

Beyond the Event Horizon: What Survives at a Rigidity Wall

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A note for reviewers. The metric in this paper is the intrinsic kinetic metric of the four-body shape space, which is the Bures metric of a real qutrit. It is not a spacetime metric and is never substituted for one. Every statement about black holes is a structural comparison, and each one carries its grade where it is made: [DERIVED] for closed forms and identities proved here, [MEASURED] for numbers computed by the scripts listed at the end, and [PROPOSED] for conjectures and ansätze.

AI-assisted. The ideas and research direction are the author's; AI tools were used for literature search, computation, drafting and typesetting, and every number was checked by the scripts in the archive.

Abstract

The four-body shape space $\Sigma_4 = \{G \succeq 0, \text{Tr } G = 1\}$ has one genuine wall, the coplanar stratum $w_3 = 0$, and its kinetic metric is the Bures metric $\sum_i dw_i^2 / (4w_i)$ on the eigenvalue block. The wall is a fold: in $\sigma_i = \sqrt{w_i}$ the metric coefficients are regular (the eigenvalue block is the round octant of the unit sphere), the wall lies at finite distance $\arcsin \sqrt{\epsilon}$, and the sign of σ_3 is the orientation, so a tetrahedron meets its mirror image there. The transport operator $A(H) = TH + HT - 2\text{Tr}(TH)G$ is the compression of the free scaling $TH + HT$ onto trace-zero tangents, and therefore, when the scaling and the configuration share principal axes, self-adjoint in the kinetic metric. In the Gram frame its back-coupling vanishes like σ_3^2 , and on a codimension-two condition it forms a Jordan block whose square-root branch is exactly the orientation double cover. Lifted to the cover, where the metric is regular, the same operator is analytic and the wall point is a plain double point at the tip of a cone. Both readings are exact, and the singular differential of the covering map converts one into the other. A complex loop gives eigenvectors a sign -1 in the Gram frame and $+1$ on the cover, with residue $\frac{1}{2}$ equal to the share of each eigenvector lying across the fold. A real loop through the mirror gives the chart-free sign, the conical -1 of a real self-adjoint family, which the Gram observer cannot see. Through every frame the pair projector and the trace-log ledger stay exact, while single eigenprojectors diverge in the Gram frame and keep norm 1 in the kinetic metric. Four further results carry the same bookkeeping. Majorana stars fix the Jordan form of the Weyl operator: Kerr is type D, diagonalisable, while type II is a Jordan block unfolded by its stars as our wall is by σ_3 . Ramanujan's mock theta function and its modular partner each reach 10^{597} near $q = -1$ while their difference tends to 4; in black-hole microstate counting, mock modularity is itself a wall-crossing effect, and orientation reversal of the Brieskorn spheres built on the trefoil separates false theta functions from mock ones. The amphichiral figure-eight knot has a real invariant growing at its volume 2.029883, while the chiral torus knots, the trefoil and 8_{19} , are conjugated by the mirror and grow like $N^{3/2}$, the second with a constant of period 16 in N . And the off-diagonal block B and its square $BB^\dagger$, a positive pullback, run through the boundary census, the memory kernel, the contractive Cayley boundary and the transport itself. These

sort walls into spectral ones, where a real pair turns complex (the Sun’s dynamo, an optical threshold), and geometric ones, where the spectrum stays real on both sides (the coplanar fold, the static base of a horizon). A black hole carries one of each: the static base of its horizon is a geometric fold, and the innermost stable circular orbit of its exterior is a spectral wall, a Jordan block between oscillation and growth, around which Levin’s whirling orbits gain azimuth like a logarithm. In the same circular-orbit generator the ISCO is the point where gravity’s effective power law, read from the epicyclic ratio, reaches the inverse cube, $p = 2 + 6M/r = 3$. Read for light, the generator reproduces the photon ring: its growth rate equals its orbital rate, so each extra half orbit dims an image by $e^{-\pi}$ and delays it by $3\sqrt{3}\pi M$, a value that belongs to the uncharged hole alone. Its ringdown has spectral walls as well: near a quasinormal exceptional point the excitation factors of the two merging overtones diverge with opposite phases while their sum, fixed by the pair projector, stays finite and the signal acquires a term linear in time. Families of such walls can climb or swap: a loop around several of the near-extremal Kerr exceptional points carries the fundamental mode up the overtones, while the Sun’s dynamo ladder, computed exactly, gives only separate swaps. The horizon’s effective surface resistance equals the impedance of empty space, so in the boundary calculus it is a matched load that returns nothing. Bell’s two results meet at the one-way boundary from opposite sides: the outside half of a Hawking pair is thermal and blind to the pair’s phase while the pair, in the ideal squeezed state, beats the CHSH bound at every frequency, and a theory that keeps only outgoing motion balances its books only with Hawking’s flux, the same shape as the re-entry of what a reduction discards. For a fermion pair the outside record’s offset, the log-odds of empty against full, is exactly $\hbar\omega/k_B T_H$ for every pair phase; read as a log-impedance it places the record at $\Gamma = \tanh(\hbar\omega/2k_B T_H)$ on the same Smith chart as the matched horizon, reaching its centre only at zero frequency. A wave on the projector metric reflects at a step through the metric along the step and the Clairaut constant alone, the transverse metric dropping out as in Fresnel’s formula, and total reflection begins exactly where a geodesic turns: the optical threshold is a turning point, at normal incidence a change of spectral gap reflects with the peel coordinate $\tanh(\frac{1}{2} \ln \Delta_2/\Delta_1)$, one Gram determinant sets the admittance and caps the Berry curvature a path can feel, and a polar stratum is a perfect mirror where a Killing horizon, polar in the same sense, is a matched load. Set beside a recent spacetime Penrose-inequality manuscript, the shared items are the fold normal form $dX^2/(4c^2 X)$ with c the surface gravity there and 1 here, the chart lesson that an apparent singularity is a coordinate effect, a non-self-adjoint horizon operator with no defect in the regular chart, and a degenerate form with a kernel. No identification of horizons is claimed, and the paper states what one would need.

1 Introduction

Black holes occupy a unique position in modern physics. They are exact solutions of general relativity, laboratories for quantum field theory in curved spacetime, and testing grounds for ideas from thermodynamics to quantum information. Many of the tools used to analyse their perturbations are operator-theoretic: linearised perturbations evolve under differential operators whose spectra are the quasinormal modes, whose resolvents control stability, and whose non-self-adjoint structure governs transient amplification [89, 90, 1, 2, 3, 4].

This motivates a simple question:

Can black-hole geometry itself be viewed as an emergent manifestation of persistent operator structure?

Rather than begin with spacetime geometry and derive operators, we reverse the direction. We begin with a transport operator on the shape space of four bodies, where everything can be computed exactly, and ask what happens when it reaches a wall of its configuration space: what fails, what survives, and which of those behaviours a black-hole horizon shares.

The answer has five parts.

The wall is a fold. The coplanar wall looks singular in the Gram coordinate, where the metric coefficient diverges like $1/w_3$. In the amplitude coordinate $\sigma_3 = \sqrt{w_3}$ it is smooth and at finite distance, and it is where a configuration meets its mirror image (Section 2).

The operator has two exact readings at the wall. When the scaling and the configuration share principal axes, the transport is self-adjoint in its own kinetic metric. The Gram observer, whose frame the fold squeezes, sees a one-way coupling and a Jordan block. On the oriented cover the same operator is regular and the wall point is a cone (Section 3).

The record survives, and so does a sign. Individual eigenprojectors depend on the frame, but the pair projector and the trace-log ledger do not. A real loop through the mirror reverses the eigenline, and only the oriented cover can make that loop. Majorana's stars, Ramanujan's radial limits and the mirror rule for knots show the same bookkeeping in three other settings, and in black-hole microstate counting the mock side of Ramanujan's functions comes from a wall crossing (Section 4). A passive boundary that returns nothing is a matched load, its contractivity is a pullback, and the horizon's surface resistance makes it one (Section 5).

Bell meets the one-way boundary from both sides. The entanglement of a Hawking pair says what the outside record cannot hold, and the anomaly that fixes Hawking's flux says what the outside books must carry (Section 7).

Walls come in two kinds. At a spectral wall a real pair turns complex and the Jordan block is in every frame. At a geometric wall the spectrum stays real on both sides and the block is a chart effect. A black hole has both: the static base of its horizon is geometric, and the innermost stable circular orbit and the exceptional points of its ringdown are spectral (Section 8). A wave on the projector metric meets the same sorting. At a step it reflects through $g_{\parallel}$ and the Clairaut constant alone, with $g_{\perp}$ dropping out, and total reflection begins exactly where a geodesic turns, which makes the optical threshold a turning point (Section 8.3).

Each part has a counterpart on the gravitational side. Section 9 sets them beside a recent spacetime Penrose-inequality manuscript [7] and the quasinormal-mode literature. The framework is a mathematical reformulation, not a replacement for general relativity. The Schwarzschild and Kerr geometries, perturbation equations and conservation laws remain intact. Throughout, established mathematical results are kept separate from speculative extensions.

What is claimed, and what is not. The premise that horizon physics involves non-normal operator theory is not claimed here. It is the settled starting point of an active programme: quasinormal modes on hyperboloidal slices are the spectrum of a non-self-adjoint operator, unstable under high-frequency perturbations [1, 3], with gravitational-wave signatures [2]. The scalar product is structural [4]. Non-orthogonal quasinormal sums produce long transient plateaus [87, 5], and transient superradiance occurs in modally stable regimes [6]. What is claimed is the relational side and its exact structure, together with a graded set of structural comparisons.

Two survival principles. When eigenvalues coalesce, contour-integrated functional calculus (Riesz projectors, trace-logs) stays exact where eigenvector arithmetic depends on the frame: *the contour keeps the ledger*. When a configuration passes through the wall, the Gram observer loses the sign of σ_3 , but the orientation and the analytic labelling of the eigenvalue branches keep it: *the orientation keeps the trace*.

2 The shape space and its wall

2.1 Shape space and its kinetic metric

Encode a four-body configuration in $\mathbb{R}^3$, after centring and mass-weighting, by the 3×3 matrix X of Jacobi vectors. Rotations act by $X \mapsto RX$, and the rotation-invariant datum is the Gram matrix $G = X^\top X$. Fixing scale by $\text{Tr } G = 1$ gives

$$\Sigma_4 = \{G \succeq 0 \text{ symmetric, } \text{Tr } G = 1\}, \quad (1)$$

a five-dimensional convex body stratified by rank. At a principal-frame point $W = \text{diag}(w_1, w_2, w_3)$ with $\sum w_i = 1$, the tangent space of traceless symmetric matrices has the Frobenius-orthonormal basis

$$E_1 = \frac{1}{\sqrt{2}} \text{diag}(1, -1, 0), \quad E_2 = \frac{1}{\sqrt{6}} \text{diag}(1, 1, -2), \quad E_{(ij)} = \frac{1}{\sqrt{2}}(e_i e_j^\top + e_j e_i^\top), \quad (2)$$

with $E_3, E_4, E_5 = E_{(12)}, E_{(13)}, E_{(23)}$. E_1, E_2 move the eigenvalues (shape); E_3, E_4, E_5 rotate the eigenframe (shear).

Theorem 1 (the kinetic metric is the Bures metric). [DERIVED] *The quotient of the flat Frobenius metric on $\{\|X\|_F = 1\}$ by rotations is, at W ,*

$$g(E_a, E_b)|_{\text{shape}} = \sum_{i=1}^3 \frac{(E_a)_{ii}(E_b)_{ii}}{4w_i}, \quad g(E_{(ij)}, E_{(ij)}) = \frac{1}{2(w_i + w_j)}, \quad (3)$$

with the shape and shear sectors orthogonal and distinct shear directions mutually orthogonal. It is the Bures metric of real density matrices, so Σ_4 with its own kinetic metric is the state space of a real qutrit, and its strata are the rank strata: rank two is the coplanar configurations and rank one the collinear ones, which are not collisions.

The proof (horizontal lifts at $X = \text{diag}(\sqrt{w_i})$) is in Appendix A. At $d = 2$ the same construction gives the real-qubit Bures geometry: with $\rho = \frac{1}{2}(I + \sin \chi (\cos \varphi \sigma_x + \sin \varphi \sigma_z))$ the metric is $\frac{1}{4}(d\chi^2 + \sin^2 \chi d\varphi^2)$, a hemisphere of radius $\frac{1}{2}$ whose boundary $\chi = \pi/2$ is the circle of pure states, of length π [DERIVED; CHECKED, V10-CORRECTIONS]. Its orientation double cover, two hemispheres glued along that circle, is Kendall's shape sphere of triangles, of radius $\frac{1}{2}$; the circle is the collinear triangles. That is the control used for all curvature computations: the code returns sectional curvature 4.0000000 there [MEASURED].

2.2 The wall is a fold

The strata of Σ_4 are: the interior (rank 3); the coplanar wall $w_3 = 0$ (rank 2, codimension 1); and the collinear stratum $w_2 = w_3 = 0$ (rank 1, codimension 3). Interior eigenvalue coincidences $w_i = w_j$ are not orbit-type strata: the stabiliser of a full-rank X is trivial whatever the spectrum of G , and the curvature is finite there.

Theorem 2 (the coplanar wall is a fold at finite distance). [DERIVED] *Let $\sigma_i = \sqrt{w_i}$. Then:*

- (i) *On the eigenvalue block the metric is $\sum_i dw_i^2 / (4w_i) = \sum_i d\sigma_i^2$ restricted to $\sum \sigma_i^2 = 1$: the round metric of the positive octant of the unit sphere, of Gaussian curvature 1. The square-root coordinate removes the $1/w_i$ divergence of the coefficients, not the curvature.*
- (ii) *Along $W(w) = \text{diag}(\frac{1-w}{2}, \frac{1-w}{2}, w)$ the metric is $ds^2 = dw^2 / (4w(1-w))$, and the distance from the wall to $w = \varepsilon$ is $\arcsin \sqrt{\varepsilon}$, finite.*

(iii) Extending σ_3 to negative values by $\sigma_3 = \text{sign}(\det X) \sqrt{w_3}$ realises the rotation-reduced shape space as the double cover of Σ_4 branched over $\det G = (\det X)^2 = 0$. The metric $d\sigma_3^2$ is smooth and even under $\sigma_3 \mapsto -\sigma_3$, which exchanges a configuration with its mirror image.

Proof. (i) $dw_i = 2\sigma_i d\sigma_i$ gives $dw_i^2/(4w_i) = d\sigma_i^2$. (ii) On the path, $w_1 = w_2 = \frac{1-w}{2}$ and $dw_1 = dw_2 = -\frac{dw}{2}$, so the two equal terms give $2 \cdot \frac{dw^2/4}{2(1-w)} = \frac{dw^2}{4(1-w)}$ and $ds^2 = \frac{dw^2}{4(1-w)} + \frac{dw^2}{4w} = \frac{dw^2}{4w(1-w)}$. With $w = \sin^2 \theta$ this is $d\theta^2$, so the length from 0 to ε is $\arcsin \sqrt{\varepsilon}$. (iii) $\det G = (\det X)^2$, and X is determined by G up to rotation and reflection. On rank-two X the rotation stabiliser is trivial, so the oriented quotient is smooth across $\det X = 0$; σ_3 is a smooth signed coordinate there, and the metric $d\sigma_3^2$ is even. $\square$

In the Gram coordinate the wall direction E_2 has coefficient $g(E_2, E_2) = \frac{1}{6} \left(\frac{1}{4w_1} + \frac{1}{4w_2} + \frac{1}{w_3} \right) \sim \frac{1}{6w_3}$. This divergence is the square-root shadow of the fold, not a barrier, and its far side is the orientation double cover of the shape space branched over $\det X = 0$, which holds for every square case $N = d + 1$ [75] (Theorem 2.4 there). The fold is Whitney's: a map whose differential has a simple zero on a codimension-one locus looks like $x \mapsto x^2/2$ there (type A_2) [77], and reading a smooth signed coordinate through its square is what produces the $1/w_3$ coefficient. On the three-body shape sphere the same normal form accounts for the $\sec^4$ rigidity law near the collinear wall, and a cusp (A_3) would give a different exponent [72]. The same holds for curvature. Metric-normalised sectional curvatures stay finite as $w_3 \rightarrow 0$; for example, along a fixed generic direction V we measure $K(E_2, V) \rightarrow 5.61$, $K(E_4, V) \rightarrow 6.43$ and $K(E_5, V) \rightarrow 7.66$ at $w_3 = 10^{-4}$; over two decades in w_3 the largest change is in $K(E_4, V)$, a factor 1.30, while $K(E_2, V)$ moves by 2% [MEASURED]. Divergences of order $1/w_3$ appear only when curvatures are measured against Frobenius-unit frames, whose metric length itself diverges.

