

The third unit in the adjoint tower of group involutions

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Abstract

The functor $G \mapsto \text{Hom}(C_2, G)$ determines an adjoint tower through successive Eilenberg–Moore comparison functors. We prove that the unit of the third adjunction need not be an isomorphism, answering a question of Bergman. For the generalized dihedral group $H = (C_4 \times C_4) \rtimes C_2$, with inversion action, we compute $F_3 V_3 H \cong H \times C_2$, with counit the first projection. The unit at $V_3 H$ is injective, with underlying domain and codomain of cardinalities 20 and 40. For an arbitrary group G , let $E(G) = F_2 V_2 G$ and let K_G be the kernel of the second counit. We prove that K_G is central and that $F_3 V_3 G \cong E(G)/K_G[2]$, where $K_G[2]$ consists of the elements of K_G whose square is one. The third unit at $V_3 G$ is an isomorphism precisely when K_G has no element of order four. A central extension of H of order 128 supplies such an element and yields the stated description of $F_3 V_3 H$.

Keywords. Adjoint tower, monadic decomposition, involutory quandle, central extension, generalized dihedral group.

1 Introduction

For a group G , put

$$\Omega(G) = \{g \in G : g^2 = 1\}.$$

We use *involution* for an element of order two, so $\Omega(G)$ consists of the identity and the involutions of G . Evaluation at the nonidentity element of C_2 identifies $\Omega(G)$ with $\text{Hom}_{\mathbf{Grp}}(C_2, G)$. Consider the functor

$$V_1 : \mathbf{Grp} \longrightarrow \mathbf{Set}, \quad V_1 G = \Omega(G),$$

and its left adjoint F_1 . Form the monad and its Eilenberg–Moore category

$$M_1 = V_1 F_1, \quad \mathcal{I}_1 = \mathbf{Set}^{M_1},$$

and let $F_2 \dashv V_2$ be the adjunction associated with the comparison functor $V_2 : \mathbf{Grp} \rightarrow \mathcal{I}_1$. Repeating the construction gives

$$M_2 = V_2 F_2, \quad \mathcal{I}_2 = \mathcal{I}_1^{M_2}, \quad F_3 \dashv V_3, \quad V_3 : \mathbf{Grp} \rightarrow \mathcal{I}_2.$$

Write η_i and ε_i for the units and counits. The forgetful functors are $U_1 : \mathcal{I}_1 \rightarrow \mathbf{Set}$ and $U_2 : \mathcal{I}_2 \rightarrow \mathcal{I}_1$. The left adjoints F_2 and F_3 exist because $\mathbf{Grp}$ has coequalizers; the construction is recalled in [Lemma 3.1](#).

The adjoint tower originates in the work of Applegate and Tierney [2]. We follow Bergman's indexing [4, pp. 290–292]; a recent treatment of iterated comparison constructions is given by Ardizzoni and Menini [3]. In his 1974 ICM lecture, published in 1975, Bergman asked whether

$$\eta_{3,A}: A \longrightarrow V_3 F_3 A$$

is an isomorphism for every $A \in \mathcal{I}_2$ [4, p. 292]. Equivalently, he asked whether F_3 is fully faithful, so that it identifies $\mathcal{I}_2$ with a full coreflective subcategory of **Grp**.

Theorem 1.1. *Let*

$$H = (C_4 \times C_4) \rtimes C_2,$$

where the generator of C_2 acts by inversion. There is an isomorphism

$$F_3 V_3 H \cong H \times C_2$$

under which the counit $\varepsilon_{3,H}$ is the first projection. The underlying set map of $\eta_{3,V_3 H}$ is injective, with domain of cardinality 20 and codomain of cardinality 40. Hence η_3 is not a natural isomorphism, and F_3 is not fully faithful.

The proof uses a description of the third unit on comparison objects. For any group G , set

$$E(G) = F_2 V_2 G, \quad \epsilon_G = \varepsilon_{2,G}: E(G) \rightarrow G, \quad K_G = \ker \epsilon_G.$$

The group K_G is central. For an abelian group K , write $K[m] = \{k \in K : k^m = 1\}$. We prove natural isomorphisms

$$F_3 V_3 G \cong E(G)/K_G[2] \tag{1.1}$$

in **Grp**, and

$$\Omega(E(G)/K_G[2]) \cong \Omega(G) \times K_G[4]/K_G[2] \tag{1.2}$$

in the category of unital keis. The second factor has the trivial operation $b \triangleright c = c$ and distinguished element 0. In these coordinates $U_2 \eta_{3,V_3 G}$ is $t \mapsto (t, 0)$. Thus an element of order four in K_G is exactly the obstruction to the third unit being an isomorphism at $V_3 G$.

To obtain the counterexample, we construct a group P of order 128 and a central quotient $P \rightarrow H$ with cyclic kernel of order four. The map on the sets of elements whose square is one has a section that preserves the identity and conjugation, and whose image generates P . The universal property of $E(H)$ therefore gives a surjection $E(H) \rightarrow P$. Since H has nilpotency class two, $E(H)$ has exponent dividing four; the chosen generator of the kernel of $P \rightarrow H$ then lifts to an element of order four in K_H . We then prove $E(H) \cong P \times C_2^3$ over H and compute the quotient in (1.1).

The first comparison category is the variety of involutory quandles with a distinguished unit, also called unital keis. This description occurs in Bergman's discussion of algebraic structure [5, §10.11] and in [14, Remark 2.14]. For a nonempty set of generators, the free kei is the union of their conjugacy classes in the corresponding free product of groups of order two; see [11, Corollary 10.3] and [14, Propositions 2.11 and 2.13]. We give the free-object argument, identify the monad, and derive (1.1) from the universal property of the comparison adjunction.

