

IMPROVED BOUNDS ON H_m FOR $2 \leq m \leq 5$

ABSTRACT. Let $H_m = \liminf_{n \rightarrow \infty} (p_{n+m} - p_n)$, where p_n denotes the n th prime. We prove

$$H_2 \leq 20,370, \quad H_3 \leq 1,380,156, \quad H_4 \leq 82,959,170, \quad H_5 \leq 5,040,082,836,$$

improving the bounds 33,118, 2,718,108, 214,099,720 and 14,541,349,288 of Roeling (2026). We use the multidimensional Selberg sieve of Maynard and Tao in the form developed by Polymath. The arithmetic input is Stadlmann's equidistribution estimate for primes in arithmetic progressions, in the extension to triply densely divisible moduli given in OpenAI's 2026 preprint proving $H_1 \leq 186$. Together with a criterion from the same preprint for the triple dense divisibility of least common multiples, it allows the sieve weights to be supported, instead of on Polymath's truncated simplex, on a region in which single coordinates may be as large as about 0.38. On this region we use product trial functions multiplied by an optimised function of the sum of the coordinates, and we certify the resulting variational quotients in ball arithmetic, computing the relevant distribution functions by Fourier inversion.

1. INTRODUCTION

For $m \geq 1$ put

$$H_m := \liminf_{n \rightarrow \infty} (p_{n+m} - p_n),$$

where p_n is the n th prime; thus H_m is the smallest H such that infinitely many intervals of length H contain $m+1$ primes. Zhang [19] proved that $H_1 < 7 \cdot 10^7$. Maynard [9] (with related work obtained independently, but not published, by Tao; see [12, §1]) introduced a multidimensional form of the Selberg sieve, with which he proved, using only the Bombieri–Vinogradov theorem, that every H_m is finite, with $H_m \ll m^3 e^{4m}$. The Polymath project [12] refined the method; using the equidistribution estimates of Zhang type from [11], it obtained the bounds for $H_2, \dots, H_5$ in the first column of Table 1 and $H_m \ll m \exp((4 - \frac{28}{157})m)$. Baker and Irving [1] brought Harman's sieve into the problem and obtained $H_m \ll \exp(3.815m)$. Stadlmann [15] proved that the primes have level of distribution $\frac{1}{2} + \frac{1}{40} - \varepsilon$, for every $\varepsilon > 0$, to smooth moduli (in the sense of §1.1); with the method of [12] this gave the second column of Table 1 and $H_m \ll \exp(3.81m)$, and combined with Harman's sieve it gave $H_m \ll \exp(3.8075m)$ [15, Thm. 2].

1.1. Background. We recall the framework of [9, 12], which is used throughout.¹ A tuple $\mathcal{H} = (h_1 < \dots < h_k)$ of integers is *admissible* if for every prime p some residue class modulo p contains none of the h_i . (If every class were hit, then for every n one of $n + h_1, \dots, n + h_k$ would be divisible by p .) Following [12, Claim 3.1], we write $\text{DHL}[k, j]$ (for “Dickson–Hardy–Littlewood”: it is a weak form of their prime tuples conjecture) for the assertion that every admissible k -tuple has infinitely many translates $n + h_1, \dots, n + h_k$ containing at least j primes. If $\text{DHL}[k, m+1]$ holds, then $H_m \leq h_k - h_1$ for every admissible k -tuple. Bounds for H_m therefore come in two steps: prove $\text{DHL}[k, m+1]$ for some k , as small as possible, and find an admissible k -tuple of small diameter $h_k - h_1$.

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¹We write ω for the parameter denoted ϖ in [11, 12], as in [10, 15].

The first step needs information about primes in arithmetic progressions. The primes have *level of distribution* ϑ if the errors in the prime number theorem for arithmetic progressions, taken for the worst residue class modulo q and summed over all $q \leq x^\vartheta$, are $O(x(\log x)^{-L})$ for every fixed L [12, Claim 2.2]. The Bombieri–Vinogradov theorem gives every level $\vartheta < \frac{1}{2}$. The sieve weights of Maynard and Tao are built from a function F on $[0, \infty)^k$, and their quality is measured by the *variational quotient*

$$Q(F) = \frac{J_1(F) + \cdots + J_k(F)}{I(F)},$$

a ratio of two quadratic functionals of F defined in §2.5. If the primes have level ϑ and $Q(F) > 2m/\vartheta$ for some F supported in the simplex $\{t_1 + \cdots + t_k \leq 1\}$, then DHL $[k, m+1]$ holds [12, Thm. 3.8]. The supremum M_k of these quotients satisfies $M_k = \log k + O(1)$ [12, Thm. 3.9(xi) and Cor. 6.4], so lowering the threshold $2m/\vartheta$ by 1 lowers the required k by a factor of about e .

Levels above $\frac{1}{2}$ are known only in restricted forms. Zhang [19] obtained one for *smooth* moduli, whose prime factors are all at most x^δ for a small fixed $\delta > 0$, and for residue classes fixed in advance rather than the worst class for each modulus. Polymath proved such estimates, denoted MPZ $[\omega, \delta]$ (for Motohashi–Pintz–Zhang [12, Claim 2.4]), at level $\frac{1}{2} + 2\omega$ for suitable $\omega, \delta > 0$ [11], and also versions for the larger class of *densely divisible* moduli, which have divisors in every range of relative width x^δ (Definition 2.1). To keep the moduli in the sieve smooth, Polymath restricted the support of F to the truncated simplex

$$\{t \in [0, \alpha]^k : t_1 + \cdots + t_k \leq 1\}, \quad \alpha := \frac{\delta}{1/4 + \omega},$$

and showed that DHL $[k, m+1]$ follows from MPZ $[\omega, \delta]$ if $Q(F) > m/(\frac{1}{4} + \omega) = 2m/(\frac{1}{2} + 2\omega)$ for some F supported there [12, Thm. 3.10]. (A coordinate t_i corresponds to a divisor of size $x^{(1/4+\omega)t_i}$ of $n + h_i$ in the sieve weights, so the cap $t_i \leq \alpha$ keeps all prime factors of the moduli below x^δ .) This forces a trade-off between the level and the truncation. For example, for H_2 Stadlmann used $\omega \approx 0.0056$ and $\delta \approx 0.021$ [15, §2.1.4], so that $\alpha \approx 0.08$ and the threshold is $2/(\frac{1}{4} + \omega) \approx 7.83$, against 8 from the Bombieri–Vinogradov theorem alone.

1.2. Results. Roeling [13] took the profile $g(t) = 1/(c + (k-1)t)$ that Polymath used in [12, Thm. 6.7] and, in place of the lower bound of that theorem, certified lower bounds for the variational quotient of the product trial function

$$(1.1) \quad F(t_1, \dots, t_k) = \mathbf{1}\{t_1 + \cdots + t_k \leq 1\} \prod_{i=1}^k g(t_i), \quad g(t) = \frac{\mathbf{1}_{[0,T]}(t)}{c + (k-1)t} \quad (c, T > 0),$$

with $T = 1$, by a Fourier-domain computation in ball arithmetic (cf. [12, §9, item 4]). Using only the Bombieri–Vinogradov theorem, this gave the third column of Table 1; for instance, [13] certifies $M_{3655} > 8$, where [12, Thm. 3.9] needed $k = 41,588$. For H_1 the current record is $H_1 \leq 186$, due to OpenAI [10]; Stadlmann independently proved $H_1 \leq 240$ [16], and Charton et al. [3] obtained $H_1 \leq 212$ building on [16].

Our main result is the following.

Theorem 1.1. *The assertions*

$$\text{DHL}[2\,355, 3], \quad \text{DHL}[111\,300, 4], \quad \text{DHL}[5\,129\,500, 5], \quad \text{DHL}[234\,360\,000, 6]$$

hold. Consequently

$$H_2 \leq 20,370, \quad H_3 \leq 1,380,156, \quad H_4 \leq 82,959,170, \quad H_5 \leq 5,040,082,836.$$

TABLE 1. Upper bounds for H_m .

	Polymath [12] (2014)	Stadlmann [15] (2025)	Roeling [13] (2026)	Theorem 1.1
H_2	398,130	396,504	33,118	20,370
H_3	24,797,814	24,407,016	2,718,108	1,380,156
H_4	1,431,556,072	1,391,051,532	214,099,720	82,959,170
H_5	80,550,202,480	77,510,685,234	14,541,349,288	5,040,082,836

1.3. **Method.** The improvement has three ingredients.

(1) *A new support region.* Polymath suggested using its estimates for densely divisible moduli to replace the truncated simplex by a more complicated region [12, §9, item 2]. We do this with two inputs from OpenAI’s proof of $H_1 \leq 186$ [10]: Stadlmann’s estimate extended to triply densely divisible moduli [10, Prop. 2.18 and Cor. 2.19] (Stadlmann had indicated that such an extension is possible [15, §2.1.1]), and a criterion [10, Prop. 2.3], a special case of which is reproduced as Proposition 2.3, for the least common multiple of two divisor products to be triply densely divisible.