The collinear stratum is different. The shear coefficient $1/(2(w_j + w_k))$ diverges when $w_j = w_k = 0$ because rotations about the common line act trivially there: the orbit dimension drops. That is a genuine singular stratum of the quotient, of codimension three, and generic trajectories miss it.

What the Gram observer sees is shown in Figure 1. A path crossing $\sigma_3 = 0$ at constant speed appears in $w_3 = \sigma_3^2$ as a bounce. The normal velocity measured in w_3 is

$$\dot{w}_3 = 2\sigma_3 \dot{\sigma}_3, \quad |\dot{w}_3| = 2\sqrt{w_3} |\dot{\sigma}_3| \rightarrow 0, \quad (4)$$

so incoming trajectories appear to flatten against the wall and skim it, while in the signed coordinate they pass through to the mirror configuration [DERIVED].

3 The transport operator

3.1 Definition and block structure

The intrinsic transport operator is

$$A(H) = TH + HT - 2\text{Tr}(TH)G, \quad T = \text{diag}(t_1, t_2, t_3) \succ 0. \quad (5)$$

Here T is a fixed anisotropic scaling of the configuration. Under $X \mapsto Xe^{sT}$ the Gram matrix moves as $G \mapsto e^{sT} G e^{sT}$, with velocity $TG + GT$; A applies the same scaling to a tangent H and returns it to the trace-zero tangents along G . The entries t_k are the three scaling rates, and nothing else about T is used. T and G are diagonal in one frame: the scaling acts along the principal axes of

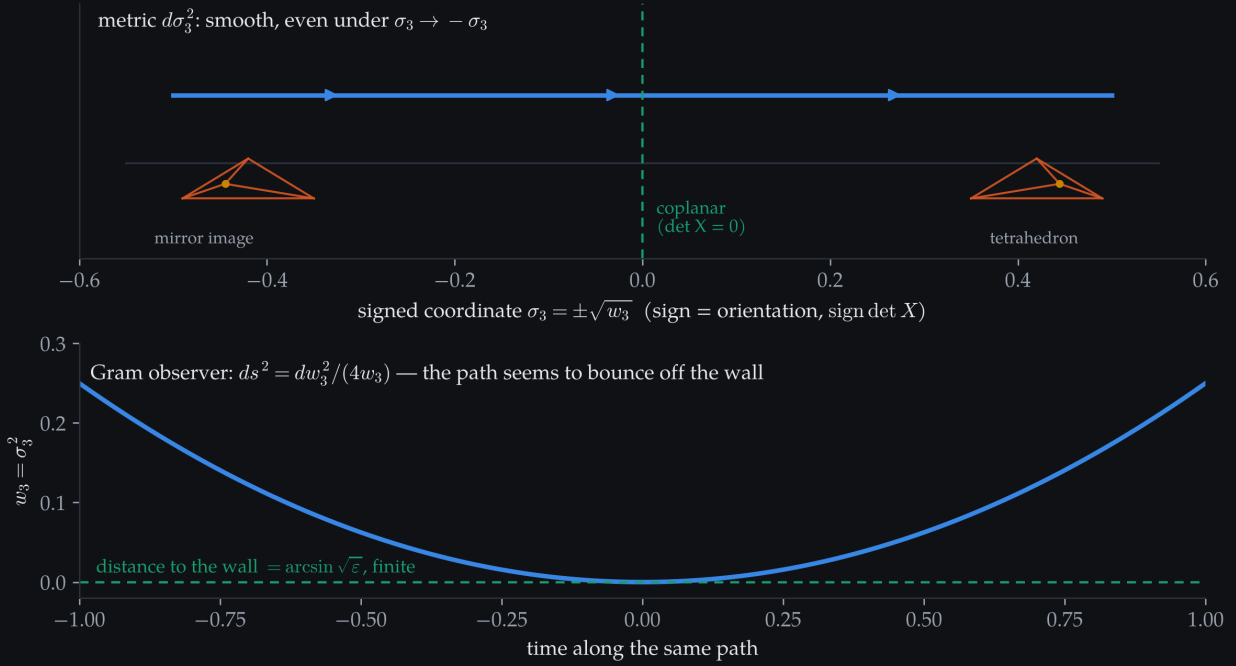


Figure 1: The coplanar wall as a fold. Top: in the signed coordinate σ_3 the metric is $d\sigma_3^2$ and a path passes from a tetrahedron to its mirror image. Bottom: the same path seen by the Gram observer $w_3 = \sigma_3^2$ appears to bounce. The coefficient $1/(4w_3)$ diverges, but the distance $\arcsin \sqrt{\varepsilon}$ is finite.

the configuration, and every result below uses this. The coefficient 2 is forced: it is the unique choice keeping $A(H)$ traceless when $\text{Tr } G = 1$, because $\text{Tr}(TH + HT) = 2 \text{Tr}(TH)$. The operator is defined without reference to any metric. It is the compression of the scaling, not the Jacobian of the normalised flow $G \mapsto e^{sT} G e^{sT} / \text{Tr}(e^{sT} G e^{sT})$; that Jacobian is $A(H) - 2 \text{Tr}(TG) H$, a uniform shift that moves every eigenvalue by the same amount and changes no eigenvector and no Jordan structure [DERIVED; CHECKED SYMBOLICALLY, V10-CORRECTIONS].

Theorem 3 (block structure). [DERIVED] *In the basis $E_1, \dots, E_5$ at W , with $A(E_b) = \sum_a A_{ab} E_a$:*

$$A_{11} = (t_1 + t_2) - (t_1 - t_2)(w_1 - w_2), \quad A_{22} = 2t_3 + w_3(t_1 + t_2 - 2t_3), \quad (6)$$

$$A_{12} = \frac{1}{\sqrt{3}} [(t_1 - t_2) - (t_1 + t_2 - 2t_3)(w_1 - w_2)], \quad A_{21} = \sqrt{3} (t_1 - t_2) w_3, \quad (7)$$

the shear block is diagonal with entries $t_1 + t_2$, $t_1 + t_3$, $t_2 + t_3$, and the shape–shear coupling is identically zero.

This is a direct computation, verified symbolically (BH-CORE-02). The vanishing shape–shear coupling matters. A defective 2×2 sub-block need not survive coupling to the rest of a matrix: the author’s filament paper gives a five-dimensional pattern whose sub-block is a Jordan block while the full matrix has only a simple eigenvalue there [73]. Here the shape block is an exact restriction of A , so whatever it shows at the wall, the full operator shows.

3.2 The operator is self-adjoint in its own metric

Write the kinetic metric of Theorem 1 on symmetric matrices as

$$g_W(H, K) = \frac{1}{2} \sum_{i,j} \frac{H_{ij} K_{ij}}{w_i + w_j}, \quad (8)$$

which reproduces $\sum_i H_{ii} K_{ii} / (4w_i)$ on the diagonal and $1 / (2(w_i + w_j))$ on $E_{(ij)}$. Let $D_T(H) = TH + HT$ be the free scaling.

Theorem 4 (the transport is a compression of a self-adjoint scaling). [DERIVED] *At every interior point, with T diagonal in the eigenframe of W :*

- (i) D_T is self-adjoint for g_W ;
- (ii) $g_W(W, H) = \frac{1}{4} \text{Tr } H$, so $P_W(Z) = Z - \text{Tr}(Z) W$ is the g_W -orthogonal projection onto the trace-zero tangent space, and $Q_W = I - P_W$ projects onto the scale direction W ;
- (iii) $A = P_W D_T P_W$ on trace-zero tangents. Hence A is self-adjoint for the kinetic metric, $g_W A = A^\top g_W$ in any basis, its eigenvalues are real, and its eigenprojectors are g_W -orthogonal, with kinetic norm exactly 1.

Proof. (i) $g_W(D_T H, K) = \frac{1}{2} \sum_{ij} (t_i + t_j) H_{ij} K_{ij} / (w_i + w_j)$ is symmetric in H, K . (ii) $g_W(W, H) = \frac{1}{2} \sum_i w_i H_{ii} / (2w_i) = \frac{1}{4} \text{Tr } H$, and P_W is idempotent with kernel span W and image the trace-zero matrices, which are g_W -orthogonal to W by this formula. (iii) $P_W D_T(H) = TH + HT - \text{Tr}(TH + HT) W = A(H)$. A compression of a self-adjoint operator by an orthogonal projection is self-adjoint. $\square$

The common frame is needed. At $W = \text{diag}(\frac{1}{2}, \frac{3}{10}, \frac{1}{5})$ with $T = I + \frac{1}{5}(e_1 e_2^\top + e_2 e_1^\top)$, which is not diagonal there, $g_W(AH, K) - g_W(H, AK) = \frac{2}{15}$ for $H = \sqrt{2}E_1$ and $K = \sqrt{2}E_{(12)}$; with T diagonal the difference stays below 2×10^{-14} on 200 random pairs [DERIVED; CHECKED, V10-CORRECTIONS]. Part (i) is where the frame enters: $t_i + t_j$ multiplies H_{ij} only when T is diagonal in the frame where g_W is. The symbolic check BH-LIFT-05 confirms $g_W A - A^\top g_W = 0$ identically on the full 5×5 operator. Theorem 4 gives a one-line reason for the reality of the interior spectrum, which the next subsection also proves by a similarity. It also places the operator's non-normality. In the Frobenius frame E_a the commutator is nonzero only when both T and W are anisotropic: $\|[A, A^\top]\|_F = 0.2314$ at $t = (2, 1, 1.3)$, $w = (0.5, 0.3, 0.2)$, and it vanishes if $T \propto I$ or $W = \frac{1}{3}I$ [DERIVED; VALUE MEASURED]. In the kinetic metric the operator is normal everywhere inside. The non-normality is a property of the Frobenius frame, which is the frame the Gram observer uses. That distinction carries through the rest of the paper.

The hidden block of the transport. Because A is a compression, its square is not the compression of D_T^2 . The difference is

$$P_W D_T^2 P_W - A^2 = P_W D_T Q_W D_T P_W = (Q_W D_T P_W)^* (Q_W D_T P_W) \succeq 0, \quad (9)$$

a positive operator of rank at most one in the kinetic metric, and zero when $T \propto I$ [DERIVED; CHECKED, BH-SPINE-09, V10-CORRECTIONS]. Its single direction is the coupling of the shape tangents to the scale direction W , which fixing $\text{Tr } G = 1$ removes from view. Section 5.4 shows that this $BB^\top$ structure is the same one that runs through the boundary census, the memory kernel and the gravity quadratic form.

3.3 Real spectrum in the interior

Theorem 5 (no exceptional points inside). [DERIVED] *If $w_1, w_2, w_3 > 0$, the shape block of A is diagonalisable with real eigenvalues.*

Proof. This follows from Theorem 4. A second proof exhibits the similarity directly. On diagonal $H = \text{diag}(h)$, A acts as the 3×3 matrix $2(D_t - wt^\top)$ with $D_t = \text{diag}(t)$, whose image is traceless. With $u_i = \sqrt{w_i t_i}$ and $S = \text{diag} \sqrt{w_i/t_i}$,

$$S(D_t - uu^\top)S^{-1} = D_t - wt^\top,$$

since $(Suu^\top S^{-1})_{ij} = \sqrt{w_i/t_i} \sqrt{w_i t_i} \sqrt{w_j t_j} \sqrt{t_j/w_j} = w_i t_j$. So $D_t - wt^\top$ is similar to a symmetric matrix, and the invariant traceless plane inherits real, diagonalisable spectrum. $\square$

Both the similarity S and the kinetic metric degenerate exactly when some $w_i = 0$. Away from the wall the discriminant $D = (A_{11} - A_{22})^2 + 4A_{12}A_{21}$ is positive at every sampled point: over 2×10^4 random interior points its minimum is 2.9×10^{-4} , and for 5×10^4 random choices of (t, δ) no root of D in w_3 falls inside the physical segment [MEASURED].

3.4 The wall as the Gram observer sees it

Theorem 6 (the wall in the Gram frame). [DERIVED] *In the frame E_1, E_2 at the coplanar wall:*

- (i) *The back-coupling vanishes quadratically in the signed coordinate, $A_{21} = \sqrt{3}(t_1 - t_2)\sigma_3^2$, while A_{12} stays finite. The shape block becomes upper-triangular: the direction across the wall (E_2) still drives the direction in the wall (E_1), but not the reverse.*
- (ii) *At $w_3 = 0$, $A_{22} = 2t_3$, and the shape block is a Jordan block if and only if $A_{11} = A_{22}$ and $A_{12} \neq 0$, that is,*

$$t_1 + t_2 - 2t_3 = (t_1 - t_2)(w_1 - w_2), \quad (10)$$

a codimension-two condition in the joint space of (G, T) . On it $A_{12} = (t_1 - t_2)(1 - \delta^2)/\sqrt{3}$ with $\delta = w_1 - w_2$, so $A_{12} \neq 0$ means $t_1 \neq t_2$ and $|\delta| < 1$; equal rates $t_1 = t_2$ give a semisimple double eigenvalue instead.

Proof. (i) is read from Theorem 3 with $w_3 = \sigma_3^2$. (ii) A 2×2 upper-triangular matrix is non-diagonalisable exactly when its diagonal entries agree and its off-diagonal entry is nonzero. At $w_3 = 0$, $A_{11} - A_{22} = (t_1 + t_2 - 2t_3) - (t_1 - t_2)(w_1 - w_2)$. $\square$

In the dynamics $\dot{x} = -Ax$ of the shape amplitudes, (i) says $\dot{x}_2 = -A_{21}x_1 - A_{22}x_2 \rightarrow -A_{22}x_2$: what happens in the wall cannot write back across it, while what arrives across the wall is written into it through A_{12} (Figure 2). The kinetic metric explains how a self-adjoint operator can show a Jordan block here. The coefficient $g_W(E_2, E_2) \sim 1/(6w_3)$ diverges at the wall, so the metric in which A is self-adjoint stops being a metric on the Gram tangent space. The Jordan block lives in that degenerate frame.

3.5 The fold unfolds the exceptional point

Fix T and $\delta = w_1 - w_2$ on the joint condition (10), and vary w_3 with $w_1 + w_2 = 1 - w_3$. Then A_{11} and $A_{12} = (t_1 - t_2)(1 - \delta^2)/\sqrt{3}$ are constant, $A_{22} - A_{11} = w_3(t_1 + t_2 - 2t_3)$, and

$$D(w_3) = (t_1 + t_2 - 2t_3)^2 w_3^2 + 4(t_1 - t_2)^2 (1 - \delta^2) w_3. \quad (11)$$

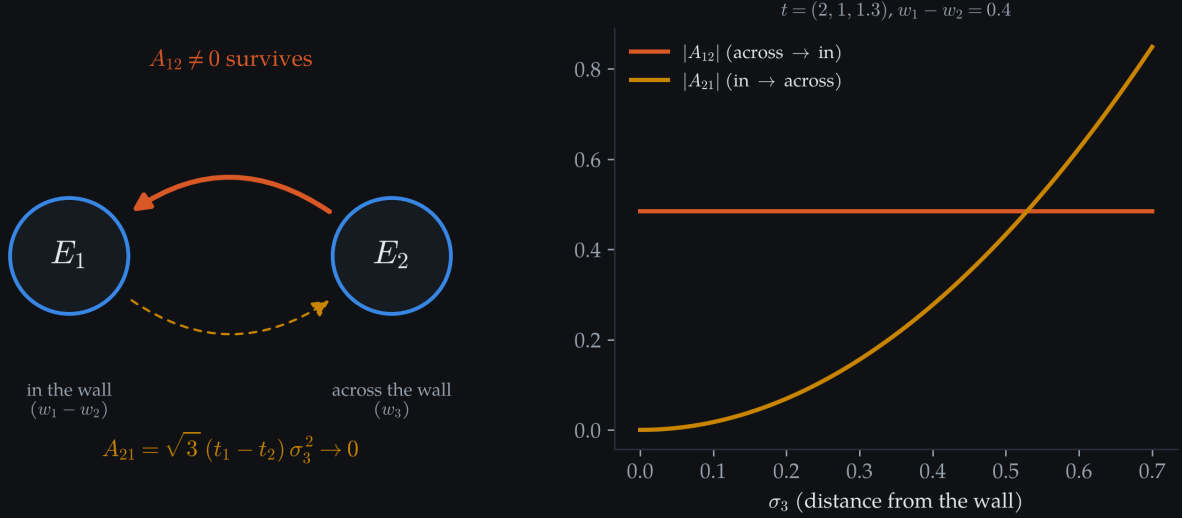


Figure 2: One-way coupling at the wall, in the Gram frame. The coupling from across the wall into the wall, A_{12} , stays finite. The reverse coupling $A_{21} = \sqrt{3}(t_1 - t_2)\sigma_3^2 \rightarrow 0$ vanishes quadratically in the distance to the wall.