The group $E(G)$ also has established descriptions in group and quandle theory. Its presentation is the active sum of the cyclic subgroups of order two in G [7, 13]. On the quandle side, it is obtained by imposing generator-square relations in an associated group and then setting the identity label equal to one. Hasegawa [9] studies the former quotient for involutive quandles, and Akita and Shikata [1] treat it within the theory of symmetric quandles. Related central-extension descriptions for structure groups of conjugation quandles appear in [12].

For the group H in [Theorem 1.1](#), the noncentral involutions form the Takasaki quandle on $(\mathbb{Z}/4)^2$, with operation $x \triangleright y = 2x - y$. It has four orbits. Associated groups of Alexander quandles were computed by Clauwens [6] under an invertibility hypothesis and by Ishikawa [10] in general. Together with the results of Hasegawa and Akita–Shikata, Ishikawa’s formula gives the order 128 of the involutory associated group in this example. [Remark 5.4](#) explains this specialization. The construction below determines the group explicitly, identifies the kernel of its map to H , and calculates the third unit.

[Section 2](#) identifies $\mathcal{I}_1$ and constructs the enveloping group. [Section 3](#) proves (1.1)–(1.2). [Section 4](#) constructs a central extension of H and obtains an element of order four in K_H . [Section 5](#) computes $E(H)$, the third counit, and the third unit. The appendices give coordinates and describe the finite computation. Throughout, commutators are $[x, y] = x^{-1}y^{-1}xy$, and kei operations use left translations.

2 Unital keis and enveloping groups

Conjugation and the identity element give $\Omega(G)$ the structure needed for the first comparison category. We identify this category with **UKei** and describe the left adjoint F_2 to $V_2: \mathbf{Grp} \rightarrow \mathbf{UKei}$ by a group presentation.

2.1 The first comparison category

Definition 2.1. A *unital kei* is a set X with an element 1_X and a binary operation $\triangleright$ satisfying

$$x \triangleright x = x, \quad x \triangleright (x \triangleright y) = y, \quad (2.1)$$

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z), \quad (2.2)$$

$$1_X \triangleright x = x, \quad x \triangleright 1_X = 1_X. \quad (2.3)$$

A morphism preserves the operation and the distinguished element. We denote this category by **UKei**.

The operation on $\Omega(G)$ is

$$s \triangleright t = sts,$$

with distinguished element 1_G . The left translation $L_x(y) = x \triangleright y$ in a unital kei is an automorphism satisfying $L_x^2 = \text{id}$. It fixes 1_X , and every automorphism α satisfies $\alpha L_x \alpha^{-1} = L_{\alpha(x)}$.

Lemma 2.2. For a set S , let

$$D(S) = \bigast_{s \in S} C_2 = \langle s \mid s \in S \mid s^2 = 1 \rangle.$$

Then $\Omega(D(S))$, with its canonical map from S , is the free unital kei on S .

Proof. If S is empty, then $D(S)$ is trivial and $\Omega(D(S))$ is the one-element unital kei, which is initial. Suppose that S is nonempty. Reduced words in the free product give two facts. Every element of order two is conjugate to a distinguished generator, and distinct distinguished generators are not conjugate. Indeed, a cyclically reduced word of length at least two has infinite order. Also

$$C_{D(S)}(s) = \langle s \rangle.$$

To see the latter assertion, write a word w as vs^ϵ , where v is empty or its last letter differs from s . Then $wsww^{-1} = vsv^{-1}$ is reduced of length $2|v| + 1$, unless v is empty. The same argument distinguishes the conjugacy classes of different generators.

Let X be a unital kei and $f: S \rightarrow X$ a function. Since $L_{f(s)}^2 = \text{id}$, there is a homomorphism

$$\rho: D(S) \longrightarrow \text{Aut}(X), \quad s \longmapsto L_{f(s)}.$$

Define

$$\tilde{f}(1) = 1_X, \quad \tilde{f}(wsww^{-1}) = \rho(w)f(s).$$

If $wsww^{-1} = vtv^{-1}$, then $s = t$ and $v^{-1}w \in \langle s \rangle$. As $L_{f(s)}$ fixes $f(s)$, the definition is independent of the chosen expression.

For $g \in D(S)$ and $q \in \Omega(D(S))$, it gives

$$\tilde{f}(gqg^{-1}) = \rho(g)\tilde{f}(q).$$

If $p = wsww^{-1}$, then

$$\rho(p) = \rho(w)L_{f(s)}\rho(w)^{-1} = L_{\tilde{f}(p)}.$$

It follows that $\tilde{f}(pqp) = \tilde{f}(p) \triangleright \tilde{f}(q)$. This identity also holds for $p = 1$. Thus $\tilde{f}$ is a unital-kei morphism. Every involution of $D(S)$ is an iterated conjugate of generators, so this extension is unique. $\square$

The free construction also identifies the algebra structure on V_2G . This requires compatibility with the monad multiplication, as well as the universal property of the free object.