The result, Theorem 3.4, is a criterion for $\text{DHL}[k, m+1]$ in which F may be supported on a region $\mathcal{R}_k(u_0, \kappa)$ (Definition 3.1): for every v with $u_0 < v \leq \max_i t_i$, the coordinates of size at least v must have total at most $\kappa - \frac{3}{2}v$. Our parameters are $u_0 \approx 1.9 \cdot 10^{-4}$ and $\kappa \approx 0.953$, so single coordinates up to $\frac{2}{5}\kappa \approx 0.381$ are allowed. The level is $\frac{1}{2} + 2\omega$ for any $\omega, \delta > 0$ with $240\omega + 80\delta < 3$. This is the same condition as in Stadlmann’s work on smooth moduli [15, Cor. 2], but δ no longer limits the support, so we can take $\delta = 10^{-4}$ and $\omega = 0.012466$, close to $\frac{1}{80}$. The threshold for $\text{DHL}[k, m+1]$ becomes $m/(\frac{1}{4} + 0.012466) \approx 3.810m$, against $4m$ from the Bombieri–Vinogradov theorem alone. For the trial functions relevant to $H_2, \dots, H_5$ the restriction to $\mathcal{R}_k(u_0, \kappa)$ removes almost no mass (Table 2), although for $m = 2$ the cap $\frac{2}{5}\kappa$ constrains the choice of trial function (§8).

(2) *A sum profile.* We use trial functions of the form

$$(1.2) \quad F_\psi(t_1, \dots, t_k) = \psi(t_1 + \dots + t_k) \prod_{i=1}^k g(t_i),$$

with g as in (1.1) and $\psi : [0, \infty) \rightarrow [0, 1]$ a step function vanishing on $(1, \infty)$, chosen by optimisation (Figure 1). Compared with (1.1), which is the case $\psi = \mathbf{1}_{[0,1]}$, this increases the quotient by about 0.05 and reduces the required k by about 5%.

(3) *Certification.* As in [13], the variational quotient of (1.2) is a ratio of two expectations involving sums of independent random variables (Lemma 4.1). We bound the distribution functions of these sums rigorously by Fourier inversion in ball arithmetic (§5). Our implementation is independent of that of [13]: we use Gauss–Legendre quadrature with an analytic error bound, where [13] uses trapezoidal sums with bounds for the aliasing error.

1.4. **Inputs.** Besides the framework of [12] and the Bombieri–Vinogradov theorem [2, 18], the proof uses the equidistribution estimate [10, Cor. 2.19] for $r = 3$, which combines Stadlmann’s bilinear estimate [15, Prop. 1], extended from smooth to triply densely divisible moduli in [10, Prop. 2.18 and §A.4], with the Type III estimate [11, Thm. 2.8(v)] and the Heath-Brown identity [6]; and a theorem of Sarajian and Weingartner on dense divisibility [14, Thm. 2]. We reprove the case of [10, Prop. 2.3] that we need. Everything else is proved here: the sieve criterion in §3, the analysis of the trial functions in §4, and the computations in §§5–6, whose programs and data are described at the end of the paper.

1.5. Organisation. Section 2 collects notation and the inputs from the literature. Section 3 proves the sieve criterion with the new support region. Section 4 treats the trial functions (1.2) and the cost of restricting them to the region. Section 5 describes the certified computation and Section 6 the admissible tuples. Section 7 completes the proof of Theorem 1.1, and Section 8 contains remarks.

2. PRELIMINARIES

2.1. Notation. The letters p and q denote primes when used as summation or product variables over prime factors; otherwise q denotes a modulus. An integer is Y -smooth if all its prime factors are at most Y . For a squarefree positive integer n and a prime p we write $n_{<p} := \prod_{q|n, q < p} q$ and $n_{\geq p} := n/n_{<p}$. For positive integers a, b we write $[a, b]$ for their least common multiple; all other square brackets with real endpoints denote intervals. We write $\theta(n) = \log n$ if n is prime and $\theta(n) = 0$ otherwise, Λ for the von Mangoldt function, and $\Lambda_J := \Lambda \mathbf{1}_J$ for an interval J . For a finitely supported $f : \mathbf{N} \rightarrow \mathbf{C}$ and $(a, q) = 1$ put

$$\Delta(f; a \bmod q) := \sum_{n \equiv a \pmod{q}} f(n) - \frac{1}{\varphi(q)} \sum_{(n, q) = 1} f(n).$$

Throughout, x is a large parameter; a quantity is *fixed* if it does not depend on x , and $o(1)$ denotes a quantity tending to 0 as $x \rightarrow \infty$ with all fixed quantities held fixed. We write $\log_x y := \log y / \log x$.

2.2. Admissible tuples. As in §1.1, $\text{DHL}[k, j]$ denotes the assertion that for every admissible k -tuple $\mathcal{H} = (h_1 < \dots < h_k)$ there are infinitely many n such that at least j of $n + h_1, \dots, n + h_k$ are prime. Admissibility only needs to be checked at primes $p \leq k$, since k integers cannot meet all p classes modulo a prime $p > k$.

2.3. Dense divisibility.

Definition 2.1 ([11, Def. 2.1], [10, Def. 2.1]). Let $Y \geq 1$. Put $\mathcal{D}^{(0)}(Y) := \mathbf{N}$. For an integer $r \geq 1$, a positive integer n belongs to $\mathcal{D}^{(r)}(Y)$, and is called r -tuply Y -densely divisible, if for all integers $j, l \geq 0$ with $j + l = r - 1$ and every $1 \leq U \leq Yn$ there is a factorisation $n = uv$ with $U/Y \leq v \leq U$, $u \in \mathcal{D}^{(j)}(Y)$ and $v \in \mathcal{D}^{(l)}(Y)$.

For example, every Y -smooth integer lies in $\mathcal{D}^{(r)}(Y)$ for every r [11, Lemma 2.10(iii)].

Lemma 2.2 ([10, Lemma 2.2]). Let $r \geq 1$ and $Y \geq 2$, and let Q be a squarefree positive integer such that $p^r \leq YQ_{<p}$ for every prime $p \mid Q$ with $p > Y$. Then $Q \in \mathcal{D}^{(r)}(Y)$.

This is a case of [14, Thm. 2], as noted in [10].

Proposition 2.3 ([10, Prop. 2.3], special case). Let $Y \geq 2$ and $X, A > 0$ with $A^2 \leq XY$. Let D, E be squarefree positive integers such that

$$\begin{aligned} p^{3/2} D_{\geq p} &\leq A && \text{for every prime } p \mid D \text{ with } p > Y, \\ p^{3/2} E_{\geq p} &\leq A && \text{for every prime } p \mid E \text{ with } p > Y. \end{aligned}$$

If $Q := [D, E] > X$, then $p^3 < YQ_{<p}$ for every prime $p \mid Q$ with $p > Y$; in particular $Q \in \mathcal{D}^{(3)}(Y)$.

Proof (from [10]). Let $p \mid Q$ with $p > Y$; by symmetry we may assume $p \mid D$, so that $p^{3/2} D_{\geq p} \leq A$. If E has a prime factor $\geq p$, let q be the least one; then $q > Y$, $E_{\geq p} = E_{\geq q}$ and $p^{3/2} E_{\geq p} \leq q^{3/2} E_{\geq q} \leq A$. Otherwise $E_{\geq p} = 1$ and $p^{3/2} E_{\geq p} = p^{3/2} \leq p^{3/2} D_{\geq p} \leq A$. In both

cases $p^3 D_{\geq p} E_{\geq p} \leq A^2 \leq XY$. Since $Q_{\geq p}$ divides $D_{\geq p} E_{\geq p}$ and $Q_{< p} Q_{\geq p} = Q > X$, we get $p^3 \leq XY/Q_{\geq p} < Y Q_{< p}$. The last assertion follows from Lemma 2.2. $\square$

2.4. Equidistribution. For a finite set I of primes put $P_I := \prod_{p \in I} p$. For $\omega, \delta, \varepsilon > 0$, a primitive residue class $a \bmod P_I$ and a finitely supported f , put

$$E_3(f; a; \omega, \delta, \varepsilon) := \sum_{\substack{q \leq x^{1/2+2\omega-\varepsilon} \\ q|P_I, q \in \mathcal{D}^{(3)}(x^\delta)}} |\Delta(f; a \bmod q)|,$$

where $a \bmod q$ denotes the reduction of a .

Theorem 2.4 ([10, Cor. 2.19], case $r = 3$). *Let $\omega, \delta > 0$ be fixed with $240\omega + 80\delta < 3$. Then for all fixed $L, \varepsilon > 0$,*

$$E_3(\Lambda_J; a; \omega, \delta, \varepsilon) \ll x(\log x)^{-L}$$

uniformly for all finite sets I of primes, all primitive classes $a \bmod P_I$ and all intervals $J \subseteq [x, 2x)$.

The underlying bilinear estimate is [15, Prop. 1], which Stadlmann proved for x^δ -smooth moduli under the conditions

$$72\omega + 24\delta < 1, \quad 48\omega + 16\delta + 4\sigma < 1, \quad 64\omega + 20\delta + 2\sigma < 1,$$

remarking that the proof can be adapted to triply densely divisible moduli [15, §2.1.1]. The adaptation is carried out in [10, Prop. 2.18 and §A.4], and [10, Cor. 2.19] combines it with the Type III estimate [11, Thm. 2.8(v)] and the Heath-Brown identity, taking σ slightly larger than $\frac{1}{10}$ (explicitly, $\sigma = \frac{1}{10} + (3 - 240\omega - 80\delta)/40$). At $\sigma = \frac{1}{10}$ the second condition is $240\omega + 80\delta < 3$, and it implies the other two, since $72\omega + 24\delta = \frac{3}{10}(240\omega + 80\delta)$ and $64\omega + 20\delta < \frac{4}{15}(240\omega + 80\delta)$. The same condition, written $80\omega + \frac{80}{3}\delta < 1$, appears in Stadlmann's estimate for smooth moduli [15, Cor. 2].

Theorem 2.5 (Bombieri–Vinogradov [2, 18]; see [12, Thm. 2.3]). *For all fixed $L, \varepsilon > 0$,*

$$\sum_{q \leq x^{1/2-\varepsilon}} \max_{(a,q)=1} |\Delta(\Lambda_J; a \bmod q)| \ll x(\log x)^{-L},$$

uniformly for all intervals $J \subseteq [x, 2x)$.

