robin [20], who characterized the conjecture through the maximal growth rate of $\sum_{d|n} d$: the Riemann Hypothesis holds if and only if

$$\sum_{d|n} d \;<\; e^\gamma\, n \log\log n \qquad \text{for every } n \geq 5041, \tag{1}$$

where γ is the Euler–Mascheroni constant, and if the hypothesis fails, then the reverse inequality holds for infinitely many n with a quantitative error term. Complementing this framework, Jeffrey C. Lagarias [18] proved that the Riemann Hypothesis is equivalent to the inequality

$$\sum_{d|n} d \;\leq\; H_n + e^{H_n}\log H_n \qquad \text{for every } n \in \mathbb{N}, \tag{2}$$

and, equivalently, to

$$\sum_{d|n} d \;<\; e^{H_n}\log H_n \qquad \text{for every } n > 60. \tag{3}$$

Both criteria express the hypothesis as an elementary inequality involving the sum-of-divisors function. The present work compiles and offers a unified presentation of all the results previously established and rigorously detailed across a series of ten foundational papers, with [4, 7, 12, 13, 5, 8, 9] published between December 2025 and January 2026, and [6, 11, 10] published between June 2026 and July 2026. We begin with a new elementary equivalence of the Riemann Hypothesis, expressed as an inequality involving the sum-of-divisors function and a partition-indexed series. The starting point is the Espinosa–Riemann Equivalence Theorem (Theorem 2.10) established in [7], which reduces the conjecture to an inequality of the form

$$\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!}\, a_k \;>\; \sum_{d|n} d \qquad \text{for every } n\in\mathbb{N}, \tag{4}$$

for an admissible sequence $\{a_k\}_{k\geq 1}$ satisfying three explicit conditions: a prescribed asymptotic behaviour of the associated generating function, a lower bound $a_k > H_k - \gamma$, and a finite base validation for $1 \leq n \leq 60$. The central contribution of all these works is the reformulation of (4) in the language of integer partitions and Young's Lattice. Using the identity

$$\frac{H_n^k}{k!} \;=\; \sum_{\boldsymbol{\lambda}\vdash k} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}, \tag{5}$$

where $H_n^{\boldsymbol{\lambda}}$ denotes the generalized harmonic number associated with a partition $\boldsymbol{\lambda} \vdash k$ and $\boldsymbol{\lambda}!$ is the partition factorial, one obtains the Riemann Hypothesis Young's Lattice Equivalence: the hypothesis holds if and only if

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;>\; \sum_{d|n} d \qquad \text{for every } n\in\mathbb{N}. \tag{6}$$

The passage from (4) to (6) is a direct application of (5); the content of the equivalence lies in the combinatorial structure that emerges from the partition decomposition, not in the algebraic manipulation itself.

In particular, two of the explicit equivalences stated in Section 2.2 can be rewritten in this partition-based form in a particularly elegant way. The first, with logarithmic coefficients, was established in [12], while the second, with fully explicit harmonic coefficients, was established in [4]. For the logarithmic coefficients $a_k = \log(k+1)$ and for the fully explicit coefficients $a_k = \frac{1}{k} + \cdots + \frac{1}{k^2}$, the corresponding partition-indexed inequalities read

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!} \log(k+1) \;>\; \sum_{d|n} d \qquad \text{for every } n\in\mathbb{N}, \tag{7}$$

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!} \left(\tfrac{1}{k} + \cdots + \tfrac{1}{k^2}\right) \;>\; \sum_{d|n} d \qquad \text{for every } n\in\mathbb{N}, \tag{8}$$

respectively. Both inequalities are equivalent to the Riemann Hypothesis, and both are obtained by the same direct substitution (5) into the corresponding logarithmic and harmonic forms of the equivalence. This shows that the partition-based reformulation is not only a restatement of the original result, but a systematic framework that accommodates the different realizations of the coefficients a_k in a unified way. The structural analysis of (6) is carried out through the Generalized Assembly Theorem (Theorem 2.37), which extends the classical Assembly Theorem to arbitrary summation intervals. For all $n, m, s \in \mathbb{N}$ with $s \leq n$, the generalized assembly sum satisfies

$$\mathfrak{S}_{s,n}^m \;:=\; \sum_{k=s}^{n} \sum_{\boldsymbol{\lambda}\in\wp_{k-s+1}^{k-s+m}} H_{s,n}^{\boldsymbol{\lambda}} \;=\; \frac{n-s+1}{s^m}, \tag{9}$$

where $\wp_{k-s+1}^{k-s+m}$ denotes the Assembly Set of all partitions of $k-s+1+m$ into exactly $k-s+1$ parts, and $H_{s,n}^{\boldsymbol{\lambda}}$ is the generalized harmonic number over the restricted index range $[s,n]$. In the particular case $s=1$, the identity reduces to the classical Assembly Theorem,

$$\mathfrak{S}_n^m \;=\; n \qquad \text{for every } m \in \mathbb{N}, \tag{10}$$

revealing a hidden invariance: independently of the excess parameter $m-1$, the ensemble of harmonic weights always reconstructs the count of indices $\{1,\ldots,n\}$.

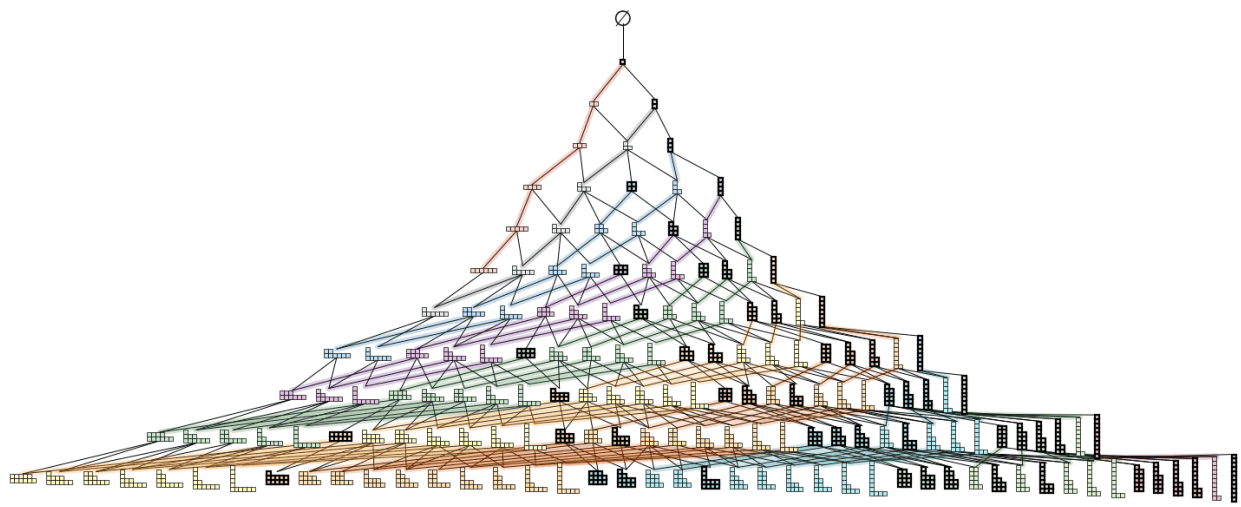


FIGURE 1. The truncated Young's Lattice up to rank $N=11$, restricted to partitions with at most $n=6$ parts. Each node is a partition $\boldsymbol{\lambda} \vdash_j j+m-1$ belonging to the Assembly Set $\wp_j^{j+m-1}$, drawn as a transposed Young diagram: the number of columns equals the number of summands $j=\ell(\boldsymbol{\lambda})$, and the height of each column equals the value of the corresponding part. The nodes are colored by the excess parameter $m-1$, and the leaders of each level, that is, the partitions without any part equal to 1 together with the partition (1), are highlighted with a double rectangular frame. The edges in marker style trace the chains of partitions that remain within the same m as j increases in the Assembly Set $\wp_j^{j+m-1}$. This is precisely the combinatorial structure that underlies the Generalized Assembly Theorem 2.37: each level m corresponds to one row of the lattice, and the accumulated contribution of all levels reconstructs the value $\frac{n-s+1}{s^m}$ that appears in equation (48).

The Generalized Assembly Theorem provides the combinatorial mechanism through which the divisors of n are assembled by the partition-indexed series. Each level m of the assembly contributes exactly n to the total sum, so that the series (6) can be re-indexed by the excess parameter $m-1$:

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;=\; \sum_{m\geq 1}\left(\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k+m-1}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_{k+m-1}\right). \tag{11}$$

The family $\wp_k^{k+m-1}$ corresponds to the partitions of $k+m-1$ into exactly k parts, and each level m reconstructs the count of indices $\{1,\ldots,n\}$ via the Assembly Theorem. Since $a_k \geq 1$ for every

$k \in \mathbb{N}$ and all terms are positive, the first two levels $m = 1, 2$ already yield the coarse bound

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;>\; \frac{n}{1!} + \frac{n}{2!}, \tag{12}$$

and the accumulated contributions of the remaining levels $m \geq 3$ refine this bound. The realization $\frac{n}{\boldsymbol{\Lambda}!} = d$ reconstructs $\sum_{d|n} d$ exactly, not asymptotically. It allows for the successive extraction of the divisors of the "peaks numbers". Since the most general and natural "peak numbers", are the highly abundant numbers n, for which $\sum_{d|n} d$ exceeds the value of $\sum_{d|m} d$ for every $m < n$, the points of maximal value of the divisor function, it suffices to verify the inequality at those integers.

2. Main Framework

2.1. A Theorem for a new family of equivalences of the Riemann Hypothesis.

2.1.1. *On* $\sum_{k\in\mathbb{N}}(H_k - \gamma)\frac{x^k}{k!}$. We first establish the following, which will provide the analytic backbone for the equivalence theorem.

Proposition 2.1. *For every* $x \in \mathbb{R}^+$,

$$\sum_{k\in\mathbb{N}_0} \psi(k+1)\frac{x^k}{k!} \;=\; e^x \log x + E_1(x), \tag{13}$$

where $\psi(k) = \frac{\Gamma'(k)}{\Gamma(k)}$ *and* $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt$.

Proof. See [3, Ch. III, p. 37]. □

Proposition 2.2. *For every* $n \in \mathbb{Z}_{\geq 1}$, $\psi(n) = -\gamma + H_{n-1}$.

Proof. See [1, Eq. 6.3.1, p. 258]. □

Then a two-sided bound of $e^x E_1(x)$:

Proposition 2.3 (Bounds for the Exponential Integral)**.** *For every* $x \in \mathbb{R}^+$ *with* $x \neq 0$,

$$\tfrac{1}{2}\log\Big(1 + \tfrac{2}{x}\Big) \;<\; e^x E_1(x) \;<\; \log\Big(1 + \tfrac{1}{x}\Big), \tag{14}$$

where $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt$.

Proof. See [1, Eq. 5.1.20, p. 229]. □

Finally, combining the previous three results we obtain a two-sided bound and the asymptotic expansion of the series $\sum_{k\in\mathbb{N}}(H_k - \gamma)\frac{x^k}{k!}$.

Proposition 2.4. *For every* $x \in \mathbb{R}^+$,

$$\sum_{k\in\mathbb{N}}(H_k - \gamma)\frac{x^k}{k!} \;=\; e^x \log x + \gamma + \mathcal{O}\Big(\frac{1}{x}\Big) \qquad (x \to \infty). \tag{15}$$

Proof. By Proposition 2.2, $H_k - \gamma = \psi(k+1)$ for $k \geq 1$, so

$$\sum_{k\geq 1}(H_k - \gamma)\frac{x^k}{k!} \;=\; \sum_{k\geq 0}\psi(k+1)\frac{x^k}{k!} - \psi(1) \;=\; e^x \log x + E_1(x) + \gamma,$$

using Proposition 2.1 and $\psi(1) = -\gamma$. Since $E_1(x) = \mathcal{O}(\frac{e^{-x}}{x})$ as $x \to \infty$ [1, Eq. 5.1.51, p. 231], the claim follows. □

Proposition 2.5. *For every $x \in \mathbb{R}^+$,*

$$e^x \log x + \tfrac{1}{2}\log\Big(1+\tfrac{2}{x}\Big) + \gamma \;<\; \sum_{k\in\mathbb{N}} (H_k - \gamma)\frac{x^k}{k!} \;<\; e^x \log x + \log\Big(1+\tfrac{1}{x}\Big) + \gamma. \tag{16}$$

Proof. From the proof of Proposition 2.4, $\sum_{k\geq 1}(H_k-\gamma)\frac{x^k}{k!} = e^x \log x + e^x E_1(x) + \gamma$. Substituting the two-sided bound (14) of Proposition 2.3 for $e^x E_1(x)$ yields (16) verbatim. □

2.1.2. *On elementary versions of the Riemann Hypothesis.* We now recall the two criteria that translate the Riemann Hypothesis into elementary inequalities involving the sum-of-divisors function. These will serve as the arithmetic bridge between the analytic results of the previous section and the equivalence theorem below.

Theorem 2.6 (Lagarias, 2002)**.** *The **Riemann Hypothesis** is equivalent to*

$$\sum_{d|n} d \;\leq\; H_n + e^{H_n}\log H_n \qquad \text{for every } n \in \mathbb{N}, \tag{17}$$

*where $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Equivalently, the **Riemann Hypothesis** is equivalent to*

$$\sum_{d|n} d \;<\; e^{H_n}\log H_n \qquad \text{for every } n > 60. \tag{18}$$

Proof. See [18, Thm. 1.1 and Cor. 1.2]. □

Theorem 2.7 (Robin, 1984)**.** *The following two statements hold.*

(i) *If the **Riemann Hypothesis** is true, then*

$$\sum_{d|n} d \;<\; e^{\gamma}\, n \log\log n \qquad \text{for every } n \geq 5041. \tag{19}$$

(ii) *If the **Riemann Hypothesis** is false, then*

$$\sum_{d|n} d \;\geq\; e^{\gamma}\, n \log\log n \tag{20}$$

for infinitely many $n \in \mathbb{N}$.

