

Dimension of the equilibrium measure on $\mathbb{P}^n$

Abstract

Consider the dynamics of endomorphisms $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ of degree $d \geq 2$. This note presents an explicit and elementary counterexample to the dimension formula for the equilibrium measure μ conjectured by Binder–DeMarco [8], coming from the same family of quadratic skew products that appeared recently in Bianchi–He [7]. In addition, the note shows that the one-sided inequality

$$\dim_{\mathrm{H}} \mu \geq (\log d) \sum_{i=1}^n \lambda_i^{-1}$$

holds for the equilibrium measure of every degree- d holomorphic endomorphism of $\mathbb{P}^n$. The proof adapts Dupont’s volume-growth method [1] to the unstable flag: combined with Gromov’s orbit-graph argument [5], it bounds the cumulative partial entropies. Qian–Xie’s extension [4] of Ledrappier–Young theory [3] then yields the inequality.

Provenance. This revision was produced by ChatGPT to clarify the sources and condense the exposition.

1 Introduction

Let $f : \mathbb{P}^n(\mathbb{C}) \rightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic endomorphism of degree $d \geq 2$, with equilibrium measure μ and Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_n > 0$, counted with complex multiplicity. Binder–DeMarco [8, Conjecture 1.3] conjectured that

$$\dim_{\mathrm{H}} \mu = (\log d) \sum_{i=1}^n \frac{1}{\lambda_i}, \quad (1.1)$$

for the Hausdorff dimension of μ , defined by

$$\dim_{\mathrm{H}} \mu = \inf \{ \dim_{\mathrm{H}} A : A \text{ is Borel, } \mu(A) = 1 \}.$$

Bianchi–He [7] recently disproved this conjecture using quadratic skew products. The Appendix proves directly that the example (from the same family)

$$f(x, y) = (x^2 - 10000, y^2 - 100 + x/2)$$

has $\dim_{\mathrm{H}} \mu \geq 2/5$, whereas the right-hand side of (1.1) is less than $23/60$.

The purpose of this note is to observe that expression in (1.1) provides a lower bound on the dimension in general.

Theorem 1.1. *For every holomorphic endomorphism f as above,*

$$\dim_{\mathbb{H}} \mu \geq (\log d) \sum_{i=1}^n \frac{1}{\lambda_i}. \quad (1.2)$$

The proof adapts the method of Dupont [1], who established this inequality for $n = 2$ and the bound $(n - 1) \log d / \lambda_1 + \log d / \lambda_n$ in arbitrary dimension. His argument counts inverse branches meeting a small ball by means of holomorphic slices, volume growth, and the normal forms of Berteloot–Dupont–Molino [2]. The same volume-growth principle applies at every level of the fast unstable flag.

The partial-entropy formulation is already suggested in Dupont’s §1.1, which compares (1.1) with the Ledrappier–Young formula and singles out the values $H_i = i \log d$. The cumulative bounds $H_j \leq p_j \log d$ suffice; here p_j is the dimension of the corresponding fast plaque. These follow by combining Dupont’s local growth estimate [1, Appendix] with Gromov’s orbit-graph packing argument [5, §§1–3] and the conditional version of the Brin–Katok local-entropy method [6, 4]. The dimension estimate then follows from Qian–Xie’s endomorphism extension [4, 14] of Ledrappier–Young theory [3]. Bianchi–He [7, Proposition 2.3] use the analogous entropy-defect identity in dimension two and, under their hypotheses, deduce $H_1 \leq \log d$ from Dupont’s inequality. Equality in (1.1) does hold for skew products whose fiber exponent dominates the base exponent, by Genes [9].

Notation. Put $M = \mathbb{P}^n(\mathbb{C})$ and equip it with the Fubini–Study form ω and its associated metric. Group equal exponents as

$$\chi_1 > \cdots > \chi_s > 0, \quad \sum_{j=1}^s m_j = n, \quad p_j = \sum_{a=1}^j m_a, \quad L = \log d,$$

where m_j is the complex multiplicity of χ_j . Thus $\sum_i \lambda_i^{-1} = \sum_j m_j / \chi_j$. The measure μ is ergodic, with

$$h_{\mu}(f) = nL, \quad \log |\det D_{\mathbb{R}} f| \in L^1(\mu), \quad \lambda_n > 0. \quad (1.3)$$

See [10, 11, 12], and Briend–Duval [13] for positivity of the exponents. In particular the critical set and its iterated preimages have zero μ -measure.