2.5. The sieve framework. Let $k \geq 2$ and $m \geq 1$ be fixed, and let $(h_1, \dots, h_k)$ be a fixed admissible tuple; after translation we may assume $h_i \geq 0$. Put $w := \log \log \log x$, $W := \prod_{p \leq w} p$ and $B := \frac{\varphi(W)}{W} \log x$, and let $b \bmod W$ be a class with $(b + h_i, W) = 1$ for all i (it exists by admissibility). For a smooth compactly supported $f : [0, \infty) \rightarrow \mathbf{R}$ put

$$\lambda_f(n) := \sum_{d|n} \mu(d) f(\log_x d), \quad S(f) := \sup(\{0\} \cup \{t \geq 0 : f(t) \neq 0\}).$$

For a square-integrable $F : [0, \infty)^k \rightarrow \mathbf{R}$ with bounded support put

$$I(F) := \int_{[0, \infty)^k} F^2, \quad J_i(F) := \int_{[0, \infty)^{k-1}} \left(\int_0^\infty F \, dt_i \right)^2 \prod_{j \neq i} dt_j \quad (1 \leq i \leq k),$$

and, if $F \neq 0$, $Q(F) := \sum_{i=1}^k J_i(F)/I(F)$. Let M_k be the supremum of $Q(F)$ over all such $F \neq 0$ supported in the simplex $\{t_1 + \dots + t_k \leq 1\}$.

Lemma 2.6 ([12, Lemma 3.4]). *Suppose that for every admissible k -tuple and every b as above there are a function $\nu : \mathbf{N} \rightarrow [0, \infty)$ and fixed $\alpha_0 > 0$ and $\alpha_1, \dots, \alpha_k \geq 0$ such that*

$$\sum_{\substack{x \leq n \leq 2x \\ n \equiv b \pmod{W}}} \nu(n) \leq (\alpha_0 + o(1)) B^{-k} \frac{x}{W},$$

$$\sum_{\substack{x \leq n \leq 2x \\ n \equiv b \pmod{W}}} \nu(n) \theta(n + h_i) \geq (\alpha_i - o(1)) B^{1-k} \frac{x}{\varphi(W)} \quad (1 \leq i \leq k),$$

and $(\alpha_1 + \dots + \alpha_k)/\alpha_0 > m$. Then $\text{DHL}[k, m+1]$ holds.

Lemma 2.7 ([12, Thm. 3.6(i)]). *Let F_i, G_i ($1 \leq i \leq k$) be fixed smooth compactly supported functions on $[0, \infty)$ with $\sum_{i=1}^k (S(F_i) + S(G_i)) < 1$. Then*

$$\sum_{\substack{x \leq n \leq 2x \\ n \equiv b \pmod{W}}} \prod_{i=1}^k \lambda_{F_i}(n + h_i) \lambda_{G_i}(n + h_i) = (c_0 + o(1)) B^{-k} \frac{x}{W}, \quad c_0 := \prod_{i=1}^k \int_0^\infty F'_i G'_i.$$

3. A SIEVE CRITERION ON A NEW REGION

3.1. The regions.

Definition 3.1. For $k \geq 1$ and $\sigma, u_0, \kappa > 0$ let $\mathcal{R}_k(\sigma; u_0, \kappa)$ be the set of $t \in [0, \infty)^k$ such that $t_1 + \dots + t_k \leq \sigma$ and

$$(3.1) \quad \frac{3}{2}v + \sum_{i: t_i \geq v} t_i \leq \kappa \quad \text{for every } v \text{ with } u_0 < v \leq \max_i t_i.$$

Put $\mathcal{R}_k(u_0, \kappa) := \mathcal{R}_k(1; u_0, \kappa)$. For a measurable $\mathcal{R} \subseteq [0, \infty)^k$ let $M_k^{\mathcal{R}}$ be the supremum of $Q(F)$ over all square-integrable $F \neq 0$ with bounded support contained in $\mathcal{R}$.

Write $t_{(1)} \geq t_{(2)} \geq \dots \geq t_{(k)}$ for the coordinates of t in decreasing order and put $t_{(k+1)} := 0$. If $t_{(j+1)} < v \leq t_{(j)}$, the indices i with $t_i \geq v$ are exactly those of $t_{(1)}, \dots, t_{(j)}$, so the left side of (3.1) is at most $\frac{3}{2}t_{(j)} + t_{(1)} + \dots + t_{(j)}$; and at $v = t_{(j)}$ it is at least this quantity. Hence (3.1) holds if and only if

$$(3.2) \quad \frac{3}{2}t_{(j)} + t_{(1)} + \dots + t_{(j)} \leq \kappa \quad \text{for every } j \text{ with } t_{(j)} > u_0.$$

Since the maps $t \mapsto t_{(j)}$ are continuous, each set $\{t : t_{(j)} \leq u_0 \text{ or } \frac{3}{2}t_{(j)} + t_{(1)} + \dots + t_{(j)} \leq \kappa\}$ is closed, and hence so is $\mathcal{R}_k(\sigma; u_0, \kappa)$. Condition (3.2) with $j = 1$ caps every coordinate larger than u_0 at $\frac{2}{5}\kappa$. In general, (3.1) says that for every v with $u_0 < v \leq \max_i t_i$ the coordinates of size at least v have total at most $\kappa - \frac{3}{2}v$; this restricts both the number and the sizes of the coordinates above u_0 . In the normalisation of Theorem 3.4 below, v is the size of a prime factor $p > x^\delta$ of a sieve modulus, measured in units of $(\frac{1}{4} + \omega) \log x$, and (3.1) is what is needed to apply Proposition 2.3.

Lemma 3.2. *Let $k \geq 1$, $\sigma, u_0, \kappa > 0$ and $t \in \mathcal{R}_k(\sigma; u_0, \kappa)$.*

- (i) *If $t' \in [0, \infty)^k$ and $t'_i \leq t_i$ for all i , then $t' \in \mathcal{R}_k(\sigma; u_0, \kappa)$.*
 - (ii) *If $\lambda > 0$, then $\lambda t \in \mathcal{R}_k(\lambda\sigma; \lambda u_0, \lambda\kappa)$.*
 - (iii) *If $e > 0$, $t' \in [0, \infty)^k$ and $|t'_i - t_i| \leq e$ for all i , then $t' \in \mathcal{R}_k(\sigma + ke; u_0 + e, \kappa + (k + \frac{3}{2})e)$.*
- Moreover $\mathcal{R}_k(\sigma; u_0, \kappa) \subseteq \mathcal{R}_k(\sigma'; u'_0, \kappa')$ whenever $\sigma \leq \sigma'$, $u_0 \leq u'_0$ and $\kappa \leq \kappa'$.*

Proof. (i) Clearly $\sum t'_i \leq \sigma$. If $u_0 < v \leq \max t'$, then $v \leq \max t$, and $t'_i \geq v$ implies $t_i \geq t'_i \geq v$, so $\frac{3}{2}v + \sum_{t'_i \geq v} t'_i \leq \frac{3}{2}v + \sum_{t_i \geq v} t_i \leq \kappa$.

(ii) If $\lambda u_0 < v \leq \max \lambda t$, then $v' := v/\lambda \in (u_0, \max t]$ and $\frac{3}{2}v + \sum_{\lambda t_i \geq v} \lambda t_i = \lambda(\frac{3}{2}v' + \sum_{t_i \geq v'} t_i) \leq \lambda \kappa$.

(iii) Clearly $\sum t'_i \leq \sigma + ke$. Let $u_0 + e < v \leq \max t'$ and put $v' := v - e$, so that $u_0 < v' \leq \max t' - e \leq \max t$. If $t'_i \geq v$ then $t_i \geq v'$; as all $t_i \geq 0$,

$$\sum_{t'_i \geq v} t'_i \leq \sum_{t'_i \geq v} (t_i + e) \leq \sum_{t_i \geq v'} t_i + ke.$$

Hence $\frac{3}{2}v + \sum_{t'_i \geq v} t'_i \leq \frac{3}{2}v' + \sum_{t_i \geq v'} t_i + (k + \frac{3}{2})e \leq \kappa + (k + \frac{3}{2})e$.

The last assertion is clear from the definition. $\square$

3.2. Prime sums on densely divisible moduli. The following proposition replaces [12, Thm. 3.5(ii)]. There the hypothesis $S(F_i), S(G_i) < \delta$ makes all moduli x^δ -smooth; here it is replaced by (3.4).