Proof. See [20, Thm. 1 and Thm. 2, pp. 192–196]. □

Theorem 2.8 (Robin, 1984, quantitative form)**.** *If the **Riemann Hypothesis** is false, then there exist absolute constants $C > 0$ and $\beta \in (0, \frac{1}{2})$ such that*

$$\sum_{d|n} d \;\geq\; e^{\gamma}\, n \log\log n \;+\; \frac{C\, n \log\log n}{(\log n)^{\beta}} \tag{21}$$

for infinitely many $n \in \mathbb{N}$.

Proof. See [20, Thm. 2, p. 194 and §4, pp. 201–207]. □

2.1.3. *On the Espinosa-Riemann equivalence theorem.* Given the preceding sections, and the two preceding families of criteria wich reduce the Riemann Hypothesis to inequalities involving the sum-of-divisors function. We now present a general framework, in the spirit of these criteria, that produces new elementary equivalences in a systematic and compact way. The starting point is the following auxiliary result.

Proposition 2.9. *Let $\mathfrak{T} : \mathbb{R}^+ \to \mathbb{R}$ be a function such that, for every $k \in \mathbb{N}$,*

$$\mathfrak{T}(k) \;>\; H_k - \gamma, \tag{22}$$

and let $\mathfrak{O} : \mathbb{R}^+ \to \mathbb{R}$ be defined by the everywhere convergent series

$$\mathfrak{O}(x) \;:=\; \sum_{k\in\mathbb{N}} \frac{x^k}{k!}\,\mathfrak{T}(k). \tag{23}$$

Then, for every $x \in \mathbb{R}^+$,

$$\mathfrak{O}(x) \;>\; e^x \log x. \tag{24}$$

Proof. Fix $x > 0$. By (22), multiplying both sides by the positive quantity $\frac{x^k}{k!}$ and summing over $k \in \mathbb{N}$ gives

$$\sum_{k\in\mathbb{N}} (H_k - \gamma)\frac{x^k}{k!} \;<\; \mathfrak{O}(x),$$

in view of (23). Proposition 2.5 provides the lower bound

$$e^x \log x + \tfrac{1}{2}\log\Big(1 + \tfrac{2}{x}\Big) + \gamma \;<\; \sum_{k\in\mathbb{N}} (H_k - \gamma)\frac{x^k}{k!}.$$

Combining both inequalities,

$$e^x \log x + \tfrac{1}{2}\log\Big(1 + \tfrac{2}{x}\Big) + \gamma \;<\; \mathfrak{O}(x).$$

Since $\frac{1}{2}\log(1 + \frac{2}{x}) > 0$ for all $x > 0$ and $\gamma > 0$, the additional terms on the left are strictly positive. Hence $e^x \log x < \mathfrak{O}(x)$. □

With this auxiliary inequality in hand, we can now state the central result of section 2.1.3. The following theorem in [7] is stated in terms of a function $\mathfrak{O}$ satisfying three explicit conditions, and it produces a new family of equivalences of the Riemann Hypothesis by varying the choice of $\mathfrak{T}$.

Theorem 2.10 (Espinosa, 2025, Espinosa–Riemann Equivalence Theorem)**.** *The **Riemann Hypothesis** is equivalent to*

$$\mathfrak{O}(H_n) \;=\; \sum_{k\in\mathbb{N}} \frac{H_n^k}{k!}\,\mathfrak{T}(k) \;>\; \sum_{d|n} d \qquad \text{for every } n \in \mathbb{N}, \tag{25}$$

where $H_n = \sum_{i=1}^{n} \frac{1}{i}$ and the functions $\mathfrak{O}, \mathfrak{T} : \mathbb{R}^+ \to \mathbb{R}$ satisfy:

(i) ***Asymptotic behaviour of $\mathfrak{O}$:***

$$\mathfrak{O}(x) \;=\; e^x \log x + \mathcal{O}\Big(\frac{e^x}{x}\Big) \qquad (x \to \infty). \tag{26}$$

(ii) ***Inner condition on $\mathfrak{T}$:***

$$\mathfrak{T}(k) \;>\; H_k - \gamma \qquad \text{for every } k \in \mathbb{N}. \tag{27}$$

(iii) ***Base validation:*** *Inequality (25) holds for every integer $1 \le n \le 60$.*

Proof. **(Riemann Hypothesis ⇒ inequality (25)).** Assume Riemann Hypothesis. By Theorem 2.6,

$$\sum_{d|n} d \;<\; e^{H_n} \log H_n \qquad \text{for every } n > 60.$$

Condition (27) and Proposition 2.9 give $\mathfrak{O}(x) > e^x \log x$ for all $x \in \mathbb{R}^+$. Evaluating at $x = H_n$ yields

$$\sum_{d|n} d \;<\; e^{H_n} \log H_n \;<\; \mathfrak{O}(H_n).$$

Together with the base validation, (25) holds for every $n \in \mathbb{N}$.

(Inequality (25) $\Rightarrow$ Riemann Hypothesis). Suppose (25) holds for every $n \in \mathbb{N}$ but Riemann Hypothesis is false. By Theorem 2.8, there exist absolute constants $C > 0$ and $\beta \in (0, \frac{1}{2})$ such that

$$\sum_{d|n} d \;\geq\; e^{\gamma} n \log\log n + \frac{C\, n \log\log n}{(\log n)^{\beta}} \qquad \text{for infinitely many } n \in \mathbb{N}. \tag{28}$$

On the other hand, by (26) there exist $M > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\mathfrak{O}(H_n) \;\leq\; e^{H_n} \log H_n + M\, \frac{e^{H_n}}{H_n}.$$

Combined with (25), this gives, for every $n \geq n_0$,

$$\sum_{d|n} d \;<\; e^{H_n} \log H_n + M\, \frac{e^{H_n}}{H_n}.$$

Lagarias' elementary bounds [18, Ineqs. (3.11), (3.9), (3.5)] state that, for $n \geq 3$,

$$e^{H_n} \log H_n \;\leq\; e^{\gamma} n \log\log n + \frac{3e^{\gamma} n}{2 \log n}, \tag{29}$$

$$e^{H_n} \;\leq\; e^{\gamma}(n+1), \tag{30}$$

$$H_n \;>\; \log n + \gamma \;>\; \log n. \tag{31}$$

From (31), $\frac{1}{H_n} < \frac{1}{\log n}$. Multiplying by (30) and by $M > 0$ yields

$$M\, \frac{e^{H_n}}{H_n} \;<\; M\, e^{\gamma}\, \frac{n+1}{\log n} \;\leq\; \frac{2Me^{\gamma} n}{\log n} \qquad (n \geq 1). \tag{32}$$

Adding (29) and (32), we obtain, for $n \geq n_1 := \max\{n_0, 3\}$,

$$\sum_{d|n} d \;<\; e^{\gamma} n \log\log n + \frac{3e^{\gamma} n}{2 \log n} + \frac{2Me^{\gamma} n}{\log n} \;=\; e^{\gamma} n \log\log n + S\, \frac{n}{\log n},$$

where $S := \frac{3e^{\gamma}}{2} + 2Me^{\gamma} > 0$ is a constant. Since $\beta < \frac{1}{2}$, we have $(\log n)^{-\beta} > (\log n)^{-\frac{1}{2}}$ for $n \geq 3$, and $C\, n \log\log n\, (\log n)^{-\frac{1}{2}} > S\, \frac{n}{\log n}$ for all sufficiently large n. Hence, for $n \geq n_2$,

$$\sum_{d|n} d \;<\; e^{\gamma} n \log\log n + \frac{C\, n \log\log n}{(\log n)^{\frac{1}{2}}}.$$

Equivalently, the reverse inequality

$$\sum_{d|n} d \;\geq\; e^{\gamma} n \log\log n + \frac{C\, n \log\log n}{(\log n)^{\frac{1}{2}}}$$

can hold only for finitely many n, namely $1 \leq n < n_2$. But by (28) and $\beta < \frac{1}{2}$,

$$\sum_{d|n} d \;>\; e^{\gamma} n \log\log n + \frac{C\, n \log\log n}{(\log n)^{\frac{1}{2}}} \qquad \text{for infinitely many } n \in \mathbb{N},$$

which contradicts the finiteness just established. Therefore Riemann Hypothesis must be true and the both implications together prove the equivalence. □

The inner condition on $\mathfrak{T}(k)$ can be relaxed without affecting the equivalence, provided the weaker lower bound is imposed directly on $\mathfrak{O}(x)$. This yields the following corollary, which strictly generalizes the theorem

Corollary 2.11 (Espinosa, 2025, Espinosa–Riemann Equivalence Corollary)**.** *The **Riemann Hypothesis** is equivalent to* (25) *under conditions* (i) *and* (iii) *of Theorem 2.10, with condition* (ii) *replaced by the weaker*

$$\mathfrak{O}(x) \;>\; e^x \log x \qquad \textit{for every } x \in \mathbb{R}^+. \tag{33}$$

Proof. The proof is identical to that of Theorem 2.10, except that condition (27) on $\mathfrak{T}$ is replaced by the direct lower bound (33) on $\mathfrak{O}$; the paragraph invoking Proposition 2.9 is therefore omitted. Since (27) implies (33) (by Proposition 2.9), but not conversely, the corollary strictly generalizes the theorem. □

This concludes the construction of the equivalence. The framework is now sufficiently general to accommodate a wide range of explicit sequences $\{a_k\}_{k\geq 1}$, some of which are exhibited in the next section 2.2.

2.2. **Three equivalences of the Riemann Hypothesis.** In Theorem 2.10 the function $\mathfrak{T} : \mathbb{R}^+ \to \mathbb{R}$ is only required to satisfy conditions (i)–(iii). Its restriction to the natural numbers yields an admissible sequence

$$a_k \;:=\; \mathfrak{T}\big|_{\mathbb{N}}(k) \;=\; H_k - \gamma + \varepsilon_k, \qquad \varepsilon_k > 0, \qquad k \in \mathbb{N}, \tag{34}$$

and each choice of such a sequence produces an explicit equivalence of the Riemann Hypothesis. In what follows we exhibit three natural realizations of $\{a_k\}_{k\geq 1}$. This reduces the conjecture to an inequality of the form

$$\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!}\, a_k \;>\; \sum_{d|n} d \qquad \text{for every } n \in \mathbb{N}, \tag{35}$$

for an admissible sequence $\{a_k\}_{k\geq 1}$ satisfying the three explicit conditions in Theorem 2.10. These three equivalences are the simplest explicit instances of the Espinosa–Riemann Equivalence Theorem. Each provides a direct combinatorial bridge to Young's Lattice, as developed in the next section.

Theorem 2.12 (Espinosa, 2025, First equivalence: logarithmic coefficients)**.** *The **Riemann Hypothesis** is equivalent to*

$$\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!} \log(k+1) \;>\; \sum_{d|n} d \qquad \textit{for every } n \in \mathbb{N}. \tag{36}$$

Proof. Apply Theorem 2.10 with $a_k := \log(k+1)$. It is straightforward to verify that this sequence satisfies the three conditions of the theorem:

(i) $\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!} \log(k+1) = e^x \log x + \mathcal{O}(\frac{e^x}{x})$ as $x \to \infty$;
Substituting Proposition 2.2, $H_k - \gamma = \psi(k+1)$ for $k \geq 1$ in [1, Eq. 6.3.18, p. 259] we obtain:

$$H_k - \gamma \;<\; \log(k+1) \;<\; H_k - \gamma + \frac{1}{k} \qquad (k \geq 1).$$

Taking the upper bound, multiplying by $\frac{x^k}{k!} > 0$ and summing over $k \in \mathbb{N}$ yields

$$0 \;<\; \sum_{k\geq 1} \frac{x^k}{k!} \log(k+1) \;<\; \sum_{k\geq 1} (H_k - \gamma) \frac{x^k}{k!} + \sum_{k\geq 1} \frac{x^k}{k!} \frac{1}{k}, \tag{37}$$

finally applying $\sum_{k\geq 1} \frac{x^k}{k!} \frac{1}{k} = \mathcal{O}\Big(\frac{e^x}{x}\Big)$ by:

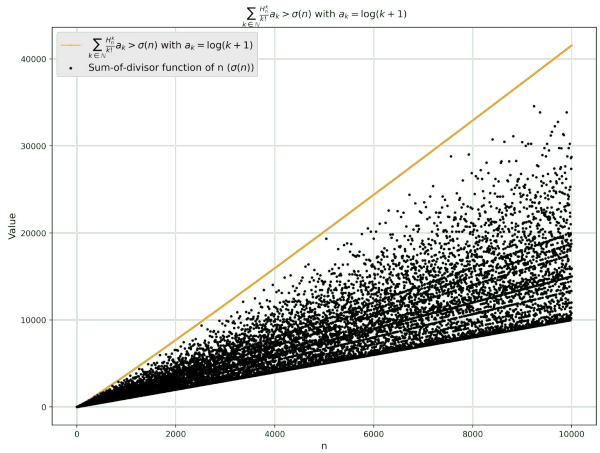
$$0 \;<\; \sum_{k\geq 1} \frac{x^k}{k!} \frac{1}{k} \;=\; \frac{1}{x} \sum_{k\geq 1} \frac{x^{k+1}}{(k+1)!} \left(1 + \frac{1}{k}\right) \;\leq\; \frac{2}{x} \sum_{k\geq 1} \frac{x^{k+1}}{(k+1)!} \;=\; \frac{2}{x}(e^x - 1 - x) \;=\; \mathcal{O}\Big(\frac{e^x}{x}\Big)$$

and the Proposition 2.4 in (37), we obtain the desired result:

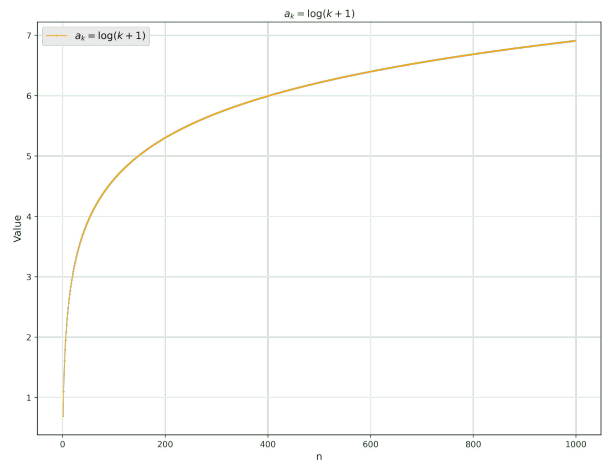
$$0 \;<\; \sum_{k\geq 1} \frac{x^k}{k!}\log(k+1) \;<\; e^x \log x + \gamma + \mathcal{O}\Big(\frac{1}{x}\Big) + \mathcal{O}\Big(\frac{e^x}{x}\Big) \;=\; e^x \log x + \mathcal{O}\Big(\frac{e^x}{x}\Big) \qquad (x \to \infty).$$

(ii) the lower bound $\log(k+1) > H_k - \gamma$ holds for every $k \in \mathbb{N}$;
(iii) the base inequality (36) is verified directly for $1 \leq n \leq 60$.