Proposition 2.3. *The monad $M_1 = V_1F_1$ is the monad of free unital keis. Consequently, $\mathcal{I}_1$ is equivalent to $\mathbf{UKei}$, and V_2G is identified with the unital kei $\Omega(G)$.*

Proof. The first left adjoint is $F_1S = D(S)$, so Lemma 2.2 identifies the underlying free-object functors and their units. The multiplication of M_1 is induced by the homomorphism

$$D(\Omega(D(S))) \longrightarrow D(S)$$

which evaluates the generator indexed by q at q . Restricting this homomorphism to elements whose square is one gives free-kei evaluation. The identifications therefore preserve monad multiplication. Algebras for this term monad are exactly the sets with operations satisfying (2.1)–(2.3). On the comparison algebra V_2G , evaluation is induced by the group counit. Its distinguished element is therefore 1_G , and its operation is $s \triangleright t = sts$. $\square$

2.2 The enveloping group

For a unital kei X , define

$$\mathcal{E}(X) = \left\langle \widehat{x} \ (x \in X) \mid \widehat{1_X} = 1, \quad \widehat{x}^2 = 1, \quad \widehat{xy} = \widehat{x}\widehat{y}\widehat{x} \right\rangle. \quad (2.4)$$

The hat distinguishes a labelled generator from the element labelling it.

Proposition 2.4. *The functor $\mathcal{E}: \mathbf{UKei} \rightarrow \mathbf{Grp}$ is left adjoint to $G \mapsto \Omega(G)$. Thus $F_2X = \mathcal{E}(X)$.*

Proof. A homomorphism $\mathcal{E}(X) \rightarrow G$ is specified by the images of its labelled generators. The relations in (2.4) say exactly that these images define a unital-kei morphism $X \rightarrow \Omega(G)$. This correspondence is natural. $\square$

For a group G , we henceforth write

$$E(G) = \langle \widehat{t} \ (t \in \Omega(G)) \mid \widehat{1} = 1, \quad \widehat{t}^2 = 1, \quad \widehat{sts} = \widehat{st\widehat{s}} \rangle. \quad (2.5)$$

The counit $\epsilon_G: E(G) \rightarrow G$ sends $\widehat{t}$ to t . Its image is the subgroup generated by $\Omega(G)$.

Lemma 2.5. *For every $g \in E(G)$ and $t \in \Omega(G)$,*

$$g\widehat{t}g^{-1} = \epsilon_G(g)\widehat{t\epsilon_G(g)^{-1}}. \quad (2.6)$$

In particular,

$$\epsilon_G^{-1}(Z(G)) \subseteq Z(E(G)), \quad K_G \subseteq Z(E(G)).$$

Proof. Equation (2.6) is a defining relation when g is a labelled generator. These generators square to one and generate $E(G)$, so induction on word length proves the equation for every g . If $\epsilon_G(g)$ is central, the equation says that g commutes with every generator. $\square$

3 The third comparison unit

The next comparison adjunction imposes relations between elements of $E(G)$ whose squares are one and the labelled generators with the same image in G . Centrality of K_G makes these relations explicit.

3.1 The comparison coequalizer

Let $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ be an adjunction, with $L \dashv R$, and assume that $\mathcal{D}$ has coequalizers. Denote the unit and counit of the adjunction by η and ε . Its monad is $T = RL$, with multiplication $\mu = R\varepsilon L$. A T -algebra is a pair (X, a) with $a: TX \rightarrow X$ satisfying $a\eta_X = \text{id}_X$ and $aTa = a\mu_X$. A morphism $g: (X, a) \rightarrow (Y, b)$ satisfies $ga = bTg$. Let

$$J: \mathcal{D} \longrightarrow \mathcal{C}^T, \quad JD = (RD, R\varepsilon_D)$$

be the comparison functor.

Lemma 3.1. *The functor J has a left adjoint L' , given on a T -algebra (X, a) by*

$$LTX \xrightarrow[\varepsilon_{LX}]{La} LX \xrightarrow{q} L'(X, a). \quad (3.1)$$

The underlying $\mathcal{C}$ -map of the unit is $Rq \circ \eta_X$.

Proof. A morphism $f: LX \rightarrow D$ factors through the displayed coequalizer precisely when $fLa = f\varepsilon_{LX}$. Under adjunction, write $g = Rf \circ \eta_X: X \rightarrow RD$. The transposes of these two composites are respectively

$$ga, \quad R\varepsilon_D \circ Tg.$$

For the second equality, the triangle identity gives the transpose Rf , while naturality of ε gives

$$R\varepsilon_D \circ Tg = Rf \circ R(\varepsilon_{LX} \circ L\eta_X) = Rf.$$

Equality is exactly the condition that g be a T -algebra morphism $(X, a) \rightarrow JD$. This gives the adjunction, and the transpose of q gives the stated unit. $\square$

Proposition 3.2. *For every group G , there is an isomorphism, natural in G ,*

$$F_3V_3G \cong E(G)/K_G[2]. \quad (3.2)$$

The counit $\varepsilon_{3,G}$ is induced by ϵ_G . The morphism $U_2\eta_{3,V_3G}$ is given by

$$\Omega(G) \longrightarrow \Omega(E(G)/K_G[2]), \quad t \longmapsto \widehat{t}K_G[2]. \quad (3.3)$$

Proof. Apply Lemma 3.1 to $L = F_2$, $R = V_2$, and the comparison algebra $V_3G = (V_2G, V_2\epsilon_G)$. The middle group is $E = E(G)$, and the left group is $E(E)$. On the generator indexed by $u \in \Omega(E)$, the two maps take the respective values

$$\widehat{\epsilon_G(u)}, \quad u.$$

Thus F_3V_3G is the quotient of E by the normal subgroup generated by

$$\widehat{u\epsilon_G(u)}^{-1} \quad (u \in \Omega(E)).$$

Each such element d lies in $K = K_G$, which is central by Lemma 2.5. Since $u = \widehat{d\epsilon_G(u)}$ and both u and the labelled generator square to one, $d^2 = 1$. All the relations therefore belong to $K[2]$.