Proposition 3.3. *Let $\omega, \delta > 0$ be fixed with $240\omega + 80\delta < 3$, and let $\varepsilon > 0$ be fixed. Let $1 \leq i_0 \leq k$, and for $i \neq i_0$ let F_i, G_i be fixed smooth compactly supported functions on $[0, \infty)$. Put $s_i := S(F_i)$ and $s'_i := S(G_i)$, and assume that*

$$(3.3) \quad \sum_{i \neq i_0} (s_i + s'_i) \leq \frac{1}{2} + 2\omega - \varepsilon,$$

$$(3.4) \quad \frac{3}{2}u + \sum_{i \neq i_0: s_i \geq u} s_i \leq \frac{1}{4} + \frac{\delta}{2} - \varepsilon \quad \text{for every } u \text{ with } \delta < u \leq \max_{i \neq i_0} s_i,$$

and that (3.4) also holds with s' in place of s . Then

$$\sum_{\substack{x \leq n \leq 2x \\ n \equiv b \pmod{W}}} \theta(n + h_{i_0}) \prod_{i \neq i_0} \lambda_{F_i}(n + h_i) \lambda_{G_i}(n + h_i) = (c_0 + o(1)) B^{1-k} \frac{x}{\varphi(W)}, \quad c_0 := \prod_{i \neq i_0} \int_0^\infty F'_i G'_i.$$

Proof. We follow [12, §§4.2–4.3]. Expanding the divisor sums, the left side equals

$$\sum_{\mathbf{d}, \mathbf{d}' \neq i_0} \left(\prod \mu(d_i) \mu(d'_i) F_i(\log_x d_i) G_i(\log_x d'_i) \right) \sum_{\substack{x \leq n \leq 2x, n \equiv b \pmod{W} \\ n + h_i \equiv 0 \pmod{[d_i, d'_i]} \forall i \neq i_0}} \theta(n + h_{i_0}),$$

where $\mathbf{d} = (d_i)_{i \neq i_0}$ and $\mathbf{d}' = (d'_i)_{i \neq i_0}$. The inner sum vanishes unless $q = q_{\mathbf{d}, \mathbf{d}'} := W \prod_{i \neq i_0} [d_i, d'_i]$ is squarefree, in which case the congruence conditions combine into $n + h_{i_0} \equiv a_{\mathbf{d}, \mathbf{d}'} \pmod{q}$ for a primitive class $a_{\mathbf{d}, \mathbf{d}'}$, with $a_{\mathbf{d}, \mathbf{d}'} \equiv b + h_{i_0} \pmod{W}$ and $a_{\mathbf{d}, \mathbf{d}'} \equiv h_{i_0} - h_i \pmod{[d_i, d'_i]}$. Exactly as in [12, (43)–(46)], the main terms give $(c_0 + o(1)) B^{1-k} x / \varphi(W)$ by [12, Lemma 4.1], and, writing $\theta_0 := \theta \mathbf{1}_{[x+h_{i_0}, 2x+h_{i_0}]}$, it remains to show that for every fixed $L > 0$

$$(3.5) \quad \sum_{\mathbf{d}, \mathbf{d}' \neq i_0} \prod |F_i(\log_x d_i) G_i(\log_x d'_i)| |\Delta(\theta_0; a_{\mathbf{d}, \mathbf{d}'} \pmod{q_{\mathbf{d}, \mathbf{d}'}})| \ll x(\log x)^{-L},$$

the sum running over $\mathbf{d}, \mathbf{d}'$ with $q_{\mathbf{d}, \mathbf{d}'}$ squarefree. For such $\mathbf{d}, \mathbf{d}'$ the numbers $[d_i, d'_i]$ are pairwise coprime and coprime to W , so with $D := \prod_{i \neq i_0} d_i$ and $E := \prod_{i \neq i_0} d'_i$ we have $q_{\mathbf{d}, \mathbf{d}'} = W[D, E]$. Only $\mathbf{d}, \mathbf{d}'$ with $\log_x d_i \leq s_i$ and $\log_x d'_i \leq s'_i$ contribute. Put $X := x^{1/2-\varepsilon}$ and $Y := x^\delta$, and split (3.5) according to whether $[D, E] \leq X$ or $[D, E] > X$.

Terms with $[D, E] \leq X$. Here $q \leq WX \leq x^{1/2-\varepsilon/2}$, and each q arises from $\ll \tau(q)^{O(1)}$ pairs $(\mathbf{d}, \mathbf{d}')$, where τ is the divisor function. As in [12, §4.2] (Cauchy–Schwarz against the

trivial bound to remove $\tau(q)^{O(1)}$, replacement of θ by Λ , and removal of the $O(1)$ integers of $[x + h_{i_0}, 2x + h_{i_0}]$ beyond $2x$, the contribution of these terms is $\ll x(\log x)^{-L}$ by Theorem 2.5.

Terms with $[D, E] > X$. We claim that then $q = W[D, E] \in \mathcal{D}^{(3)}(Y)$. Let $p \mid D$ with $p > Y$ and put $u := \log_x p$, so $u > \delta$. Since the d_i are pairwise coprime, p divides exactly one d_i , and then $u \leq \log_x d_i \leq s_i$; in particular $u \leq \max_{i \neq i_0} s_i$. Every prime factor of D that is $\geq p$ divides some d_i with $d_i \geq p$, and such d_i satisfy $s_i \geq \log_x d_i \geq u$. Hence $D_{\geq p} \leq \prod_{i \neq i_0: s_i \geq u} d_i \leq x^{\sum_{i \neq i_0: s_i \geq u} s_i}$, and by (3.4), with $A := (XY)^{1/2}$,

$$\log_x(p^{3/2} D_{\geq p}) \leq \frac{3}{2}u + \sum_{i \neq i_0: s_i \geq u} s_i \leq \frac{1}{4} + \frac{\delta}{2} - \varepsilon < \frac{1}{4} + \frac{\delta}{2} - \frac{\varepsilon}{2} = \log_x A.$$

The same holds for E . By Proposition 2.3, every prime $p \mid [D, E]$ with $p > Y$ satisfies $p^3 < Y[D, E]_{< p}$. If $p \mid q$ and $p > Y$, then $p \nmid W$ (as $w < Y$), so $p \mid [D, E]$ and $q_{< p} \geq [D, E]_{< p}$; hence $p^3 < Y q_{< p}$. As q is squarefree, Lemma 2.2 gives $q \in \mathcal{D}^{(3)}(Y)$. Moreover $q \leq W x^{1/2+2\omega-\varepsilon} \leq x^{1/2+2\omega-\varepsilon/2}$ by (3.3).

Let I be the set of primes up to $2x$. Let $\mathcal{A}$ be the set of primitive classes $a \bmod P_I$ with $a \equiv b + h_{i_0} \pmod{W}$ and, for each $p \in I$ with $p > w$, $a \equiv h_{i_0} - h_i \pmod{p}$ for some $i \neq i_0$; these classes are primitive because $|h_i - h_j| < w$ for large x . Every $a_{\mathbf{d}, \mathbf{d}'}$ is the reduction modulo $q_{\mathbf{d}, \mathbf{d}'}$ of an element of $\mathcal{A}$, and $\mathcal{A}$ is a product of local sets with at most $k-1$ elements each. By the averaging argument of [12, §4.3] (see also [10, Props. 2.11 and 2.14]), the contribution of these terms is

$$\ll \frac{1}{|\mathcal{A}|} \sum_{a \in \mathcal{A}} \sum_{\substack{q \leq x^{1/2+2\omega-\varepsilon/2} \\ q \mid P_I, q \in \mathcal{D}^{(3)}(x^\delta)}} \tau(q)^{O(1)} |\Delta(\theta_0; a \bmod q)|,$$

and it suffices to bound the inner sum by $x(\log x)^{-L}$ uniformly in $a \in \mathcal{A}$. After removing $\tau(q)^{O(1)}$ by Cauchy–Schwarz, replacing θ by Λ and discarding the $O(1)$ integers beyond $2x$, as before, this follows from Theorem 2.4 with $\varepsilon/2$ in place of ε . $\square$

3.3. The criterion.

Theorem 3.4. *Let $k \geq 2$ and $m \geq 1$. Let $\omega, \delta > 0$ with $240\omega + 80\delta < 3$, and put $\varrho := \frac{1}{4} + \omega$. Let $u_0, \kappa > 0$ satisfy*

$$u_0 < \frac{\delta}{\varrho} \quad \text{and} \quad \kappa < \frac{1/4 + \delta/2}{\varrho},$$

and put $\mathcal{R} := \mathcal{R}_k(u_0, \kappa)$. If $M_k^{\mathcal{R}} > m/\varrho$, then $\text{DHL}[k, m+1]$ holds.

Proof. Let F be square-integrable with bounded support contained in $\mathcal{R}$, with $Q(F) > m/\varrho$. We follow [12, §§5.1–5.2] and indicate the changes. Below, supp denotes the closed (essential) support; all regions $\mathcal{R}_k(\cdot; \cdot, \cdot)$ are closed, so they contain the supports of the functions vanishing outside them. Let $\delta_1 > 0$ be small and fixed, and let $\delta_2 \in (0, \delta_1)$ be fixed and small in terms of δ_1 ; the precise requirements are collected in Step 5.

Step 1 (rescaling). Let $\lambda := \varrho - \delta_1$ and $F_1(t) := F(t/\lambda)$. Since $I(F_1) = \lambda^k I(F)$ and $J_i(F_1) = \lambda^{k+1} J_i(F)$, we have $Q(F_1) > m$ if δ_1 is small, and $\text{supp } F_1 \subseteq \mathcal{R}_k(\lambda; \lambda u_0, \lambda \kappa)$ by Lemma 3.2(ii). In these coordinates t_i will be $\log_x d_i$.

Step 2 (shift and mollification). As in [12, §5.1], translating F_1 by δ_2 in every coordinate and convolving with a smooth approximation to the identity supported in $[-\delta_2/2, \delta_2/2]^k$ gives a smooth compactly supported F_3 with $Q(F_3) > m$ (if δ_2 is small), supported in $[\delta_2/2, \infty)^k$, every point of whose support lies within sup-distance $\frac{3}{2}\delta_2$ of $\text{supp } F_1$. By Lemma 3.2(iii), $\text{supp } F_3 \subseteq \mathcal{R}_k(\lambda + \frac{3}{2}k\delta_2; \lambda u_0 + \frac{3}{2}\delta_2, \lambda \kappa + \frac{3}{2}(k + \frac{3}{2})\delta_2)$.