2 Conditional entropy and dimension

Let $(\widehat{M}, \widehat{f}, \widehat{\mu})$ be the natural extension, with time-zero projection $\pi : \widehat{M} \rightarrow M$. A regular inverse history determines a strong unstable flag $W^1 \subset \cdots \subset W^s$, where $\dim_{\mathbb{C}} W^j = p_j$. The local spatial projections of these plaques are holomorphic, by holomorphic graph transform; compare the normal forms in [2]. Their parametrizations extend to slightly larger polydiscs after shrinking. The flag is kept on the natural extension and may depend on the inverse history.

Qian–Xie’s endomorphism version of Ledrappier–Young theory gives exact conditional dimensions and their entropy relations [4, Proposition 2.4 and equation (19)], and conditional local entropy [4, Proposition 4.4]; see also [14, Chapter IX].

Proposition 2.1 (Qian–Xie framework). *There are cumulative partial entropies $0 = H_0 \leq H_1 \leq \dots \leq H_s = h_\mu(f)$ and exact conditional dimensions δ_j on the strong unstable plaques such that*

$$\delta_j - \delta_{j-1} = \frac{H_j - H_{j-1}}{\chi_j}, \quad 1 \leq j \leq s, \quad \delta_0 = 0. \quad (2.1)$$

Dimensions are measured in the intrinsic spatial metric on a plaque. For a typical plaque conditional σ , define

$$B_N^j(y, r) = \{z : \max_{0 \leq t < N} d_{W_t^j}(f^t y, f^t z) < r\}$$

inside its conditional-measure domain. The forward leaves and their intrinsic metrics retain the prescribed histories. Then

$$\lim_{r \downarrow 0} \liminf_{N \rightarrow \infty} -\frac{1}{N} \log \sigma(B_N^j(y, r)) = H_j \quad \text{for } \sigma\text{-almost every } y. \quad (2.2)$$

The hypotheses are supplied by (1.3). It remains to prove

$$H_j \leq p_j L \quad (1 \leq j < s). \quad (2.3)$$

3 Volume growth and orbit graphs

Gromov’s orbit graphs and minimal-volume argument [5, §§1–3] are localized by Dupont’s bounded-potential method [1, Appendix]. Intrinsic graph distances are needed to control the intrinsic leafwise dynamical metric.

3.1 The orbit-graph volume estimate

Write $\mathbb{D}^p(a) = \{z \in \mathbb{C}^p : \max_i |z_i| < a\}$. Let $\eta : \mathbb{D}^p(2) \rightarrow M$ be a holomorphic embedding whose image is contained in a bounded affine coordinate chart. For $N \geq 1$, define

$$G_N(z) = (\eta(z), f\eta(z), \dots, f^{N-1}\eta(z)) \in M^N.$$

The first component makes G_N an embedding. Give M^N the product Kähler form $\Omega_N = \sum_{t=0}^{N-1} \text{pr}_t^* \omega$, and set $\Gamma_N = G_N(\mathbb{D}^p(1))$, with its induced intrinsic metric.

Lemma 3.1 (Orbit-graph volume). *There is a constant C_η , independent of N , such that*

$$\text{Vol}_{2p}(\Gamma_N) \leq C_\eta d^{pN}. \quad (3.1)$$

Proof. Let $T = \omega + dd^c g$ be the Green current of f , where g is continuous and bounded, and $f^*T = dT$. It follows that

$$(f^t)^* \omega = d^t \omega + dd^c(d^t g - g \circ f^t).$$

Set $A_N = \sum_{t=0}^{N-1} d^t$. Choose a smooth local potential ψ for ω on the chart containing the plaque. On $\mathbb{D}^p(2)$, put

$$v_N = \psi \circ \eta + g \circ \eta - \frac{1}{A_N} \sum_{t=0}^{N-1} g \circ f^t \circ \eta.$$