Theorem 7 (the branch point lives on the orientation double cover). [DERIVED] *On the joint condition, D has a simple zero at $w_3 = 0$, so the two eigenvalues $\lambda_{\pm} = \frac{1}{2}(A_{11} + A_{22}) \pm \frac{1}{2}\sqrt{D}$ have a square-root branch point in w_3 . In the signed coordinate,*

$$\lambda_{\pm}(\sigma_3) = \lambda_* + \frac{1}{2}(t_1 + t_2 - 2t_3)\sigma_3^2 \pm \frac{1}{2}\sigma_3\sqrt{4(t_1 - t_2)^2(1 - \delta^2) + (t_1 + t_2 - 2t_3)^2\sigma_3^2}, \quad (12)$$

which is analytic in σ_3 . Consequently:

- (i) a loop of w_3 around 0 exchanges the pair, a loop of σ_3 around 0 does not, and the mirror $\sigma_3 \mapsto -\sigma_3$ exchanges the two analytic branches;
- (ii) the gap is $|\lambda_+ - \lambda_-| = 2|t_1 - t_2|\sqrt{1 - \delta^2}|\sigma_3| + O(\sigma_3^3)$, linear in the distance to the wall; at the wall $1 - \delta^2 = 4w_1w_2$;
- (iii) for real σ_3 of either sign the two eigenvalues are real.

Proof. Substitute $w_3 = \sigma_3^2$ into (11): $D = \sigma_3^2 [4(t_1 - t_2)^2(1 - \delta^2) + (t_1 + t_2 - 2t_3)^2\sigma_3^2]$. The bracket is positive near 0, so $\sqrt{D} = \sigma_3\sqrt{[\cdot]}$ with an analytic square root, which gives (12). A loop of w_3 is half a loop of σ_3 , which carries $\sigma_3 \mapsto -\sigma_3$ and exchanges $\lambda_{\pm}$. A full loop of σ_3 returns each branch. For real σ_3 the bracket and σ_3^2 are positive, so (iii) holds. $\square$

The braid gate BH-N-01 confirms each statement on the full operator at $t = (2, 1, 1.3)$, $\delta = 0.4$, where $\lambda_* = 2.6$ and $A_{12} = 0.484974$. One loop in w_3 (radius 10^{-3}) swaps the pair and two loops restore it. The discriminant winds $+1$ per loop, and one loop in σ_3 does not swap. The measured gap/ σ_3 is 1.833030 against $2\sqrt{0.84} = 1.833030$ [MEASURED]. Figure 3 shows the two loops. Off the joint condition, $D(0) = (A_{11} - A_{22})^2 > 0$, and every zero of D lies outside the closed physical segment, beyond the wall (negative w_3 , that is, imaginary σ_3) or beyond the opposite walls [DERIVED AT THE WALL; MEASURED ELSEWHERE]. Part (iii) is the property that separates this wall from a spectral exceptional point (Section 8): along every physical path the spectrum stays real on both sides, including through the mirror. Over 41 values of t_3 and 241 values of $\sigma_3 \in [-0.6, 0.6]$ the largest imaginary part is exactly 0 (BH-LIFT-05) [MEASURED].

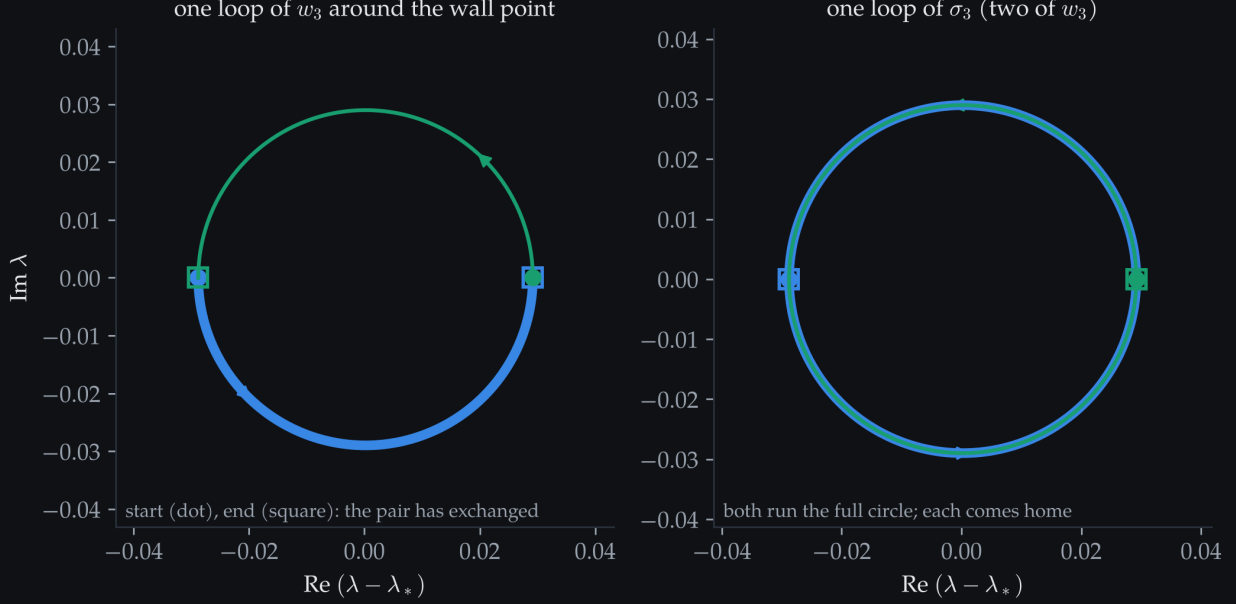


Figure 3: The braid gate at the joint point $t = (2, 1, 1.3)$, $w_1 - w_2 = 0.4$. Left: one loop of w_3 around the wall point exchanges the two eigenvalues. Right: one loop of σ_3 (two loops of w_3) returns each eigenvalue to itself. The branch cover is the orientation cover.

3.6 The operator on the cover: a plain double point

The Gram frame E_1, E_2 is a frame of the tangent space of Σ_4 . On the oriented cover the natural coordinates near the wall are (δ, σ_3) , and their coordinate vectors are

$$\partial_\delta = \frac{1}{\sqrt{2}} E_1, \quad \partial_{\sigma_3} = -\sqrt{6} \sigma_3 E_2, \quad J = \text{diag}\left(\frac{1}{\sqrt{2}}, -\sqrt{6} \sigma_3\right). \quad (13)$$

The covering map has differential J , which is singular exactly at the wall. The transport lifted to the cover is $B = J^{-1} A J$.

Theorem 8 (the lift is regular and the wall point is diabolic). [DERIVED] *With $c = t_1 + t_2 - 2t_3$,*

$$B(\sigma_3) = \begin{pmatrix} (t_1 + t_2) - \delta(t_1 - t_2) & 2\sigma_3(\delta c - (t_1 - t_2)) \\ -\frac{1}{2}\sigma_3(t_1 - t_2) & 2t_3 + c\sigma_3^2 \end{pmatrix}, \quad (14)$$

which is analytic in σ_3 , including at the wall. B is self-adjoint for the pulled-back metric $J^\top g_W J$, which is finite and positive at the wall, with $(J^\top g_W J)_{\sigma_3 \sigma_3} \rightarrow 1$. On the joint condition $B(0) = \lambda_* I$. Near that point

$$B = \lambda_* I + \sigma_3 B_1 + \varepsilon B_2 + O(\sigma_3^2, \varepsilon \sigma_3), \quad B_1 = (t_1 - t_2) \begin{pmatrix} 0 & 2(\delta^2 - 1) \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad (15)$$

with $\varepsilon = t_3 - t_3^*$ the departure from the joint condition. The two eigenvalues therefore meet at a cone: the gap is $2|t_1 - t_2|\sqrt{1 - \delta^2}|\sigma_3|$ along the wall direction and $2|\varepsilon|$ across the joint condition, linear in both.

Proof. $B = J^{-1} A J$ has entries A_{11} , $J_{11}^{-1} A_{12} J_{22} = -\sqrt{12} \sigma_3 A_{12}$, $J_{22}^{-1} A_{21} J_{11} = -A_{21}/(\sqrt{12} \sigma_3)$ and A_{22} . Substituting Theorem 3 with $w_1 - w_2 = \delta$ and $w_3 = \sigma_3^2$ gives (14); the factor σ_3^2 in A_{21}

cancels the pole. Self-adjointness transfers through any change of frame. The pulled-back metric is $J^\top g_W J$; its $\sigma_3 \sigma_3$ entry is $6\sigma_3^2 g_W(E_2, E_2) \rightarrow 6\sigma_3^2 / (6w_3) = 1$, and its off-diagonal entry is $O(\sigma_3)$. At $\sigma_3 = 0$ the diagonal entries are A_{11} and $2t_3$, equal on (10). The expansion (15) follows from (14), using $c - (t_1 - t_2)\delta = -2\varepsilon$ on the shifted condition. The off-diagonal entries of B_1 have product $(t_1 - t_2)^2(1 - \delta^2)$, which gives the stated gap in σ_3 , and B_2 gives $2|\varepsilon|$. $\square$

BH-LIFT-05 checks (14) against $J^{-1}AJ$ to 6×10^{-17} at $\sigma_3 = 0.3, 0.05, -0.2$, finds $B(0) = 2.6 I$ exactly at the paper point, and measures the cone: gap 1.833030×10^{-3} at $\sigma_3 = 10^{-3}$ and 2.000000×10^{-3} at $\varepsilon = 10^{-3}$ [MEASURED]. The closed form agrees: at this point $2|t_1 - t_2|\sqrt{1 - \delta^2} = 2\sqrt{0.84} = 1.833030278$, and BH-QUTRIT-18 measures the slope 1.833030278 [DERIVED; CHECKED, BH-QUTRIT-18].

So the wall point has two honest descriptions. To the Gram observer, who uses the tangent space of Σ_4 , it is a Jordan block with a square-root branch in w_3 . On the oriented cover, where the operator is self-adjoint in a regular metric, it is a diabolic point: a plain double eigenvalue at the tip of a cone. The singular differential J of the covering map converts the second description into the first. That J has a simple zero at the wall is the fold of Theorem 2, now seen by the operator. Throughout, “Gram frame” and “cover” name these two spaces, not two bases of one space: no similarity changes a Jordan form, and none is claimed. Away from the wall A and B are similar; at the wall J is singular, so A on the tangent space of Σ_4 and B on the tangent space of the cover are different operators, and A there is genuinely defective.

3.7 The declared operator decides

A Jordan block in a given frame is a property of the declared operator and of that frame, not of the wall alone. Replace the Frobenius-closed choice $T = \text{diag}(t)$ by the volume structure $T = G^{-1}$, whose metric is the affine-invariant $\langle H, K \rangle_G = \text{tr}(G^{-1}HG^{-1}K)$.

Proposition 9. [DERIVED] *With $T = G^{-1}$, A is self-adjoint for $\langle \cdot, \cdot \rangle_G$ at every interior point, hence diagonalisable with real spectrum. Approaching the wall, three eigenvalues diverge like $(1, 1, 2)/w_3$ and two stay finite. No Jordan block forms in any frame.*

Proof. $\langle H, A(K) \rangle_G = \text{tr}(G^{-2}HG^{-1}K) + \text{tr}(G^{-1}HG^{-2}K) - 2\text{tr}(G^{-1}K)\text{tr}(G^{-1}H)$. The first two terms exchange under $H \leftrightarrow K$ by cyclicity and the third is symmetric. The divergence rates are measured: eigenvalues $\times w_3 \rightarrow (1.000, 1.000, 2.000)$ at $w_3 = 10^{-6}$ [MEASURED]. $\square$

At the same wall, one declared operator gives a diabolic point that the Gram frame reads as a Jordan block, and the other gives a pole. The words “the wall” therefore always need the operator and the frame named alongside. This matters for any comparison with horizons, where the operator is fixed by physics.

4 What survives the wall

4.1 The contour keeps the ledger

For a contour Γ enclosing the colliding pair and nothing else, the Riesz projector and the trace-log ledger

$$P_{\text{pair}} = \frac{1}{2\pi i} \oint_{\Gamma} (z - A)^{-1} dz, \quad \mathcal{L}_c = \frac{1}{2\pi i} \oint_{\Gamma} \log(c + z) \text{Tr}(z - A)^{-1} dz = \log \det(c + A_{\text{pair}}) \quad (16)$$

are analytic in the matrix entries for as long as Γ stays clear of the spectrum. Neither quantity uses a metric, and neither notices whether the enclosed eigenvalues are distinct, form a Jordan block in one frame, or meet at a cone in another. Individual eigenprojectors depend on the metric used to measure them. In the Gram frame with the Frobenius norm, a real 2×2 block with off-diagonal entries A_{12}, A_{21} and real gap g has exactly $\|P_{\pm}\|^2 = 1 + (A_{12} - A_{21})^2/g^2$ (Appendix A), and since $A_{21} = O(\sigma_3^2)$, $\|P_{\pm}\| g \rightarrow |A_{12}|$. In the kinetic metric the same projectors are orthogonal (Theorem 4) and their norm is exactly 1.

w_3	gap	Gram norm·gap	kinetic norm	count in Γ	$\text{Tr } P_{\text{pair}}$	ledger
10^{-2}	1.83×10^{-1}	0.502311	1.000000	2	2.0000000000	2.562330546753
10^{-4}	1.83×10^{-2}	0.485147	1.000000	2	2.0000000000	2.561872320543
10^{-6}	1.83×10^{-3}	0.484976	1.000000	2	2.0000000000	2.561867737220
0	0	—	—	2	2.0000000000	2.561867690924

The ledger column is $\log \det(1 + A_{\text{pair}})$. At $t = (2, 1, 1.3)$, $\delta = 0.4$, with Γ the circle of radius 0.25 about $\lambda_* = 2.6$ [MEASURED]. In the Gram frame the product $\|P_{\pm}\| \cdot \text{gap}$ tends to $|A_{12}| = 0.484974$: the divergence constant of a single projector is the Jordan coupling. In the kinetic metric nothing diverges (the Gram-frame norms themselves are 2.74, 26.47 and 264.58 at $\sigma_3 = 0.1, 0.01, 0.001$). The ledger tends to $2 \log(1 + \lambda_*) = 2 \log 3.6 = 2.561867690924$, its exact value at the wall point. The ledger is smooth, not constant: it is a record, not a conserved charge. It needs no metric at all, and in that sense the record outlives the metric.