Step 3 (primitive). Put $f_3(t) := \int_{s \geq t} F_3(s) ds$, the integral over $s \in \mathbf{R}^k$ with $s_i \geq t_i$ for all i . Then f_3 is smooth. Let Σ_3 be the closure of $\{t \in [0, \infty)^k : f_3(t) \neq 0\}$. Every point of Σ_3 lies coordinatewise below a point of the compact set $\text{supp } F_3$ (the set of such points of $[0, \infty)^k$ is closed), so by Lemma 3.2(i) Σ_3 lies in the same region as $\text{supp } F_3$. As in [12, §5.1], $I(F_3) = \tilde{I}(f_3)$ and $J_i(F_3) = \tilde{J}_i(f_3)$, where $\tilde{I}(f) := \int_{[0, \infty)^k} |\partial_1 \cdots \partial_k f|^2$ and $\tilde{J}_i(f) := \int_{[0, \infty)^{k-1}} |(\prod_{j \neq i} \partial_j) f|_{t_i=0}|^2 \prod_{j \neq i} dt_j$.

Step 4 (tensor products). As in [12, §5.1], f_3 is the limit on $[0, \infty)^k$, in the topology of uniform convergence of all derivatives, of finite sums $\sum_j c_j \prod_{i=1}^k f_{i,j}(t_i)$ of tensor products of smooth compactly supported functions. Moreover each component $\prod_i f_{i,j}(t_i)$ may be taken to be supported in a box contained in the δ_2 -neighbourhood of Σ_3 in the sup-norm. (Use a smooth partition of unity of tensor-product form subordinate to a grid of cubes of side $\delta_2/2$; approximate each localised piece by tensor products via the Stone–Weierstrass theorem on a slightly larger cube; multiply by a tensor-product cutoff; and mollify with a tensor-product approximate identity, as in [12].) Fix such a sum $f_4 = \sum_{j=1}^J c_j \prod_i f_{i,j}(t_i)$ with $\sum_i \tilde{J}_i(f_4)/\tilde{I}(f_4) > m$, discarding components that vanish identically on $[0, \infty)^k$. The box supporting the j th component then meets $[0, \infty)^k$, so its upper corner lies in $[0, \infty)^k$, within sup-distance δ_2 of Σ_3 ; and $(S(f_{1,j}), \dots, S(f_{k,j}))$ lies coordinatewise below that corner. Put

$$\sigma_2 := \lambda + 3k\delta_2, \quad u_2 := \lambda u_0 + 3\delta_2, \quad \kappa_2 := \lambda\kappa + 3(k + \tfrac{3}{2})\delta_2.$$

By Lemma 3.2, for every j ,

$$(3.6) \quad (S(f_{1,j}), \dots, S(f_{k,j})) \in \mathcal{R}_k(\sigma_2; u_2, \kappa_2).$$

Step 5 (the sieve weights). Put

$$\nu(n) := \left(\sum_{j=1}^J c_j \prod_{i=1}^k \lambda_{f_{i,j}}(n + h_i) \right)^2 \geq 0.$$

Let $\varepsilon := \min\{\delta_1, \frac{1}{2}(\frac{1}{4} + \frac{\delta}{2} - \varrho\kappa)\} > 0$, and choose δ_2 so small that $6k\delta_2 \leq \delta_1$, $\varrho u_0 + 3\delta_2 < \delta$ and $3(k + \frac{3}{2})\delta_2 \leq \varepsilon$. Since $\sum_i (S(f_{i,j}) + S(f_{i,j'})) \leq 2\sigma_2 < 1$, Lemma 2.7 applied to each pair (j, j') gives, as in [12, §5.1],

$$\sum_{\substack{x \leq n \leq 2x \\ n \equiv b \pmod{W}}} \nu(n) = (\tilde{I}(f_4) + o(1)) B^{-k} \frac{x}{W}.$$

Fix i_0 . If $n + h_{i_0}$ is prime, then $\log_x(n + h_{i_0}) \geq 1 > S(f_{i_0,j})$, so $\lambda_{f_{i_0,j}}(n + h_{i_0}) = f_{i_0,j}(0)$; hence the sum of $\nu(n)\theta(n + h_{i_0})$ expands into sums as in Proposition 3.3 with $F_i = f_{i,j}$ and $G_i = f_{i,j'}$ ($i \neq i_0$). We check its hypotheses. By (3.6), $\sum_{i \neq i_0} (s_i + s'_i) \leq 2\sigma_2 = 2\varrho - 2\delta_1 + 6k\delta_2 \leq \frac{1}{2} + 2\omega - \varepsilon$, which is (3.3). Let $\delta < u \leq \max_{i \neq i_0} s_i$. Since $u_2 \leq \varrho u_0 + 3\delta_2 < \delta < u$ and $u \leq \max_i s_i$, condition (3.1) for the point (3.6) applies at $v = u$ and gives

$$\frac{3}{2}u + \sum_{i \neq i_0: s_i \geq u} s_i \leq \frac{3}{2}u + \sum_{i: s_i \geq u} s_i \leq \kappa_2 \leq \varrho\kappa + \varepsilon \leq \frac{1}{4} + \frac{\delta}{2} - \varepsilon,$$

which is (3.4); the same holds for s' . Proposition 3.3 therefore applies, and as in [12, §5.1] it gives

$$\sum_{\substack{x \leq n \leq 2x \\ n \equiv b \pmod{W}}} \nu(n)\theta(n + h_{i_0}) = (\tilde{J}_{i_0}(f_4) + o(1)) B^{1-k} \frac{x}{\varphi(W)}.$$

Lemma 2.6 with $\alpha_0 = \tilde{I}(f_4)$ and $\alpha_i = \tilde{J}_i(f_4)$ now gives $\text{DHL}[k, m+1]$. $\square$

Remark 3.5. Theorem 3.4 is the analogue of [12, Thm. 3.10], in which the region is the truncated simplex $\{t_1 + \dots + t_k \leq 1, \max_i t_i \leq \delta/\varrho\}$ and the input is $\text{MPZ}[\omega, \delta]$ for smooth moduli. There, coordinates above δ/ϱ are forbidden. Here every coordinate up to $\frac{2}{5}\kappa \approx 0.381$ is allowed, and (3.1) constrains, for each v with $u_0 < v \leq \max_i t_i$, the total of the coordinates of size at least v . Supports defined through [10, Prop. 2.3], of a more elaborate shape, are also used in [10] for H_1 .

4. THE TRIAL FUNCTIONS

Fix $k \geq 2$ and real parameters $\gamma, \beta > 0$, and put

$$c := \frac{\gamma}{\log k}, \quad T := \frac{\beta}{\log k}, \quad g(t) := \frac{\mathbf{1}_{[0,T]}(t)}{c + (k-1)t}.$$

(In [12, §6] and in our programs, γ is called θ .) Let

$$m_2 := \int_0^T g^2 = \frac{T}{c(c + (k-1)T)},$$

and let $G(y) := \int_0^{\min(y,T)} g = \log(1 + (k-1)\min(y,T)/c)/(k-1)$ for $y \geq 0$ and $G(y) := 0$ for $y < 0$. Let $X_1, X_2, \dots$ be independent random variables with density g^2/m_2 on $[0, T]$, and $S_n := X_1 + \dots + X_n$. For a measurable $\psi : [0, \infty) \rightarrow [0, 1]$ vanishing on $(1, \infty)$ let F_ψ be as in (1.2) and

$$\Psi(s) := \int_0^T \psi(s+t) g(t) dt \quad (s \geq 0).$$

Lemma 4.1. *We have $I(F_\psi) = m_2^k \mathbb{E}[\psi(S_k)^2]$ and $J_i(F_\psi) = m_2^{k-1} \mathbb{E}[\Psi(S_{k-1})^2]$ for $1 \leq i \leq k$. Hence, if $F_\psi \neq 0$,*

$$Q(F_\psi) = \frac{k}{m_2} \cdot \frac{\mathbb{E}[\Psi(S_{k-1})^2]}{\mathbb{E}[\psi(S_k)^2]}.$$

Proof. The first identity is the definition of I . For the second, $\int_0^\infty F_\psi dt_k = \prod_{i < k} g(t_i) \cdot \Psi(t_1 + \dots + t_{k-1})$, so $J_k(F_\psi) = m_2^{k-1} \mathbb{E}[\Psi(S_{k-1})^2]$; the other J_i agree by symmetry. $\square$

For $\psi = \mathbf{1}_{[0,1]}$ one has $\Psi(s) = G(1-s)$; for $T = 1$, Lemma 4.1 is then equivalent to [13, (24)].

4.1. Restriction to a region. Let $\mathcal{R} \subseteq [0, \infty)^k$ be measurable, invariant under permutations of the coordinates, and a down-set: $t \in \mathcal{R}$ and $0 \leq t' \leq t$ coordinatewise imply $t' \in \mathcal{R}$. For $t' \in [0, T]^{k-1}$ the fibre $\{t \in [0, T] : (t', t) \in \mathcal{R}\}$ is then an interval with left end 0, or empty; let $t^*(t')$ be its right end ($t^*(t') := 0$ if it is empty). Put

$$\Psi_{\mathcal{R}}(t') := \int_{\{t \in [0, T] : (t', t) \in \mathcal{R}\}} \psi(t'_1 + \dots + t'_{k-1} + t) g(t) dt.$$

Lemma 4.2. *With $X' := (X_1, \dots, X_{k-1})$ and $S' := X_1 + \dots + X_{k-1}$,*

$$Q(F_\psi \mathbf{1}_{\mathcal{R}}) \geq Q(F_\psi) - \frac{k}{m_2} \cdot \frac{\mathbb{E}[\Psi(S')^2 - \Psi_{\mathcal{R}}(X')^2]}{\mathbb{E}[\psi(S_k)^2]},$$

provided the right side is positive. Moreover $0 \leq \Psi(s') - \Psi_{\mathcal{R}}(t') \leq G(T) - G(t^*(t'))$ and $\Psi(s') \leq G(T)$, where $s' = t'_1 + \dots + t'_{k-1}$.