Then

$$dd^c v_N = A_N^{-1} G_N^* \Omega_N = A_N^{-1} \eta^* \sum_{t=0}^{N-1} (f^t)^* \omega \geq 0. \quad (3.2)$$

Thus v_N is plurisubharmonic. On any fixed intermediate polydisc, for instance $\mathbb{D}^p(3/2)$, it is bounded uniformly in N , because g is bounded and $N/A_N \leq 1$. The Chern–Levine–Nirenberg estimate [15], applied between $\mathbb{D}^p(1)$ and $\mathbb{D}^p(3/2)$, gives

$$\int_{\mathbb{D}^p(1)} (dd^c v_N)^p \leq C.$$

By (3.2),

$$\text{Vol}_{2p}(\Gamma_N) = \frac{A_N^p}{p!} \int_{\mathbb{D}^p(1)} (dd^c v_N)^p \leq C' A_N^p \leq C_\eta d^{pN}.$$

□

3.2 A uniform lower volume bound for intrinsic balls

The following intrinsic version of the minimal-volume estimate is proved explicitly to keep the constants independent of the orbit length.

Lemma 3.2 (Intrinsic balls in the orbit graph). *Fix a compact set $K \Subset \mathbb{D}^p(1)$. There exist $r_* > 0$ and $c > 0$, independent of N , such that*

$$\text{Vol}_{2p}(B_{\Gamma_N}(G_N(z), r)) \geq cr^{2p} \quad (z \in K, 0 < r < r_*). \quad (3.3)$$

Proof. Every Γ_N is a complex submanifold of the Kähler manifold M^N , hence is minimal. The product metrics on M^N have a sectional-curvature upper bound independent of N , and their injectivity radii have a positive lower bound independent of N . There is also a uniform boundary margin for the centres in question. Indeed,

$$a := \text{dist}_M(\eta(K), \eta(\partial\mathbb{D}^p(1))) > 0.$$

The length of every graph path is at least the length of its first coordinate projection. Consequently, the intrinsic graph distance from $G_N(K)$ to the boundary of Γ_N is at least a . Write $m = 2p$, fix a centre $q = G_N(z)$, and let R be ambient product distance from q . For sufficiently small r , an intrinsic ball of radius r lies in the ambient ball of radius r . Hessian comparison for the squared ambient distance, together with minimality, gives

$$\Delta_{\Gamma_N} R^2 \geq 2m(1 - Cr^2) \quad \text{on } B_{\Gamma_N}(q, r).$$

The constant C depends on the uniform ambient curvature bound and is independent of N . Also $|\nabla_{\Gamma_N} R^2| \leq 2r$ there. Let $V(r) = \text{Vol}_m(B_{\Gamma_N}(q, r))$. Divergence and coarea, valid for almost every radius (or by approximation across the intrinsic cut locus), imply

$$2m(1 - Cr^2)V(r) \leq 2rV'(r).$$

Thus

$$\frac{d}{dr} \log \frac{V(r)}{r^m} \geq -mCr.$$

At each smooth centre, $\lim_{r \downarrow 0} V(r)/r^m = \omega_m$, the Euclidean unit-ball volume. Integrating the last inequality gives

$$V(r) \geq \omega_m r^m \exp(-mCr^2/2).$$

Choose r_* smaller than the uniform ambient comparison radius and the boundary margin a . This proves the lemma. $\square$

3.3 Packing with the leafwise dynamical metric

Suppose now that η parametrizes a local strong unstable plaque, with its fixed inverse history. Let W_t denote its forward leaf at time t , endowed with intrinsic spatial distance. For $z, w \in \mathbb{D}^p(1)$, write

$$\rho_N(z, w) := \max_{0 \leq t < N} d_{W_t}(f^t \eta(z), f^t \eta(w)).$$

Distances are taken in the parametrized leaves, retaining histories when a spatial projection has several sheets.