Conversely, every $k \in K[2]$ is in $\Omega(E)$, and its corresponding relation is

$$\widehat{k\epsilon_G(k)}^{-1} = k\widehat{1}^{-1} = k.$$

The normal subgroup is exactly $K[2]$, proving (3.2). The map ϵ_G coequalizes the two arrows and is the $F_2 \dashv V_2$ transpose of id_{V_2G} . Its induced factor therefore corresponds, under the adjunction in Lemma 3.1, to id_{V_3G} , and is $\varepsilon_{3,G}$. The unit formula follows from the last assertion of Lemma 3.1 and the unit $t \mapsto \widehat{t}$ of $F_2 \dashv V_2$. For a homomorphism $f: G \rightarrow G'$, the map $E(f)$ sends $\widehat{t}$ to $\widehat{f(t)}$ and maps $K_G[2]$ into $K_{G'}[2]$. The quotient isomorphism therefore commutes with the maps induced by f . $\square$

3.2 The obstruction from elements of order four

Quotienting by $K_G[2]$ can create new elements whose square is one. To describe them, write a lift of an element of $\Omega(G)$ as its labelled generator times an element of the central kernel.

For an abelian group K , the quotient $K[4]/K[2]$ is an elementary abelian two-group. We regard it as a unital kei with distinguished element $0 = K[2]$ and operation $b \triangleright c = c$.

Theorem 3.3. *For every group G , the map*

$$\Theta_G: \Omega(G) \times K_G[4]/K_G[2] \longrightarrow \Omega(E(G)/K_G[2]), \quad (t, kK_G[2]) \longmapsto \widehat{t}kK_G[2] \quad (3.4)$$

is an isomorphism of unital keis, natural in G . Under this isomorphism, $U_2\eta_{3,V_3G}$ is $t \mapsto (t, 0)$. The following conditions are equivalent.

- (i) η_{3,V_3G} is an isomorphism in $\mathcal{I}_2$;
- (ii) $K_G[4] = K_G[2]$;
- (iii) K_G has no element of order four.

Proof. Abbreviate $K = K_G$, $E = E(G)$, and $\epsilon = \epsilon_G$. Centrality gives

$$(\widehat{t}k)^2 = k^2 \in K[2] \quad (k \in K[4]),$$

so the displayed map is well defined. Conversely, if $uK[2] \in \Omega(E/K[2])$, then $u^2 \in K[2]$. Put $t = \epsilon(u)$, which lies in $\Omega(G)$, and write $u = \widehat{t}k$ with $k \in K$. Then

$$k^2 = u^2 \in K[2], \quad k^4 = 1.$$

This proves surjectivity.

If $\widehat{s}kK[2] = \widehat{t}\ell K[2]$, applying the induced map to G gives $s = t$, after which cancellation gives $kK[2] = \ell K[2]$. Thus the map is bijective. In the quotient,

$$(\widehat{s}k)(\widehat{t}\ell)(\widehat{s}k) = \widehat{s}\widehat{t}\widehat{s}k^2\ell = \widehat{sts}\ell.$$

This proves compatibility with the product-kei operation. The distinguished elements also correspond.

For a homomorphism $f: G \rightarrow G'$, naturality of the counit gives $\epsilon_{G'}E(f) = f\epsilon_G$. Thus $E(f)$ induces maps on both $E(G)/K_G[2]$ and $K_G[4]/K_G[2]$. Naturality of Θ_G follows from

$$E(f)(\widehat{t}k) = \widehat{f(t)}E(f)(k).$$

The formula for the unit follows from (3.3). It is bijective precisely when $K[4]/K[2]$ is trivial. The forgetful functors U_2 and U_1 reflect isomorphisms, so bijectivity of this unit is equivalent to its being an isomorphism in $\mathcal{I}_2$. Finally, $K[4] \setminus K[2]$ consists exactly of the elements of order four. $\square$

4 A finite central extension

We construct a central extension of the group in Theorem 1.1 whose kernel contains an element of order four. A section of $\Omega(P) \rightarrow \Omega(H)$ preserving conjugation then yields an element of order four in K_H .

4.1 The extension and its quotient

Define P by

$$\begin{aligned} P = \langle x_1, x_2, x_3, x_4, a, b, c \mid & x_i^2 = a^2 = b^2 = c^2 = 1, \quad a, b, c \text{ central}, \\ & [x_1, x_2] = a, \quad [x_1, x_3] = b, \quad [x_2, x_3] = c, \\ & [x_1, x_4] = ab, \quad [x_2, x_4] = ac, \quad [x_3, x_4] = bc \rangle. \end{aligned} \quad (4.1)$$

Since a, b, c are commutators of the x_i , the four involutions x_1, x_2, x_3, x_4 generate P .

Lemma 4.1. *The group P has order 128, and $W = \langle a, b, c \rangle \cong C_2^3$. The element*

$$r = x_1x_2x_3x_4$$

is central of order four, with

$$r^2 = n := abc, \quad \langle r \rangle \cap W = \langle n \rangle.$$

Proof. The relations put every element into the form

$$x_1^{v_1} x_2^{v_2} x_3^{v_3} x_4^{v_4} a^\alpha b^\beta c^\gamma, \quad v_i, \alpha, \beta, \gamma \in \mathbb{F}_2. \quad (4.2)$$

To prove that these 128 forms are distinct, set $V = \mathbb{F}_2^4$ and use additive notation for $W_0 = \mathbb{F}_2^3$, with basis a, b, c . Define

$$\begin{aligned} b_{12} &= a, & b_{13} &= b, & b_{14} &= a + b, \\ b_{23} &= c, & b_{24} &= a + c, & b_{34} &= b + c, \end{aligned}$$

and

$$f_P(v, u) = \sum_{i < j} v_j u_i b_{ij}.$$

On $V \times W_0$, put

$$(v, w)(u, z) = (v + u, w + z + f_P(v, u)). \quad (4.3)$$

Bilinearity of f_P gives

$$f_P(v, u) + f_P(v + u, t) = f_P(u, t) + f_P(v, u + t),$$

which proves associativity. The identity is $(0, 0)$, and $(v, w)^{-1} = (v, w + f_P(v, v))$. The elements $x_i = (e_i, 0)$ and the central basis vectors satisfy (4.1). They generate all of $V \times W_0$, so the normal forms are distinct.

