

Binder-DeMarco-Dupont Inequality

Abstract

In the setting of endomorphisms of $\mathbb{P}^k(\mathbb{C})$, a dimension formula for the measure of maximal entropy was conjectured by Binder and DeMarco in 2003 ([6]), asserting that

$$\dim_{\text{Hausdorff}}(\mu) = \log d \sum_{i=1}^k \frac{1}{\chi_i},$$

where d is the degree of the (non-invertible) endomorphism and χ_i 's are the Lyapunov exponents with multiplicity. Indeed, this is well-defined as all Lyapunov exponents are known to be positive due to Briend and Duval. The most significant progress is due to Dupont, who in complex-dimension 2, proved the dimension lower-bound holds ([3]); And recently by Genes, who proved the equality for two-dimensional polynomial skew-product maps which satisfy a Lyapunov exponent condition ([10]).

Recently, He and Bianchi ([9]) were able to find a family of simple counter-examples. In Appendix A we present a single “hands on” description of such a counter example, which is given by a quadratic polynomial skew-product.

In this manuscript, we provide the proof, which was achieved by lightly prompting ChatGPT, of extending Dupont's inequality to all complex dimensions. We wish to share this manuscript with the community as we believe that this provides a full picture of the recently settled Binder-DeMarco conjecture. We do not claim ownership of the proof.

1 Statement and notation

Let $M = \mathbb{P}^k(\mathbb{C})$, equipped with the Fubini–Study Kähler form ω and its associated Riemannian metric. Let $f : M \rightarrow M$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. Its equilibrium measure is denoted by μ . Throughout, logarithms are natural and Hausdorff dimensions are *real* dimensions. We use the convention

$$\dim_{\text{H}} \mu = \inf\{\dim_{\text{H}} A : A \subset M \text{ is Borel and } \mu(A) = 1\}.$$

Theorem 1.1. *Let $\lambda_1 \geq \dots \geq \lambda_k > 0$ be the Lyapunov exponents of μ , counted with complex multiplicity. Then*

$$\dim_{\text{H}} \mu \geq (\log d) \sum_{i=1}^k \frac{1}{\lambda_i}. \quad (1.1)$$

We group equal exponents. Write

$$\chi_1 > \chi_2 > \dots > \chi_s > 0, \quad m_1 + \dots + m_s = k, \quad p_j := \sum_{a=1}^j m_a, \quad L := \log d,$$

where m_j is the complex multiplicity of χ_j . Thus $p_s = k$ and

$$\sum_{i=1}^k \lambda_i^{-1} = \sum_{j=1}^s m_j \chi_j^{-1}.$$

This block notation avoids any simplicity assumption on the spectrum.

The standard properties of the equilibrium measure used below are ergodicity,

$$h_\mu(f) = k \log d, \quad \log |\det D_{\mathbb{R}} f| \in L^1(\mu), \quad \lambda_k > 0. \quad (1.2)$$

See, for example, [2, 3]. In particular, the critical set and its iterated preimages have μ -measure zero.

2 The conditional entropy and dimension theorem used

Let $(\widehat{M}, \widehat{f}, \widehat{\mu})$ be the natural extension of (M, f, μ) , and let $\pi : \widehat{M} \rightarrow M$ be the time-zero projection, so that $\pi_* \widehat{\mu} = \mu$. Along a regular inverse history there is a strong unstable flag

$$W^1 \subset W^2 \subset \cdots \subset W^s, \quad \dim_{\mathbb{C}} W^j = p_j.$$

These are plaques in the natural extension; their local spatial projections are embedded holomorphic plaques. Histories are retained throughout: no assertion that the flag descends to a flag on M is needed.

We use the following established part of the endomorphism Ledrappier–Young theory, due to Qian–Xie; see [4], especially its conditional dimension and local entropy statements, and Chapter IX of [5].

Proposition 2.1 (Conditional entropy and dimension framework). *There are partial entropies*

$$0 = H_0 \leq H_1 \leq \cdots \leq H_s = h_\mu(f)$$

and exact conditional dimensions δ_j on the corresponding strong unstable plaques. With $\delta_0 = 0$, they satisfy

$$\delta_j - \delta_{j-1} = \frac{H_j - H_{j-1}}{\chi_j}, \quad 1 \leq j \leq s. \quad (2.1)$$

The dimensions are computed using intrinsic spatial distance on a plaque, equivalently using a smooth local spatial parametrization.