The equivalence follows. See [12, Thm. 3]. □



(A) Plot of the left-hand side of (36) against the right-hand side $\sum_{d|n} d$, for small values of n. The logarithmic coefficients $\log(k+1)$ produce a sequence that dominates the divisor sum from the very first integers, illustrating the base validation of condition (iii).



(B) Comparison of the coefficients $\log(k+1)$ with the lower bound $H_k - \gamma$, for k ranging over the first integers. The logarithmic growth of $\log(k+1)$ lies strictly above $H_k - \gamma$, confirming condition (ii) of the theorem.

FIGURE 2. Graphical verification of the First Equivalence.
(a) The $\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!}\log(k+1)$ series on the left-hand side of (36) dominates the divisor sum for small n, validating the base case. **(b)** The coefficients $\log(k+1)$ lie strictly above the lower bound $H_k - \gamma$, confirming the inner condition of the theorem.

Theorem 2.13 (Espinosa, 2025, Second equivalence: alternating generalized-harmonic coefficients)**.** *The **Riemann Hypothesis** is equivalent to*

$$\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!} a_k \;>\; \sum_{d|n} d \qquad \textit{for every } n \in \mathbb{N}, \tag{38}$$

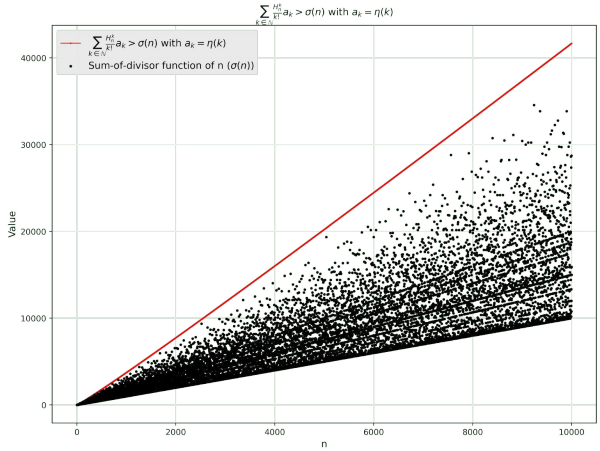
where

$$a_k \;:=\; \sum_{m=1}^{k} \frac{(-1)^{m+1}}{m} H_{k-m+1}^{(m)}, \qquad H_p^{(m)} \;:=\; \sum_{j=1}^{p} \tfrac{1}{j^m}. \tag{39}$$

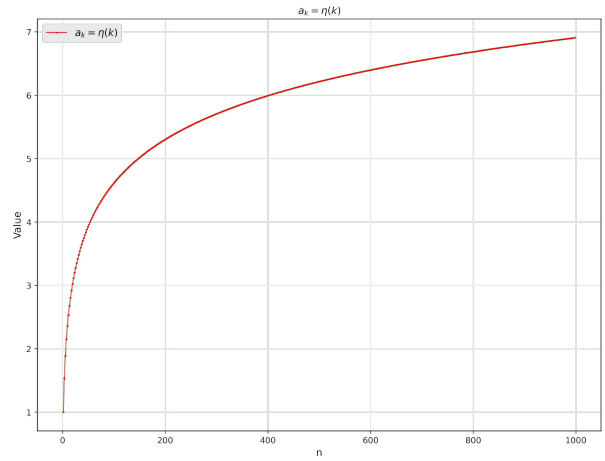
Proof. Apply Theorem 2.10 with a_k defined as in (39). It is straightforward to verify that this sequence satisfies the three conditions of the theorem:

(i) $\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!} a_k = e^x \log x + \mathcal{O}(\frac{e^x}{x})$ as $x \to \infty$;
(ii) the lower bound $a_k > H_k - \gamma$ holds for every $k \in \mathbb{N}$;
(iii) the base inequality (38) is verified directly for $1 \leq n \leq 60$.

The equivalence follows. See [13, Thm. 4]. □



(A) Plot of the left-hand side of (38) against the right-hand side $\sum_{d|n} d$, for small values of n. The alternating generalized-harmonic coefficients a_k defined in (39) produce a sequence that dominates the divisor sum from the very first integers, illustrating the base validation of condition (iii).



(B) Comparison of the coefficients a_k defined in (39) with the lower bound $H_k - \gamma$, for k ranging over the first integers. The alternating harmonic layers accumulate so that a_k lies strictly above $H_k - \gamma$, confirming condition (ii) of the theorem.

FIGURE 3. Graphical verification of the Second Equivalence.
(a) The series on the left-hand side of (38) dominates the divisor sum for small n, validating the base case. **(b)** The coefficients a_k lie strictly above the lower bound $H_k - \gamma$, confirming the inner condition of the theorem.

Theorem 2.14 (Espinosa, 2025, Third equivalence: fully explicit coefficients)**.** *The* ***Riemann Hypothesis*** *is equivalent to*

$$\sum_{k\in\mathbb{N}} \frac{\left(\frac{1}{1}+\cdots+\frac{1}{n}\right)^k}{k!}\left(\frac{1}{k}+\cdots+\frac{1}{k^2}\right) > \sum_{d|n} d \qquad \textit{for every } n\in\mathbb{N}. \tag{40}$$

Proof. Apply Theorem 2.10 with $a_k := \frac{1}{k}+\cdots+\frac{1}{k^2}$. It is straightforward to verify that this sequence satisfies the three conditions of the theorem, the proof is exactly the same of Theorem 2.12 replacing with $H_k - \gamma < \frac{1}{k}+\cdots+\frac{1}{k^2} < H_k - \gamma + \frac{1}{k}$ for $k \geq 1$, the equivalence follows. See [4, Thm. 7]. □

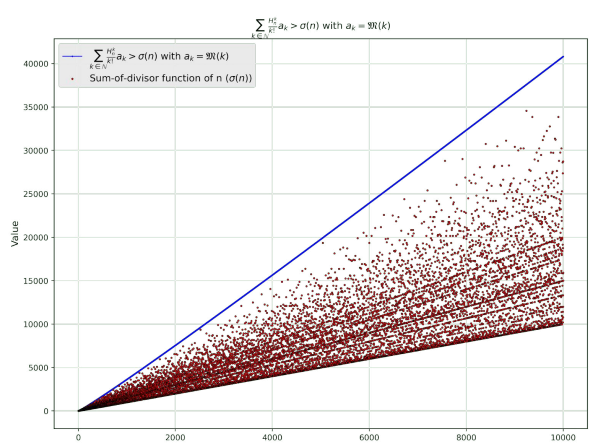
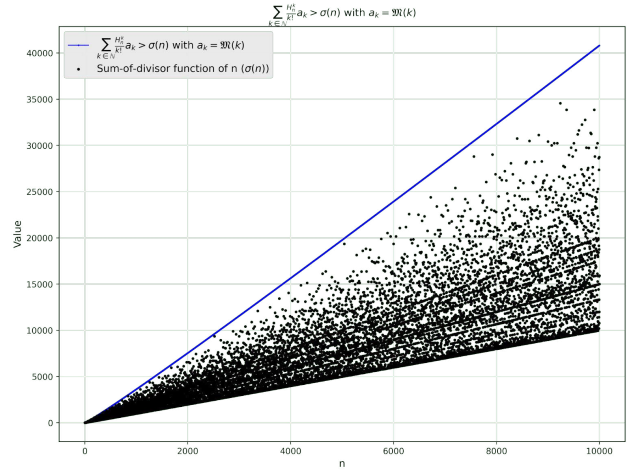
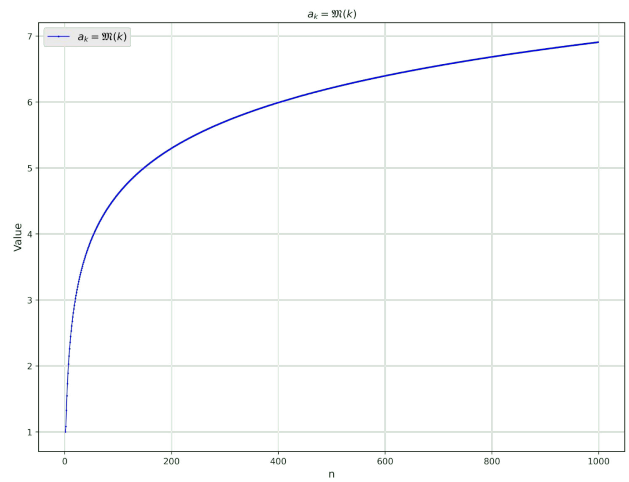


FIGURE 4. Third equivalence of the Riemann Hypothesis, with $a_k = \frac{1}{k}+\cdots+\frac{1}{k^2}$.



(A) Plot of the left-hand side of (40) against the right-hand side $\sum_{d|n} d$, for small values of n. The fully explicit coefficients $a_k = \frac{1}{k} + \cdots + \frac{1}{k^2}$ produce a sequence that dominates the divisor sum from the very first integers, illustrating the base validation of condition (iii).



(B) Comparison of the coefficients $a_k = \frac{1}{k} + \cdots + \frac{1}{k^2}$ with the lower bound $H_k - \gamma$, for k ranging over the first integers. The harmonic series of coefficients accumulates so that a_k lies strictly above $H_k - \gamma$, confirming condition (ii) of the theorem.

FIGURE 5. Graphical verification of the Third Equivalence.
(a) The series on the left-hand side of (40) dominates the divisor sum for small n, validating the base case. **(b)** The coefficients a_k lie strictly above the lower bound $H_k - \gamma$, confirming the inner condition of the theorem.

2.3. On the generalized harmonic number of a partition $\boldsymbol{\lambda}$ and a k-tuple λ.

2.3.1. *About $H_n^{\boldsymbol{\lambda}}$ and H_n^{λ}.* We now develop the partition-theoretic machinery that underlies the combinatorial reformulation of the Riemann Hypothesis. The central objects are the generalized harmonic numbers associated with partitions and tuples, which extend the classical harmonic numbers to the multi-index setting, see [8]. We begin by establishing the formal language of partitions.

Definition 2.15 (Partition of an integer)**.** *Let $m \in \mathbb{N}$. A* **partition $\boldsymbol{\lambda}$** *of m of length k is a k-tuple of integers $\lambda_i \in \mathbb{N}$ for $1 \leq i \leq k$ such that*

$$\boldsymbol{\lambda} \;=\; (\lambda_1, \lambda_2, \ldots, \lambda_k), \qquad \lambda_1 \;\geq\; \lambda_2 \;\geq\; \ldots \;\geq\; \lambda_k \;\geq\; 1, \qquad \sum_{r=1}^{k} \lambda_r \;=\; m.$$

We write $\boldsymbol{\lambda} \vdash m$ to denote that $\boldsymbol{\lambda}$ is a partition of m.

A crucial aspect of our proposal is the distinction between a partition (where order is fixed) and the set of its possible arrangements. To capture the full symmetry of the harmonic power, we define the set of all distinct permutations as follows.

Definition 2.16 (Set of distinct permutations)**.** *For a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \vdash m$, let $\Psi(\boldsymbol{\lambda})$ denote the* **set of all distinct rearrangements of $\boldsymbol{\lambda}$**, *equivalently, the set of all distinct permutations of the k-tuple $(\lambda_1, \ldots, \lambda_k)$:*

$$\Psi(\boldsymbol{\lambda}) \;:=\; \Big\{(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbb{N}^k : \forall j \in \mathbb{N},\ \operatorname{card}\{r : \sigma_r = j\} = \operatorname{card}\{r : \lambda_r = j\}\Big\}.$$

Each $\Theta = (\sigma_1, \ldots, \sigma_k) \in \Psi(\boldsymbol{\lambda})$ is a k-tuple representing a distinct rearrangement of the parts of $\boldsymbol{\lambda}$.

Once the partition is defined, it is necessary to introduce the algebraic weighting associated with it. The following definitions for the factorial and the multinomial coefficient of a partition are essential for our final formula.

Definition 2.17 (Partition factorial)**.** *For a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \vdash m$, we define the* **partition factorial***:*

$$\boldsymbol{\lambda}! \;:=\; \lambda_1!\,\lambda_2! \cdots \lambda_k!.$$

Definition 2.18 (Multinomial coefficient for partitions)**.** *For a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \vdash m$, we define the* **partition multinomial coefficient***:*

$$\binom{m}{\boldsymbol{\lambda}} \;:=\; \binom{m}{\lambda_1, \ldots, \lambda_k} \;=\; \frac{m!}{\lambda_1!\,\lambda_2! \cdots \lambda_k!} \;=\; \frac{m!}{\boldsymbol{\lambda}!}.$$

With these tools at hand, we introduce the two core objects of this study: the partition-based harmonic number $H_n^{\boldsymbol{\lambda}}$ (Definition 2.19), which exploits the full symmetry of the partition, and the tuple-based harmonic number H_n^{λ} (Definition 2.20), whose dependence on the order of the exponents is essential.