The product of the commutators of any x_i with the other three generators is one. Hence r is central. The square formula in a group of class at most two gives

$$r^2 = \prod_{i < j} [x_i, x_j] = abc = n \neq 1.$$

In the coordinate model, $r = ((1, 1, 1, 1), 0)$. Its odd powers are outside W , and its even powers are $1, n$, proving the intersection assertion. $\square$

We now realize the group H of Theorem 1.1 as a quotient of P . Set

$$H = P / \langle r \rangle, \quad \pi: P \longrightarrow H, \quad h_i = \pi(x_i). \quad (4.4)$$

Bars will denote the images of central elements of P , so $\overline{W} = W / \langle n \rangle$.

Lemma 4.2. *The group H is isomorphic to $(C_4 \times C_4) \rtimes C_2$, with inversion action. More precisely, set*

$$s = h_1, \quad u = h_2 h_1, \quad v = h_3 h_1.$$

Then $\langle u, v \rangle \cong C_4^2$, and

$$sus = u^{-1}, \quad svs = v^{-1}, \quad Z(H) = [H, H] = \overline{W} = \langle u^2, v^2 \rangle.$$

The set $\Omega(H)$ is the disjoint union

$$\Omega(H) = \overline{W} \sqcup h_1 \overline{W} \sqcup h_2 \overline{W} \sqcup h_3 \overline{W} \sqcup h_4 \overline{W}. \quad (4.5)$$

Each of the last four sets is a conjugacy class of size four.

Proof. By Lemma 4.1, $|H| = 32$. The relations imply

$$u^2 = \bar{a}, \quad v^2 = \bar{b}, \quad [u, v] = \overline{abc} = 1,$$

as well as the two inversion identities. Since $h_4 = (h_1 h_2 h_3)^{-1}$, the group H is generated by s, u, v . The subgroup $A = \langle u, v \rangle$ is an abelian quotient of C_4^2 and is normalized by s . Hence

$$32 = |H| \leq 2|A| \leq 32.$$

Both inequalities are equalities. Thus $A \cong C_4^2$, $s \notin A$, and $H = A \rtimes \langle s \rangle$.

In this description,

$$h_1 = s, \quad h_2 = us, \quad h_3 = vs, \quad h_4 = u^{-1}vs.$$

The central subgroup is the two-torsion subgroup $\langle u, v \rangle[2] = \langle u^2, v^2 \rangle$, which is also the commutator subgroup. All sixteen elements of $\langle u, v \rangle s$ square to one, whereas exactly four elements of $\langle u, v \rangle$ do so. Conjugating a reflection changes its translation coordinate by an element of $2(\mathbb{Z}/4)^2$. The four parity classes give the four conjugacy classes in (4.5). $\square$

4.2 A section preserving conjugation

Each noncentral conjugacy class in $\Omega(H)$ lifts bijectively to a conjugacy class of involutions in P . Choosing central lifts for the remaining elements gives a section compatible with conjugation.

Proposition 4.3. *There is a unital-kei morphism*

$$\sigma: \Omega(H) \longrightarrow \Omega(P)$$

such that $\pi\sigma = \text{id}$ and $\sigma(h_i) = x_i$ for $1 \leq i \leq 4$. More generally,

$$\sigma(\pi(p)t\pi(p)^{-1}) = p\sigma(t)p^{-1} \quad (p \in P, t \in \Omega(H)). \quad (4.6)$$

Proof. The four subgroups $[x_i, P]$ are

$$\begin{aligned} W_1 &= [x_1, P] = \langle a, b \rangle, & W_2 &= [x_2, P] = \langle a, c \rangle, \\ W_3 &= [x_3, P] = \langle b, c \rangle, & W_4 &= [x_4, P] = \langle ab, ac \rangle. \end{aligned} \quad (4.7)$$

Each has order four and does not contain $n = abc$. Since $\ker(W \rightarrow \overline{W}) = \langle n \rangle$, restriction gives

$$W_i \xrightarrow{\sim} \overline{W}.$$

The conjugacy class $x_i W_i$ consequently maps bijectively to the conjugacy class $h_i \overline{W}$.

Define σ on $h_i \overline{W}$ to be the inverse of this bijection. On $\overline{W}$, take the unique lift in $\langle a, b \rangle$, which is a complement to $\langle n \rangle$ in W . All selected elements square to one, $\sigma(1) = 1$, and $\sigma(h_i) = x_i$.

Each class bijection is equivariant for the action of P by conjugation on the source and through π on the target. Its inverse is therefore equivariant. The selected lifts of central elements are central, so the same statement holds there. This proves (4.6). Taking $p = \sigma(s)$ gives

$$\sigma(sts) = \sigma(s)\sigma(t)\sigma(s) \quad (s, t \in \Omega(H)),$$

as required. $\square$

4.3 An element of order four in the kernel

The section induces a surjection $E(H) \rightarrow P$. We first bound the exponent of $E(H)$ so that a preimage of r has order exactly four.