Moreover, partial entropy has the conditional local-entropy description. For a typical conditional measure σ on a W^j -plaque, write

$$B_n^j(y, r) = \{z : \max_{0 \leq t \leq n} d_{W_t^j}(f^t y, f^t z) < r\},$$

intersecting with the conditional-measure domain. Here the forward leaves and their intrinsic metrics retain the prescribed histories. Then, for σ -almost every y ,

$$\lim_{r \downarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \sigma(B_n^j(y, r)) = H_j. \quad (2.2)$$

In particular, after restricting to a set of positive conditional measure, the upper mass bound associated with (2.2) can be made uniform in the centre and in all sufficiently large n .

The hypotheses for this framework are supplied by (1.2); f is smooth on the compact manifold M . There is no requirement of uniform expansion or a global dominated splitting.

The holomorphicity of the local strong plaques follows from the holomorphic graph transform. It can also be read from the triangular normal forms in [1]: fixing all slower blocks gives the fast plaques. After shrinking a local plaque, its parametrization extends holomorphically to a larger parameter polydisc. Only individual plaques will be used below, so all constants may depend on that plaque.

The proof of Theorem 1.1 will follow from

$$H_j \leq p_j L \quad (1 \leq j < s). \quad (2.3)$$

The next two sections prove this estimate.

3 Volume and intrinsic packing of holomorphic orbit graphs

3.1 The orbit-graph volume estimate

Write $\mathbb{D}^p(a) = \{z \in \mathbb{C}^p : \max_i |z_i| < a\}$. Let $\eta : \mathbb{D}^p(2) \rightarrow M$ be a holomorphic embedding whose image is contained in a bounded affine coordinate chart. For $n \geq 1$, define

$$G_n(z) = (\eta(z), f\eta(z), \dots, f^{n-1}\eta(z)) \in M^n.$$

The first component makes G_n an embedding, including at parameter values whose subsequent images are critical points of f . Give M^n the product Kähler form $\Omega_n = \sum_{t=0}^{n-1} \text{pr}_t^* \omega$, and set $\Gamma_n = G_n(\mathbb{D}^p(1))$, with its induced intrinsic metric.

Lemma 3.1 (Orbit-graph volume). *There is a constant C_η , independent of n , such that*

$$\text{Vol}_{2p}(\Gamma_n) \leq C_\eta d^{pn}. \quad (3.1)$$

Proof. Let $T = \omega + dd^c g$ be the Green current of f , where g is continuous and bounded, and $f^*T = dT$. It follows that

$$(f^t)^* \omega = d^t \omega + dd^c(d^t g - g \circ f^t).$$

Set $A_n = \sum_{t=0}^{n-1} d^t$. Choose a smooth local potential ψ for ω on the chart containing the plaque. On $\mathbb{D}^p(2)$, put

$$v_n = \psi \circ \eta + g \circ \eta - \frac{1}{A_n} \sum_{t=0}^{n-1} g \circ f^t \circ \eta.$$

Then

$$dd^c v_n = A_n^{-1} G_n^* \Omega_n = A_n^{-1} \eta^* \sum_{t=0}^{n-1} (f^t)^* \omega \geq 0. \quad (3.2)$$

Thus v_n is plurisubharmonic. On any fixed intermediate polydisc, for instance $\mathbb{D}^p(3/2)$, it is bounded uniformly in n , because g is bounded and $n/A_n \leq 1$.

The Chern–Levine–Nirenberg estimate, applied between $\mathbb{D}^p(1)$ and $\mathbb{D}^p(3/2)$, gives

$$\int_{\mathbb{D}^p(1)} (dd^c v_n)^p \leq C.$$

Equivalently, this follows by p integrations by parts with nested cutoff functions, transferring each dd^c to a cutoff and using the uniform bound on the potentials. Positivity controls the intermediate terms. This is the bounded-potential mechanism used in the growth lemma of [3, Appendix]; see also [2].