Proof. Since $0 \leq \psi \leq 1$, the difference $\Psi(s') - \Psi_{\mathcal{R}}(t')$ is an integral of $\psi g \leq g$ over a subset of $[t^*(t'), T]$, which gives the pointwise bounds. As in Lemma 4.1, $J_k(F_\psi \mathbf{1}_{\mathcal{R}}) = m_2^{k-1} \mathbb{E}[\Psi_{\mathcal{R}}(X')^2]$, and by symmetry of F_ψ and $\mathcal{R}$ every J_i decreases by $m_2^{k-1} \mathbb{E}[\Psi(S')^2 - \Psi_{\mathcal{R}}(X')^2]$. Since $I(F_\psi \mathbf{1}_{\mathcal{R}}) \leq I(F_\psi)$, the claim follows. $\square$

4.2. The cost of restriction. We now bound the expectation in Lemma 4.2 for $\mathcal{R} = \mathcal{R}_k(u_0, \kappa)$. Fix finitely many points $u_0 = v_0 < v_1 < \dots$, the last of which is T , and put $V_l := [v_l, v_{l+1})$, except that the last cell is taken closed; these cells partition $[u_0, T]$. For each cell V_l put

$$\begin{aligned} A_l &:= \kappa - \frac{3}{2}v_{l+1}, & n_l^a &:= \lfloor A_l/T \rfloor, & n_l^b &:= \lfloor (A_l - T)/T \rfloor, \\ b_l^a &:= \min\{1 - A_l, 1 - (n_l^a + 1)v_l\}, & b_l^b &:= \min\{1 - A_l, 1 - (n_l^b + 2)v_l\}, \\ N_l &:= \#\{i \leq k-1 : X_i \geq v_l\}, & R_l &:= \sum_{i \leq k-1 : X_i < v_l} X_i. \end{aligned}$$

Lemma 4.3. For $\mathcal{R} = \mathcal{R}_k(u_0, \kappa)$,

$$\begin{aligned} \mathbb{E}[\Psi(S')^2 - \Psi_{\mathcal{R}}(X')^2] &\leq \sum_l G(T)^2 \mathbb{P}(N_l \geq n_l^a + 1, R_l \leq b_l^a) \\ &\quad + \sum_l 2G(T)(G(T) - G(v_l)) \mathbb{P}(N_l \geq n_l^b + 1, R_l \leq b_l^b). \end{aligned}$$

Moreover $\mathbb{E}[\Psi(S')^2 - \Psi_{\mathcal{R}}(X')^2] \leq G(T)^2 \sum_l \mathbb{P}(N_l \geq n_l^b + 1, R_l \leq 1 - A_l)$.

Proof. Fix $t' \in [0, T]^{k-1}$ with $\Psi_{\mathcal{R}}(t') \neq \Psi(s')$, and write $t_i := t'_i$ for $i < k$. Then $s' \leq 1$ (otherwise $\psi(s' + t) = 0$ for all $t \geq 0$), and some $t \in [0, \min(T, 1 - s')]$ has $(t', t) \notin \mathcal{R}$. For such t the condition $\sum t_i \leq 1$ holds, so with $t_k := t$ there is $v \in (u_0, \max(t', t)]$ with

$$(4.1) \quad \frac{3}{2}v + \sum_{i \leq k : t_i \geq v} t_i > \kappa.$$

Case (a): (4.1) holds for some $v \in (u_0, \max t']$ with the sum restricted to $i < k$. Then $(t', 0) \notin \mathcal{R}$, the fibre is empty, $\Psi_{\mathcal{R}}(t') = 0$, and $\Psi(s')^2 - \Psi_{\mathcal{R}}(t')^2 \leq G(T)^2$. Let V_l be the cell containing v . Then $\sum_{i < k, t_i \geq v} t_i > \kappa - \frac{3}{2}v \geq A_l$; since each $t_i \leq T$, more than A_l/T of the t_i with $i < k$ are $\geq v \geq v_l$, so $N_l(t') \geq n_l^a + 1$. Moreover

$$R_l(t') = s' - \sum_{i < k, t_i \geq v_l} t_i \leq s' - \sum_{i < k, t_i \geq v} t_i < 1 - A_l \quad \text{and} \quad R_l(t') \leq s' - (n_l^a + 1)v_l \leq 1 - (n_l^a + 1)v_l.$$

Case (b): otherwise. Let (t, v) satisfy (4.1) with $t \leq \min(T, 1 - s')$. If $v > t$, then t does not contribute to the sum in (4.1), and $v \leq \max(t', t)$ forces $v \leq \max t'$, which is excluded in case (b). Hence $v \leq t$. Every t in the interval $(t^*(t'), \min(T, 1 - s'))$ is the first member of such a pair, and this interval is nonempty because $\Psi_{\mathcal{R}}(t') \neq \Psi(s')$. Taking such pairs (t, v) with t tending to $t^*(t')$ and passing to a subsequence with v in a fixed cell V_l , we obtain l with $t^*(t') \geq v_l$ and a pair (t, v) with $v_l \leq v \leq t$. Then $\Psi(s') - \Psi_{\mathcal{R}}(t') \leq G(T) - G(v_l)$, so $\Psi(s')^2 - \Psi_{\mathcal{R}}(t')^2 \leq 2G(T)(G(T) - G(v_l))$. From (4.1) and $t \leq T$, $\sum_{i < k, t_i \geq v} t_i > \kappa - \frac{3}{2}v - T \geq A_l - T$, so $N_l(t') \geq n_l^b + 1$. Moreover, using $s' \leq 1 - t$ and $t + \sum_{i < k, t_i \geq v} t_i > \kappa - \frac{3}{2}v$,

$$R_l(t') \leq (1 - t) - \sum_{i < k, t_i \geq v} t_i < 1 - \kappa + \frac{3}{2}v \leq 1 - A_l, \quad R_l(t') \leq (1 - t) - (n_l^b + 1)v_l \leq 1 - (n_l^b + 2)v_l.$$

Bounding the pointwise loss by the sum over l of the indicated terms and taking expectations gives the first inequality. For the second, note that $n_l^a \geq n_l^b$ and that in both cases $R_l(t') < 1 - A_l$ and the loss is at most $G(T)^2$. $\square$

The probabilities in Lemma 4.3 are bounded as follows. Put $p_l := \mathbb{P}(X_1 \geq v_l)$, so that N_l is binomial with parameters $(k-1, p_l)$. Given the set of indices i with $X_i \geq v_l$, the remaining X_i are independent with the law of X_1 conditioned on $X_1 < v_l$. Hence for all integers n and $0 \leq n_{\max} < k-1$ and every real b' ,

$$(4.2) \quad \mathbb{P}(N_l \geq n+1, R_l \leq b') \leq \mathbb{P}(N_l \geq n+1) \mathbb{P}(Z_1 + \cdots + Z_{k-1-n_{\max}} \leq b') + \mathbb{P}(N_l > n_{\max}),$$

where the Z_j are independent with the law of X_1 given $X_1 < v_l$; indeed if $N_l = j \leq n_{\max}$ then R_l is a sum of $k-1-j \geq k-1-n_{\max}$ such variables, all nonnegative. We use $\mathbb{P}(N_l \geq n) \leq \exp(-(k-1)D(\frac{n}{k-1} \| p_l))$ for $p_l < \frac{n}{k-1} \leq 1$, where $D(y \| p) = y \log \frac{y}{p} + (1-y) \log \frac{1-y}{1-p}$ (with $0 \log 0 := 0$) [7, Thm. 1], and $\mathbb{P}(\sum_{j \leq M} Z_j \leq b') \leq e^{\zeta b'} (\mathbb{E} e^{-\zeta Z_1})^M$ for every $\zeta > 0$, with

$$\mathbb{E} e^{-\zeta Z_1} = \frac{1}{m_2 \mathbb{P}(X_1 < v_l)} \int_0^{v_l} \frac{e^{-\zeta t}}{(c + (k-1)t)^2} dt,$$

an integral expressible in closed form through the exponential integral E_1 .

5. CERTIFIED EVALUATION

All quantities below are computed in midpoint-radius (ball) interval arithmetic using Arb [8], now part of FLINT [5], through python-flint, except for a few sums of floating-point numbers, which carry explicit rounding-error bounds. We used python-flint 0.9.0 with FLINT 3.6.0. This version of FLINT computes some products of integer polynomials with coefficients of mixed signs incorrectly in its FFT code (for example the product of the polynomials $\sum_{j < 16} (2^{509} - 1)x^j$ and $-\sum_{j < 16} (2^{510} - 1)x^j$). Our computations do not multiply polynomials: we intercepted FLINT's polynomial multiplication routines during complete runs of the four certificates, and none of them was called. The function ψ is a step function with $N = 201$ cells on a lattice $\eta \mathbf{Z}$, where $\eta := 2^{-18}$ for $m = 2$ and $\eta := 2^{-19}$ for $m \geq 3$: there are $0 = s_0 < s_1 < \cdots < s_N = 1$ in $\eta \mathbf{Z}$ with

$$\psi = \sum_{j=0}^{N-1} \psi_j \mathbf{1}_{[s_j, s_{j+1})}, \quad \psi_0 = 0, \quad \psi_j \in [0, 1] \cap 2^{-40} \mathbf{Z}.$$

The point s_1 lies at the left edge of the bulk of the distribution of S_k (we took $\mathbb{P}(S_k < s_1) \approx 10^{-9}$ for $m \leq 4$ and $\approx 10^{-6}$ for $m = 5$), the other 200 cells have equal lengths up to rounding to $\eta \mathbf{Z}$, and (ψ_j) was taken to be the top generalised eigenvector of the discretised pair of quadratic forms in Lemma 4.1, computed in floating point. Any choice of ψ yields a valid bound, so this optimisation needs no certification. Figure 1 shows the four profiles.