One hypothesis is visible in the data. At $w_3 = 10^{-1}$ the pair spreads to $[2.33, 2.91]$ and the shear eigenvalue $t_2 + t_3 = 2.3$ enters any circle wide enough to hold the pair. The contour must isolate the cluster with a spectral margin, which is exactly the hypothesis of the persistence bounds (Appendix A, Lemma 18 and Theorem 19). With the margin in place, the map from operators to their Grassmannian projector data is locally Lipschitz, in any frame. The cluster holds where its vectors do not. By the Davis–Kahan theorem a perturbation E turns the cluster’s subspace by an angle whose sine is at most $\|E\|/\delta$, with δ the gap to the rest of the spectrum [97], while single eigenvectors inside a tight cluster can turn by order one. On 2000 random symmetric 6×6 matrices with a three-member cluster split by 10^{-3} and $\|E\| = 0.02$, the subspace turned by a median 0.27° and its vectors by 74° ; split the cluster by 0.4 instead and the vectors turn only 2.7° ; the bound was never exceeded [MEASURED, V10-PUSH].

The same law holds on a real orbit. Along the Chenciner–Montgomery figure-eight of three equal masses, the shape passes through the collinear wall six times per period. There the metric layer collapses: the rigidity $R = \lambda_{\min}^{-2}$ diverges and reaches 2.5×10^{16} at the closest sampled point. Through all six crossings the contour count holds at 2 to 5.1×10^{-15} and the shifted trace-log equals $\log(1 + \lambda_{\min}) + \log 2$ to 2.2×10^{-15} . With no shift the branch point of the logarithm

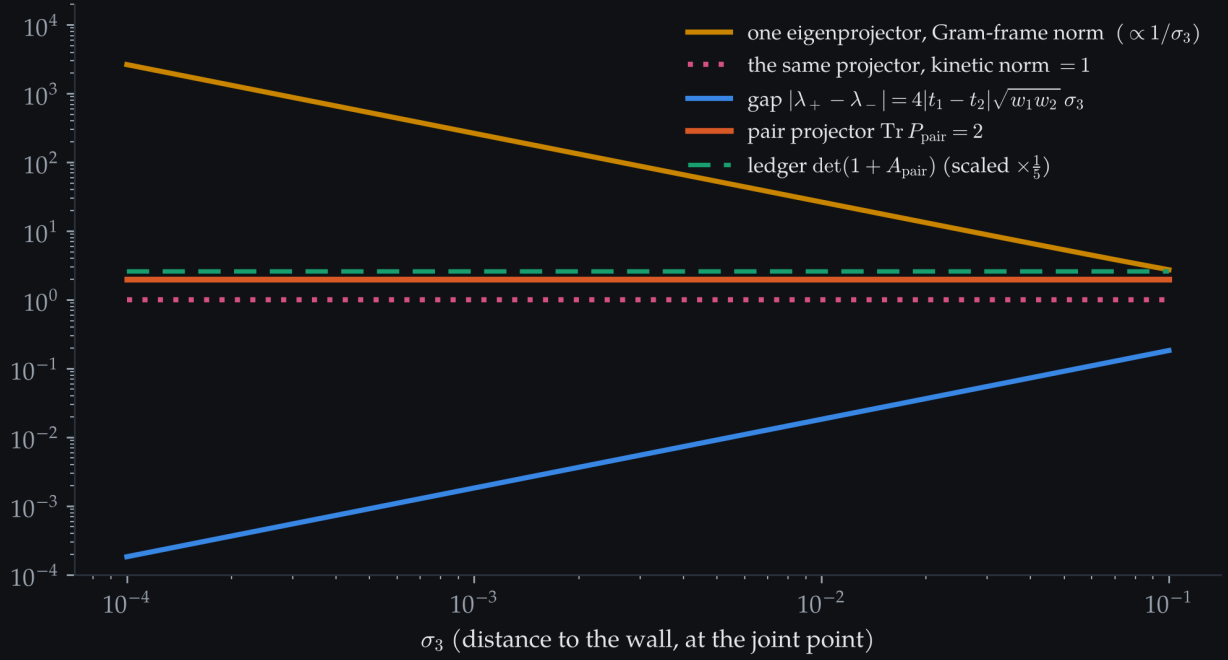


Figure 4: The record outlives the metric. Approaching the joint point along σ_3 : one eigenprojector measured in the Gram frame diverges like $1/\sigma_3$ while the gap closes linearly; measured in the kinetic metric the same projector has norm 1. The pair projector keeps trace 2 and the ledger $\det(1 + A_{\text{pair}})$ stays finite all the way to the wall, in every frame.

sits inside the contour and the ledger fails, reaching -0.580 against the correct minimum $\log 2$ [72] [CITED MEASUREMENT]. The hypothesis doing the work is a branch condition, not positivity: $\log(c + z)$ must have a single-valued holomorphic branch on a region containing Γ and the enclosed eigenvalues, which keeping the contour off the spectrum does not by itself give.

4.2 The orientation keeps the trace

A configuration crossing the coplanar wall changes orientation. The Gram observer cannot see it, because G depends on σ_3^2 only, but two other objects carry the record. The sign of $\det X$ is one. The other is the analytic labelling of the eigenvalue pair: by Theorem 7 the branch that was λ_+ before the crossing continues analytically into the branch the Gram observer calls λ_- . The encounter leaves a $\mathbb{Z}_2$ record, readable as a monodromy, while every Gram-level quantity returns to its starting value. The same fold appears one dimension down. For three bodies in the plane, the three normalised mutual distances keep the shape of the triangle up to reflection and lose the sign of its oriented area: a triangle and its mirror image share one distance record, and their two sheets meet at the collinear configurations [41].

The handed record cancels across the mirror. The pairing $N - \text{Tr}(A^\dagger B)$ splits into a symmetric part, which is a distance and cannot tell a configuration from its mirror image, and an antisymmetric part. On projector tangents the antisymmetric part is $\Omega(V, W) = \text{Tr}(P[V, W])$. For a real orthogonal projector family it vanishes identically, since $\text{Tr}(PVW) = \text{Tr}((PVW)^\top) = \text{Tr}(WVP) = \text{Tr}(PWV)$.

In the Gram frame the wall's eigenprojector P_+ is oblique (it is orthogonal only for the kinetic metric, which varies from point to point), and there Ω does not vanish. Because P_+ depends on σ_3 only through $w_3 = \sigma_3^2$, $\partial_{\sigma_3} P_+ = 2\sigma_3 \partial_{w_3} P_+$, so $\Omega(\partial_{\sigma_3} P_+, \partial_{\delta} P_+)$ is odd under the mirror [DERIVED]. At $t = (2, 1, 1.3)$, $\delta = 0.4$:

$ \sigma_3 $	Ω at $+\sigma_3$	Ω at $-\sigma_3$	symmetrised operator
0.1	-27.257764	+27.257764	$< 10^{-11}$
0.2	-6.799872	+6.799872	$< 10^{-11}$
0.4	-1.685527	+1.685527	$< 10^{-11}$

[MEASURED]. Each side of the wall is handed in the Gram frame, the two sides cancel exactly, and the handedness lives only in the split of the pair: on the pair projector, which is the identity on the shape block, $\Omega = 0$. This is the behaviour of an object fixed by an involution, where every reading odd under the involution sums to zero while the even readings survive. Section 4.6 shows the same rule in knot invariants.

4.3 The sign the cover keeps

Theorems 7 and 8 settle the eigenvalues. The eigenvectors carry one more piece of information, a sign, and where that sign lives depends on which loop is taken and in which chart. Transport right and left eigenvectors biorthogonally, keeping $u \cdot v = 1$ and moving v by the connection

$$a = \frac{u \cdot dv}{u \cdot v}, \quad v \mapsto v e^{-\int a}. \quad (17)$$

A change of frame $v \mapsto Fv$, $u \mapsto F^{-\top} u$ leaves $u \cdot v$ unchanged and adds to the connection

$$a \mapsto a + \frac{u \cdot (F^{-1} dF) v}{u \cdot v}. \quad (18)$$

The added term need not be exact, even for single-valued invertible F : a rotation $F(\theta)$ with constant $u = (1, 0)$, $v = (1, 1)$ adds $\oint = -2\pi$ [DERIVED; CHECKED, V10-CORRECTIONS]. So the transported factor belongs to the operator and the frame together, and each chart below is named. For a diagonal frame change the added term is $\sum_i \frac{u_i v_i}{u \cdot v} d \log F_i$, the logarithmic derivative of each frame entry weighted by the share of the pairing it carries. The frame J of (13) is diagonal with a simple zero at the wall, and this term is where the two charts part company.

Theorem 10 (two loops, two charts). [DERIVED] *On the joint condition (10), write $c = t_1 + t_2 - 2t_3$, $Q(\sigma_3) = (t_1 - t_2)^2(1 - \delta^2) + \frac{1}{4}c^2\sigma_3^2$ and $p = \sqrt{Q} + \frac{1}{2}c\sigma_3$.*

- (i) *Gram frame, complex loop. The analytic eigenvectors $v = (A_{12}, \lambda - A_{11})^\top$ and $u = (A_{21}, \lambda - A_{11})^\top$ of the branch $\lambda = \frac{1}{2}(A_{11} + A_{22}) + \sigma_3\sqrt{Q}$ satisfy*

$$u \cdot v = 2\sigma_3^2 \sqrt{Q} p, \quad a_G = \frac{p + \sigma_3 p'}{2\sigma_3 \sqrt{Q}} d\sigma_3, \quad (19)$$

a simple pole with residue exactly $\frac{1}{2}$. One loop of σ_3 returns each eigenvector multiplied by -1 ; one loop of w_3 acts by a monodromy M with $M^2 = -I$ and $M^4 = I$.

- (ii) The same loop on the cover. For the lifted operator B the connection a_{reg} is holomorphic at the wall point and the same loop returns $+1$. The two connections differ by

$$a_G = a_{\text{reg}} + \left. \frac{u_2 v_2}{u \cdot v} \right|_B \frac{d\sigma_3}{\sigma_3}, \quad (20)$$

where $u_2 v_2 / (u \cdot v)$ is the share of the eigenvector pairing carried by the folded direction ∂_{σ_3} . At the wall point it equals $\frac{1}{2}$ exactly, because B_1 in (15) has zero diagonal. The residue $\frac{1}{2}$ in (i) is therefore the simple zero of $\det J$ weighted by the half of each eigenvector that lies across the fold.

- (iii) Real loop through the mirror. On the cover, take a real loop in (σ_3, ε) that encircles the wall point. It must pass through $\sigma_3 < 0$, the mirror configurations. Along it the eigenvalues stay real and distinct, and a continuously chosen real eigenvector returns reversed: the eigenline has holonomy -1 . A real loop that does not encircle the point returns $+1$.
- (iv) The Gram observer cannot see (iii). The Gram image of the loop in (iii) lies in $w_3 \geq 0$ and retraces itself, so it returns $+1$. No loop in $w_3 \geq 0$ encircles the wall point.

Proof. (i) On the joint condition A_{12} is constant and $A_{22} - A_{11} = c \sigma_3^2$, so $\lambda - A_{11} = \sigma_3 p$ and $\sqrt{3}(t_1 - t_2)A_{12} = (t_1 - t_2)^2(1 - \delta^2) = Q - \frac{1}{4}c^2\sigma_3^2$. Then $u \cdot v = A_{12}A_{21} + \sigma_3^2 p^2 = \sigma_3^2(2Q + c\sigma_3\sqrt{Q}) = 2\sigma_3^2\sqrt{Q}p$. Since $dv = (0, d(\sigma_3 p))$, $u \cdot dv = \sigma_3 p(p + \sigma_3 p') d\sigma_3$. The residue is $p(0)/(2\sqrt{Q(0)}) = \frac{1}{2}$, so $\oint a_G = i\pi$ and $e^{-\oint a_G} = -1$. A loop of w_3 is half a loop of σ_3 , carrying λ_+ into λ_- , so $M^2 = -I$ and $M^4 = I$. (ii) The eigenvectors of $A = JBJ^{-1}$ are $v_G = Jv$ and $u_G = J^{-\top}u$ for eigenvectors v, u of B , so $u_G \cdot v_G = u \cdot v$ and $u_G \cdot dv_G = u \cdot dv + u \cdot (J^{-1}dJ)v$. This is (18) with $F = J$, and $J^{-1}dJ = \text{diag}(0, d\sigma_3/\sigma_3)$ gives (20). B is analytic with $B(0) = \lambda_* I$, so its eigenvectors near 0 are analytic continuations of those of B_1 , the pairing is nonzero, and a_{reg} is holomorphic. For a 2×2 matrix with zero diagonal and off-diagonal entries β, γ , with eigenvalue μ , $\mu^2 = \beta\gamma$, $v = (\beta, \mu)$ and $u = (\gamma, \mu)$, so $u \cdot v = 2\mu^2$ and $u_2 v_2 = \mu^2$, a share of $\frac{1}{2}$. (iii) In coordinates orthonormal for the pulled-back metric at the wall point, $\sigma_3 B_1 + \varepsilon B_2$ has traceless part $-\varepsilon \sigma_z + \kappa \sigma_3 \sigma_x$ (Pauli matrices) with $\kappa = |t_1 - t_2|\sqrt{1 - \delta^2}$, a real symmetric family whose eigenvector turns through half the angle that $(\varepsilon, \kappa \sigma_3)$ turns. One full turn of the parameter point reverses the eigenvector. (iv) $w_3 = \sigma_3^2$ maps the loop onto an arc in $w_3 \geq 0$ traversed out and back, which is contractible away from the wall point. $\square$

The check BH-LIFT-05 confirms each part at $t = (2, 1, 1.3)$, $\delta = 0.4$. For a complex loop of radius 0.02 in σ_3 : Gram frame -1.00012 , regular frame $+1.00000$. The folded share at the wall point is 0.500000000000 on both branches. For a real loop of radius 0.05 in (σ_3, t_3) around $(0, 1.3)$, the eigenline orientation in the kinetic metric is -1.0000000000 on the cover and $+1.0000000000$ in the Gram frame. For the same loop moved to centre $\sigma_3 = 0.2$ it is $+1$ in both [MEASURED]. Figure 5 shows all three.

So the cover does keep a sign. It is not the complex-loop sign, which is the trace the fold leaves on the Gram observer's eigenvectors, half a unit of flux per simple zero of $\det J$. It is the sign of a conical intersection of a real self-adjoint family. That is the Herzberg–Longuet–Higgins sign of molecular physics [21], the real case of Berry's phase [22]. Here it is carried around a point that only the oriented configuration space can encircle, because the loop has to pass through the mirror. The order-four monodromy around a generic exceptional point, with its geometric phase, is known and has been observed in microwave cavities [16, 17]. What is specific here is that both signs occur at the same point and are told apart exactly by the fold.

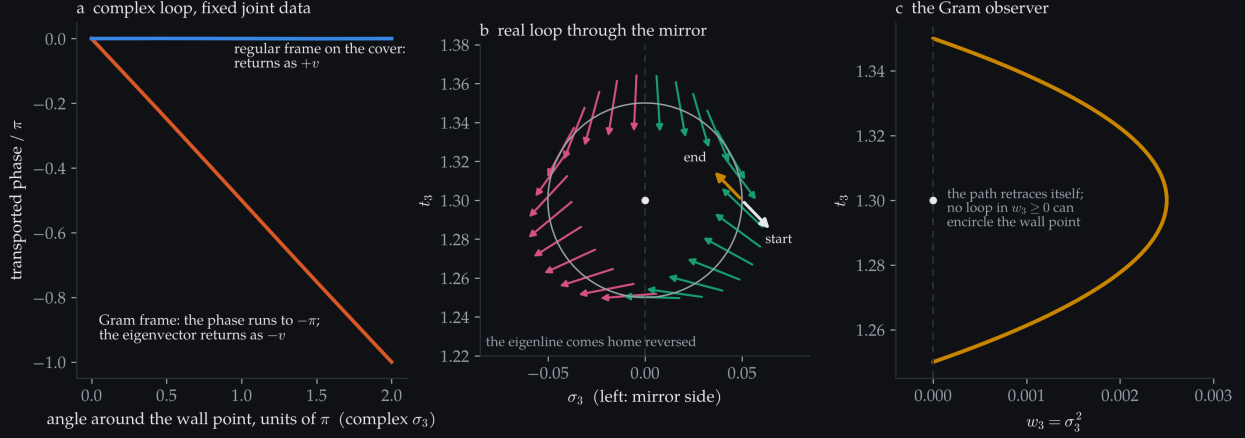


Figure 5: Two loops, two charts, at $t = (2, 1, 1.3)$, $\delta = 0.4$. (a) A complex loop of σ_3 about the wall point: the transported phase runs to $-\pi$ in the Gram frame and stays at 0 on the cover. (b) A real loop on the cover in (σ_3, t_3) about the wall point, passing through the mirror side $\sigma_3 < 0$. The eigenline (kinetic-orthonormal coordinates) turns half a turn and comes home reversed. (c) The Gram observer's image of the same loop lies in $w_3 \geq 0$ and retraces itself, so it carries no sign.