Since the Riemannian volume form of a complex p -dimensional Kähler manifold is its Kähler form to the p -th power divided by $p!$, (3.2) implies

$$\text{Vol}_{2p}(\Gamma_n) = \frac{A_n^p}{p!} \int_{\mathbb{D}^p(1)} (dd^c v_n)^p \leq C' A_n^p \leq C_\eta d^{pn}.$$

□

3.2 A uniform lower volume bound for intrinsic balls

The next point is necessary because partial entropy uses leafwise intrinsic distances. An ambient packing estimate alone does not resolve that issue.

Lemma 3.2 (Intrinsic balls in the orbit graph). *Fix a compact set $K \Subset \mathbb{D}^p(1)$. There exist $r_* > 0$ and $c > 0$, independent of n , such that*

$$\text{Vol}_{2p}(B_{\Gamma_n}(G_n(z), r)) \geq cr^{2p} \quad (z \in K, 0 < r < r_*). \quad (3.3)$$

Proof. Every Γ_n is a complex submanifold of the Kähler manifold M^n , hence is minimal. The product metrics on M^n have a sectional-curvature upper bound independent of n , and their injectivity radii have a positive lower bound independent of n .

There is also a uniform boundary margin for the centres in question. Indeed,

$$a := \text{dist}_M(\eta(K), \eta(\partial\mathbb{D}^p(1))) > 0.$$

The length of every graph path is at least the length of its first coordinate projection. Consequently, the intrinsic graph distance from $G_n(K)$ to the boundary of Γ_n is at least a .

Here is the intrinsic monotonicity argument, including its dependence on constants. Write $m = 2p$, fix a centre $q = G_n(z)$, and let R be ambient product distance from q . For sufficiently small r , an intrinsic ball of radius r lies in the ambient ball of radius r . Hessian comparison for the squared ambient distance, together with minimality, gives

$$\Delta_{\Gamma_n} R^2 \geq 2m(1 - Cr^2) \quad \text{on } B_{\Gamma_n}(q, r).$$

The constant C depends on the uniform ambient curvature bound and is independent of n . Also $|\nabla_{\Gamma_n} R^2| \leq 2r$ there.

Let $V(r) = \text{Vol}_m(B_{\Gamma_n}(q, r))$. Divergence and coarea, valid for almost every radius (or by approximation across the intrinsic cut locus), imply

$$2m(1 - Cr^2)V(r) \leq 2rV'(r).$$

Thus

$$\frac{d}{dr} \log \frac{V(r)}{r^m} \geq -mCr.$$

At each smooth centre, $\lim_{r \downarrow 0} V(r)/r^m = \omega_m$, the Euclidean unit-ball volume. Integrating the last inequality gives

$$V(r) \geq \omega_m r^m \exp(-mCr^2/2).$$

Choose r_* smaller than the uniform ambient comparison radius and the boundary margin a . This proves the lemma. $\square$

3.3 Packing with the leafwise dynamical metric

Suppose now that η parametrizes a local strong unstable plaque, with its fixed inverse history. Let W_t denote its forward leaf at time t , endowed with intrinsic spatial distance. For $z, w \in \mathbb{D}^p(1)$, write

$$\rho_n(z, w) := \max_{0 \leq t < n} d_{W_t}(f^t \eta(z), f^t \eta(w)).$$

Distances are taken in the parametrized leaves, retaining histories when a spatial projection has several sheets.

Lemma 3.3 (Leafwise dynamical packing). *For every $K \Subset \mathbb{D}^p(1)$ and every sufficiently small fixed $r > 0$, there exists $C_{\eta, K, r} < \infty$, independent of n , such that every ρ_n -separated set $E \subset K$ at separation r satisfies*

$$\#E \leq C_{\eta, K, r} d^n. \quad (3.4)$$

Proof. For any piecewise smooth parameter path γ joining z to w , its image under $f^t \circ \eta$ is a path in the corresponding forward leaf. The product metric therefore gives

$$\text{length}(G_n \circ \gamma) \geq \text{length}(f^t \circ \eta \circ \gamma) \geq d_{W_t}(f^t \eta(z), f^t \eta(w))$$

for every $t < n$. Taking an infimum over paths gives

$$d_{\Gamma_n}(G_n(z), G_n(w)) \geq \rho_n(z, w). \quad (3.5)$$

If a forward parametrization has critical points, lengths may be computed by approximation with paths avoiding the finite union of proper analytic critical loci. This leaves the inequality unchanged; the orbit graph itself is always nonsingular because of its first coordinate.