Lemma 4.4. *If G has nilpotency class at most two, then $E(G)$ has nilpotency class at most two and exponent dividing four.*

Proof. For $a, b \in E(G)$,

$$\epsilon_G([a, b]) \in [G, G] \subseteq Z(G).$$

[Lemma 2.5](#) gives $[E(G), E(G)] \subseteq Z(E(G))$. In a group of class at most two, commutators of involutory generators satisfy

$$[a, b]^2 = [a^2, b] = 1.$$

Hence the commutator subgroup has exponent dividing two. The square of a word in the involutory generators belongs to that subgroup, so every fourth power is one. $\square$

Let $E = E(H)$, $K = K_H$, and write

$$y_i = \hat{h}_i, \quad k = y_1 y_2 y_3 y_4.$$

Proposition 4.5. *The element k belongs to K and has order four. Consequently, $\eta_{3, V_3 H}$ is not an isomorphism.*

Proof. The section in [Proposition 4.3](#) induces a homomorphism

$$\Phi: E \longrightarrow P, \quad \hat{t} \longmapsto \sigma(t).$$

Its image contains all four generators x_i , so it is surjective. The identity $\pi\Phi = \epsilon_H$ holds on every labelled generator. In particular,

$$\epsilon_H(k) = h_1 h_2 h_3 h_4 = 1, \quad \Phi(k) = r.$$

Hence $k \in K$. The group H has nilpotency class at most two, so [Lemma 4.4](#) gives $k^4 = 1$. On the other hand,

$$\Phi(k^2) = r^2 = n \neq 1.$$

Thus $k^2 \neq 1$, and k has order four. [Theorem 3.3](#) now applies. $\square$

5 The enveloping group and the third unit

The homomorphism $\Phi: E(H) \rightarrow P$ in [Proposition 4.5](#) determines a direct-product decomposition of $E(H)$. Its kernel is generated by three central involutions arising from the nonidentity elements of $Z(H)$. This decomposition determines both the third counit and its unit.

5.1 The enveloping group

Proposition 5.1. *There is an isomorphism over H*

$$E(H) \cong P \times C_2^3, \tag{5.1}$$

under which $\epsilon_H(p, b) = \pi(p)$. In particular,

$$K_H \cong \langle r \rangle \times C_2^3, \quad K_H[2] \cong \langle n \rangle \times C_2^3.$$

Proof. We first split Φ and identify a central complement, then prove that the complement has rank three. By Lemma 4.4, the group $E = E(H)$ has nilpotency class at most two. Put

$$a_E = [y_1, y_2], \quad b_E = [y_1, y_3], \quad c_E = [y_2, y_3].$$

These elements are central and have square one. The element $k = y_1 y_2 y_3 y_4$ lies in the central kernel K_H . Expanding $[y_i, k] = 1$ gives

$$[y_1, y_4] = a_E b_E, \quad [y_2, y_4] = a_E c_E, \quad [y_3, y_4] = b_E c_E.$$

The relations of P therefore define a homomorphism

$$f: P \rightarrow E, \quad f(x_i) = y_i.$$

The composite Φf fixes each x_i , so $\Phi f = \text{id}_P$.

Consider the elements

$$z_a = \widehat{a} a_E, \quad z_b = \widehat{b} b_E, \quad z_{ab} = \widehat{ab} a_E b_E. \quad (5.2)$$

They are central and have square one by Lemma 2.5. The choice of σ on $\overline{W}$ also gives $\Phi(z_a) = \Phi(z_b) = \Phi(z_{ab}) = 1$. Thus

$$B_0 = \langle z_a, z_b, z_{ab} \rangle \subseteq Z(E) \cap \ker \Phi$$

is an elementary abelian two-group of rank at most three.

Every noncentral element of $\Omega(H)$ is conjugate to one of the h_i . Since h_1, h_2, h_3 generate H , (2.6) places its labelled generator in $f(P)$. The three remaining nonidentity labels lie in $f(P)B_0$ by (5.2). Consequently,

$$E = f(P)B_0.$$

The identity $\Phi f = \text{id}_P$ implies $f(P) \cap B_0 = 1$. Since B_0 is central, this proves $E \cong P \times B_0$, with Φ as the first projection.

To determine the rank of B_0 , let $B = C_2^3$ have basis e_a, e_b, e_{ab} . Define $\tau: \Omega(H) \rightarrow \Omega(P \times B)$ by $\tau(t) = (\sigma(t), 0)$ on the noncentral elements and on the identity, and by

$$\tau(\bar{a}) = (a, e_a), \quad \tau(\bar{b}) = (b, e_b), \quad \tau(\overline{ab}) = (ab, e_{ab})$$

on the three remaining elements. The added B -coordinate is constant on conjugacy classes. Since B is central of exponent two, τ is a unital-kei morphism and induces

$$\Psi: E \rightarrow P \times B.$$

For example,

$$\Psi(z_a) = (a, e_a)(a, 0) = (1, e_a).$$

The other two generators similarly map to $(1, e_b)$ and $(1, e_{ab})$. They are therefore independent, and $B_0 \cong C_2^3$.