Beside a Majorana exchange. Exchanging two Majorana modes γ_1, γ_2 acts by $R = (1 + \gamma_1 \gamma_2) / \sqrt{2}$, that is $R = e^{\pi \gamma_1 \gamma_2 / 4}$ [18]. Then $R^2 = \gamma_1 \gamma_2$, $(R^2)^2 = -I$ and $R^8 = I$, relations checked on four modes in the Jordan–Wigner representation [MEASURED]. The Gram-frame monodromy M of a w_3 loop obeys the relations of the double exchange R^2 : it exchanges, $M^2 = -I$ and $M^4 = I$. A Majorana pair is fixed by charge conjugation, and the wall point is fixed by the mirror. The comparison is structural. Unlike a Majorana sign, which belongs to the representation, the Gram monodromy belongs to a chart. The chart-free sign at this point is the real conical one of part (iii) [STRUCTURAL COMPARISON].

4.4 Majorana stars and the Petrov types

Majorana represented a spin- j state by $2j$ points on the sphere, the roots of a polynomial [23]. A Weyl tensor is a spin-2 object, and its four points are its principal null directions. The algebraic classification of spacetimes is the classification of how those four stars coincide [24, 25]. This gives the black-hole side a precise object with which to compare the two readings of our wall point.

Represent the Weyl spinor by the traceless complex symmetric 3×3 matrix Q acting on self-dual bivectors, and its stars by the roots of

$$\psi_Q(z) = n(z)^\top Q n(z), \quad n(z) = (1 - z^2, i(1 + z^2), -2z), \quad n \cdot n = 0. \quad (21)$$

Because $n(z)$ is null, the trace part of Q drops out, and $Q \mapsto \psi_Q$ is linear from the five traceless directions to binary quartics.

Proposition 11 (stars fix the Jordan form). [DERIVED; CHECKED, BH-PETROV-06] *The map $Q \mapsto \psi_Q$ is invertible; in the basis $\text{diag}(1, -1, 0)$, $\text{diag}(1, 1, -2)$, $e_i e_j^\top + e_j e_i^\top$ its determinant is 3072. The multiplicities of the four stars fix the Jordan structure of Q :*

<i>stars</i>	<i>Petrov type</i>	<i>Jordan structure of Q</i>
<i>four distinct</i>	<i>I</i>	<i>three distinct eigenvalues, diagonalisable</i>
<i>one double</i>	<i>II</i>	<i>one 2×2 Jordan block</i>
<i>two double</i>	<i>D</i>	<i>eigenvalues $(\lambda, \lambda, -2\lambda)$, diagonalisable</i>
<i>one triple</i>	<i>III</i>	<i>one nilpotent 3×3 block</i>
<i>one quadruple</i>	<i>N</i>	<i>nilpotent, $Q^2 = 0$, blocks $(2, 1)$</i>

The correspondence is the classical one [26]; the check builds Q from prescribed stars and reads the Jordan structure from ranks of powers.

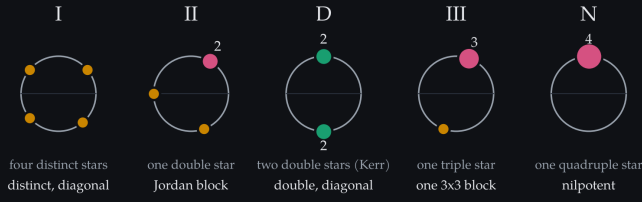
How each degeneracy unfolds. Split the double star of a type II quartic by a separation δ . The quartic's coefficients move by $\epsilon \propto \delta^2$, and the gap of the colliding pair of Q grows like $\sqrt{\epsilon}$, that is, linearly in δ : $\text{gap}/\delta = 0.800000$ at $\delta = 10^{-1}, \dots, 10^{-4}$ for the configuration in the check [MEASURED]. The coefficients are the even observable and the star separation is the signed coordinate. This is the fold again, now in the space of Weyl tensors. Type II is a genuine Jordan block of Q ; the stars unfold its square-root branch without removing the block. Split both double stars of a type D quartic symmetrically instead, and the double eigenvalue of Q splits quadratically: $\text{gap}/\delta^2 = 0.250000$ [MEASURED]. Type D is diagonalisable, so its gap is linear in the quartic's coefficients, and those move by δ^2 : the degeneracy is protected to first order.

Kerr. In the Kinnersley tetrad the only nonzero Weyl component of the Kerr metric is $\Psi_2 = -M/(r - ia \cos \theta)^3$ [27], so its stars are two doubled pairs at every point. The check builds Q at three points with $M = 1, a = 0.7$ and finds type D each time, eigenvalue ratio -2 to 10^{-9} [MEASURED]. The Kerr exterior, horizon included, is a type D object: a degenerate but diagonalisable Weyl operator, a diabolic rather than an exceptional degeneracy. The marginally trapped surface operator says less: its principal eigenvalue is always real and simple [8], which rules out a defect at that eigenvalue only.

The comparison now reads cleanly. Our wall point read in the Gram frame looks like type II: a Jordan block with a square-root branch in the even coordinate, unfolded by the signed one. Read on the cover it is a diabolic point, like type D. The gap laws differ, which is a real difference: our cover point is a cone, linear in both directions, while D splits quadratically under the symmetric splitting (Figure 6). The two operators are different objects. The comparison is between Jordan structures and unfolding exponents, not an identification of a Weyl tensor with a transport operator [STRUCTURAL COMPARISON].

The three-body shape sphere shows why the two gap laws differ. With equal masses, the equilateral configuration is fixed by the $\mathbb{Z}_3$ relabelling of the bodies, and there the linear splitting vanishes: the gap closes quadratically ($\text{gap}/\vartheta^2 \rightarrow 1.000000$) and the soft eigendirection returns with holonomy $+1$. Break the symmetry with masses $(1, 2, 3)$ and the degeneracy becomes a genuine cone: the gap closes linearly ($\text{gap}/r \rightarrow 2.16280$) and the eigenvector returns negated, -1.000000 at every radius tested [72] [CITED MEASUREMENT]. The two quadratic laws have different causes. For Kerr's type D it is the one above, a diagonalisable degeneracy whose coefficients move by δ^2 . On the shape sphere it is a property of this operator at a $\mathbb{Z}_3$ -fixed point, not of the symmetry alone: a degeneracy fixed by $\mathbb{Z}_3$ can also split linearly, as in the $E \otimes e$ Jahn–Teller problem [94], and the holonomy $+1$ is the mark that the linear term vanishes here. Breaking the symmetry gives the generic cone, which is what our cover point is [STRUCTURAL COMPARISON].

a four Majorana stars fix the Jordan form of the Weyl operator



Kerr: only $\Psi_2 = -M/(r - i a \cos \theta)^3$ is nonzero, so the stars sit in two doubled pairs at every point (type D), and the Weyl operator has eigenvalues $(\lambda, \lambda, -2\lambda)$ with no Jordan block.

Small circles: the celestial sphere seen edge-on; dot size grows with multiplicity.

b how each degeneracy unfolds

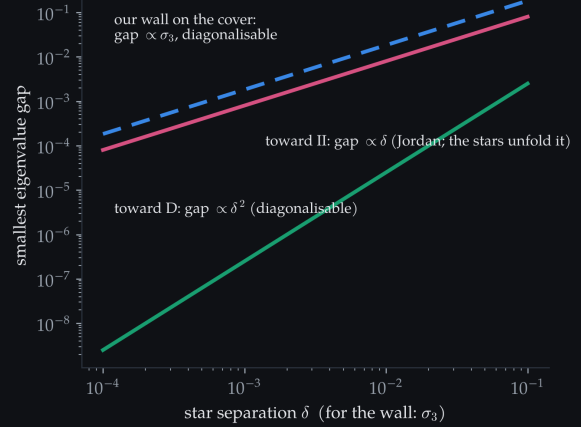


Figure 6: Majorana stars and Petrov types. (a) The multiplicities of the four principal null directions fix the Jordan form of the Weyl operator Q ; Kerr is type D at every point. (b) How the degeneracies unfold: toward type II the gap is linear in the star separation (the stars unfold a genuine Jordan block); toward type D it is quadratic (diagonalisable). The wall point read on the cover is diagonalisable with a linear gap, a cone.

4.5 Ramanujan's shadow: the divergence cancels in the pair

In his last letter to Hardy, Ramanujan described functions he called mock theta functions by their behaviour at the roots of unity. One of them is

$$f(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2}, \quad b(q) = \frac{(q; q)_\infty}{(-q; q)_\infty} = (1 - q)(1 - q^3)(1 - q^5) \cdots (1 - 2q + 2q^4 - \cdots). \quad (22)$$

As q approaches a root of unity ζ of even order $2k$ along the radius, f and the modular function b each blow up, but the combination $f(q) - (-1)^k b(q)$ stays bounded. Folsom, Ono and Rhoades proved this and identified the limit as a finite sum at the root of unity [28]:

$$\lim_{q \rightarrow \zeta} (f(q) - (-1)^k b(q)) = L(\zeta) = -4 \sum_{n \geq 0} (-\zeta; \zeta)_n^2 \zeta^{n+1}, \quad (23)$$

which terminates as soon as a factor $1 + \zeta^j$ vanishes. The bounded value is the visible trace of the mock theta function's shadow, the modular correction that makes it a harmonic Maass form [29, 30].

BH-SHADOW-07 computes both functions along the radius at working precision set from their size, using Gauss's and Euler's series for b [MEASURED]:

ζ	$1 - q $	$\log_{10} f $	$f - (-1)^k b$	$L(\zeta)$
-1	10^{-3}	180.26	3.996012	4
-1	3×10^{-4}	597.24	3.998801	
i	3×10^{-4}	150.67	$0.002390 + 3.994025 i$	$4i$
$e^{i\pi/3}$	3×10^{-4}	67.91	$-1.988709 - 3.450721 i$	$-2 - 2\sqrt{3} i$

Two numbers of size 10^{597} cancel to 3.9988. A three-point extrapolation in $1 - |q|$ gives 3.9999999, $0.0000022 + 3.9999947i$ and $-1.9999547 - 3.4640476i$, against $L(\zeta) = 4$, $4i$ and $4e^{-2\pi i/3}$.

The pattern is the one of Section 4. Each member of a pair diverges at the special point, and the pair has a finite, exact value there. At the wall the individual eigenprojectors diverge in the Gram frame and the pair projector and ledger stay finite. At the root of unity f and b diverge and their signed difference lands on a finite sum. For f itself the analogy is one of bookkeeping, not mechanism: this mock theta function has no transport operator behind it [STRUCTURAL COMPARISON]. In the counting of black-hole microstates, however, mock modularity does come from a wall.

Where black holes meet mock modularity. Dabholkar, Murthy and Zagier showed that the function counting quarter-BPS black-hole states in $\mathcal{N} = 4$ string theory splits canonically into two parts [58]. One is a mock Jacobi form, whose coefficients count single-centred black holes, which are immortal. The other is an Appell–Lerch sum, which counts multi-centred bound states. Those bound states come apart when the moduli cross walls of marginal stability, and that wall-crossing is where the mock behaviour comes from. Adding the shadow’s non-holomorphic completion restores the modular symmetry that holography expects. Their family contains almost all known mock modular forms, Ramanujan’s among them. So the pattern of this subsection has a black-hole instance with a mechanism: a single piece fails a symmetry, and the piece that restores it is exactly what a wall crossing removes. These walls lie in the moduli space of the theory, not in spacetime and not at a horizon. They are a third kind of wall, beside the two of Section 8, and no identification with the coplanar fold is made [CITED RESULT; STRUCTURAL COMPARISON].

The mirror joins this subsection to the next. Reversing the orientation of a three-manifold sends q to q^{-1} in its quantum invariants, which is the knot mirror of Section 4.6. For the Brieskorn spheres the two orientations land on two different kinds of function. For one orientation the q -series invariant $\hat{Z}$ is a false theta function. For the reversed orientation it is conjectured to be a mock theta function, with $\Sigma(2, 3, 7)$ worked out in detail [59, 60]. On the false-theta side, the strong quantum-modularity conjecture for the Witten–Reshetikhin–Turaev invariants of every Brieskorn homology sphere is now proved [61]. These spheres are built from the knots of the next subsection: $1/n$ surgery on the torus knot $T(p, q)$ gives $\Sigma(p, q, |pqn - 1|)$ [62], so ± 1 surgery on the trefoil gives the Poincaré sphere $\Sigma(2, 3, 5)$ and $\Sigma(2, 3, 7)$ [DERIVED; BH-KNOT-14]. The orientation reversal that separates the false side from the mock side is the same sign flip that carries a configuration to its mirror image at the coplanar wall (Theorems 2 and 10). Two cautions: the mock functions in this link are of Ramanujan’s seventh order, not the third-order f above, and the mock side is a conjecture supported by computation. Neither is a transport operator [CITED RESULT; STRUCTURAL COMPARISON].

4.6 The mirror rule in knots

Quantum knot invariants evaluated at roots of unity show the mirror rule of Section 4 with no transport at all. Take $q = e^{2\pi i/N}$. The Kashaev invariant of the figure-eight knot is $\langle 4_1 \rangle_N = \sum_{k=0}^{N-1} |(q; q)_k|^2$, and the Kontsevich–Zagier series $F(q) = \sum_{n \geq 0} (q; q)_n$ terminates at a root of unity and gives the Kashaev invariant of the trefoil up to conjugation [32, 31]. The mirror replaces q by q^{-1} , which conjugates both.

Proposition 12 (the mirror rule, checked). [MEASURED, BH-KNOT-08]

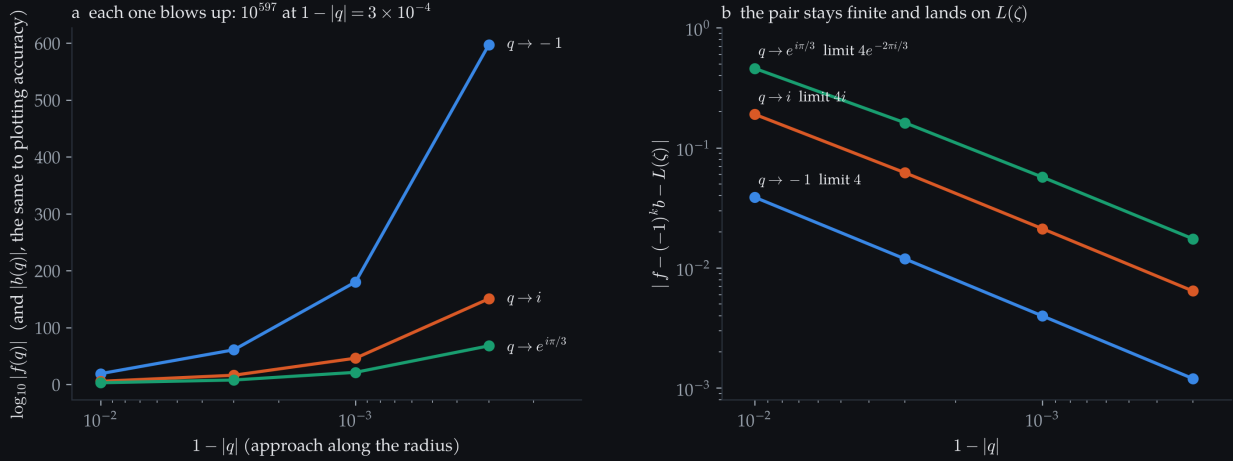


Figure 7: Ramanujan's radial limits. (a) Along the radius toward -1 , i and $e^{i\pi/3}$, the mock theta function f and its modular partner b each blow up (to 10^{597} at $1 - |q| = 3 \times 10^{-4}$ toward -1). (b) The signed difference $f - (-1)^k b$ converges to the finite sum $L(\zeta)$, with error proportional to $1 - |q|$.