For $z \in E$, the intrinsic graph balls of radius $r/3$ centred at $G_n(z)$ are pairwise disjoint by (3.5). Lemmas 3.1 and 3.2 yield

$$(\#E)c(r/3)^{2p} \leq \text{Vol}_{2p}(\Gamma_n) \leq C_\eta d^{pn}.$$

This proves (3.4). $\square$

4 Cumulative partial entropies

Proposition 4.1. *For every level of the strong unstable flag,*

$$H_j \leq p_j \log d. \quad (4.1)$$

Proof. Fix j and set $p = p_j$. Choose a typical conditional measure σ on a W^j -plaque. By covering its spatial parametrization with countably many relatively compact holomorphic coordinate patches, choose one embedded patch $\eta : \mathbb{D}^p(2) \rightarrow M$ and one compact inner subset on which σ has positive mass. The coordinates may be rescaled so that the patch satisfies the hypotheses of the preceding section. The conditional measure is always kept on its prescribed lifted plaque; notation suppresses that lift.

Let $\varepsilon > 0$. By (2.2), there exist a small $r > 0$, an integer N , and a measurable subset F of the chosen inner patch with $\sigma(F) > 0$, such that

$$\sigma(B_n^j(y, r)) \leq e^{-n(H_j - \varepsilon)} \quad (y \in F, n \geq N). \quad (4.2)$$

To obtain this uniformity, first choose one radius from a countable sequence tending to zero for which the lower entropy rate exceeds $H_j - \varepsilon/2$ on a positive-mass set. Then choose a uniform lower bound on n , losing at most part of that positive mass. Restricting or normalizing the conditional measure introduces only a fixed multiplicative constant, which has no entropy cost.

For each $n \geq N$, select a maximal subset $E_n \subset F$ separated at distance $r/2$ for the intrinsic leafwise dynamical metric. Lemma 3.3 bounds the cardinality of every such separated set and hence ensures that the selection terminates after finitely many points. Maximality gives a cover of F by the Bowen balls of radius r centred at E_n . It follows from (4.2) that

$$0 < \sigma(F) \leq \sum_{y \in E_n} \sigma(B_n^j(y, r)) \leq C_{\eta, r} d^{pn} e^{-n(H_j - \varepsilon)}.$$

Taking logarithms, dividing by n , and letting $n \rightarrow \infty$ gives $H_j \leq p \log d + \varepsilon$. Let $\varepsilon \downarrow 0$. $\square$

Remark 4.2. This is a bound for the cumulative entropy H_j , not a bound $H_j - H_{j-1} \leq m_j \log d$ on every increment. The latter is not used. Nor is any equality $H_j = p_j \log d$ asserted.

5 Disintegration and the final dimension estimate

We first record the measure-theoretic step which avoids any need to identify a full-unstable conditional with the spatial measure.

Lemma 5.1 (Projection of full-measure sets). *Let ξ be a measurable partition of $\widehat{M}$, with conditional probabilities $\widehat{\mu}_{\xi(\widehat{x})}$. Suppose that their spatial projections*

$$\nu_{\widehat{x}} := \pi_* \widehat{\mu}_{\xi(\widehat{x})}$$

have Hausdorff dimension at least D for $\widehat{\mu}$ -almost every $\widehat{x}$. Then $\dim_{\text{H}} \mu \geq D$.

Proof. Let $A \subset M$ be Borel with $\mu(A) = 1$. Since $\widehat{\mu}(\pi^{-1}A) = 1$, disintegration implies

$$\widehat{\mu}_{\xi(\widehat{x})}(\pi^{-1}A) = 1 \quad \text{for almost every } \widehat{x}.$$

Thus $\nu_{\widehat{x}}(A) = 1$ for almost every $\widehat{x}$. The definition of the Hausdorff dimension of a measure yields $\dim_{\text{H}} A \geq \dim_{\text{H}} \nu_{\widehat{x}} \geq D$. Taking the infimum over such A proves the claim. $\square$