The identity $\epsilon_H = \pi\Phi$ gives $\epsilon_H(p, b) = \pi(p)$. Since $\ker \pi = \langle r \rangle$ and $r^2 = n$, the two formulas for the kernel follow. $\square$

5.2 The unit in direct-product coordinates

Proposition 5.2. *There is an isomorphism over H*

$$F_3V_3H \cong P/\langle n \rangle \cong H \times C_2.$$

Write

$$H = \langle u, v \rangle \rtimes \langle s \rangle$$

as in [Lemma 4.2](#), and identify C_2 with $\mathbb{F}_2$. Under this isomorphism, the underlying set map of the third unit is

$$t \mapsto (t, \delta(t)), \tag{5.3}$$

where

$$\delta(t) = \begin{cases} 1, & t \in h_4\overline{W}, \\ 0, & t \in \overline{W} \cup h_1\overline{W} \cup h_2\overline{W} \cup h_3\overline{W}. \end{cases} \tag{5.4}$$

Equivalently, for $u^i v^j s^e \in \Omega(H)$,

$$\delta(u^i v^j s^e) = (i \bmod 2)(j \bmod 2).$$

Proof. [Propositions 3.2](#) and [5.1](#) give

$$F_3V_3H \cong (P \times C_2^3)/(\langle n \rangle \times C_2^3) \cong P/\langle n \rangle.$$

Define $\ell: P \rightarrow \mathbb{F}_2$ by

$$\ell(x_4) = 1, \quad \ell(x_1) = \ell(x_2) = \ell(x_3) = \ell(a) = \ell(b) = \ell(c) = 0.$$

The relations show that this is a homomorphism; in [\(4.3\)](#) it is the fourth coordinate of v . Consider

$$(\pi, \ell): P \longrightarrow H \times \mathbb{F}_2.$$

Its kernel is

$$\langle r \rangle \cap \ker \ell = \langle r^2 \rangle = \langle n \rangle,$$

since $\ell(r) = 1$. The elements x_1, x_2, x_3 map to generators of $H \times \{0\}$, while r maps to $(1, 1)$. Thus (π, ℓ) is surjective, and

$$p\langle n \rangle \mapsto (\pi(p), \ell(p))$$

is the required isomorphism over H .

Under the isomorphism of [Proposition 5.1](#), a labelled generator $\hat{t}$ has first coordinate $\sigma(t)$. Its second coordinate in C_2^3 is killed on passing to F_3V_3H . Thus [\(3.3\)](#) becomes

$$t \mapsto (t, \ell(\sigma(t))).$$

The chosen representatives in [Proposition 4.3](#) have fourth coordinate one exactly on $h_4\overline{W}$. This proves [\(5.3\)](#) and [\(5.4\)](#). Since $h_4 = u^{-1}vs$, this class consists of the reflections with both translation coordinates odd. $\square$

Proof of Theorem 1.1. The isomorphism over H and the description of the counit follow from [Proposition 5.2](#). By [Lemma 4.2](#), $|\Omega(H)| = 20$, whereas

$$\Omega(H \times C_2) = \Omega(H) \times C_2$$

has cardinality 40. Formula [\(5.3\)](#) is injective, and the element $(1, 1)$ is absent from its image. The unit is therefore not an isomorphism. A left adjoint is fully faithful if and only if its unit is an isomorphism, so F_3 is not fully faithful. $\square$

Remark 5.3. The group coordinates in [Proposition 5.2](#) express the unit as the graph of δ . Identify $K_H[4]/K_H[2]$ with $\mathbb{F}_2$ by sending $kK_H[2]$ to 1. The natural kei coordinates of [Theorem 3.3](#) then express the unit as $t \mapsto (t, 0)$. The change from kei coordinates to group coordinates is

$$(t, b) \mapsto (t, b + \delta(t)).$$

5.3 Associated groups and cellularity

Remark 5.4. The order of P also follows from general results on associated groups. Let $M = (\mathbb{Z}/4)^2$, and let Q be the kei on M with $x \triangleright y = 2x - y$. Its ordinary associated group is

$$A = \text{As}(Q) = \langle e_x \ (x \in Q) \mid e_{x \triangleright y} = e_x e_y e_x^{-1} \rangle.$$

Ishikawa uses the right operation $x * y = y \triangleright x$. His associated group is identified with A by sending each labelled generator to its inverse. Put $R = \text{As}(Q, \text{id})$, the quotient obtained by imposing $e_x^2 = 1$. Identifying M with $\langle u, v \rangle$, the map $x \mapsto xs$ identifies Q with the reflections of H . It induces a homomorphism $i: R \rightarrow E(H)$, given by $i(e_x) = \widehat{xs}$. Define $\rho: E(H) \rightarrow R$ on labels by $\rho(\widehat{xs}) = e_x$ and $\rho(\widehat{z}) = 1$ for $z \in \overline{W}$. The reflection relations become the defining relations of R , and relations involving a central label become identities. Thus ρ is well defined and $\rho i = \text{id}_R$, so i is injective. The preceding proof identifies $i(R)$ with $f(P)$, and $\Phi f = \text{id}_P$ identifies this subgroup with P . Hence $R \cong P$.

The kei Q has four orbits, and its inner automorphism group is generated by the maps $z \mapsto -z + 2x$. Thus $\text{Inn}(Q) \cong C_2^3$. If α is the abelianization of A and ψ its action on Q , then $\ker(\alpha, \psi) = A'$, because $\text{Inn}(Q)$ is abelian. Put $T = -\text{id}_M$ and $N = (1 - T)M = 2M$. Ishikawa's formula [[10](#), arXiv version, Theorem 4.1] specializes to

$$A' \cong (\Lambda^2 N)_T \oplus N^T.$$

Here the subscript denotes coinvariants and the superscript denotes fixed points for the action of T . Since $N \cong C_2^2$ and T acts trivially on N , this gives

$$A' \cong \Lambda^2(C_2^2) \oplus C_2^2 \cong C_2^3.$$

The passage from A to R preserves the commutator subgroup and gives $R_{\text{ab}} \cong C_2^4$ [[9](#), Lemma 3.5 and Theorem 3.8]; see also [[1](#)]. In this case these facts follow directly from the presentation. Because Q is a kei, conjugation by e_x^2 fixes every generator of A . Thus each e_x^2 is central, and the relation $e_{x \triangleright y}^2 = e_y^2$ makes these squares equal within each orbit. Let S be the subgroup generated by the four orbit squares. Abelianizing the presentation gives $A_{\text{ab}} \cong \mathbb{Z}^4$, with one basis vector per orbit. The images of the four generators of S are twice these basis vectors, so $S \rightarrow A_{\text{ab}}$ is injective and $S \cap A' = 1$. Since $R = A/S$, it follows that $R' \cong A'$ and $R_{\text{ab}} \cong C_2^4$. Consequently, $|R| = 2^3 2^4 = 128$. The presentation (4.1) specifies the map to H and its cyclic kernel $\langle r \rangle$, which determine the third unit.