- (i) The figure-eight knot is amphichiral. Its invariant is real (imaginary part 10^{-13} of the real part at $N = 200$), its signature is 0, and it grows exponentially with rate set by its hyperbolic volume: with the known $N^{3/2}$ and $3^{-1/4}$ factors removed, $\frac{2\pi}{N} \log \langle 4_1 \rangle_N$ gives 2.029889 at $N = 800$, against $\text{Vol}(4_1) = 2.029883$ [33].
- (ii) The trefoil is chiral. Its invariant is not real, the mirror conjugates it, its signature changes sign (-2 and $+2$), and it grows only like a power: $|F(e^{2\pi i/N})|/N^{3/2} = 1.000012$ at $N = 800$. Its complement has no hyperbolic volume.
- (iii) The torus knot $T(3, 4)$, which is 8_{19} , is chiral as well, and it shows that the trefoil's row is a class. Its Alexander polynomial is $t^6 - t^5 + t^3 - t + 1$, its genus is 3, and its Jones polynomial is $q^3 + q^5 - q^8$ up to the mirror [DERIVED]. Its invariant at $q = e^{2\pi i/N}$, computed from the Rosso–Jones formula after the formula reproduces the unknot, both Jones polynomials and the trefoil series above, grows like a power with no volume, and the mirror conjugates it. The constant in front of $N^{3/2}$ is where it differs from the trefoil: it depends on N modulo 16, taking sixteen values from 0.0858 at $N \equiv 0$ to 2.9142 at $N \equiv 8$, symmetric under $r \mapsto 16 - r$ and repeating to 10^{-4} along each class [MEASURED, BH-KNOT-14]. The trefoil has a single constant. Its ± 1 surgeries give $\Sigma(3, 4, 11)$ and $\Sigma(3, 4, 13)$ [62]. A molecular 8_{19} has been made by self-assembly [64], and the modular structure of torus-knot invariants is treated in [63].

At a configuration fixed by the mirror, every reading odd under the mirror vanishes (the signature, the imaginary part, the handed part Ω summed across the wall), while the even readings survive, and the volume and the ledger are among them. The same rule appears in a fermion chain: the boundary offset of a half-filled Rice–Mele chain is exactly the part odd in the staggering, and a boundary-preserving particle–hole conjugacy protects its zero [57]. Figure 8 shows the three knots.

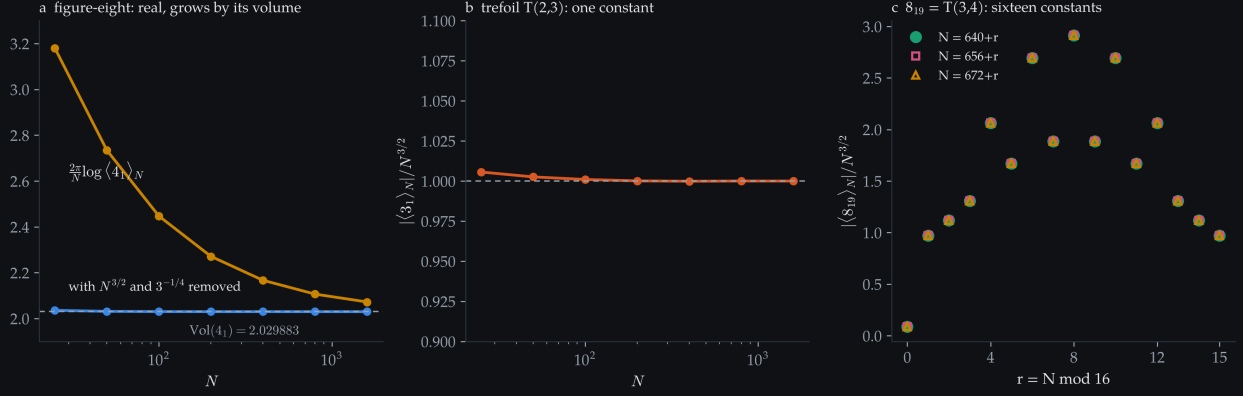


Figure 8: Knots at roots of unity. (a) The figure-eight knot, fixed by the mirror: its Kashaev invariant is real and grows at the rate of its hyperbolic volume 2.029883. (b) The trefoil, chiral: its invariant grows like $N^{3/2}$ with a single constant and no exponential rate, and the mirror conjugates it. (c) The torus knot $8_{19} = T(3,4)$: also power growth with no volume, but the constant depends on N modulo 16; three runs of sixteen consecutive N fall on the same sixteen values.

4.7 Transients and the transverse oscillator

On the black-hole side the transient story is the literature’s [1, 5, 4, 6]. On the shape side, a near-defective block in a given frame sizes its own transients exactly. For $J_\mu = \begin{pmatrix} b & \mu \\ 0 & b \end{pmatrix}$,

$$\|e^{-J_\mu t}\|_2 = e^{-bt} \frac{\sqrt{4 + \mu^2 t^2} + |\mu|t}{2}, \quad (24)$$

so transient growth above 1 occurs exactly when $|\mu|/b > 2$ (Appendix A). At the joint point, read in the Gram frame, $|\mu|/b = 0.4850/2.6 = 0.1865$: there is no transient amplification, only a slowed decay [DERIVED; VALUE MEASURED]. In the kinetic metric the operator is self-adjoint and $\|e^{-A t}\| = e^{-\lambda_{\min} t}$, with no transient at all. The choice of norm is structural on both sides [4]. The classical bound on such transients is Kreiss’s [88].

Three refinements come from companion papers. The ε -pseudospectrum of J_μ is exactly the disk $|\lambda - b| < \sqrt{\varepsilon|\mu| + \varepsilon^2}$. An input confined to the interface coordinate, the model of a prescribed hit rather than the optimal state, grows above its starting size only for $|\mu|/b > \kappa_r = 2.48527$, above the optimal-state threshold 2; a black hole struck by infalling matter receives a prescribed kick, so κ_r is the relevant test. And the same block is approached at two rates: diagonal detuning collapses the eigenvector linearly, a reverse coupling collapses it as a square root [73] [CITED; DERIVED THERE]. On a real orbit, the figure-eight amplifies perturbations 45.7-fold in energy within one period, with mean gain 10.58 in the quartile of the orbit nearest the collinear wall and 26.10 in the quartile farthest from it [72] [CITED MEASUREMENT]. Transients live away from the geometric wall, as they do here. Persistence of structure over vectors is measured by the overlap $C(\Delta) = \text{Tr } P(t)P(t + \Delta)$: observable channels are those with $C(\Delta) \rightarrow \text{rank } P$.

Where a confining stiffness ω^2 acts across the wall, the transverse motion $\ddot{y} + \omega^2 y = 0$ at fixed energy $E = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\omega^2 y^2$ has displacement amplitude $\sqrt{2E}/\omega$, velocity amplitude $\sqrt{2E}$, and accumulated phase $\int \omega dt$ [DERIVED]. A stiffness growing without bound therefore suppresses the displacement as ω^{-1} at fixed energy, while the velocity amplitude is fixed and the phase keeps counting. On Σ_4 the kinetic metric supplies no such stiffness, since curvature is finite in σ_3 . What

looks like suppression in the Gram coordinate is the fold law (4). A physical potential that stiffens toward coplanarity would be extra input, and the statement above is its conditional form.

Ordinary physics supplies that input. Any potential built from the mutual distances is a function of G , because the squared distances are linear in its entries, and in the fold chart G depends on σ_3 only through $w_3 = \sigma_3^2$. Such a potential is therefore even under orientation reversal. The wall is always a transverse critical point. Along the normal lift, which moves only σ_3 and keeps the in-plane shape and the moment of inertia fixed, the stiffness is

$$\omega^2 = \frac{\partial^2 V}{\partial \sigma_3^2} \Big|_{\sigma_3=0} = 2 \frac{\partial V}{\partial w_3} \Big|_{w_3=0} \quad (25)$$

[DERIVED]. The factor 2 is the chain rule through $w_3 = \sigma_3^2$, and since the metric in the signed coordinate is $d\sigma_3^2$ at the wall (Theorem 2), ω is a frequency in the kinetic metric. Two realisations follow.

A thin disk. Bodies held near a plane by an external field feel $V = \frac{1}{2}\nu^2 \sum_i m_i z_i^2$, with the centre of mass in the plane. The sum is the zz entry of $XX^\top$, whose eigenvalues are those of G , so its minimum over orientations is the smallest of them, w_3 . At that orientation $V = \frac{1}{2}\nu^2 \sigma_3^2$: the transverse frequency of the shape is exactly the disk's vertical frequency ν , and at fixed energy the out-of-plane amplitude falls as ν^{-1} [DERIVED; CHECKED, BH-QUTRIT-18]. The check confirms the minimum to 2×10^{-16} and the identity $\sum_i m_i z_i^2 = (XX^\top)_{zz}$ on explicit bodies.

Newtonian gravity. At fixed moment of inertia the shape potential of four equal masses, $U = -\sum_{i<j} 1/r_{ij}$ with unit masses and unit gravitational constant, stiffens toward coplanarity on an open set of flat shapes: $\partial U/\partial w_3 > 0$ at all 400 sampled flat configurations, the smallest value +1.89, and at all of 20 000 more [MEASURED, BH-QUTRIT-18, BH-STIFF-19]. Like charges do the opposite, $\partial U/\partial w_3 < 0$ at every sample, so for them the wall is a transverse maximum. The sign is decided shape by shape. A flat shape of four unit masses has exactly one out-of-plane shape direction c , orthogonal to the constant vector and to both in-plane coordinates; lifting along c and rescaling back to $I = 1$ gives

$$\frac{\partial U}{\partial w_3} = \frac{1}{2} \left[\sum_{i<j} \frac{(c_i - c_j)^2}{|c|^2 r_{ij}^3} - \sum_{i<j} \frac{1}{r_{ij}} \right] \quad (26)$$

[DERIVED; CHECKED, BH-STIFF-19]. The first term is the lift stretching the pairs that c separates, the second the cost of shrinking back to fixed I . The rhombus shows that the second term can win. With diagonals in ratio $k \leq 1$, every edge has length $1/\sqrt{2}$ at $I = 1$, and (26) becomes $\frac{1}{2} [4\sqrt{2} - \sqrt{(1+k^2)/2} (1+1/k)]$. It vanishes where $(1+k^2)(1+k)^2 = 64k^2$, at $k_* = 0.144557$ [DERIVED; CHECKED, BH-STIFF-19]. Gravity therefore stiffens the rhombus toward its plane only for $k > k_*$; an elongated rhombus with one close pair, $k < k_*$, sits on a transverse maximum, as like charges do (Figure 9b). The square is an exact control. Lifting alternate corners by $\pm h$ at $I = 1$ moves along the path of Theorem 2(ii) with $w_3 = 4h^2$, and expanding the six distances gives $U = U_0 + (2\sqrt{2} - 1) w_3 + O(w_3^2)$. By (25) the transverse frequency is

$$\omega = \sqrt{2(2\sqrt{2} - 1)} = 1.912290 \quad (27)$$

[DERIVED; CHECKED, BH-QUTRIT-18]. On the lift path the potential is exact,

$$U(\sigma_3) = -\frac{4\sqrt{2}}{\sqrt{1+\sigma_3^2}} - \frac{2}{\sqrt{1-\sigma_3^2}} = U_0 + (2\sqrt{2} - 1) \sigma_3^2 - \left(\frac{3\sqrt{2}}{2} + \frac{3}{4}\right) \sigma_3^4 + O(\sigma_3^6), \quad (28)$$

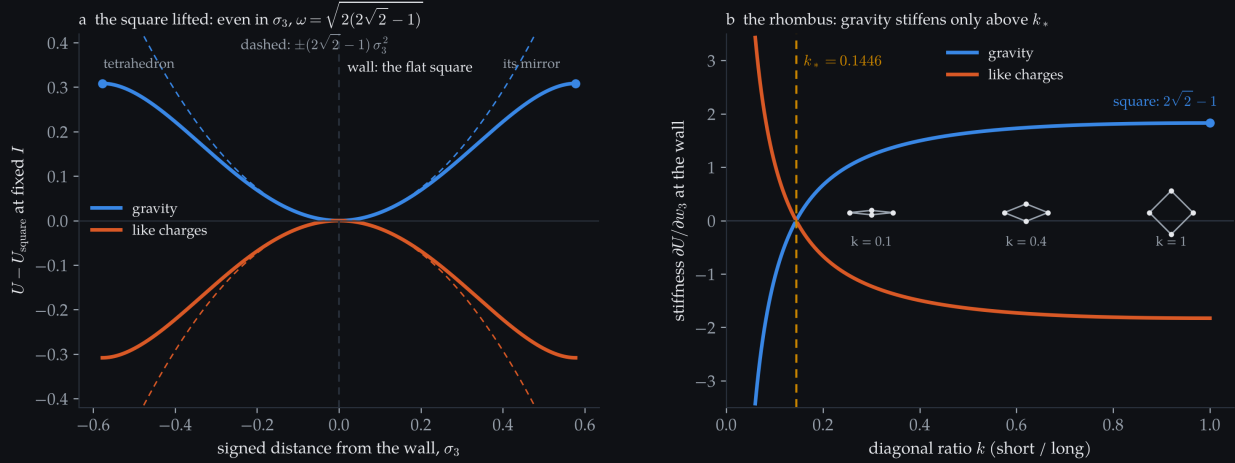


Figure 9: A potential that stiffens toward coplanarity. (a) The square lifted at fixed moment of inertia, in the signed coordinate σ_3 . Gravity (blue) and like charges (orange) are both even in σ_3 , as every distance potential is; at the wall they follow $\pm(2\sqrt{2}-1)\sigma_3^2$ (dashed), so gravity holds the square with $\omega = \sqrt{2(2\sqrt{2}-1)}$ and like charges push it out. The ends are the regular tetrahedron and its mirror image, which have the same Gram matrix. (b) The rhombus with diagonals in ratio k , lifted by raising one diagonal and lowering the other: the stiffness (26) at the wall. Gravity stiffens only for $k > k_* = 0.144557$; like charges have the opposite sign throughout. Computed by BH-FIGS-V8-5 from the formulas of BH-QUTRIT-18 and BH-STIFF-19.

from the four edges of length $\sqrt{(1+\sigma_3^2)/2}$ and the two diagonals of length $\sqrt{1-\sigma_3^2}$. The quartic term is negative: gravity softens as the square lifts. A small kick of energy E lifts the square to $|\sigma_3| \approx \sqrt{2E}/\omega$, and a larger one lifts it further: $E = 0.1420$ reaches $\sigma_3 = 0.300$, where the harmonic estimate gives 0.279. The exact turning point solves $U(\sigma_3) - U_0 = E$ [DERIVED; CHECKED, V10-CORRECTIONS]. The climb ends at the regular tetrahedron, $\sigma_3 = \pm 1/\sqrt{3}$, a critical point of U at fixed I (a central configuration) lying $2 + 4\sqrt{2} - 6\sqrt{3}/2 = 0.3084$ above the square, one on each side of the wall: a mirror pair with the same G (Figure 9).

4.8 What fidelity measures on the shape space

Theorem 1 makes Σ_4 the state space of a real qutrit, with $\rho = G$. Two quantities of quantum information then acquire mechanical meanings, and a third shows which observer they belong to.

Fisher information is kinetic energy. The quantum Fisher information of a curve of states of constant rank is four times its Bures metric [91], so along any shape motion that keeps full rank $F_Q = 4g(\dot{G}, \dot{G})$. At unit moment of inertia the shape kinetic energy is $K = \frac{1}{2}g(\dot{G}, \dot{G})$, and therefore

$$F_Q = 8K : \quad (29)$$

the quantum Fisher information of a moving configuration is eight times its shape kinetic energy [DERIVED; CHECKED, BH-QUTRIT-18]. The check computes g without any formula, as the smallest $|\dot{X}|^2$ that produces $\dot{G}$, and finds $F_Q/g = 4$ to 10^{-11} on 200 random motions. At the wall the rank drops, and there the two part. Along $G(s) = \text{diag}(\frac{1-s^2}{2}, \frac{1-s^2}{2}, s^2)$, which crosses the wall at $s = 0$ at unit speed in $\sigma_3 = s$, $8K$ is 4.0404, 4.0004, 4.000004 and 4 at $s = 0.1, 0.01, 0.001$ and 0, and

the Fisher information of the symmetric logarithmic derivative matches it at every $s \neq 0$ and is exactly 0 at $s = 0$ [DERIVED; CHECKED, V10-CORRECTIONS]. This is the known discontinuity of the quantum Fisher information where the rank changes [98]. Kinetic energy is continuous through the fold and Fisher information is not: it reads only G , so it belongs to the Gram observer.