Definition 2.19 (Generalized harmonic number of a partition)**.** *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of length k with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. For $n \in \mathbb{N}$ with $n \geq k$, the n-th* **generalized harmonic number of order $\lambda_1 + \lambda_2 + \ldots + \lambda_k$ associated with the partition $\boldsymbol{\lambda}$** *is defined as*

$$H_n^{\boldsymbol{\lambda}} \;:=\; \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{r=1}^{k} \tfrac{1}{i_r^{\sigma_r}},$$

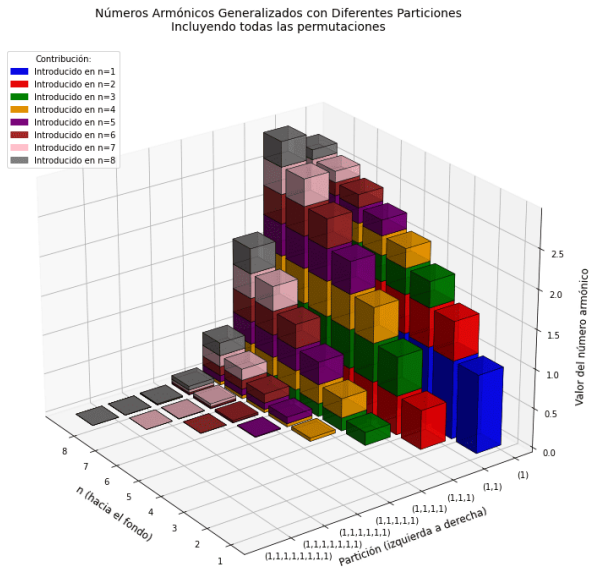
where $\Psi(\boldsymbol{\lambda})$ is as in Definition 2.16. For $n < k$, we set $H_n^{\boldsymbol{\lambda}} := 0$.

Definition 2.20 (Generalized harmonic number for a k-tuple)**.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a k-tuple of positive integers. For $n \in \mathbb{N}$ with $n \geq k$, the n-th* **generalized harmonic number of order $\lambda_1 + \lambda_2 + \ldots + \lambda_k$ associated with the k-tuple** *λ is defined as*

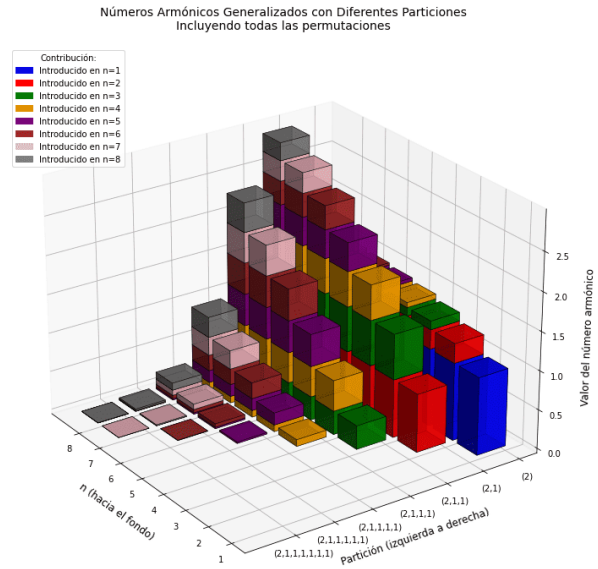
$$H_n^{\lambda} \;:=\; \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{r=1}^{k} \frac{1}{i_r^{\lambda_r}}.$$

In this definition the **order of the exponents matters***: distinct orderings of the same multiset generally yield distinct harmonic numbers. For $n < k$, we set $H_n^{\lambda} := 0$.*

The elegance of this framework lies in the direct correspondence between these two definitions: the harmonic number of a partition is the sum of the harmonic numbers of all its distinct tuple arrangements.

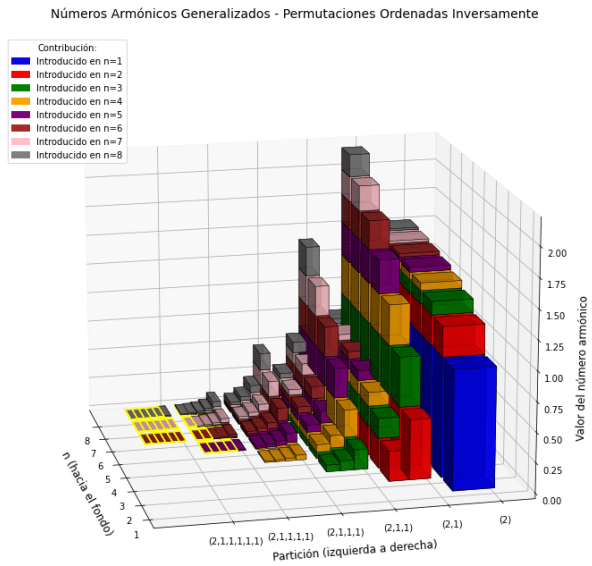


(A) Incremental contributions to the generalized harmonic numbers $H_n^{\boldsymbol{\lambda}}$ for partitions of the form $(1, 1, 1, \dots)$, that is, partitions with all parts equal to 1. Each bar represents the increment $\Delta H_n^{\boldsymbol{\lambda}} = H_n^{\boldsymbol{\lambda}} - H_{n-1}^{\boldsymbol{\lambda}}$ added by the index n to the partial sum, so that the accumulated value $H_n^{\boldsymbol{\lambda}}$ is recovered by stacking the bars of colors $1, 2, \dots, n$. The horizontal axis lists the partitions $\boldsymbol{\lambda} = (1)$, $(1, 1)$, $\dots$, $(1, 1, 1, 1, 1, 1, 1, 1)$, from right to left; the depth axis shows the upper limit n from 1 to 8; and the height of each bar is the contribution of the level n to the total sum. The colors follow the natural order of n: blue for $n = 1$, red for $n = 2$, green for $n = 3$, yellow for $n = 4$, purple for $n = 5$, dark red for $n = 6$, pink for $n = 7$, and gray for $n = 8$.

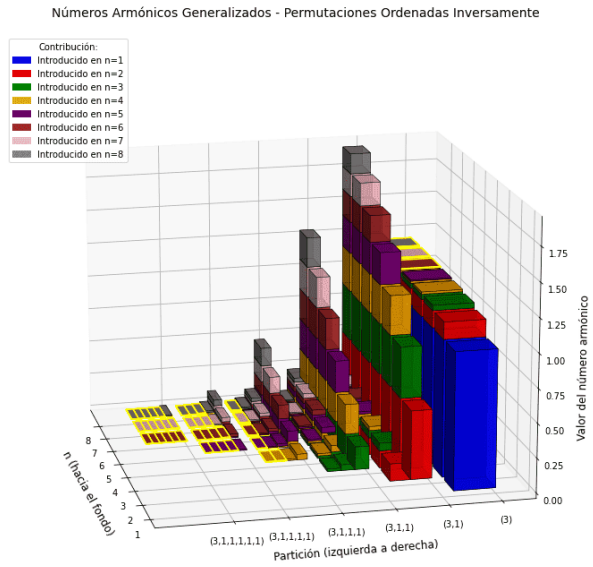


(B) Incremental contributions to the generalized harmonic numbers $H_n^{\boldsymbol{\lambda}}$ for general partitions, including the forms (2), $(2, 1)$, $(2, 1, 1)$, $(2, 1, 1, 1)$, and so on. As in panel (a), each bar represents the increment $\Delta H_n^{\boldsymbol{\lambda}} = H_n^{\boldsymbol{\lambda}} - H_{n-1}^{\boldsymbol{\lambda}}$ contributed by the index n, and the total value $H_n^{\boldsymbol{\lambda}}$ is recovered by stacking the bars of colors $1, 2, \dots, n$. The additional partitions with parts greater than 1 generate contributions that cannot be captured by the one-part partition (k) alone, and they become significant at higher levels of n. The color encoding follows the same convention as in panel (a).

FIGURE 6. Incremental contributions to the generalized harmonic numbers $H_n^{\boldsymbol{\lambda}}$ for different partitions $\boldsymbol{\lambda}$ and different values of the upper limit n. **(a)** Partitions of the form $(1, 1, 1, \dots)$. **(b)** Partitions of the form $(2, 1, 1, \dots)$. In both panels, each bar represents the increment $\Delta H_n^{\boldsymbol{\lambda}} = H_n^{\boldsymbol{\lambda}} - H_{n-1}^{\boldsymbol{\lambda}}$ added by the index n to the partial sum $H_{n-1}^{\boldsymbol{\lambda}}$, so that the accumulated value at level n is obtained by stacking the bars of colors $1, 2, \dots, n$. The horizontal axis lists the partitions, the depth axis shows the upper limit n from 1 to 8, and the height of each bar is the contribution of the level n to the total sum. The colors encode the value of n: blue for $n = 1$, red for $n = 2$, green for $n = 3$, yellow for $n = 4$, purple for $n = 5$, dark red for $n = 6$, pink for $n = 7$, and gray for $n = 8$.

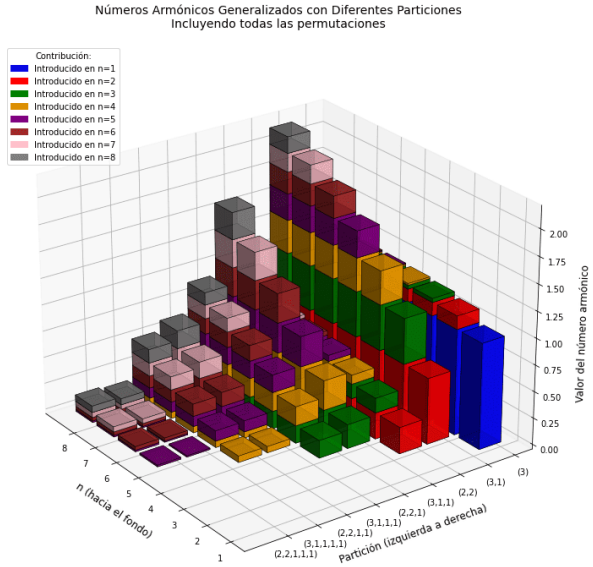


(A) Family $(2,1,1,\dots)$: partitions of the form $(2,1,1,\dots)$. Each bar represents the generalized harmonic number H_n^{Θ} associated with a distinct permutation $\Theta \in \Psi(\boldsymbol{\lambda})$ of the partition $\boldsymbol{\lambda}$, decomposed into its tuple-based harmonic components according to Proposition 2.21. The permutations are ordered from largest to smallest value of the tuple, so that $(2,1,1,1,1,1)$ appears at the right and $(1,1,1,1,1,2)$ at the left. The horizontal axis lists the partitions; the depth axis shows the upper limit n from 1 to 8; and the height of each bar is the value of H_n^{Θ}. The colors encode the value of n at which each contribution is added to the accumulated sum, so that stacking the bars of colors $1,2,\dots,n$ recovers the total harmonic number $H_n^{\boldsymbol{\lambda}} = \sum_{\Theta\in\Psi(\boldsymbol{\lambda})} H_n^{\Theta}$.

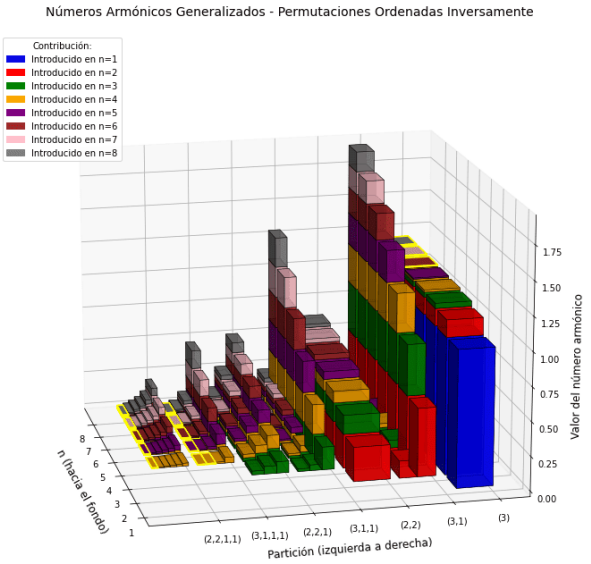


(B) Family $(3,1,1,\dots)$: partitions of the form $(3,1,1,\dots)$. Each bar represents the generalized harmonic number H_n^{Θ} associated with a distinct permutation $\Theta \in \Psi(\boldsymbol{\lambda})$ of the partition $\boldsymbol{\lambda} = (3)$, $(3,1)$, $(3,1,1)$, $\dots$. As in panel (a), the permutations are ordered from largest to smallest tuple value, and the color encoding follows the same convention, with the cumulative sum at level n recovering the full harmonic number $H_n^{\boldsymbol{\lambda}}$. The additional partitions with parts equal to 3 generate contributions of a different magnitude, and the visualization makes transparent how each tuple-based harmonic accumulates at successive levels of n.

FIGURE 7. Generalized harmonic numbers H_n^{Θ} for the tuple-based permutations $\Theta \in \Psi(\boldsymbol{\lambda})$ of the partitions in the families $(2,1,1,\dots)$ and $(3,1,1,\dots)$, evaluated at different values of the upper limit n. **(a)** Family $(2,1,1,\dots)$. **(b)** Family $(3,1,1,\dots)$. In both panels, the horizontal axis lists the partitions, the depth axis shows the upper limit n from 1 to 8, and the height of each bar is the value of H_n^{Θ}. The colors encode the value of n: blue for $n=1$, red for $n=2$, green for $n=3$, yellow for $n=4$, purple for $n=5$, dark red for $n=6$, pink for $n=7$, and gray for $n=8$. Each level n contributes the incremental amount added to the partial sum, so that the accumulated value at level n is obtained by stacking the bars of colors $1,2,\dots,n$. This follows directly from Proposition 2.21, which expresses each partition-based harmonic number as the sum of the tuple-based harmonic numbers of its distinct permutations.



(A) Partition blocks for the families $(2, 2, 1, 1, 1, \dots)$ and $(3, 1, 1, 1, \dots)$: the partitions obtained by adding trailing 1's to the base partitions $(2, 2)$ and (3). Each bar represents the full generalized harmonic number $H_n^{\boldsymbol{\lambda}}$ associated with the partition $\boldsymbol{\lambda}$, without decomposing it into its tuple-based permutations. The horizontal axis lists the partitions; the depth axis shows the upper limit n from 1 to 8; and the height of each bar is the value of $H_n^{\boldsymbol{\lambda}}$. The colors encode the value of n at which each contribution is added to the partial sum, so that stacking the bars of colors $1, 2, \dots, n$ recovers the accumulated value $\sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H_n^{\Theta} = H_n^{\boldsymbol{\lambda}}$ at level n. This panel shows the harmonic numbers directly, as they appear in the partition-based formulation of the Riemann Hypothesis.