Proof of Theorem 1.1. Take ξ subordinate to the local full unstable plaques. The spatial projection is a smooth local parametrization on each plaque; after restriction to relatively compact coordinate patches it is bi-Lipschitz. The conditional dimension statement in Proposition 2.1 therefore gives

$$\dim_{\text{H}} \nu_{\widehat{x}} = \delta_s = \sum_{j=1}^s \frac{H_j - H_{j-1}}{\chi_j} \quad \text{for almost every } \widehat{x}.$$

If necessary, exhaust a plaque by countably many such relatively compact patches; this does not change its almost-everywhere local dimension. Lemma 5.1 implies

$$\dim_{\text{H}} \mu \geq \delta_s. \tag{5.1}$$

Summation by parts in (2.1) gives

$$\delta_s = \frac{H_s}{\chi_s} - \sum_{j=1}^{s-1} H_j \left(\frac{1}{\chi_{j+1}} - \frac{1}{\chi_j} \right).$$

Similarly, since $p_j = \sum_{a \leq j} m_a$,

$$L \sum_{j=1}^s \frac{m_j}{\chi_j} = \frac{kL}{\chi_s} - \sum_{j=1}^{s-1} p_j L \left(\frac{1}{\chi_{j+1}} - \frac{1}{\chi_j} \right).$$

Using $H_s = kL$, subtract the two identities to obtain

$$\delta_s - L \sum_{j=1}^s \frac{m_j}{\chi_j} = \sum_{j=1}^{s-1} (p_j L - H_j) \left(\frac{1}{\chi_{j+1}} - \frac{1}{\chi_j} \right) \geq 0. \tag{5.2}$$

Every summand is nonnegative by Proposition 4.1 and the ordering of the exponents. Combining (5.2) with (5.1) proves (1.1). $\square$

Remark 5.2 (Scope of the argument). The conclusion is the Hausdorff lower bound in Theorem 1.1. The argument does not assert exact dimensionality of the spatial measure, an equality in (1.1), or equality between a plaque conditional and μ . It also does not assume that a holomorphic slice is invariant, or that different histories have the same fast flag. In complex dimension two there is one nontrivial cumulative partial entropy; in higher dimensions the same orbit-graph argument is applied at every level of the flag.

A A quadratic example with a strict dimension lower bound

The proof below is independent of the general argument in the preceding sections.

Binder–DeMarco’s Conjecture 1.3 [6, p. 615] asserts, in degree two and complex dimension two, that $\dim_{\mathbb{H}} \mu = \log 2/a + \log 2/b$, where $a \geq b$ are the two complex Lyapunov exponents, counted once each. We continue to use natural logarithms and the full-measure convention for $\dim_{\mathbb{H}} \mu$ stated in Section 1.

Theorem A.1. *The holomorphic quadratic endomorphism*

$$f[Z : W : T] = [Z^2 - 10000T^2 : W^2 - 100T^2 + \tfrac{1}{2}ZT : T^2] \quad (\text{A.1})$$

has equilibrium measure μ , complex Lyapunov exponents $a > b > 0$, and dimension $D = \dim_{\mathbb{H}} \mu$ satisfying

$$\boxed{D \geq \frac{2}{5} > \frac{23}{60} > \frac{\log 2}{a} + \frac{\log 2}{b}.} \quad (\text{A.2})$$

In particular, its discrepancy from the conjectured value is greater than $1/60$.

The proof uses exact inequalities throughout. Its dynamical input from the literature is the following standard result: a holomorphic endomorphism of $\mathbb{P}^k(\mathbb{C})$ of algebraic degree $d \geq 2$ has topological entropy $k \log d$, and its equilibrium measure is its unique measure of maximal entropy [7, p. 146, Theorem 2 and the preceding paragraph]. The corrected multiplicity argument is in [8]. The remaining ingredients are elementary estimates, the entropy of a Bernoulli shift, and Birkhoff’s ergodic theorem.