The wall distance is a Bures angle. With the fidelity $F(\rho, \sigma) = (\text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}})^2$ [92, 93], the coplanar configuration $\text{diag}(\frac{1}{2}, \frac{1}{2}, 0)$ and $W(\varepsilon)$ of Theorem 2(ii) commute, so $\sqrt{F} = 2\sqrt{\frac{1}{2} \cdot \frac{1-\varepsilon}{2}} = \sqrt{1-\varepsilon}$ and

$$\arccos \sqrt{F} = \arcsin \sqrt{\varepsilon}. \quad (30)$$

The finite distance to the wall is the Bures angle between the flat configuration and the lifted one [DERIVED; CHECKED, BH-QUTRIT-18], to 3×10^{-14} .

Fidelity is the Gram observer. A configuration and its mirror image have the same G , so their fidelity is exactly 1. On the orientation cover they are $2 \arcsin |\sigma_3|$ apart along the fold path through the wall [DERIVED; CHECKED, BH-QUTRIT-18]. Every function of G , fidelity included, is blind to orientation; only the cover separates the pair, as it keeps the sign of Section 4.3. The two tetrahedra at the ends of Figure 9 are such a pair.

5 A one-way boundary

5.1 Interface Schur complement

Partition a transport operator into a boundary block and the two sides it separates,

$$K = \begin{pmatrix} A & B & 0 \\ C & D_\Sigma & F \\ 0 & G & E \end{pmatrix}.$$

Eliminating both sides gives the effective boundary operator

$$K_\Sigma(z) = D_\Sigma - z - C(A - z)^{-1}B - F(E - z)^{-1}G, \quad (31)$$

in which the interior and exterior loadings enter additively (Appendix A). A boundary is one-way when one loading vanishes because one coupling does, that is, when the operator is block-triangular across it. That is what Theorem 6(i) gives at the coplanar wall in the Gram frame. The Schur load $A_{12}A_{21}/(z - A_{22})$ that the transverse direction places on the in-wall direction vanishes like σ_3^2 . Equivalently, the resolvent entry $(z - A)_{21}^{-1} = A_{21}/[(z - A_{11})(z - A_{22}) - A_{12}A_{21}]$ vanishes, while $(z - A)_{12}^{-1}$ does not [DERIVED].

The same elimination has a determinant version, proved in the author's offset paper [57]. For a Gaussian fermion state with regional covariance $0 < C < I$, the offset $\tau = \log \det(I - C) / \det C$ is the log-odds of an empty region against a full one. When one site is added to a protected even block, the determinant of the whole interior cancels, and $e^\tau = s_0/s_1$ is the ratio of two scalar Schur complements on the added site, its conditional vacancy and occupation. There the boundary holds the record exactly, as a ratio of two boundary numbers. The interior still enters those numbers through the inverse of its block, so this is a reduction to boundary data, not a proof that only a few sites matter [CITED RESULT, [57], THEOREM 8].

5.2 Matched load: the boundary that returns nothing

Let L be passive, $\text{Herm } L = \frac{1}{2}(L + L^*) \succeq 0$, with boundary and interior blocks b, i .

Proposition 13 (Schur–Cayley boundary). [DERIVED] *The boundary impedance $Z_b = L_{bb} - L_{bi}L_{ii}^{-1}L_{ib}$ satisfies $Z_b = W^*LW$ with $W = \begin{pmatrix} I \\ -L_{ii}^{-1}L_{ib} \end{pmatrix}$, so $\text{Herm } Z_b = W^*(\text{Herm } L)W \succeq 0$ with no normality assumption. The scattering matrix $S_b = (Z_b - I)(Z_b + I)^{-1}$ obeys*

$$I - S_b^*S_b = 4(WV_b)^* \text{Herm } L(WV_b) \succeq 0, \quad V_b = (Z_b + I)^{-1}, \quad (32)$$

so $\|S_b\| \leq 1$. The power not returned is the pullback of the interior dissipation, and $S_b = 0$ exactly for the matched load $Z_b = I$.

The proof is in Appendix A; the identities hold to 2.4×10^{-15} on random passive samples, with largest $\|S_b\| = 0.901$ [MEASURED, BH-SPINE-09]. A perfect absorber is a matched load. Two cautions come with this. No reflection is not full absorption, since transmitted, reflected and absorbed power must be counted separately. And a matched load is a statement about an impedance, not about causality. As a model of a one-way membrane it is the honest one, at the structural grade.

The horizon is a matched load. In the membrane description of a black hole, the stretched horizon carries an effective local surface resistivity $R_H = 4\pi$ in Gaussian units with $c = 1$, which is the impedance of empty space, 376.73Ω , and its tangential fields obey $\mathbf{B}_H = \mathbf{E}_H \times \mathbf{n}$, the relation of a plane wave in vacuum [42, 43, 44]. This is the resistance that closes the circuit in Levin’s black-hole pulsar, where a black hole moving through a magnetic field drives currents through its own horizon [45]. Normalised by the vacuum impedance the horizon load is $Z_b = 1$, and the Cayley map gives $S_b = 0$: the horizon returns nothing, the matched load of Proposition 13. Converting $4\pi/c$ from Gaussian units gives $376.7303135 \Omega = \mu_0 c$, a unit identity rather than a measurement, and $S_b = \mp \frac{1}{3}$ at $Z_b = \frac{1}{2}$ and 2 [DERIVED; CHECKED, BH-MATCHED-12]. The membrane sources state the resistivity and the plane-wave relation; reading them as zero reflection is the Cayley statement $S_b(1) = 0$, so the comparison is structural [STRUCTURAL COMPARISON, SOURCED]. Read the Smith chart as a Poincaré disk, where the distance from the centre, $2 \operatorname{artanh}|\Gamma|$, measures mismatch [76]: the matched horizon sits at the centre, at zero distance. The match belongs to the horizon surface, not to the hole seen from far away: outside the horizon the potential barrier reflects most long-wavelength radiation, and the greybody factors that count what crosses it vanish as $\omega M \rightarrow 0$ [95]. A wave that reaches the horizon is not returned; a wave from infinity mostly never reaches it. Rotation adds a separate effect: a spinning hole can return more than it receives in superradiant modes, which the local resistivity does not describe. The same chart carries the step reflection of Section 8.3 and the fermion record of Section 7, at other points (Figure 16b).

5.3 The contractive pullback, and one tensor doing two jobs

The scattering matrix S_b is a contraction, and there is a second way to say so that does not mention the interior. Let the incoming and outgoing waves be the two halves of an ambient space with the indefinite form $\mathcal{J} = \operatorname{diag}(I, -I)$, and let $\mathcal{G} = \begin{pmatrix} I \\ S_b \end{pmatrix}$ map an input to the pair (input, response), the graph of the response. Then

$$\mathcal{G}^* \mathcal{J} \mathcal{G} = I - S_b^*S_b = 4(WV_b)^* \text{Herm } L(WV_b). \quad (33)$$

[DERIVED; CHECKED, BH-SPINE-09], to 2.4×10^{-15} on random passive samples. The boundary's response graph pulls the ambient form back onto itself, and contractivity is the statement that this pullback is positive. The same quadratic form is also the pullback of the interior dissipation through the lift WV_b . Two pullbacks, one through the boundary's own graph and one through the interior, give the same form. In this precise sense the boundary subspace has a pullback of its own, and it agrees with the one the interior supplies. The graph reading is the operator-graph language of Nag and Sullivan [36]; a period-map interpretation would need more structure than is used here.

One tensor, two jobs. In general relativity one tensor, the metric $g_{\mu\nu}$, is both the ruler and the potential whose derivatives give the connection and the curvature. The premetric programme separates these, keeping the field equations metric-free and adding the metric later as a constitutive law [34]. The projector geometry used here takes the other route. For a family of projectors $P(x)$ the quantum geometric tensor

$$\mathcal{Q}_{ab} = \text{Tr}(P \partial_a P \partial_b P) = g_{ab} - \frac{i}{2} \Omega_{ab} \quad (34)$$

carries the metric in its real part and the Berry curvature in its imaginary part [35]. It is positive semidefinite, and positivity ties the two jobs together:

$$|\Omega_{12}| \leq 2\sqrt{\det g} \quad (35)$$

in any two parameters [DERIVED; CHECKED, BH-SPINE-09]; on 200 random two-parameter families of states the smallest slack is 1.0×10^{-3} . An independently assigned Hermitian form $g - \frac{i}{2} \Omega$ need not be positive even when g is, so a biorthogonal application has to verify positivity rather than inherit it from the notation. With an orthogonal projector family it is automatic. For two-level families (35) is an equality, because the geometric tensor of a rank-one projector in $\mathbb{C}^2$ has rank one; for three-level families it is strict, with median slack 37% on random samples [MEASURED, V10-PUSH]. Section 8.3 uses it on a pair made of a path direction and a symmetry direction. The contractive boundary (33) and the tensor (34) are the same idea at two scales: in each, a positive form is a pullback, and the positivity is what makes it a ruler.

5.4 The off-diagonal block and its square

Wherever a space is split into a kept part and a hidden part, the off-diagonal block B appears, and with it the positive operator $BB^\dagger$. Four exact identities, each one line, are used in this paper and in the author's related work [DERIVED; CHECKED ON RANDOM INSTANCES, BH-SPINE-09]:

- (a) For an orthogonal projector $P = \begin{pmatrix} C & B \\ B^\dagger & C' \end{pmatrix}$, the upper block of $P^2 = P$ gives $C - C^2 = BB^\dagger$, so $\text{Tr}(C^2 - C) = -\text{Tr} BB^\dagger \leq 0$. This is the boundary census of *Light Keeps the Ledger* (Theorem 5.1 there) [37], and the negative quadratic form $q(C) = -\text{Tr} BB^\dagger$ of the gravity edition [40].
- (b) For $H = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}$, eliminating the hidden block gives the memory kernel $K(t) = Be^{-iDt}B^\dagger$, whose footprint is $K(0) = BB^\dagger$, positive; for $t \neq 0$ the kernel need not be Hermitian [39].
- (c) With P the kept projection and $Q = I - P$, $PH^2P - (PHP)^2 = PHQHP = BB^\dagger$: an exit and return in operator composition [41].
- (d) For the transport operator itself, $P_W D_T^2 P_W - A^2 = P_W D_T Q_W D_T P_W$, equation (9), is of rank at most one and positive in the kinetic metric. The hidden direction is the scale.

The Schur loads of (31) and the impedance of Proposition 13 are built from the same off-diagonal blocks, divided by the hidden resolvent instead of squared. At the wall, the off-diagonal block that couples back across the wall is A_{21} , and it vanishes. Every form in this list is a pullback, and every one is positive.

5.5 Delay is not trapping

A one-way coupling and a slow coupling look alike for a while. Two exact models separate them.

One-way readout. Let $L_b = \begin{pmatrix} -1 & 0 \\ 0.3 & -b \end{pmatrix}$ act on (exterior, hidden) amplitudes, with the exterior read through $(1, 0)$. The exterior signal is $e^{-t}x_1(0)$ for every hidden rate b and every hidden initial state [DERIVED; CHECKED, BH-SPINE-09]; over $b \in \{0.1, 1, 7\}$ and three hidden states the exterior signals agree to 10^{-14} . A record that reaches the hidden block and never returns is invisible from outside, however it evolves. The hidden-block ambiguity says the same of compressions: $P(X + QYQ)P = PXP$ for every Y , so exterior data do not determine the hidden block without additional structure. Silence can also be certified in finitely many steps. A hidden block of dimension q , entered through C and read through B , never passes a signal if and only if $BD^jC = 0$ for $j = 0, \dots, q-1$, because the Cayley–Hamilton theorem carries those zeros to every later power; checking fewer can miss a delayed response [DERIVED]. In the readout above, $(1, 0)L_b^k(x, h)^\top = (-1)^k x$ for every k , by induction, so the certificate holds for every b .

Retention and return. Let $x, y \geq 0$ be accessible and retained energy, with input power $\mathcal{P}$, release rate κ , capture rate a and return rate $b \geq 0$:

$$\dot{x} = \mathcal{P} - (\kappa + a)x + by, \quad \dot{y} = ax - by, \quad \frac{d}{dt}(x + y) = \mathcal{P} - \kappa x. \quad (36)$$

After an impulse $x(0) = E_0$, for every $b > 0$ all of the energy eventually leaves and its mean residence time is $(1 + a/b)/\kappa$, which grows without bound as $b \downarrow 0$. At $b = 0$ a fraction $a/(\kappa + a)$ stays forever [DERIVED; CHECKED SYMBOLICALLY, BH-SPINE-09]. The limits $t \rightarrow \infty$ and $b \downarrow 0$ do not commute. A very long delay and an absent return channel look the same over any finite observation and differ in their eventual outcome. Storage decides how long energy stays; the return structure decides whether it can come back through that channel at all. This is a conditional energy model, not a simulation of a star or a black hole.

6 Hair, as boundary compression

Classical no-hair behaviour motivates a precise information question: how much of a configuration's history is recoverable from a restricted record at the boundary? On the operator side the natural compression is $\mathcal{D}(X) = PXP$ with P the Riesz projector of a retained cluster. It keeps the cluster's trace and ledger. $\mathcal{D}$ is idempotent for every projector P , and it is an orthogonal projection of the operator space exactly when P is Hermitian; for the oblique Riesz projector of a non-normal cluster it is a projection that is not orthogonal, and it is not automatically a quantum channel [DERIVED; CHECKED, V10-PUSH]. Recording only the cluster's invariants, and not its individual eigenprojectors, whose norms depend on the frame at the wall, is a further choice of what to keep. The interface therefore performs rank compression, not destruction. What crosses the wall is the cluster record (Section 4) and the orientation record. What is lost is the splitting into individual vectors, and the hidden block, which the compression cannot see (Section 5.5).

The informal name for this mechanism is the *hairstylist*, and it is imagery only. Its *scissors* are the exact vanishing back-coupling $A_{21} \propto \sigma_3^2$. Its *fingers* are the fold law (4), which makes trajectories skim the wall in the Gram picture. Its *mirror* is the reflection $\sigma_3 \mapsto -\sigma_3$, together with the ledger that is exact at the wall point. Its *light* is the contour that keeps the enclosed invariants finite. No claim is made that this is a no-hair theorem. In general relativity, uniqueness at equality proceeds through Killing initial data, staticity and doubling [7], a different route.

Two 2026 results sit closest to this section’s question, what a horizon’s exterior keeps after a process. Hou and Zhu show that a change in a Kerr black hole’s soft hair, its charges under the generalised BMS symmetries, shifts the black hole’s image in the observer’s sky, an image memory effect [85]. Ashtekar, Paraizo and Shu extend the first and second laws to black holes far from equilibrium, relating the change in dynamical-horizon area to the infalling energy flux [86]. Neither uses the compression $\mathcal{D}$, and nothing here reproduces their results [CITED; STRUCTURAL].

7 Bell at the horizon: two routes, compared

Bell left two results that reach a horizon from opposite sides. His inequality limits the correlations that any locally carried hidden record can produce [48, 49]. His anomaly with Jackiw shows that a conservation law of the full theory can fail once a theory distinguishes one handedness of motion [50]. That is the axial anomaly. At a horizon the same kind of failure appears as a different anomaly, the gravitational anomaly of the one-way near-horizon theory, which Robinson and Wilczek used [54]; the two are distinct, and the second is the one that fixes Hawking’s flux. A third result, with Leinaas, reads the temperature of an accelerated frame from the spin of a stored electron [51]. It sets a scale, which projector data cannot do (Proposition 16, and Proposition 13 of [57]), so it stays outside this paper. The first two are worked out below, side by side.