(B) Tuple-based decompositions for the same two families $(2, 2, 1, 1, 1, \dots)$ and $(3, 1, 1, 1, \dots)$. Each bar represents the generalized harmonic number H_n^{Θ} associated with a distinct permutation $\Theta \in \Psi(\boldsymbol{\lambda})$ of the partition $\boldsymbol{\lambda}$. The permutations are ordered from largest to smallest tuple value, so that the tuple with the largest parts first appears at the right. The height of each bar is H_n^{Θ} and the colors encode the value of n, following the same convention as in panel (a). The cumulative stacking of colors $1, 2, \dots, n$ recovers the full harmonic number $H_n^{\boldsymbol{\lambda}} = \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H_n^{\Theta}$, as stated in Proposition 2.21. This panel exposes the individual contributions of each tuple arrangement, showing how the harmonic number of a partition is assembled from its constituent permutations.

FIGURE 8. Two complementary visualizations of the generalized harmonic numbers for the partition families $(2, 2, 1, 1, 1, \dots)$ and $(3, 1, 1, 1, \dots)$. **(a)** Partition blocks: the harmonic numbers $H_n^{\boldsymbol{\lambda}}$ are plotted directly for each partition $\boldsymbol{\lambda}$, without decomposition. **(b)** Tuple-based decompositions: the harmonic numbers are decomposed into their constituent permutations $\Theta \in \Psi(\boldsymbol{\lambda})$ according to Proposition 2.21, and the permutations are ordered from largest to smallest tuple value. In both panels, the horizontal axis lists the partitions, the depth axis shows the upper limit n from 1 to 8, and the height of each bar is the value of the corresponding harmonic number. The colors encode the value of n: blue for $n = 1$, red for $n = 2$, green for $n = 3$, yellow for $n = 4$, purple for $n = 5$, dark red for $n = 6$, pink for $n = 7$, and gray for $n = 8$. The two panels are complementary: the first shows the harmonic numbers as they appear in the partition-based formulation of the Riemann Hypothesis, while the second exhibits the internal structure that produces them, via the tuple-based decomposition of Proposition 2.21.

Proposition 2.21 (Relation between the two definitions)**.** *For any partition* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *of length* k *and any* $n \in \mathbb{N}$*,*

$$H_n^{\boldsymbol{\lambda}} = \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H_n^{\Theta}.$$

Proof. By Definition 2.19, for $n \geq k$,

$$H_n^{\boldsymbol{\lambda}} = \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{r=1}^{k} \frac{1}{i_r^{\sigma_r}} = \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H_n^{\Theta},$$

where the last equality uses Definition 2.20. For $n < k$ both sides vanish by definition, so the identity holds trivially. □

2.3.2. *On* $\frac{H_n^k}{k!}$. The expansion of the harmonic power H_n^k relies on the classical multinomial theorem, which we recall first.

Theorem 2.22 (Multinomial Theorem)**.** *For* $m \in \mathbb{N}$ *and commuting variables* $x_1, \ldots, x_n$*,*

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{\substack{\alpha_1, \ldots, \alpha_n \geq 0 \\ \alpha_1 + \cdots + \alpha_n = m}} \binom{m}{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where $\binom{m}{\alpha_1, \ldots, \alpha_n} = \dfrac{m!}{\alpha_1! \cdots \alpha_n!}$ *is the multinomial coefficient.*

Proof. See [21, Sec. 1.4]. □

We now present the central decomposition of this section: the normalized harmonic power $\frac{H_n^k}{k!}$ as a weighted sum over partitions of k.

Proposition 2.23 (Partition expansion of the harmonic power I)**.** *For all* $n, k \in \mathbb{N}$*,*

$$\frac{H_n^k}{k!} = \sum_{\boldsymbol{\lambda} \vdash k} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}.$$

Proof. By the Multinomial Theorem (Theorem 2.22) applied to $x_i = \frac{1}{i}$,

$$H_n^k = \sum_{\substack{\alpha_1, \ldots, \alpha_n \geq 0 \\ \alpha_1 + \cdots + \alpha_n = k}} \frac{k!}{\alpha_1! \cdots \alpha_n!} \prod_{j=1}^{n} \frac{1}{j^{\alpha_j}}.$$

Group the terms of this sum according to the partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \vdash k$ formed by the positive exponents α_j, arranged in non-increasing order. For a fixed partition $\boldsymbol{\lambda}$ and a fixed set of indices $1 \leq j_1 < \cdots < j_m \leq n$, each assignment of $\lambda_1, \ldots, \lambda_m$ to these positions is a permutation $\Theta = (\sigma_1, \ldots, \sigma_m) \in \Psi(\boldsymbol{\lambda})$, whose multinomial coefficient satisfies

$$\frac{k!}{\sigma_1! \cdots \sigma_m!} = \frac{k!}{\boldsymbol{\lambda}!}.$$

Summing over $\Theta \in \Psi(\boldsymbol{\lambda})$ and over all strictly increasing m-tuples recovers $\frac{k!}{\boldsymbol{\lambda}!} H_n^{\boldsymbol{\lambda}}$ by Definition 2.19. Summing over all partitions yields $H_n^k = \sum_{\boldsymbol{\lambda} \vdash k} \frac{k!}{\boldsymbol{\lambda}!} H_n^{\boldsymbol{\lambda}}$, and dividing by $k!$ proves the claim. □

Substituting the relation between partition-based and tuple-based harmonic numbers (Proposition 2.21) gives a more granular expansion.

Proposition 2.24 (Partition expansion of the harmonic power II)**.** *For all* $n, k \in \mathbb{N}$*,*

$$\frac{H_n^k}{k!} = \sum_{\boldsymbol{\lambda} \vdash k} \frac{1}{\boldsymbol{\lambda}!} \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H_n^{\Theta}.$$

Proof. Combine Proposition 2.23 with Proposition 2.21 and substitute $H_n^{\boldsymbol{\lambda}} = \sum_{\Theta\in\Psi(\boldsymbol{\lambda})} H_n^{\Theta}$. □

The raw power H_n^k admits an analogous form in terms of the partition multinomial coefficient of Definition 2.18.

Proposition 2.25 (Partition expansion of the harmonic power III)**.** *For all $n, k \in \mathbb{N}$,*

$$H_n^k \;=\; \sum_{\boldsymbol{\lambda}\vdash k} \binom{k}{\boldsymbol{\lambda}} H_n^{\boldsymbol{\lambda}}.$$

Proof. Multiply the identity of Proposition 2.23 by $k!$ and apply Definition 2.18. □

Finally, the most detailed decomposition is obtained by combining Propositions 2.25 and 2.21.

Proposition 2.26 (Partition expansion of the harmonic power IV)**.** *For all $n, k \in \mathbb{N}$,*

$$H_n^k \;=\; \sum_{\boldsymbol{\lambda}\vdash k} \binom{k}{\boldsymbol{\lambda}} \sum_{\Theta\in\Psi(\boldsymbol{\lambda})} H_n^{\Theta}.$$

Proof. Substitute $H_n^{\boldsymbol{\lambda}} = \sum_{\Theta\in\Psi(\boldsymbol{\lambda})} H_n^{\Theta}$ into Proposition 2.25. □

2.3.3. *A Partition Decomposition of the Multinomial Theorem.* We close this section with a partition-symmetric restatement of the multinomial theorem. This form makes the underlying combinatorial structure transparent and provides the natural bridge to Young's Lattice.

Theorem 2.27 (Partition-symmetric form of the Multinomial Theorem)**.** *Let $m, n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be commuting variables. Then*

$$(x_1 + \cdots + x_n)^m \;=\; \sum_{\substack{\boldsymbol{\lambda}\vdash m\\ \ell(\boldsymbol{\lambda})\le n}} \binom{m}{\boldsymbol{\lambda}} \sum_{\Theta\in\Psi(\boldsymbol{\lambda})} \sum_{1\le j_1<\cdots<j_k\le n} \prod_{r=1}^{k} x_{j_r}^{\sigma_r}, \tag{41}$$

where $k = \ell(\boldsymbol{\lambda})$, $\Theta = (\sigma_1, \ldots, \sigma_k)$, and the inner sums run over all strictly increasing k-tuples and all distinct permutations $\Theta \in \Psi(\boldsymbol{\lambda})$.

Proof. By the classical Multinomial Theorem,

$$(x_1 + \cdots + x_n)^m \;=\; \sum_{\substack{\alpha_1,\ldots,\alpha_n\ge 0\\ \alpha_1+\cdots+\alpha_n=m}} \frac{m!}{\alpha_1!\cdots\alpha_n!} \prod_{j=1}^{n} x_j^{\alpha_j}.$$

Group the summands according to the partition determined by their positive exponents. For a fixed multi-index $(\alpha_1, \ldots, \alpha_n)$ let

$$\{j_1, \ldots, j_k\} \;=\; \{j : \alpha_j > 0\}, \qquad j_1 < \cdots < j_k,$$

and set $\mu_r = \alpha_{j_r}$. Arranging the positive integers $\mu_1, \ldots, \mu_k$ in non-increasing order yields a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \vdash m$ with length $k \le n$. The assignment of these values to the ordered positions $j_1, \ldots, j_k$ is precisely a permutation

$$\Theta \;=\; (\sigma_1, \ldots, \sigma_k) \in \Psi(\boldsymbol{\lambda}).$$

Since Θ is a rearrangement of $\boldsymbol{\lambda}$, we have

$$\frac{m!}{\alpha_1!\cdots\alpha_n!} \;=\; \frac{m!}{\sigma_1!\cdots\sigma_k!} \;=\; \frac{m!}{\lambda_1!\cdots\lambda_k!} \;=\; \binom{m}{\boldsymbol{\lambda}}.$$

The corresponding monomial is $\prod_{r=1}^{k} x_{j_r}^{\sigma_r}$.

The map

$$(\alpha_1, \ldots, \alpha_n) \;\longleftrightarrow\; \big(\boldsymbol{\lambda}\vdash m,\ (j_1, \ldots, j_k),\ \Theta \in \Psi(\boldsymbol{\lambda})\big)$$

is bijective: given the triple one recovers the multi-index by setting $\alpha_{j_r} = \sigma_r$ and $\alpha_j = 0$ otherwise. Summing over all such triples therefore exhausts the multinomial expansion. Because the coefficient $\binom{m}{\boldsymbol{\lambda}}$ depends neither on Θ nor on the choice of indices, it factors out of the two inner sums, yielding (41). □

2.4. **The Riemann Hypothesis Young's Lattice Equivalence.** Building on the partition decomposition of the harmonic power (Proposition 2.23) and the equivalence theorem of Section 2.1.3, we can now restate the Riemann Hypothesis as a growth condition indexed by Young's Lattice [11].

Theorem 2.28 (Espinosa, 2025, Riemann Hypothesis Young's Lattice Equivalence)**.** *The **Riemann Hypothesis** is equivalent to*

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;>\; \sum_{d|n} d \qquad \textit{for every } n\in\mathbb{N}, \tag{42}$$

where $\boldsymbol{\lambda}\vdash k$ denotes that $\boldsymbol{\lambda}$ is a partition of k, $H_n^{\boldsymbol{\lambda}}$ is the n-th generalized harmonic number of order $\lambda_1+\cdots+\lambda_m = k$ associated with the partition $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_m)\vdash k$, $\boldsymbol{\lambda}! = \lambda_1!\,\lambda_2!\cdots\lambda_m!$ is the partition factorial associated with $\boldsymbol{\lambda}\vdash k$, and $a_k = \frac{1}{k}+\cdots+\frac{1}{k^2}$.

Proof. By Proposition 2.23, the inner sum satisfies

$$\sum_{\boldsymbol{\lambda}\vdash k} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!} \;=\; \frac{H_n^k}{k!}.$$

Substituting this into (42) yields

$$\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!}\, a_k \;>\; \sum_{d|n} d \qquad \text{for every } n\in\mathbb{N},$$

which, for $a_k = \frac{1}{k}+\cdots+\frac{1}{k^2}$, is the Third Equivalence (Theorem 2.14). □

Theorem 2.29 (Espinosa, 2025, Riemann Hypothesis Young's Lattice Equivalence — Logarithmic Form)**.** *The **Riemann Hypothesis** is equivalent to*

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, \log(k+1) \;>\; \sum_{d|n} d \qquad \textit{for every } n\in\mathbb{N}, \tag{43}$$

where $\boldsymbol{\lambda}\vdash k$ denotes that $\boldsymbol{\lambda}$ is a partition of k, $H_n^{\boldsymbol{\lambda}}$ is the n-th generalized harmonic number of order $\lambda_1+\cdots+\lambda_m = k$ associated with the partition $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_m)\vdash k$, and $\boldsymbol{\lambda}! = \lambda_1!\,\lambda_2!\cdots\lambda_m!$ is the partition factorial associated with $\boldsymbol{\lambda}\vdash k$.

Proof. By Proposition 2.23, the inner sum satisfies

$$\sum_{\boldsymbol{\lambda}\vdash k} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!} \;=\; \frac{H_n^k}{k!}.$$

Substituting this into (43) yields

$$\sum_{k\in\mathbb{N}} \frac{H_n^k}{k!}\, \log(k+1) \;>\; \sum_{d|n} d \qquad \text{for every } n\in\mathbb{N},$$

which is the First Equivalence (Theorem 2.12). □

2.5. **The Generalized Assembly Theorem.** We close the combinatorial part of this work with the structural identity that governs the assembly of natural numbers by partition-indexed harmonic weights. This result in [10] extends the classical Assembly Theorem in [9] to arbitrary summation intervals and connects the harmonic objects of the previous sections to Young's Lattice in a quantitative way. We first fix the restricted partition by length and to analyze the specific configurations required by the Assembly Theorem, we define a restricted subspace of partitions where the length is fixed, leading to the construction of the following Assembly Set.

Definition 2.30 (Restricted partition by length)**.** *Let $m, n \in \mathbb{N}$ with $n \leq m$. A* **partition of m into exactly n parts** *is a tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ with*

$$\lambda_1 \;\geq\; \lambda_2 \;\geq\; \ldots \;\geq\; \lambda_n \;\geq\; 1, \qquad \sum_{i=1}^{n} \lambda_i \;=\; m.$$

We write $\boldsymbol{\lambda} \vdash_n m$ to denote that $\boldsymbol{\lambda}$ is a partition of m into exactly n parts.