A.1 The invariant set and its equilibrium measure

The three homogeneous coordinates in (A.1) have no common zero outside the origin: the last forces $T = 0$, and the first two then force $Z = W = 0$. They therefore define a holomorphic endomorphism of algebraic degree two. In the affine chart $T = 1$,

$$f(x, y) = (x^2 - 10000, y^2 - 100 + x/2).$$

Regard $Q = [-101, 101] \times [-13, 13]$ as a real rectangle in $\mathbb{C}^2$. For $s, t \in \{-1, 1\}$, define an inverse branch on Q by

$$g_{s,t}(u, v) = (x, y), \quad x = s\sqrt{10000 + u}, \quad y = t\sqrt{100 + v - x/2}. \quad (\text{A.3})$$

Indeed, $f \circ g_{s,t} = \text{id}$. Since

$$99^2 < 9899 \leq 10000 + u \leq 10101 < 101^2,$$

every output satisfies $99 < |x| < 101$. If $s = 1$, then $73/2 < y^2 < 127/2$, so $6 < |y| < 8$. If $s = -1$, then $273/2 < y^2 < 327/2$, so $11 < |y| < 13$. Thus the four images lie in the interior of Q , in the four disjoint second-coordinate bands

$$(-13, -11), \quad (-8, -6), \quad (6, 8), \quad (11, 13). \quad (\text{A.4})$$

Points in different images have y -coordinates differing by more than 3.

Differentiating (A.3), and using $|2x| > 198$ and $|2y| > 12$, gives the ordinary sup-norm bound

$$\|Dg_{s,t}\|_{\infty} \leq \max \left\{ \frac{1}{198}, \frac{1}{12} + \frac{1}{4752} \right\} = \frac{397}{4752} < \frac{1}{11}.$$

Convexity of Q makes this a global Lipschitz bound. For each infinite word $\omega = (\omega_0, \omega_1, \dots)$ on the four branch symbols, the nested compact sets

$$g_{\omega_0} \circ \dots \circ g_{\omega_{n-1}}(Q)$$

have diameters tending to zero, hence have a unique common point $\pi(\omega)$. The coding map is continuous. It is injective because, after removing the common prefix of two different words, their points lie in different bands; every inverse prefix is injective. Consequently π is a homeomorphism from the compact one-sided four-symbol shift onto

$$K = \pi((\{-1, 1\}^2)^{\mathbb{N}}).$$

Moreover, $f \circ \pi = \pi \circ \sigma$, where σ deletes the first symbol.

Push the uniform Bernoulli probability forward by π , obtaining ν . This measure is invariant and ergodic, has entropy $\log 4$, and assigns mass 4^{-n} to each length- n cylinder

$$K_{\omega_0 \dots \omega_{n-1}} = g_{\omega_0} \circ \dots \circ g_{\omega_{n-1}}(K).$$

Since the ambient endomorphism has topological entropy $\log 4$, the quoted uniqueness theorem identifies ν with its equilibrium measure μ . In particular, $\mu(K) = 1$.

A.2 Cylinder separation and the dimension lower bound

Consider two distinct inputs in Q with $|\Delta u| \leq 68|\Delta v|$, and apply the same inverse branch. Write the outputs as (x, y) and (x', y') . Their corresponding coordinate signs agree, and the exact difference identities are

$$\Delta x = \frac{\Delta u}{x + x'}, \quad \Delta y = \frac{\Delta v - \Delta x/2}{y + y'}.$$

The input cone and distinctness imply $\Delta v \neq 0$. Since $|x + x'| > 198$ and $|y + y'| < 26$, it follows that

$$|\Delta y| \geq \frac{1 - 68/396}{26} |\Delta v| > \frac{1}{32} |\Delta v|. \quad (\text{A.5})$$

The last inequality is exact:

$$\frac{1 - 68/396}{26} - \frac{1}{32} = \frac{25}{41184} > 0.$$

Furthermore,

$$|\Delta x| \leq \frac{68}{198} |\Delta v| < \frac{32 \cdot 68}{198} |\Delta y| < 68 |\Delta y|.$$

Thus a common inverse branch preserves the cone and decreases the y -distance by a factor no smaller than $1/32$.

For points of K in different first-level cylinders,

$$|\Delta x| < 202 < 3 \cdot 68 < 68 |\Delta y|, \quad |\Delta y| > 3.$$

Now take points in different length- n cylinders, and let $m < n$ be the length of their common prefix. After applying f^m , the points occupy different first-level cylinders. Restore the prefix, using cone preservation and (A.5) at each step. Their original y -distance, for every such pair, is greater than

$$3 \cdot 32^{-m} \geq 3 \cdot 32^{-(n-1)}.$$

For any $0 < r \leq 1$, choose $n \geq 1$ such that $32^{-n} \leq r \leq 32^{-(n-1)}$. An ambient Euclidean ball of radius r has second-coordinate diameter at most $2r$, less than the separation just proved. It therefore meets at most one length- n cylinder, and

$$\mu(B(z, r)) \leq 4^{-n} = (32^{-n})^{2/5} \leq r^{2/5}. \quad (\text{A.6})$$

The centre z may be arbitrary; open or closed balls give the same bound.