5.1. Distribution functions. Let $\Phi(\xi) := \mathbb{E} e^{i\xi X_1}$ and $C_n(y) := \mathbb{P}(S_n \leq y)$. With $a := \xi/(k-1)$ and $u = c + (k-1)t$,

$$\Phi(\xi) = \frac{e^{-iac}}{(k-1)m_2} \int_c^{c+(k-1)T} \frac{e^{iau}}{u^2} du,$$

where, for $0 < u_1 < u_2$,

$$\int_{u_1}^{u_2} \frac{e^{iau}}{u^2} du = \left[-\frac{e^{iau}}{u} \right]_{u_1}^{u_2} + ia(E_1(-iau_1) - E_1(-iau_2)),$$

which we evaluate in ball arithmetic, raising the working precision when E_1 is ill-conditioned. We use the following facts, in which $f_X = g^2/m_2$ and F_X are the density and distribution function of X_1 .

- (a) $|\Phi'(\xi)| \leq \mathbb{E} X_1$ for real ξ .

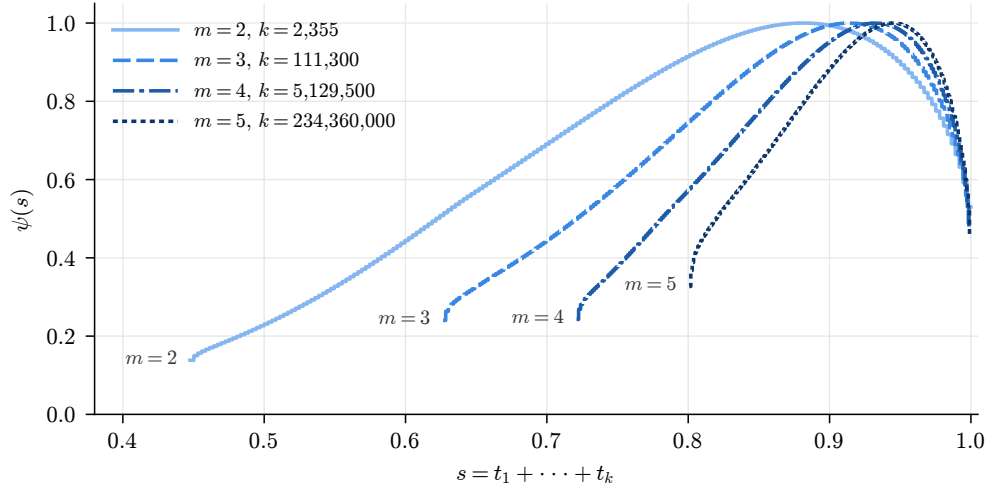


FIGURE 1. The profiles ψ used in Table 2, plotted from the left edge s_1 of the bulk of the distribution of S_k (§5); $\psi = 0$ on $[0, s_1)$. Each profile rises to 1 near $s \approx 0.88$ – 0.94 and falls to about 0.45 – 0.53 on the last cell, just below $s = 1$.

- (b) For real $\xi > 0$ and $0 < a' \leq T$: $|\Phi(\xi)| \leq F_X(a') + 2\sqrt{2}f_X(a')/\xi$. (Split the integral at a' and apply the second mean value theorem to the real and imaginary parts on $[a', T]$, using that f_X is decreasing.) In particular $|\Phi(\xi)| \leq 6\sqrt{2}f_X(0)/\xi$ for all $\xi > 0$: if $4\sqrt{2}/\xi \leq T$, take $a' = 4\sqrt{2}/\xi$ and use $F_X(a') \leq f_X(0)a'$; otherwise the right side exceeds $\frac{3}{2}f_X(0)T \geq \frac{3}{2}$.
- (c) For complex ξ with $|\operatorname{Im} \xi| \leq \frac{1}{2}$: $|\Phi(\xi)| \leq \mathbb{E}e^{X_1/2} \leq 1 + \mathbb{E}X_1 \cdot (e^{T/2} - 1)/T$ by convexity, and $K_y(\xi) := (1 - e^{-i\xi y})/(i\xi) = \int_0^y e^{-i\xi s} ds$ satisfies $|K_y(\xi)| \leq y e^{y/2}$; for real $\xi > 0$, $|K_y(\xi)| \leq \min(y, 2/\xi)$.

Lemma 5.1. For $n \geq 2$ and $y > 0$, $C_n(y) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty \Phi(\xi)^n K_y(\xi) d\xi$, the integral converging absolutely.

Proof. By (b), Φ^n is integrable for $n \geq 2$, so S_n has the continuous density

$$f_n(s) = \frac{1}{2\pi} \int_{\mathbf{R}} \Phi(\xi)^n e^{-i\xi s} d\xi,$$

which vanishes for $s < 0$. Integrating over $s \in [0, y]$ and using Fubini's theorem gives $C_n(y) = \frac{1}{2\pi} \int_{\mathbf{R}} \Phi^n K_y$, and the integrand at $-\xi$ is the complex conjugate of the integrand at ξ . $\square$

We evaluate the integral in Lemma 5.1 for $n \in \{k-1, k\}$ and all y needed simultaneously, as follows.

- On $[0, \Xi_1]$, where Ξ_1 is an integer with $|\Phi(\Xi_1)|^n < e^{-130}$ ($\Xi_1 = 817, 981, 1414, 1697$ for $m = 2, 3, 4, 5$), we use the 40-point Gauss–Legendre rule on each interval $[j, j+1]$. The integrand is entire. The Bernstein ellipse for $[j, j+1]$ with parameter $1 + \sqrt{2}$ (the ellipse with foci j and $j+1$ whose semi-axes sum to $(1 + \sqrt{2})/2$) has semi-minor axis $\frac{1}{2}$, so by (c) the integrand is bounded on it by $M := (1 + \mathbb{E}X_1(e^{T/2} - 1)/T)^n y e^{y/2}$, where y is the largest point used. By [17, Thm. 4.5, (4.14)], applied after the affine map of $[j, j+1]$ onto $[-1, 1]$, the error on each interval is at most $\frac{1}{2} \cdot \frac{64}{15} M (1 + \sqrt{2})^{-80} / (1 - (1 + \sqrt{2})^{-2})$. (The

proof expands the integrand in Chebyshev polynomials T_l , whose coefficients are at most $2M(1+\sqrt{2})^{-l}$ in absolute value; the 40-point rule integrates polynomials of degree < 80 and odd T_l exactly, and $|\int_{-1}^1 T_l| + |\sum_r w_r T_l(x_r)| \leq \frac{32}{15}$ for even $l \geq 80$ [17, (5.6)]. The nodes are rounded to binary64; the resulting error is bounded using $|\frac{d}{d\xi}(\Phi^n K_y)| \leq n \mathbb{E}X_1 y + y^2/2$ for real ξ .

- On $[\Xi_1, \infty)$ we bound the integral of $|\Phi|^n |K_y| \leq 2|\Phi|^n/\xi$. First, on consecutive cells $[\xi_r, \xi_{r+1}]$ with exactly representable endpoints and lengths about $\max\{1, (1 - |\Phi(\xi_r)|)/(4\mathbb{E}X_1)\}$, by $2\log(\xi_{r+1}/\xi_r)(|\Phi(\xi_r)| + \mathbb{E}X_1 \cdot (\xi_{r+1} - \xi_r))^n$, using (a). Once the bound (b) with $a' = 4\sqrt{2}/\xi$ has n th power below e^{-150} , we use (b) on cells of ratio about 1.01, bounding $F_X(a')$ at the left end of each cell and $f_X(a')$ at the right end. Finally, from the first point ξ_* with $6\sqrt{2}f_X(0)/\xi_* < \frac{1}{2}$, we use $\int_{\xi_*}^{\infty} (6\sqrt{2}f_X(0)/\xi)^n 2 d\xi/\xi = \frac{2}{n}(6\sqrt{2}f_X(0)/\xi_*)^n$, from the last assertion of (b).
- The values $\Phi(\xi)^n$ at the nodes are computed in ball arithmetic at 256 bits or more. The sums over the nodes, for many values of y at once, are formed in binary64 with a rigorous bound for the accumulated rounding error. This bound assumes that the binary64 sine and cosine functions used are accurate to within four units in the last place; on 10^5 test arguments in the relevant range, those we used were accurate to within 0.52 units.

This yields numbers $C_n^-(y) \leq C_n(y) \leq C_n^+(y)$, with $C_n^+(y) - C_n^-(y)$ at most 2.2, 3.0, 5.7 and $7.5 \cdot 10^{-11}$ for $m = 2, 3, 4, 5$ at every y used. These widths come almost entirely from the rounding of the nodes and the floating-point summation; the quadrature and tail terms are below 10^{-26} .