Definition 2.31 (Assembly Set)**.** *Let $m, n \in \mathbb{N}$ with $n \leq m$. The* **Assembly Set** *$\wp_n^m$ is the collection of all distinct partitions $\boldsymbol{\lambda}$ of m into exactly n parts:*

$$\wp_n^m \;:=\; \left\{ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1,\; \sum_{i=1}^{n} \lambda_i = m \right\}.$$

Equivalently, $\wp_n^m = \{\boldsymbol{\lambda} : \boldsymbol{\lambda} \vdash_n m\}$.

We now introduce the generalized harmonic numbers over an arbitrary index interval $[s, n]$.

Definition 2.32 (Generalized harmonic number of a partition over an closed interval)**.** *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of length k with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. For $n, s \in \mathbb{N}$ with $s \leq n$ and $n \geq k$, the* **$[s, n]$-th generalized harmonic number of order $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ associated with the partition $\boldsymbol{\lambda}$** *is defined as*

$$H_{s,n}^{\boldsymbol{\lambda}} \;:=\; \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} \; \sum_{s \leq i_1 < i_2 < \cdots < i_k \leq n} \; \prod_{r=1}^{k} \frac{1}{i_r^{\sigma_r}}, \tag{44}$$

where $\Psi(\boldsymbol{\lambda})$ is the set of distinct rearrangements of $\boldsymbol{\lambda}$ given in Definition 2.16.

Each $\Theta = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in \Psi(\boldsymbol{\lambda})$ is a k-tuple representing a distinct rearrangement of the parts of $\boldsymbol{\lambda}$. For $n < k$ or $s > n$, we set $H_{s,n}^{\boldsymbol{\lambda}} := 0$.

Definition 2.33 (Generalized harmonic number for a k-tuple over an closed interval)**.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a k-tuple of positive integers. For $n, s \in \mathbb{N}$ with $s \leq n$ and $n \geq k$, the* **$[s, n]$-th generalized harmonic number of order $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ associated with the k-tuple λ** *is defined as*

$$H_{s,n}^{\lambda} \;:=\; \sum_{s \leq i_1 < i_2 < \cdots < i_k \leq n} \; \prod_{r=1}^{k} \frac{1}{i_r^{\lambda_r}}. \tag{45}$$

In this definition the order of the exponents matters: distinct orderings of the same multiset generally yield distinct harmonic numbers. For $n < k$ or $s > n$, we set $H_{s,n}^{\lambda} := 0$.

The following remark clarifies the relation between the interval-based notation and the unshifted notation introduced in the previous section.

Remark 2.34 (Relation with the unshifted notation)**.** *When $s = 1$, the $[1, n]$-th generalized harmonic numbers $H_{1,n}^{\boldsymbol{\lambda}}$ and $H_{1,n}^{\lambda}$ coincide with the unshifted harmonic numbers $H_n^{\boldsymbol{\lambda}}$ and H_n^{λ} introduced in Definitions 2.32 and 2.33. Throughout the paper we use the unshifted notation unless an arbitrary starting point s is explicitly required.*

The elegance of this framework lies in the direct correspondence between the two representations. The harmonic number of a partition is precisely the aggregate of the harmonic numbers of its constituent tuple arrangements.

Proposition 2.35 (Relation between the two definitions)**.** *For any partition* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *of length* k *and any* $n, s \in \mathbb{N}$ *with* $s \leq n$*,*

$$H^{\boldsymbol{\lambda}}_{s,n} \;=\; \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H^{\Theta}_{s,n}. \tag{46}$$

Proof. By Definition 2.32, for $n \geq k$ and $s \leq n$,

$$H^{\boldsymbol{\lambda}}_{s,n} \;=\; \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} \sum_{s \leq i_1 < \cdots < i_k \leq n} \prod_{r=1}^{k} \frac{1}{i_r^{\sigma_r}} \;=\; \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H^{\Theta}_{s,n},$$

where the last equality uses Definition 2.33. For $n < k$ or $s > n$ both sides vanish by definition, so the identity holds trivially. □

To state the theorem, we define the following sum:

Definition 2.36 (Generalized Assembly Sum)**.** *For* $n, m, s \in \mathbb{N}$ *with* $s \leq n$*, the* **generalized assembly sum** *is*

$$\mathfrak{S}^{m}_{s,n} \;:=\; \sum_{k=s}^{n} \sum_{\boldsymbol{\lambda} \in \wp^{k-s+m}_{k-s+1}} H^{\boldsymbol{\lambda}}_{s,n}, \tag{47}$$

where $\boldsymbol{\lambda} \vdash_{k-s+1} k-s+m$ *and* $H^{\boldsymbol{\lambda}}_{s,n}$ *is the* $[s,n]$*-th generalized harmonic number of order* $\lambda_1 + \cdots + \lambda_{k-s+1}$ *associated with* $\boldsymbol{\lambda}$*.*

We can now state the central result of this section: $\mathfrak{S}^{m}_{s,n}$ is invariant with respect to the excess parameter $m-1$ and depends only on the interval endpoints. See [10]. Although we originally proposed a rigorous proof for this result using strong induction in [10], a remarkably beautiful and short proof was subsequently suggested by Bîsceanu in [2], which we present below.

Theorem 2.37 (Generalized Assembly Theorem I)**.** *For all* $n, m, s \in \mathbb{N}$ *with* $s \leq n$*,*

$$\mathfrak{S}^{m}_{s,n} \;:=\; \sum_{k=s}^{n} \sum_{\boldsymbol{\lambda} \in \wp^{k-s+m}_{k-s+1}} H^{\boldsymbol{\lambda}}_{s,n} \;=\; \frac{n-s+1}{s^m}. \tag{48}$$

Proof. Fix $m \geq 1$ and put $j = k-s+1$, so that $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_j)$ ranges over the partitions of $j+m-1$ into j positive parts. For a fixed j, the partitions of $j+m-1$ into j parts, together with all their distinct permutations, enumerate each ordered composition $(\sigma_1, \ldots, \sigma_j)$ with $\sum_{r=1}^{j} \sigma_r = j+m-1$ exactly once. Writing $\sigma_r = a_r + 1$ with $a_r \geq 0$, we obtain, as an identity of formal power series in t,

$$\sum_{m \geq 1} \mathfrak{S}^{m}_{s,n}\, t^{m-1} \;=\; \sum_{\varnothing \neq I \subseteq \{s,\ldots,n\}} \prod_{i \in I} \sum_{a \geq 0} \frac{t^a}{i^{a+1}}.$$

Each inner geometric series evaluates to $\sum_{a \geq 0} \frac{t^a}{i^{a+1}} = \frac{1}{i-t}$, so

$$\sum_{m \geq 1} \mathfrak{S}^{m}_{s,n}\, t^{m-1} \;=\; \prod_{i=s}^{n} \left(1 + \frac{1}{i-t}\right) - 1 \;=\; \prod_{i=s}^{n} \frac{i+1-t}{i-t} - 1.$$

The product telescopes, $\prod_{i=s}^{n} \frac{i+1-t}{i-t} = \frac{n+1-t}{s-t}$, hence

$$\sum_{m \geq 1} \mathfrak{S}^{m}_{s,n}\, t^{m-1} \;=\; \frac{n+1-t}{s-t} - 1 \;=\; \frac{n-s+1}{s-t} \;=\; \frac{n-s+1}{s} \cdot \frac{1}{1-\frac{t}{s}}.$$

Expanding $\frac{1}{1-\frac{t}{s}} = \sum_{m\geq 1} \left(\frac{t}{s}\right)^{m-1}$,

$$\sum_{m\geq 1} \mathfrak{S}^m_{s,n}\, t^{m-1} \;=\; \sum_{m\geq 1} \frac{n-s+1}{s^m}\, t^{m-1}.$$

Comparing coefficients of t^{m-1} gives $\mathfrak{S}^m_{s,n} = \frac{n-s+1}{s^m}$, as claimed. □

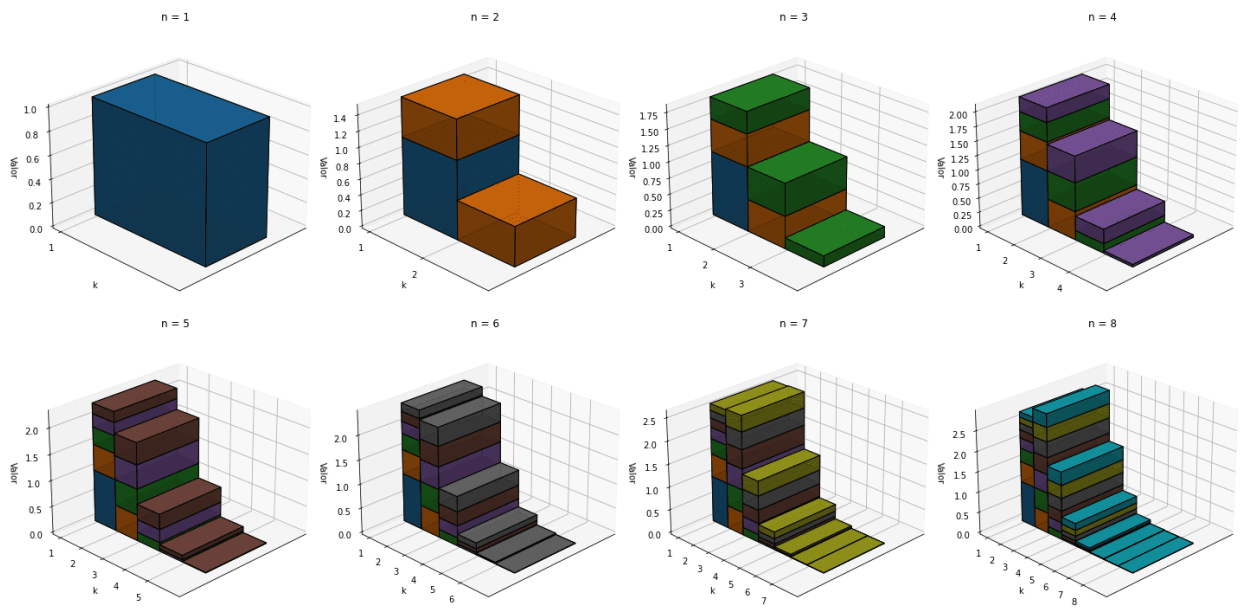


FIGURE 9. The assembly process for the classical Assembly Theorem $\mathfrak{S}^m_n = n$ with $s = 1$ and excess parameter $m - 1 = 0$. Each panel shows the partial sum of the generalized harmonic numbers $H^{\boldsymbol{\lambda}}_n$ over the Assembly Set $\wp^k_k$, as the level k increases from 1 to n. For each $n \in \{1, 2, \ldots, 8\}$, the bars represent the cumulative contributions $\sum_{k=1}^{j} \sum_{\boldsymbol{\lambda}\in\wp^k_k} H^{\boldsymbol{\lambda}}_n$ for $j = 1, \ldots, n$, and the horizontal axis lists the levels k. The colors distinguish the contributions of the successive levels, so that each new layer added at level k refines the partial sum. The identity $\mathfrak{S}^1_n = n$ is recovered as the total height of each column at level $k = n$, confirming that the ensemble of harmonic weights over the Assembly Set $\wp^k_k$ always reconstructs the count of indices $\{1, \ldots, n\}$, independently of the excess parameter $m - 1$.

The figure above visualizes the accumulation process underlying the classical Assembly Theorem

$$\mathfrak{S}^m_n \;=\; n \qquad \text{for every } m \in \mathbb{N},$$

in the particular case $m = 1$, corresponding to the branch $m = 1$ of Young's Lattice. For each value $n = 1, 2, \ldots, 8$, the panel displays the partial sums

$$\sum_{k=1}^{j} \sum_{\boldsymbol{\lambda}\in\wp^k_k} H^{\boldsymbol{\lambda}}_n, \qquad j = 1, 2, \ldots, n,$$

where $\wp^k_k$ denotes the Assembly Set of partitions of k into exactly k parts, that is, the singleton $\{(1, 1, \ldots, 1)\}$ corresponding to the partition with k parts all equal to 1. The horizontal axis lists the levels k, the depth of the bars reflects the cumulative contribution of each level, and the colors distinguish the successive layers that are added as k increases. Each panel exhibits a

staircase structure in which the total height at level $k = n$ recovers the value $\mathfrak{S}_n^1 = n$, in agreement with the theorem. This visualization makes transparent the two key features of the Assembly Theorem. First, the accumulated sum at level $k = n$ always reconstructs the count of indices $\{1, \ldots, n\}$, regardless of the particular path taken through the assembly. Second, the incremental contributions of each level k are precisely the harmonic weights $H_n^{(1,1,\ldots,1)}$ of the Assembly Set $\wp_k^k$, which by Proposition 2.35 coincide with the tuple-based harmonic numbers H_n^{Θ} for the unique permutation $\Theta = (1, 1, \ldots, 1)$ of the partition $(1, 1, \ldots, 1)$. The staircase structure of each panel therefore encodes the progressive refinement of the assembly sum as new levels are incorporated, illustrating the combinatorial mechanism that makes the identity $\mathfrak{S}_n^m = n$ hold for every value of n and m. Finally the tuple-based version of the theorem follows by direct application of the relation between partition-based and tuple-based harmonic numbers.