Lemma 3.3 (Leafwise dynamical packing). *For every $K \Subset \mathbb{D}^p(1)$ and every sufficiently small fixed $r > 0$, there exists $C_{\eta, K, r} < \infty$, independent of N , such that every ρ_N -separated set $E \subset K$ at separation r satisfies*

$$\#E \leq C_{\eta, K, r} d^{pN}. \quad (3.4)$$

Proof. For any piecewise smooth parameter path γ joining z to w , its image under $f^t \circ \eta$ is a path in the corresponding forward leaf. The product metric therefore gives

$$\text{length}(G_N \circ \gamma) \geq \text{length}(f^t \circ \eta \circ \gamma) \geq d_{W_t}(f^t \eta(z), f^t \eta(w))$$

for every $t < N$. Taking an infimum over paths gives

$$d_{\Gamma_N}(G_N(z), G_N(w)) \geq \rho_N(z, w). \quad (3.5)$$

If a forward parametrization has critical points, lengths may be computed by approximation with paths avoiding the finite union of proper analytic critical loci. This leaves the inequality unchanged; the orbit graph itself is always nonsingular because of its first coordinate. For $z \in E$, the intrinsic graph balls of radius $r/3$ centred at $G_N(z)$ are pairwise disjoint by (3.5). Lemmas 3.1 and 3.2 yield

$$(\#E)c(r/3)^{2p} \leq \text{Vol}_{2p}(\Gamma_N) \leq C_\eta d^{pN}.$$

$\square$

4 The cumulative entropy bound

Proposition 4.1. *For each level of the strong unstable flag, $H_j \leq p_j \log d$.*

Proof. Fix j , set $p = p_j$, and choose a typical conditional σ on a lifted W^j -plaque. A countable cover by holomorphic coordinate patches provides an embedding $\eta : \mathbb{D}^p(2) \rightarrow M$ as above and an inner compact set of positive σ -measure. Keep this conditional on its fixed lifted plaque.

For $\varepsilon > 0$, (2.2) gives a small $r > 0$, an integer N_0 , and a positive-measure subset F of the inner compact set such that

$$\sigma(B_N^j(y, r)) \leq e^{-N(H_j - \varepsilon)} \quad (y \in F, N \geq N_0). \quad (4.1)$$

Indeed, first choose a radius from a countable sequence tending to zero on which the lower entropy rate exceeds $H_j - \varepsilon/2$ on a set of positive measure, then choose a uniform N_0 . Restriction or normalization contributes only a fixed multiplicative constant.

A maximal $(r/2)$ -separated set $E_N \subset F$ for the leafwise dynamical metric has at most $C_{\eta, r} d^{pN}$ points by Lemma 3.3. Its radius- r Bowen balls cover F , so

$$0 < \sigma(F) \leq \sum_{y \in E_N} \sigma(B_N^j(y, r)) \leq C_{\eta, r} d^{pN} e^{-N(H_j - \varepsilon)}.$$

Letting $N \rightarrow \infty$ and then $\varepsilon \downarrow 0$ proves the claim. This is the conditional version of the local-entropy covering argument of Brin–Katok [6], with the leafwise input of [4]. $\square$

Remark 4.2. The estimate is cumulative. Bounds $H_j - H_{j-1} \leq m_j \log d$ for individual increments do not follow and are not needed.

5 Proof of the dimension inequality

Lemma 5.1. *Let ξ be a measurable partition of $\widehat{M}$. If the projected conditionals $\nu_{\widehat{x}} = \pi_* \widehat{\mu}_{\xi(\widehat{x})}$ have Hausdorff dimension at least D for almost every $\widehat{x}$, then $\dim_{\text{H}} \mu \geq D$.*

Proof. For any Borel set A with $\mu(A) = 1$, disintegration of $\widehat{\mu}(\pi^{-1}A) = 1$ gives $\nu_{\widehat{x}}(A) = 1$ almost everywhere. Hence $\dim_{\text{H}} A \geq D$; take the infimum over A . $\square$

Proof of Theorem 1.1. Take ξ subordinate to full unstable plaques. Projection to M is bi-Lipschitz on relatively compact spatial coordinate patches, so Proposition 2.1 and Lemma 5.1 give

$$\dim_{\text{H}} \mu \geq \delta_s = \sum_{j=1}^s \frac{H_j - H_{j-1}}{\chi_j}. \quad (5.1)$$

Use a countable exhaustion by patches if necessary. Summation by parts, using $H_s = nL$ and $p_j = \sum_{a \leq j} m_a$, yields

$$\delta_s - L \sum_{j=1}^s \frac{m_j}{\chi_j} = \sum_{j=1}^{s-1} (p_j L - H_j) \left(\frac{1}{\chi_{j+1}} - \frac{1}{\chi_j} \right) \geq 0. \quad (5.2)$$

Each summand is nonnegative by Proposition 4.1. Together with (5.1), this proves the theorem. $\square$

The identity (5.2) extends the two-exponent entropy defect in [7, Proposition 2.3]. Equality in (1.2) would require $H_j = p_j L$ for every $j < s$, together with equality in (5.1).