Remark 5.5. A group is C_2 -cellular if it belongs to the smallest class containing C_2 and closed under colimits in **Grp**. Romero's theorem on active sums implies that every group $E(G)$ is C_2 -cellular [[13](#), Definition 1 and Theorem 5]. Every finite 2-group generated by involutions is also C_2 -cellular [[8](#), Corollary 4.4]. Thus H is C_2 -cellular. Its third counit is the projection $H \times C_2 \rightarrow H$ computed above.

A Coordinate formulas

The model (4.3) uses $v \in \mathbb{F}_2^4$ and $w \in W_0 = \mathbb{F}_2^3$. Write $e_1, \dots, e_4$ for the standard basis of the first factor and a, b, c for the basis of W_0 . The quotient H has a model on $\mathbb{F}_2^3 \times \mathbb{F}_2^2$. Write f_1, f_2, f_3 for the standard basis of its first factor and $\bar{a}, \bar{b}$ for the basis of its second factor. These coordinates represent the normal form

$$h_1^{v_1} h_2^{v_2} h_3^{v_3} \bar{a}^\alpha \bar{b}^\beta.$$

The commutators $[h_1, h_2] = \bar{a}$, $[h_1, h_3] = \bar{b}$ and $[h_2, h_3] = \bar{a}\bar{b}$ give the bilinear map

$$f_H(v, u) = v_2 u_1 \bar{a} + v_3 u_1 \bar{b} + v_3 u_2 (\bar{a} + \bar{b})$$

and the multiplication

$$(v, w)(u, z) = (v + u, w + z + f_H(v, u)).$$

Let $\rho: W_0 \rightarrow \mathbb{F}_2^2$ be given by

$$\rho(a) = \bar{a}, \quad \rho(b) = \bar{b}, \quad \rho(c) = \bar{a} + \bar{b}.$$

In H we have $(h_1 h_2 h_3)^2 = \overline{abc} = 1$, so $h_4 = h_1 h_2 h_3$. Substituting this expression for each occurrence of x_4 in the image of (4.2) and collecting the first three generators gives the quotient map

$$\pi(v, w) = ((v_1, v_2, v_3) + v_4(1, 1, 1), \rho(w) + v_4(v_2 + v_3)\bar{a}). \quad (\text{A.1})$$

It sends x_i to $(f_i, 0)$ for $i \leq 3$, and sends x_4 to $((1, 1, 1), 0)$.

The square of $(v, w) \in P$ is $(0, q(v))$, where

$$q(v) = f_P(v, v) = \sum_{i < j} v_i v_j b_{ij}.$$

The value is zero when v has weight zero or one. At vectors of weight two, the six values are

$$a, \quad b, \quad a + b, \quad c, \quad a + c, \quad b + c.$$

At vectors of weight three or four, the value is $a + b + c$. Consequently,

$$\Omega(P) = W \sqcup x_1 W \sqcup x_2 W \sqcup x_3 W \sqcup x_4 W, \quad |\Omega(P)| = 40.$$

The section of Proposition 4.3 is given explicitly by the following table, where $\alpha, \beta \in \mathbb{F}_2$.

$t \in \Omega(H)$	$\sigma(t) \in \Omega(P)$
$(0, \alpha\bar{a} + \beta\bar{b})$	$(0, \alpha a + \beta b)$
$(f_1, \alpha\bar{a} + \beta\bar{b})$	$(e_1, \alpha a + \beta b)$
$(f_2, \alpha\bar{a} + \beta\bar{b})$	$(e_2, (\alpha + \beta)a + \beta c)$
$(f_3, \alpha\bar{a} + \beta\bar{b})$	$(e_3, (\alpha + \beta)b + \alpha c)$
$((1, 1, 1), \alpha\bar{a} + \beta\bar{b})$	$(e_4, \beta a + \alpha b + (\alpha + \beta)c)$

B Finite computation

The program `anc/verify.py` implements the coordinate models over $\mathbb{F}_2$ using integers and bitwise addition. It also constructs H independently in generalized dihedral coordinates over $\mathbb{Z}/4$. All finite data are generated from the displayed formulas, using only Python's standard library. From the directory containing the manuscript, the command

```
python3 anc/verify.py
```

checks the cocycle identities, the quotient map (A.1), the central cyclic kernel $\langle r \rangle$, and the section on all of $\Omega(H)$. In particular, it checks all $128 \cdot 20$ conjugation equivariance identities and all 20^2 kei identities. Using the description $E(H) \cong P \times C_2^3$ proved in Proposition 5.1, it checks all defining relations for the labelled generators and verifies that these labels generate the group. It also checks the quotient and the unit in Proposition 5.2.

The finite groups appearing in the calculation have the following sizes.

Group	Order	$ \Omega(-) $
P	128	40
$H = P/\langle r \rangle$	32	20
$E(H) \cong P \times C_2^3$	1024	320
$K_H \cong C_4 \times C_2^3$	32	16
$F_3V_3H \cong H \times C_2$	64	40

Generative AI statement

ChatGPT was used for proofreading and language polishing.

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