Corollary 2.38 (Generalized Assembly Theorem II)**.** *For all $n, m, s \in \mathbb{N}$ with $s \leq n$,*

$$\mathfrak{S}_{s,n}^m = \sum_{k=s}^{n} \sum_{\boldsymbol{\lambda} \in \wp_{k-s+1}^{k-s+m}} \sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H_{s,n}^{\Theta} = \frac{n-s+1}{s^m}. \tag{49}$$

Proof. By Proposition 2.35, $\sum_{\Theta \in \Psi(\boldsymbol{\lambda})} H_{s,n}^{\Theta} = H_{s,n}^{\boldsymbol{\lambda}}$ for every partition $\boldsymbol{\lambda}$. Substituting this identity into (49) reduces the left-hand side to $\mathfrak{S}_{s,n}^m$, and Theorem 2.37 then gives $\frac{n-s+1}{s^m}$. □

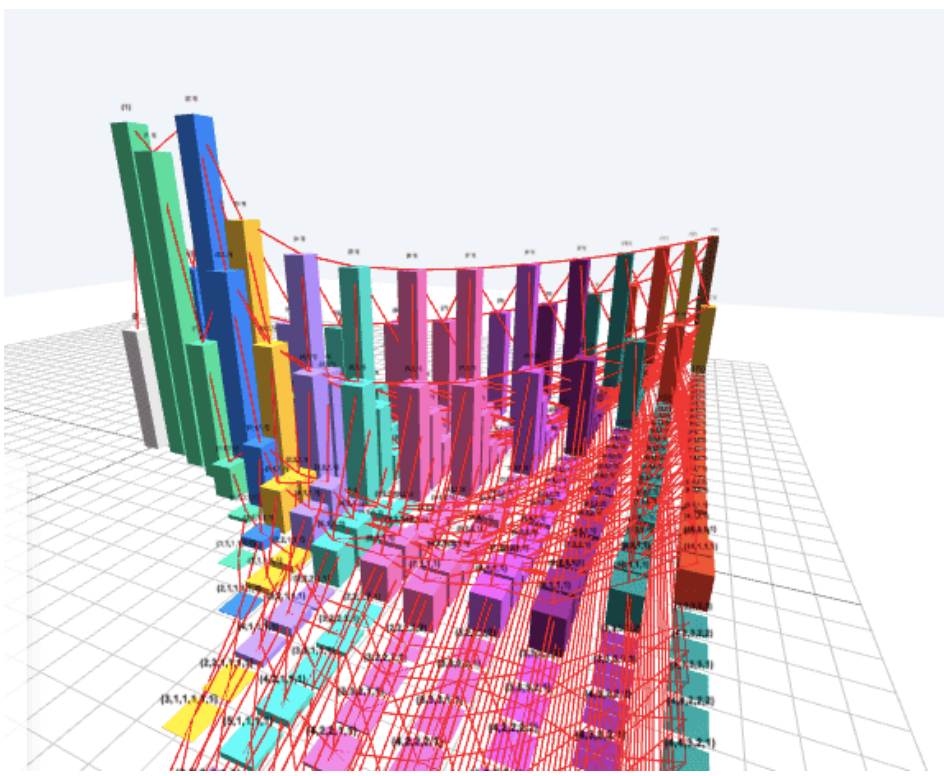


FIGURE 10. Assembly Sums $\mathfrak{S}_{1,n}^m$ for $n = 6$. Each bar is the generalized harmonic number $H_n^{\boldsymbol{\lambda}}$ of one partition $\boldsymbol{\lambda}$ in the Assembly Set $\wp_k^{k+m-1}$. The bars of one color form a single branch, that is, one Assembly Sum $\mathfrak{S}_{1,n}^m = \sum_{k=1}^{n} \sum_{\boldsymbol{\lambda} \in \wp_k^{k+m-1}} H_n^{\boldsymbol{\lambda}} = n$, where the equality follows from the Generalized Assembly Theorem 2.37. The depth axis lists all the partitions $\boldsymbol{\lambda} \vdash (k+m-1)$, The horizontal axis encodes the parameter $k+m-1$, the vertical axis shows the value $H_n^{\boldsymbol{\lambda}}$ and the red lines connect partitions. Each branch independently reconstructs the count of indices $\{1, \ldots, n\}$.

The figure above displays the Assembly Sums for $n = 6$ with $s = 1$. Each bar is the generalized harmonic number $H_n^{\boldsymbol{\lambda}}$ of one partition $\boldsymbol{\lambda}$ in the Assembly Set $\wp_k^{k+m-1}$, and the bars of one color form the Assembly Sum

$$\mathfrak{S}_{1,n}^m \;=\; \sum_{k=1}^{n} \sum_{\boldsymbol{\lambda}\in\wp_k^{k+m-1}} H_n^{\boldsymbol{\lambda}} \;=\; n,$$

by the Generalized Assembly Theorem 2.37. Two features are apparent. First, the contributions of the partitions are ordered by the excess parameter $m-1$. Second, the accumulated value of every branch, obtained by stacking the bars of that color, recovers the count $\mathfrak{S}_{1,n}^m = n$ for every m.

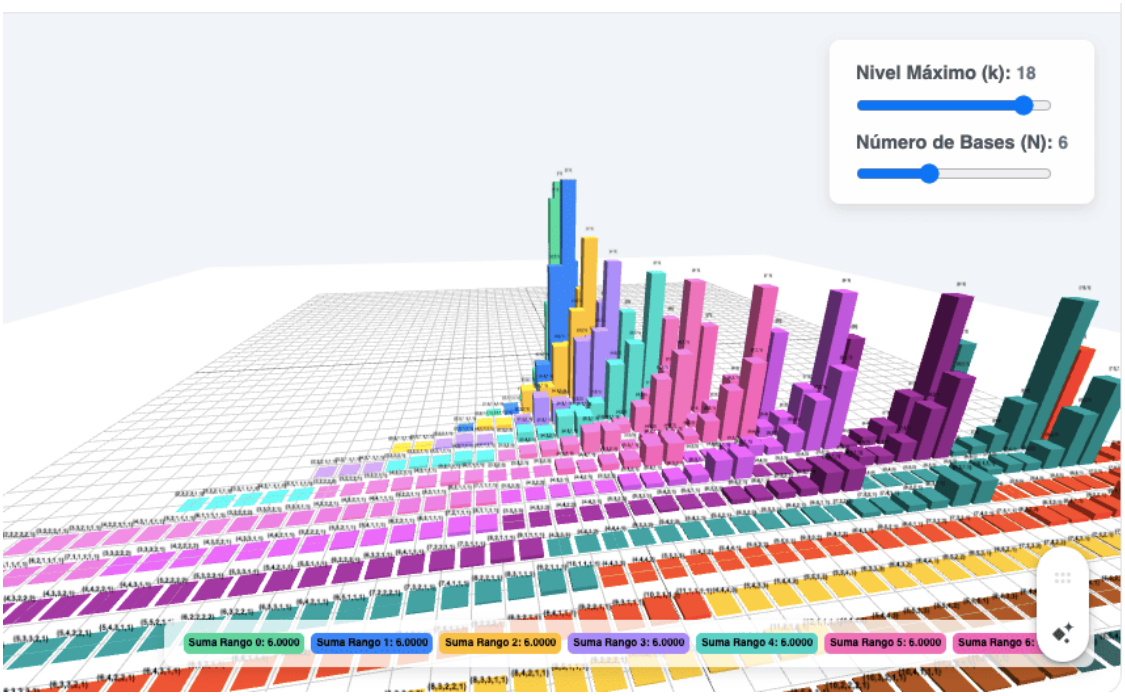


FIGURE 11. Young's Lattice representation of the assembly sets $\wp_k^{k+m-1}$ for $n = 6$. The Assembly Sums $\mathfrak{S}_{1,n}^m$ for $n = 6$, with $s = 1$. Each bar is the generalized harmonic number $H_n^{\boldsymbol{\lambda}}$ of one partition $\boldsymbol{\lambda}$ in the truncated lattice (partitions with at most n summands) and the Assembly Set $\wp_k^{k+m-1}$. The bars are arranged in horizontal levels: each level corresponds to a fixed value of $k+m-1$, so that all partitions $\boldsymbol{\lambda} \vdash (k+m-1)$ lie on the same line. The bars of a single color form one Assembly Sum (one branch of the assembly), namely $\mathfrak{S}_{1,n}^m = \sum_{k=1}^{n} \sum_{\boldsymbol{\lambda}\in\wp_k^{k+m-1}} H_n^{\boldsymbol{\lambda}}$. The horizontal axis lists the partitions of $k+m-1$, the vertical axis shows the value $H_n^{\boldsymbol{\lambda}}$, and the depth axis encodes the parameter $k+m-1$. By the Generalized Assembly Theorem 2.37, every branch satisfies $\mathfrak{S}_{1,n}^m = n$, so each Assembly Sum reconstructs the count of indices $\{1,\ldots,n\}$ independently of the excess parameter $m-1$.

2.6. **Hierarchical expansion of e^γ via Multiple Zeta Values.** We close the main framework with a expansion of the exponential of the Euler–Mascheroni constant in terms of multiple zeta values. This intuition is a natural consecuence of thinking like in the Assembly Theorem, by with the weights $\frac{1}{\boldsymbol{\lambda}!}$, because $e^\gamma n = e^{\gamma+\ln(n)} \approx e^{H_n} = \sum_{k\in\mathbb{N}} \frac{H_n^k}{k!} - 1 = \sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!} - 1$, we take in the Young's Lattice the partitions without 1's, "the leaders", and factoring the partitions which are

their "extension" of each leader by adding 1's, these are the partitions wich have the same partition factorial $\frac{1}{\lambda!}$, because 1! do not affect the product. A sketch of a proof of this expansion was provided by Hucht in [16].

Theorem 2.39 (Hierarchical Expansion of e^γ)**.** *The constant e^γ admits the hierarchical expansion*

$$\begin{aligned} e^\gamma \;=\; 1 \;&+\; \sum_{a\geq 2}\frac{1}{a!}\sum_{m\geq 0}(-1)^m\zeta^{\mathrm{A}}(a+m) \\ &+\; \sum_{a,b\geq 2}\frac{1}{a!\,b!}\sum_{m,n\geq 0}(-1)^{m+n}\zeta^{\mathrm{A}}(a+m,b+n) \\ &+\; \sum_{a,b,c\geq 2}\frac{1}{a!\,b!\,c!}\sum_{m,n,p\geq 0}(-1)^{m+n+p}\zeta^{\mathrm{A}}(a+m,b+n,c+p) \;+\; \cdots, \end{aligned} \tag{50}$$

or, equivalently,

$$e^\gamma \;=\; \sum_{k=0}^{\infty} S_k, \tag{51}$$

where the depth-k sum is defined by

$$S_k \;=\; \sum_{\substack{a_1,\dots,a_k\geq 2\\ m_1,\dots,m_k\geq 0}} \frac{(-1)^{m_1+\cdots+m_k}}{a_1!\cdots a_k!}\,\zeta^{\mathrm{A}}(A_1,\dots,A_k), \qquad A_i := a_i+m_i \tag{52}$$

(with the convention $S_0=1$), and ζ^{A} denotes the Abel-regularized multiple zeta value

$$\zeta^{\mathrm{A}}(s_1,\dots,s_k) \;=\; \lim_{r\to 1^-}\sum_{n_1>n_2>\cdots>n_k\geq 1} r^{n_1}\,n_1^{-s_1}\cdots n_k^{-s_k}. \tag{53}$$

Proof. By definition, the right-hand side of (50) is precisely $\sum_{k=0}^{\infty} S_k$, where S_k is given by (52). Thus it is enough to prove (51). Define the auxiliary sums of ordinary Riemann zeta values

$$\Sigma_k \;=\; \sum_{\substack{a_1,\dots,a_k\geq 2\\ m_1,\dots,m_k\geq 0}} \frac{(-1)^{m_1+\cdots+m_k}}{a_1!\cdots a_k!}\,\zeta(A_1+\cdots+A_k), \qquad A_i := a_i+m_i. \tag{54}$$

The series defining each Σ_k converges absolutely (the general term is $O(j^{-2k})$).

Abel regularization of series is classical (Hardy [14]). When applied to multiple zeta values via (53), it yields a morphism of algebras that preserves both the shuffle product and the stuffle product (Ihara–Kaneko–Zagier [17]; see also Hoffman [15]). Consequently, the symmetrized shuffle relations imply that each S_k coincides with the complete Bell polynomial evaluated at $\Sigma_1,\dots,\Sigma_k$. The first few polynomials may be written explicitly: $S_1=\Sigma_1, S_2=\frac{1}{2}(\Sigma_1^2-\Sigma_2), S_3=\frac{1}{6}(\Sigma_1^3-3\Sigma_1\Sigma_2+2\Sigma_3)$. These are precisely the complete Bell polynomials evaluated at the sequence $(\Sigma_m)_{m\geq 1}$ with the natural sign pattern. In general one has the identity

$$S_k \;=\; \frac{1}{k!}B_k(\Sigma_1,\,-1!\,\Sigma_2,\,2!\,\Sigma_3,\,\dots,\,(-1)^{k-1}(k-1)!\,\Sigma_k),$$

where B_k denotes the k-th complete exponential Bell polynomial. Equivalently, the exponential generating function of the sequence $(S_k)_{k\geq 0}$ is given by the formal power series identity

$$\sum_{k=0}^{\infty} S_k\,x^k \;=\; \exp\!\left(-\sum_{k=1}^{\infty}\Sigma_k\frac{(-x)^k}{k}\right). \tag{55}$$

We now evaluate the exponent at $x = 1$. Substitute the Dirichlet series $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$ into (54):

$$\begin{aligned}\Sigma_k &= \sum_{\substack{a_1,\dots,a_k\geq 2\\ m_1,\dots,m_k\geq 0}} \frac{(-1)^{m_1+\cdots+m_k}}{a_1!\cdots a_k!} \sum_{j=1}^{\infty} j^{-(A_1+\cdots+A_k)} \\ &= \sum_{j=1}^{\infty}\left(\sum_{a\geq 2}\sum_{m\geq 0} \frac{(-1)^m}{a!} j^{-(a+m)}\right)^k.\end{aligned}$$

The inner double sum is elementary:

$$\sum_{a\geq 2} \frac{j^{-a}}{a!} \sum_{m\geq 0} (-j^{-1})^m = \frac{e^{\frac{1}{j}} - 1 - j^{-1}}{1 + j^{-1}} = \frac{e^{\frac{1}{j}}}{1+\frac{1}{j}} - 1.$$

Thus we obtain the explicit formula

$$\Sigma_k = \sum_{j=1}^{\infty}\left(\frac{e^{\frac{1}{j}}}{1+\frac{1}{j}} - 1\right)^k. \tag{56}$$