A An explicit counterexample

The following counterexample to [8, Conjecture 1.3] belongs to the quadratic skew-product family of Bianchi–He [7]. The proof is independent of the conditional entropy argument above.

Theorem A.1. *The holomorphic quadratic endomorphism*

$$f[Z : W : T] = [Z^2 - 10000T^2 : W^2 - 100T^2 + \tfrac{1}{2}ZT : T^2] \quad (\text{A.1})$$

has equilibrium measure μ , complex Lyapunov exponents $a > b > 0$, and dimension $D = \dim_{\mathbb{H}} \mu$ satisfying

$$D \geq \frac{2}{5} > \frac{23}{60} > \frac{\log 2}{a} + \frac{\log 2}{b}. \quad (\text{A.2})$$

The proof uses uniqueness of the measure of maximal entropy [11, Theorem 2], with its erratum [12].

A.1 The invariant set and its equilibrium measure

The three homogeneous coordinates in (A.1) have no common zero outside the origin: the last forces $T = 0$, and the first two then force $Z = W = 0$. They therefore define a holomorphic endomorphism of algebraic degree two. In the affine chart $T = 1$,

$$f(x, y) = (x^2 - 10000, y^2 - 100 + x/2).$$

Regard $Q = [-101, 101] \times [-13, 13]$ as a real rectangle in $\mathbb{C}^2$. For $s, t \in \{-1, 1\}$, define an inverse branch on Q by

$$g_{s,t}(u, v) = (x, y), \quad x = s\sqrt{10000 + u}, \quad y = t\sqrt{100 + v - x/2}. \quad (\text{A.3})$$

These satisfy $f \circ g_{s,t} = \text{id}$ and $99 < |x| < 101$. For $s = 1$, $6 < |y| < 8$; for $s = -1$, $11 < |y| < 13$. Thus the images are compactly contained in Q and occupy the disjoint bands

$$(-13, -11), \quad (-8, -6), \quad (6, 8), \quad (11, 13). \quad (\text{A.4})$$

Points in different images have y -coordinates differing by more than 3. Differentiating (A.3), and using $|2x| > 198$ and $|2y| > 12$, gives the ordinary sup-norm bound

$$\|Dg_{s,t}\|_{\infty} \leq \max \left\{ \frac{1}{198}, \frac{1}{12} + \frac{1}{4752} \right\} = \frac{397}{4752} < \frac{1}{11}.$$

Since Q is convex, these are contractions. Their disjoint images give a homeomorphic coding of the invariant compact set K by the one-sided shift on four symbols. Pushing forward the uniform Bernoulli measure gives an invariant ergodic probability of entropy $\log 4$, hence the equilibrium measure μ by uniqueness. Every cylinder $K_{\omega_0 \dots \omega_{N-1}} = g_{\omega_0} \circ \dots \circ g_{\omega_{N-1}}(K)$ has mass 4^{-N} .

A.2 Cylinder separation and the dimension lower bound

Consider two distinct inputs in Q with $|\Delta u| \leq 68|\Delta v|$, and apply the same inverse branch. Write the outputs as (x, y) and (x', y') . Their corresponding coordinate signs agree, and the exact difference identities are

$$\Delta x = \frac{\Delta u}{x + x'}, \quad \Delta y = \frac{\Delta v - \Delta x/2}{y + y'}.$$

The input cone and distinctness imply $\Delta v \neq 0$. Since $|x + x'| > 198$ and $|y + y'| < 26$, it follows that

$$|\Delta y| \geq \frac{1 - 68/396}{26} |\Delta v| > \frac{1}{32} |\Delta v|. \quad (\text{A.5})$$