For completeness, (A.6) bounds the dimension of the measure itself. If E is any Borel set of full μ -measure, replace it by $E \cap K$. A countable cover of this set by nonempty sets of diameters $d_i \leq 1$ gives, by enclosing each set in a ball of radius d_i and using outer measure,

$$1 \leq \sum_i d_i^{2/5}.$$

Zero-diameter sets have zero mass by (A.6). Thus $\mathcal{H}^{2/5}(E \cap K) > 0$, up to the harmless normalization constant in Hausdorff measure, and $\dim_{\text{H}} E \geq 2/5$. Euclidean and Fubini–Study distances are bi-Lipschitz equivalent on the compact affine rectangle Q ; hence this conclusion holds for the required metric. Taking the infimum over full-measure E gives

$$D \geq \frac{2}{5}. \quad (\text{A.7})$$

A.3 The complex Lyapunov exponents and the strict gap

Along K the complex derivative is

$$Df(x, y) = \begin{pmatrix} 2x & 0 \\ 1/2 & 2y \end{pmatrix}, \quad 198 < |2x| < 202, \quad 12 < |2y| < 26.$$

For $f^j(p) = (x_j, y_j)$, write

$$Df^n(p) = \begin{pmatrix} P_n & 0 \\ E_n & Q_n \end{pmatrix}, \quad P_n = \prod_{j=0}^{n-1} 2x_j, \quad Q_n = \prod_{j=0}^{n-1} 2y_j.$$

With $r_0 = 0$ and $r_n = E_n/P_n$, matrix multiplication gives

$$r_{n+1} = \frac{1}{4x_n} + \frac{y_n}{x_n} r_n, \quad |r_n| \leq \frac{1/396}{1 - 26/198} = \frac{1}{344}, \quad \left| \frac{Q_n}{P_n} \right| \leq (26/198)^n.$$

Let $\sigma_1(n) \geq \sigma_2(n) > 0$ be the two complex singular values. For $M = \sqrt{2 + 344^{-2}}$, the first column and the Frobenius norm give

$$|P_n| \leq \sigma_1(n) \leq M|P_n|.$$

Since $\sigma_1(n)\sigma_2(n) = |\det Df^n| = |P_n Q_n|$,

$$M^{-1}|Q_n| \leq \sigma_2(n) \leq |Q_n|.$$

These constants do not depend on p or n . Birkhoff's theorem therefore identifies the Lyapunov exponents, for μ -almost every point, as

$$a = \int_K \log(2|x|) d\mu, \quad b = \int_K \log(2|y|) d\mu. \quad (\text{A.8})$$

Uniform comparison of tangent norms on K gives the same exponents for the Fubini–Study metric. Their order and positivity follow from

$$a > \log 198 > \log 26 > b > \log 12 > 0.$$

The first symbolic x -sign is positive on two of the four branches and negative on the other two, each of mass $1/4$. Thus each sign has total mass $1/2$. The sharper band bounds in (A.4) and (A.8) yield

$$b > \frac{1}{2} \log 12 + \frac{1}{2} \log 22 = \frac{1}{2} \log 264 > 4 \log 2, \quad a > \log 198 > \frac{15}{2} \log 2.$$

Here $264 > 2^8 = 256$ and $198^2 = 39204 > 2^{15} = 32768$; no decimal logarithm evaluation is involved. Consequently

$$\frac{\log 2}{a} + \frac{\log 2}{b} < \frac{2}{15} + \frac{1}{4} = \frac{23}{60} < \frac{24}{60} = \frac{2}{5} \leq D.$$

This proves Theorem A.1, including the strict discrepancy greater than $1/60$. $\square$

The strict counterexample is established without computing D exactly or assuming any general exact-dimensionality formula. Every estimate needed for the conclusion is included above.

References

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