(The interchange of sums is justified by absolute convergence.) Insert (56) into the series for the exponent:

$$\sum_{k=1}^{\infty} \Sigma_k \frac{(-1)^{k+1}}{k} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z_j^k,$$

where we have set

$$z_j := \frac{e^{\frac{1}{j}}}{1+\frac{1}{j}} - 1.$$

For every $j \geq 1$ one has $0 < z_j < 1$, so the Taylor series of the logarithm converges absolutely:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z_j^k = \log(1 + z_j).$$

A direct computation gives

$$\log(1 + z_j) = \log\left(\frac{e^{\frac{1}{j}}}{1+\frac{1}{j}}\right) = \frac{1}{j} + \log j - \log(j+1).$$

Therefore

$$\sum_{k=1}^{\infty} \Sigma_k \frac{(-1)^{k+1}}{k} = \sum_{j=1}^{\infty}\left(\frac{1}{j} + \log j - \log(j+1)\right). \tag{57}$$

The right-hand side of (57) is the classical series representation of the Euler–Mascheroni constant (Abramowitz–Stegun [1]; originally due to Euler):

$$\gamma = \sum_{j=1}^{\infty}\left(\frac{1}{j} + \log j - \log(j+1)\right).$$

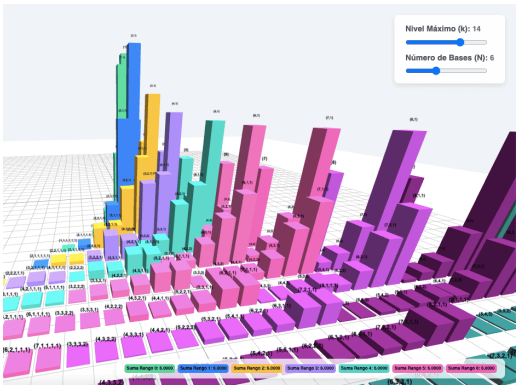
Hence the exponent evaluated at $x = 1$ equals γ. Substituting into the generating function (55) yields

$$\sum_{k=0}^{\infty} S_k = e^{\gamma},$$

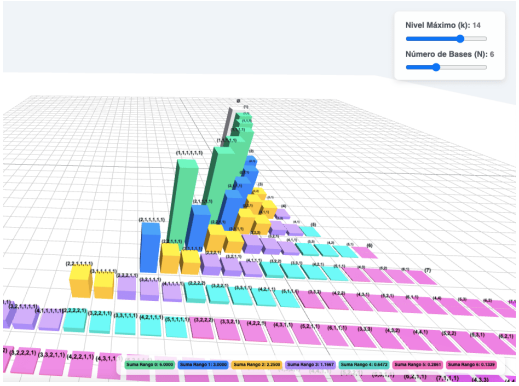
which is precisely (51) (and therefore also (50)). □

3. Espinosa's branches and exact, not asymptotic, divisor realization

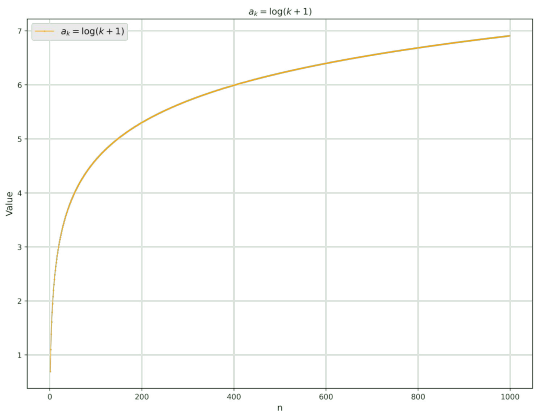
3.1. The path toward the proof of the Riemann Hypothesis: divisor assembly via the Young's Lattice. The Young's Lattice equivalence (42) is built from four structural components that, taken together, produce the inequality. The first two components are the lattice of partition factorials $\frac{1}{\boldsymbol{\lambda}!}$ and the lattice of generalized harmonic numbers $H_n^{\boldsymbol{\lambda}}$, whose interaction defines the assembly mechanism over the assembly sets (Figures 12b and 12a). The third component is the sequence of coefficients a_k, whose growth profile is essentially logarithmic and controls the analytic behaviour of the partition-indexed series (Figure 12c). The fourth component is $\sum_{d|n} d$: the sum-of-divisors function, which constitutes the arithmetic target of the equivalence (Figure 12d). Each of these four structures plays a distinct and indispensable role in the construction, and the content of the equivalence lies not in any of them individually but in their deep mutual interaction. we must study each component to be able to understand and the mechanism by which they combine produce the inequality becomes transparent.



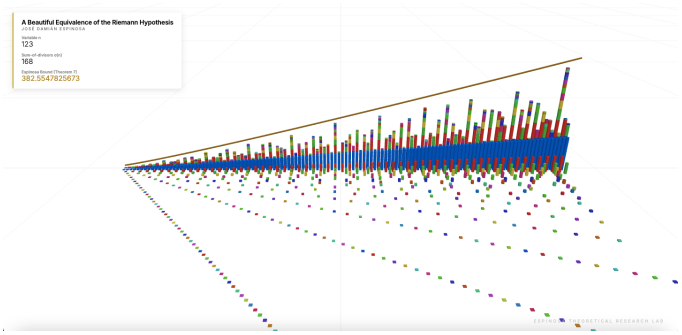
(A) Generalized harmonic numbers $H_n^{\boldsymbol{\lambda}}$: The lattice of generalized harmonic numbers. Each node carries a harmonic value, and their aggregation over the assembly sets produces the assembly mechanism.



(B) Partition factorials $\frac{1}{\boldsymbol{\lambda}!}$: the normalization weights of the assembly. The lattice of partition factorials. These weights normalize the contribution of each partition to the assembled sum.



(C) Coefficients a_k: the smooth analytic growth profile of the series, essentially logarithmic matched by $\log(k+1)$.



(D) $\sum_{d|n} d$: the sum-of-divisors function, the arithmetic target of the equivalence.

Figure 12. The four structural components of the Riemann Hypothesis Young's Lattice Equivalence: $\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k > \sum_{d|n} d$.

We now turn to the structural analysis of the partition-indexed series that appears in the Young's Lattice equivalence. The goal of this section is to exhibit the mechanism by which the divisors of n are assembled through the levels of Young's Lattice, see [6], using the Generalized Assembly Theorem of Section 2.5. This analysis reveals how each level of the partition decomposition contributes to the divisor sum. We analyze the partition-indexed series

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k, \qquad a_k \;=\; \frac{1}{k}+\cdots+\frac{1}{k^2}. \tag{58}$$

Since $a_k \geq 1$ for every $k \in \mathbb{N}$ and all terms are positive, we decompose the series according to the excess parameter $m-1$ and the length k of the partition:

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;=\; \sum_{m\geq 1}\left(\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k+m-1}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_{k+m-1}\right). \tag{59}$$

We isolate the first two levels $m=1,2$ from the tail $m\geq 3$:

$$\begin{aligned}\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;&=\; \sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \\ &\quad+ \sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k+1}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_{k+1} \\ &\quad+ \sum_{m\geq 3}\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k+m-1}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_{k+m-1}.\end{aligned} \tag{60}$$

For the first two levels, the partition factorial takes a single value. At level $m=1$, the Assembly Set $\wp_k^k$ consists of the sole partition $(1,\ldots,1)$, so $\boldsymbol{\lambda}! = (1!)^k = 1$, and

$$\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;=\; \frac{1}{1!}\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k}} H_n^{\boldsymbol{\lambda}}\, a_k. \tag{61}$$

At level $m=2$, the Assembly Set $\wp_k^{k+1}$ consists of the sole partition $(2,1,\ldots,1)$, so $\boldsymbol{\lambda}! = 2!\cdot(1!)^{k-1} = 2$, and

$$\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k+1}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_{k+1} \;=\; \frac{1}{2!}\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k+1}} H_n^{\boldsymbol{\lambda}}\, a_{k+1}. \tag{62}$$

Using $a_k \geq 1$ and $a_{k+1}\geq 1$, together with the positivity of the tail $m\geq 3$ in (60), we obtain

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;>\; \frac{1}{1!}\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k}} H_n^{\boldsymbol{\lambda}} \;+\; \frac{1}{2!}\sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k+1}} H_n^{\boldsymbol{\lambda}}. \tag{63}$$

By the Generalized Assembly Theorem (2.37) with $s=1$,

$$\mathfrak{S}_n^m \;=\; \sum_{k=1}^{n}\sum_{\boldsymbol{\lambda}\in\wp_k^{k+m-1}} H_n^{\boldsymbol{\lambda}} \;=\; n \qquad \text{for every } m\in\mathbb{N}. \tag{64}$$

Substituting (64) into (63) for $m=1$ and $m=2$,

$$\sum_{\substack{k\in\mathbb{N}\\ \boldsymbol{\lambda}\vdash k}} \frac{H_n^{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}!}\, a_k \;>\; \frac{n}{1!}+\frac{n}{2!}. \tag{65}$$

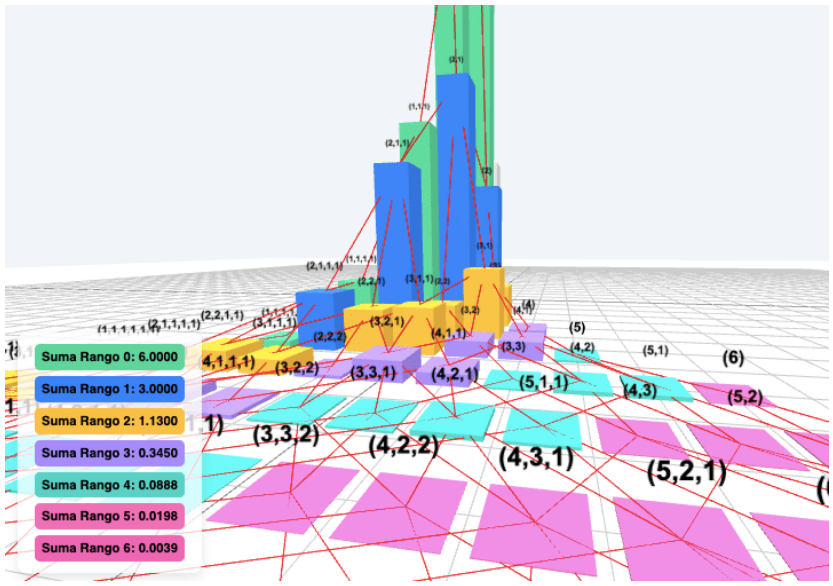


FIGURE 13. Young's Lattice representation of the assembly branches. Each colored region groups the partitions $\boldsymbol{\lambda} \in \wp_k^{k+m-1}$ that belong to the same single branch. The assembly sums along each colored branch reproduce the corresponding divisor of n, making explicit the correspondence between the divisor branches of Figure 14 and assembly branches along Young's Lattice. Each branch assembles the divisors of n assembled at a fixed excess parameter $m-1$, so that the first color is the first branch ($m=1$) wich covers the divisor $\frac{n}{1}$ (level $m=1$) and the second color is the second, it covers the divisor $\frac{n}{2}$ (level $m=2$). The assembly sums along each colored branch reproduce the corresponding divisor of n.

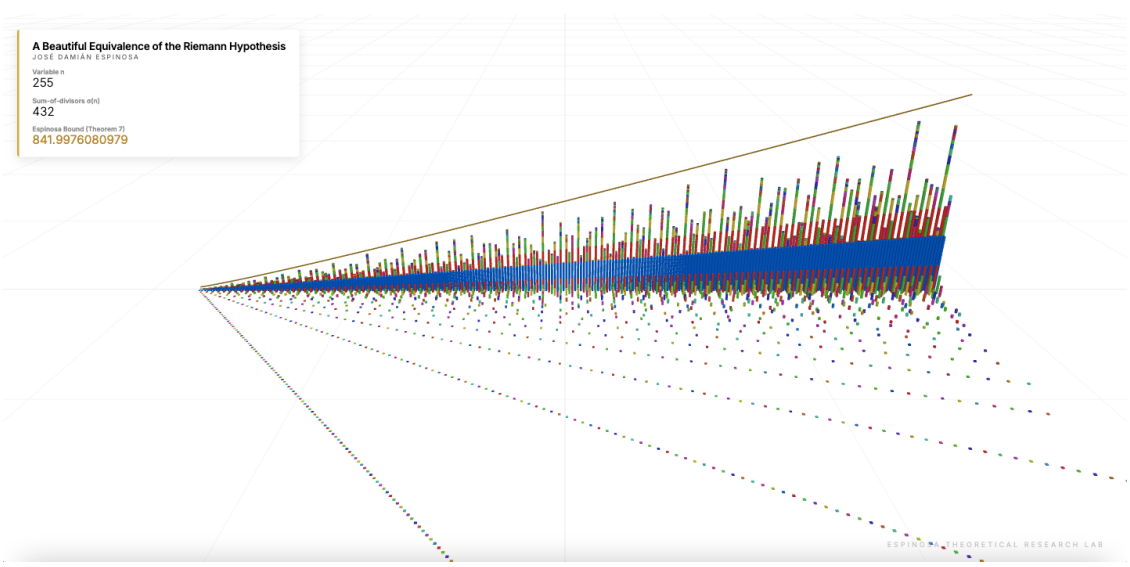


FIGURE 14. Divisor branches $\frac{n}{k}$. Each branch collects the divisors of the form $\frac{n}{k}$. The bars represent the contribution of each branch to the sum-of-divisors function of n. The first branch is $\frac{n}{1}$, while the second branch is $\frac{n}{2}$ and so on.

4. Concluding Remarks

This unified framework bridges partition theory, permutations and symmetric functions, Multiple Zeta Values, and the Young's Lattice to the study of the Riemann Hypothesis. The author firmly believe that the proof of the Riemann Hypothesis can very possibly be obtained by adding the divisors branches simultaneously with adding the assembly sums along the partitions of Young's Lattice. By creating an association of branches of divisors of the form $\frac{n}{k}$—as illustrated in the figure—with assembly branches along the assembly sets weighted by coefficients in the lattice.

Responsibility Notice

The author accepts full responsibility for the integrity, originality, and accuracy of all content, data, and text presented in this manuscript.

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