Furthermore,

$$|\Delta x| \leq \frac{68}{198} |\Delta v| < \frac{32 \cdot 68}{198} |\Delta y| < 68 |\Delta y|.$$

Thus a common inverse branch preserves the cone and decreases the y -distance by a factor no smaller than $1/32$. For points of K in different first-level cylinders,

$$|\Delta x| < 202 < 3 \cdot 68 < 68 |\Delta y|, \quad |\Delta y| > 3.$$

Now take points in different length- N cylinders, and let $m < N$ be the length of their common prefix. After applying f^m , the points occupy different first-level cylinders. Restore the prefix, using cone preservation and (A.5) at each step. Their original y -distance, for every such pair, is greater than

$$3 \cdot 32^{-m} \geq 3 \cdot 32^{-(N-1)}.$$

For any $0 < r \leq 1$, choose $N \geq 1$ such that $32^{-N} \leq r \leq 32^{-(N-1)}$. An ambient Euclidean ball of radius r has second-coordinate diameter at most $2r$, less than the separation just proved. It therefore meets at most one length- N cylinder, and

$$\mu(B(z, r)) \leq 4^{-N} = (32^{-N})^{2/5} \leq r^{2/5}. \quad (\text{A.6})$$

For any full-measure Borel set $E \subset K$, a cover by sets of diameters $d_i \leq 1$ therefore satisfies $1 \leq \sum_i d_i^{2/5}$. This proves $\dim_{\text{H}} E \geq 2/5$. Euclidean and Fubini–Study distances are bi-Lipschitz equivalent on Q , so

$$D \geq \frac{2}{5}. \quad (\text{A.7})$$

A.3 The complex Lyapunov exponents and the strict gap

Along K the complex derivative is

$$Df(x, y) = \begin{pmatrix} 2x & 0 \\ 1/2 & 2y \end{pmatrix}, \quad 198 < |2x| < 202, \quad 12 < |2y| < 26.$$

For $f^j(p) = (x_j, y_j)$, write

$$Df^N(p) = \begin{pmatrix} P_N & 0 \\ E_N & Q_N \end{pmatrix}, \quad P_N = \prod_{j=0}^{N-1} 2x_j, \quad Q_N = \prod_{j=0}^{N-1} 2y_j.$$

With $r_0 = 0$ and $r_N = E_N/P_N$, matrix multiplication gives

$$r_{N+1} = \frac{1}{4x_N} + \frac{y_N}{x_N}r_N, \quad |r_N| \leq \frac{1/396}{1 - 26/198} = \frac{1}{344}, \quad \left| \frac{Q_N}{P_N} \right| \leq (26/198)^N.$$

Let $\sigma_1(N) \geq \sigma_2(N) > 0$ be the two complex singular values. For $M = \sqrt{2 + 344^{-2}}$, the first column and the Frobenius norm give

$$|P_N| \leq \sigma_1(N) \leq M|P_N|.$$

Since $\sigma_1(N)\sigma_2(N) = |\det Df^N| = |P_N Q_N|$,

$$M^{-1}|Q_N| \leq \sigma_2(N) \leq |Q_N|.$$

Birkhoff's theorem therefore gives

$$a = \int_K \log(2|x|) d\mu, \quad b = \int_K \log(2|y|) d\mu. \quad (\text{A.8})$$

Uniform comparison of tangent norms on K gives the same exponents for the Fubini–Study metric. Their order and positivity follow from

$$a > \log 198 > \log 26 > b > \log 12 > 0.$$

Each x -sign has mass $1/2$. The band bounds give

$$b > \frac{1}{2} \log 12 + \frac{1}{2} \log 22 = \frac{1}{2} \log 264 > 4 \log 2, \quad a > \log 198 > \frac{15}{2} \log 2.$$

Since $264 > 2^8$ and $198^2 > 2^{15}$,

$$\frac{\log 2}{a} + \frac{\log 2}{b} < \frac{2}{15} + \frac{1}{4} = \frac{23}{60} < \frac{24}{60} = \frac{2}{5} \leq D.$$

This proves Theorem A.1. □

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