5.2. The lower bound for $Q(F_\psi)$. Since $\psi_0 = 0$ and $s_N = 1$, summation by parts gives

$$\Psi(s) = \psi_{N-1}G(1-s) + \sum_{j=1}^{N-1} (\psi_{j-1} - \psi_j) G(s_j - s).$$

Each $G(s_j - s)$ is nonincreasing in s , as G is nondecreasing on $\mathbf{R}$. Hence on each lattice cell $[r\eta, (r+1)\eta]$ with $s_1/\eta \leq r < 1/\eta$ we have $\Psi(s) \geq \ell_r$, where ℓ_r is obtained by evaluating each term with a positive coefficient at $s = (r+1)\eta$ and each term with a negative coefficient at $s = r\eta$, using a table of rigorous enclosures of G on $\eta\mathbf{Z}$ (and ℓ_r is replaced by $\max(\ell_r, 0)$). Therefore

$$\begin{aligned} \mathbb{E}[\Psi(S_{k-1})^2] &\geq \sum_{r=s_1/\eta}^{1/\eta-1} \ell_r^2 (C_{k-1}((r+1)\eta) - C_{k-1}(r\eta)), \\ \mathbb{E}[\psi(S_k)^2] &= \psi_{N-1}^2 C_k(1) + \sum_{j=1}^{N-1} (\psi_{j-1}^2 - \psi_j^2) C_k(s_j). \end{aligned}$$

Both sums are evaluated by summation by parts in ball arithmetic, using C^- or C^+ according to the sign of each coefficient. This gives a rigorous lower bound Q_{low} for $Q(F_\psi)$ and a rigorous lower bound for $\mathbb{E}[\psi(S_k)^2]$.

5.3. The restriction loss. We take

$$(5.1) \quad \begin{aligned} \omega &= 0.012466, & \delta &= 10^{-4}, & \varrho &= \frac{1}{4} + \omega = 0.262466, \\ u_0 &= \frac{\delta}{2\varrho} = 1.905 \dots \cdot 10^{-4}, & \kappa &= \frac{1/4 + \delta/2}{\varrho} - 10^{-4} = 0.95259 \dots, \end{aligned}$$

so that $240\omega + 80\delta = 2.99984 < 3$ and the hypotheses of Theorem 3.4 hold. We bound the loss in Lemma 4.2 for $\mathcal{R} = \mathcal{R}_k(u_0, \kappa)$ by Lemma 4.3, with $v_{l+1} = 1.02 v_l$ (the last point being T), $n_{\max} = \min\{k - 2, \lfloor (k - 1)p_l + 14\sqrt{(k - 1)p_l(1 - p_l)} + 20 \rfloor\}$ in (4.2), and ζ chosen by a floating-point search; each term is then evaluated in ball arithmetic, with the floors rounded down and an additional slack 10^{-4} added to b_l^a , b_l^b and $1 - A_l$. For $m = 2$ we use the first inequality of Lemma 4.3, for $m \geq 3$ the second. In all four cases $T < \frac{2}{5}\kappa$, so that $A_l - T > 0$.

5.4. Results. Table 2 lists the certified values. The exact binary64 values of γ and β , and the cells and values of ψ , are included in the ancillary files.

TABLE 2. Certified lower bounds for $M_k^{\mathcal{R}}$, $\mathcal{R} = \mathcal{R}_k(u_0, \kappa)$ with (5.1). Here $Q_{\text{low}} \leq Q(F_\psi)$, “loss” bounds the term subtracted in Lemma 4.2, and the margin is $Q_{\text{low}} - \text{loss} - m/\rho$. The lattice spacing is $\eta = 2^{-18}$ for $m = 2$ and $\eta = 2^{-19}$ for $m \geq 3$. All bounds are rounded in the safe direction.

m	k	γ	β	Q_{low}	loss	m/ρ	margin
2	2,355	1.028016828	2.8	7.6201538	$3.3 \cdot 10^{-7}$	7.6200346	$1.18 \cdot 10^{-4}$
3	111,300	1.021466024	3.929974771	11.4301517	$2.9 \cdot 10^{-19}$	11.4300519	$9.98 \cdot 10^{-5}$
4	5,129,500	1.017798184	3.310796618	15.2401167	$3.6 \cdot 10^{-34}$	15.2400692	$4.75 \cdot 10^{-5}$
5	234,360,000	1.014304631	3.373928705	19.0501196	$3.9 \cdot 10^{-34}$	19.0500865	$3.31 \cdot 10^{-5}$

The certified lower bounds lie $1\text{--}4 \cdot 10^{-4}$ below floating-point evaluations of $Q(F_\psi)$. The difference comes almost entirely from replacing Ψ by its lower bound ℓ_r on lattice cells, and a finer lattice would allow slightly smaller k .

6. ADMISSIBLE TUPLES

Table 3 lists the admissible tuples used.

TABLE 3. Admissible k -tuples.

m	k	construction	diameter
2	2,355	database [4]	20,370
3	111,300	greedy sieve on $[-558\,707, 838\,061]$, $z = 300$	1,380,156
4	5,129,500	greedy sieve on $[-38\,025\,000, 46\,475\,000]$, $z = 10^4$	82,959,170
5	234,360,000	the first k primes greater than k	5,040,082,836

For $m = 2$ we used the admissible 2355-tuple of diameter 20,370 from the database of narrow admissible tuples [4]. For $m = 3, 4$ we used a variant of the shifted greedy sieve of [12, §10.2.2]: starting from the even integers in the interval of Table 3, remove those divisible by an odd prime $p \leq z$; then, for each prime $z < p \leq k$ in increasing order, if every residue class modulo p meets the survivors, remove the survivors lying in a least populated class; finally take k consecutive survivors of smallest diameter. The output is determined by k , the interval and z (with ties broken as in our program). For $m = 5$ we used Zhang’s construction: the first k primes greater than k form an admissible k -tuple, since no prime $p \leq k$ divides any of them and k numbers cannot meet all p classes when $p > k$. For $k = 234,360,000$ these primes run from 234,360,017 to 5,274,442,853, since $\pi(234,360,000) = 12,868,673$ and $\pi(5,274,442,853) = 247,228,673$. We computed them with a segmented sieve and confirmed these values with an independent prime-counting computation.

Admissibility of the tuples for $m = 2, 3, 4$ was checked at every prime $p \leq k$ by two independently written programs. One counts residues for small p and probes for an empty class for large p . The other finds, for every $p \leq k$, a residue class modulo p and verifies directly that it contains no element of the tuple.

7. PROOF OF THEOREM 1.1

Let $\omega, \delta, \varrho, u_0, \kappa$ be as in (5.1); then $240\omega + 80\delta < 3$, $u_0 < \delta/\varrho$ and $\kappa < (\frac{1}{4} + \frac{\delta}{2})/\varrho$. Let (m, k) be one of the rows of Table 2, with the corresponding γ, β, ψ , and let $\mathcal{R} := \mathcal{R}_k(u_0, \kappa)$, a closed, permutation-invariant down-set by §3 and Lemma 3.2(i). The function $F_\psi \mathbf{1}_{\mathcal{R}}$ is square-integrable, has bounded support contained in $\mathcal{R}$, and by Lemma 4.2 and Table 2 satisfies

$$M_k^{\mathcal{R}} \geq Q(F_\psi \mathbf{1}_{\mathcal{R}}) \geq Q_{\text{low}} - \text{loss} > \frac{m}{\varrho}.$$

(In particular $F_\psi \mathbf{1}_{\mathcal{R}} \neq 0$.) By Theorem 3.4, DHL[$k, m+1$] holds. Combined with the admissible k -tuples of Table 3, this gives the bounds for H_m stated in Theorem 1.1. $\square$

8. REMARKS

- (1) The cost of the region $\mathcal{R}_k(u_0, \kappa)$ is negligible for these trial functions: the restriction loss is at most $3.3 \cdot 10^{-7}$ for $m = 2$ and below $3 \cdot 10^{-19}$ for $m \geq 3$. For $m = 2$ the relevant constraint is rather the cap $\max_i t_i \leq \frac{2}{5}\kappa \approx 0.381$: coordinates above it are excluded from the support, and we kept T below it, using $\beta = 2.8$, i.e. $T \approx 0.361$, whereas without the cap the optimal β for this family would be about 3.8, i.e. $T \approx 0.49$.
- (2) The threshold m/ϱ is limited by the condition $240\omega + 80\delta < 3$ of Theorem 2.4. The quotients of our trial functions are $\log k + O(1)$: they are at most $M_k \leq \frac{k}{k-1} \log k$ [12, Cor. 6.4], and in our range they exceed by about 0.05 the quotients of the product trial function (1.1), which are $\log k - O(1)$ [12, §6], [13]. Hence an equidistribution estimate for triply densely divisible moduli at a level $\frac{1}{2} + 2\omega'$ with $\varrho' = \frac{1}{4} + \omega' > \varrho$ would reduce the required k by a factor of roughly $\exp(m/\varrho - m/\varrho')$.
- (3) Since $240\omega + 80\delta < 3$ forces $\varrho < \frac{1}{4} + \frac{1}{80} = \frac{21}{80}$, the threshold m/ϱ in Theorem 3.4 exceeds $\frac{80}{21}m \approx 3.8095m$, while $M_k^{\mathcal{R}} \leq M_k \leq \frac{k}{k-1} \log k$. So for large m the method needs $\log k \geq (\frac{80}{21} + o(1))m$ and cannot improve Stadlmann's asymptotic bound $H_m \ll \exp(3.8075m)$ [15, Thm. 2]; its gain is for small m .
- (4) The margins in Table 2 are small but rigorous. The values of k were found by a coarse search, and slightly smaller values may be attainable with the same method.

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DATA AND CODE

The ancillary files contain: the programs used for the certificates (§5) and their exact inputs (γ, β , the cells and the values of ψ for each m); the output of a complete re-run of the four certificates with these programs; the admissible tuples for $m = 2, 3$; the program that constructs the tuple for $m = 4$, with the SHA-256 checksum of its output; the program that

computes the tuple for $m = 5$ and an independent check of it; the two admissibility checkers; a test of the floating-point sine and cosine and the program used to intercept FLINT's polynomial multiplication (§5); and logs of all these computations. The certificates were run with python-flint 0.9.0 (FLINT 3.6.0), NumPy 2.5.3 and SciPy 1.18.1 under Python 3.13.

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