7.1 The entanglement route: what the outside record cannot hold

A Hawking mode of frequency ω leaves its partner behind the horizon in the two-mode squeezed state [52]

$$|\psi\rangle = \sum_{n \geq 0} \frac{\tanh^n r}{\cosh r} |n\rangle_{\text{out}} |n\rangle_{\text{in}}, \quad \tanh r = e^{-\omega/(2T_H)}. \quad (37)$$

- Proposition 14** (the outside record and the pair). (i) *The outside half alone is exactly thermal, with Bose weights $\tanh^{2n} r / \cosh^2 r$, and it is the same for every phase put on the pair [DERIVED].*
(ii) *In the ideal two-mode squeezed state, measured with the parity pseudospin operators, the largest CHSH value of the pair is $2\sqrt{1 + \tanh^2 2r}$, above the bound 2 for every frequency [53] [REPRODUCTION]. No cross-horizon measurement is implied.*
(iii) *Two states with identical one-mode records, $(|10\rangle \pm i|01\rangle) / \sqrt{2}$, evolve under the same exchange coupling g into opposite imbalances $\pm \sin 2gt$ [DERIVED].*

Proof. (i) Tracing out the inside mode of (37) removes every off-diagonal term, since each $|n\rangle_{\text{out}}$ is paired with its own $|n\rangle_{\text{in}}$; a phase $e^{in\phi}$ cancels in $|c_n|^2$. (ii) The pseudospin operators obey the spin- $\frac{1}{2}$ algebra, so the Horodecki criterion gives the maximum from the two largest singular values of the correlation matrix, which are 1 and $\tanh 2r$. (iii) The one-mode states are both $\frac{1}{2}$ of each occupation. The relative phase $\pm i$ sets the sign of the population flow under $g(|10\rangle\langle 01| + |01\rangle\langle 10|)$. $\square$

BH-BELL-11 confirms (i) to machine precision, (ii) to 10^{-10} at four squeezings (2.0385848200 at $r = 0.1$, 2.8054348865 at $r = 1.2$), and (iii) with imbalances $\pm 0.531186 = \pm \sin 0.56$ [MEASURED]. In terms of the horizon, the CHSH value is 2.785688 at $\omega / T_H = \frac{1}{2}$, 2.383254 at 2 and 2.000182 at 10: it exceeds 2 at every frequency and approaches it only for modes far above the Hawking temperature (Figure 10a).

The comparison with Section 5.5 is direct. There, the identity $P(X + QYQ)P = PXP$ says the outside record does not determine the hidden block. Here, Bell's theorem says something stronger in the other direction. The correlation between the outside half and the inside half is one that no record carried locally on either side could produce. The outside record is complete as a record and blind to the pair's phase (i), and that phase decides what the pair does next (iii) [STRUCTURAL COMPARISON]. A study of Bell violation in an analogue black hole is in [56].

For a fermion mode the outside record has an exact zero point. Replace (37) by the Dirac pair $\cos r |00\rangle + e^{i\phi} \sin r |11\rangle$ with $\tan^2 r = e^{-\omega/T_H}$. The outside half is Fermi–Dirac thermal with occupation $C = \sin^2 r$, and its offset [57] is

$$\tau = \log \frac{\det(I - C)}{\det C} = \log \frac{P(\text{empty})}{P(\text{full})} = \frac{\hbar\omega}{k_B T_H}, \quad (38)$$

the same for every pair phase ϕ , and for several independent modes $\tau = \sum_k \hbar\omega_k / k_B T_H$ [DERIVED]. The offset is a log-impedance. With $z = e^\tau = (1 - C)/C$ the Smith coordinate is

$$\Gamma = \frac{z - 1}{z + 1} = 1 - 2C = \tanh \frac{\hbar\omega}{2k_B T_H}, \quad (39)$$

exact for every pair phase (to 3×10^{-16} by partial trace over six frequencies and seven phases) [DERIVED; CHECKED, V10-PUSH]. The particle–hole exchange $C \mapsto 1 - C$ sends $\Gamma \mapsto -\Gamma$, and its fixed point $\Gamma = 0$ is reached only at $\omega = 0$: the outside record of a fermion mode sits at the centre of the chart, where the matched horizon of Section 5 sits, only in the zero-frequency limit. This is a statement about one chart, not an identification of the two. BH-OFFSET-15 computes the outside state by partial trace of the full pair and finds $\tau = \omega/T_H$ to 6×10^{-16} over six frequencies and seven phases, and 13.26 both ways for four modes [MEASURED]. The identity is the logarithm of a Boltzmann factor, so what it adds is the reading: the outside record's zero point is fixed by ω/T_H , and the phase that decides the pair's future never enters it. It needs $0 < C < 1$, so it holds for fermion modes and not for the bosonic pair (37), where a full mode is not defined. It also fixes only the ratio ω/T_H , never an absolute scale: rescaling ω and T_H together leaves τ unchanged, as Proposition 16 and its twin in the offset paper require. Routes from a horizon record to Einstein's equation are developed elsewhere: Jacobson's, from local horizon heat flux and the Unruh temperature [82], and Dorau and Much's, from the relative entropy of horizon quantum-field states [83], with an axisymmetric extension by Dorau, Much and Verch noted in [40]. The offset line supplies none of their premises: it fixes a ratio, not an area or a flux.

7.2 The anomaly route: what the one-way books must carry

Near a horizon the modes that fall in cannot affect the outside. Robinson and Wilczek removed them from the near-horizon theory. What remains is a theory with one direction of motion, in which energy balance fails by an anomaly, and they showed that the balance is restored exactly by Hawking's flux [54, 55]. For $f(r) = 1 - 2M/r$, the flux fixed by the anomaly is $f'(r_H)^2 / (192\pi)$, which equals the flux $\pi T_H^2 / 12$ of one chiral mode at $T_H = f'(r_H) / (4\pi) = 1 / (8\pi M)$ [REPRODUCTION; CHECKED SYMBOLICALLY, BH-BELL-11].

Each step of that argument has an exact counterpart on our side.

- (a) *Removing one direction.* At the coplanar wall in the Gram frame the back-coupling A_{21} vanishes (Theorem 6): the reduced dynamics carries information into the wall and none back.

- (b) *A law fails in the reduced theory.* For a reduction $x \star y = P(xy)$ of an associative product, the associativity defect is exactly the discarded part coming back:

$$(x \star y) \star z - x \star (y \star z) = P[x Q(yz) - Q(xy) z], \quad Q = I - P. \quad (40)$$

For the transport, the reduced square misses $P_W D_T^2 P_W - A^2 = P_W D_T Q_W D_T P_W$, the term of rank at most one in (9) [DERIVED; CHECKED, BH-BELL-11], to 1.1×10^{-14} on random instances.

- (c) *The flux restores it.* In the retention model with no return, $\dot{x} = \mathcal{P} - \kappa x - ax$: the outside books close only when the one-way flux ax into the retained stock is counted (Section 5.5, Figure 10b) [DERIVED].

In each case the reduced description is exact once the one-way term is carried and fails without it. That is the shape of the anomaly argument. No anomaly coefficient and no temperature come out of our side [STRUCTURAL COMPARISON].

7.3 The two routes compared

	entanglement route	anomaly route
Bell's result	inequality (1964)	anomaly, axial with Jackiw (1969); at a horizon its gravitational form (Robinson–Wilczek)
what it says at a horizon	the pair beats any local record	a one-way theory keeps its balance only with Hawking's flux
our object	hidden-block identity; same record, opposite futures	vanishing back-coupling; associator; reduced square
what is exact here	thermal outside record; CHSH $2\sqrt{1 + \tanh^2 2r}$	the defect identities; the one-way balance
what is not ours grade	the quantum state of the pair reproduction plus structural comparison	the anomaly coefficient and T_H reproduction plus structural comparison

The two routes meet at the same place, the one-way boundary. The entanglement route says what the outside record cannot contain, and the anomaly route says what the outside books must carry. Read together, they are the quantum versions of the two models of Section 5.5: the hidden block nobody outside can see, and the one-way flux nobody outside can return.

8 Three walls

The word “wall” has been used for several different failures, and they must have different names [37]: a pullback metric losing rank, a smallest singular value tending to zero, a spectral cluster meeting its contour, a numerator cancelling in a response, and a loss of positivity. The last can happen with no exceptional point at all: the gravity edition gives an analytic rank-one projector

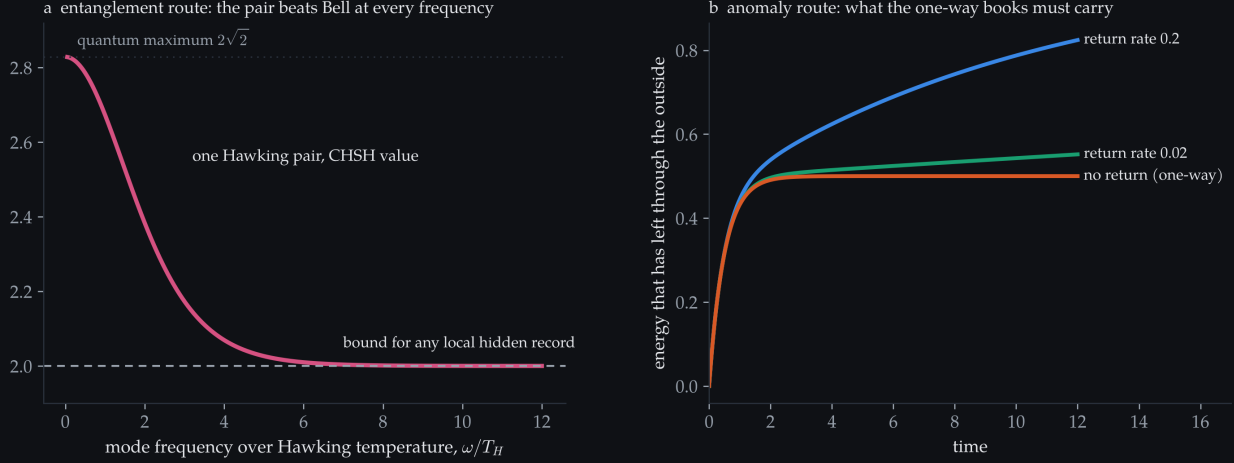


Figure 10: Bell at the horizon. (a) Entanglement route: the largest CHSH value of one Hawking pair, $2\sqrt{1 + \tanh^2 2r}$ with $\tanh r = e^{-\omega/2T_H}$, against the bound 2 for any local hidden record. (b) Anomaly route: energy that has left through the outside after an impulse, in the retention model. With any return the outside eventually receives all of it; with none, the one-way flux keeps a fixed share, and the outside books close only if that flux is counted.

family whose tangent form passes from positive definite to indefinite across a line, while the generator's two eigenvalues stay separated and the contour count and trace-log remain regular [40]. Their loci need not agree, and no codimension slogan settles in advance whether a geometric wall and an exceptional point are distinct; the operators decide. Four examples, all computed, sort into two kinds, and a black hole carries one of each. A third kind lies outside this paper's operators: walls of marginal stability in the moduli space of a string compactification, where bound states of black holes come apart and the count of states jumps (Section 4.5). Those walls live in a space of theories, not in a configuration space or in spacetime, and they are not compared with the other two here.

8.1 Spectral walls

The Sun's dynamo. The minimal Parker $\alpha\Omega$ operator on colatitude $x \in [0, \pi]$,

$$\partial_t A = \alpha_0 \cos x B + \partial_x^2 A, \quad \partial_t B = G \partial_x A + \partial_x^2 B, \quad (41)$$

with Dirichlet conditions and $G = \eta = 1$, was studied in the author's solar paper in its energy norm $E = \int (|\partial_x A|^2 + |B|^2)$ [38]. There the oscillatory dynamo-wave branch is born at a genuine exceptional point, $\alpha_0^* = 3.8717$, where a real eigenvalue pair collides and leaves as a complex pair. The eigenvector condition number obeys the square-root law $\kappa\sqrt{\delta} = 1.842, 1.850, 1.853, 1.853$ at $\delta = 0.2, 0.05, 0.0125, 0.003125$. A contour enclosing the pair alone (count 2.000000) gives a shifted trace-log ledger equal to $\sum \log(c + \lambda_{\text{pair}})$ to 9×10^{-14} at the exceptional point and beside it. A loop $\alpha_0 = \alpha_0^* + 0.15e^{i\varphi}$ exchanges the pair after one turn and restores it after two, with discriminant winding $+1.0000$ and $+2.0000$, while the ledger returns single-valued to 10^{-8} [MEASURED THERE]. This exceptional point is distinct from the steady instability onset at $\alpha_{0c} = 12.696$.

An optical threshold. For a lossless TE half-space with $D = R_{\text{opt},2} - (k_{\parallel}/k_0)^2$, the normal propagation equation $u'' + Du = 0$ has the first-order generator $M(D) = \begin{pmatrix} 0 & 1 \\ -D & 0 \end{pmatrix}$ with $M^2 = -DI$. At

$D = 0$ it is defective, its two projectors $\frac{1}{2}(I \pm M/\sqrt{-D})$ are singular, and their sum I is regular [37]. The reflection coefficient in the Smith chart is $\Gamma = 1 - 2\sqrt{D}/\sqrt{D_1} + O(D)$, with D_1 the same quantity on the incident side, a square-root threshold. In the frame $(u, u'/k_z)$, with $k_z = k_0\sqrt{D}$, the generator becomes $\sqrt{D} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, regular in $\sqrt{D}$. But $\sqrt{D}$ is real on one side and imaginary on the other: the eigenvalues $\pm i\sqrt{D}$ are oscillatory for $D > 0$ and growing or decaying for $D < 0$ [DERIVED].

The Sun's acoustic cutoff. The Sun has a spectral wall at its surface as well as in its dynamo. In Lamb's isothermal atmosphere, with scale height H and sound speed c , vertical acoustic waves obey $w_{tt} = c^2 w_{zz} - (c^2/H)w_z$. Writing $w = e^{z/2H}\psi e^{-i\omega t}$ gives

$$\psi'' + k_z^2 \psi = 0, \quad k_z^2 = \frac{\omega^2 - \omega_{ac}^2}{c^2}, \quad \omega_{ac} = \frac{c}{2H} \quad (42)$$

[80, 81] [DERIVED, BH-ISCO-CUBE-17]. This is the optical-threshold equation with k_z^2 in the place of D . Below the cutoff the wave is evanescent and, in this model, the atmosphere reflects it; above the cutoff it propagates out; at $\omega = \omega_{ac}$ the generator $\begin{pmatrix} 0 & 1 \\ -k_z^2 & 0 \end{pmatrix}$ is a nonzero nilpotent. In the solar photosphere the cutoff lies near 5 mHz [81]. In the boundary calculus of Section 5, the photosphere reflects on one side of a spectral wall in frequency, while the horizon surface is a matched load that returns nothing at every frequency; the reflection a distant observer sees comes from the barrier outside it (Section 5) [STRUCTURAL COMPARISON].

A two-level model. $H(\gamma) = \sigma_x + i\gamma\sigma_z$ has eigenvalues $\pm\sqrt{1-\gamma^2}$, real below $\gamma = 1$ and imaginary above it. The individual projectors fail at $\gamma = 1$ while the pair projector is the identity [37].

The innermost stable circular orbit. A black hole has a spectral wall of its own, in the orbits around it. For a test body in the Schwarzschild exterior ($G = c = M = 1$, proper time), radial motion obeys $\dot{r}^2 + V(r) = E^2$ with $V = (1 - 2/r)(1 + L^2/r^2)$. Circular orbits sit at $L^2 = r^2/(r - 3)$, and a small push about a circular orbit at r_0 evolves under

$$\frac{d}{d\tau} \begin{pmatrix} \delta r \\ \delta \dot{r} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}V''(r_0) & 0 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta \dot{r} \end{pmatrix}, \quad \frac{1}{2}V''(r_0) = \frac{r_0 - 6}{r_0^3(r_0 - 3)}. \quad (43)$$

Proposition 15 (the ISCO is a spectral wall). [DERIVED] For $r_0 > 6$ the eigenvalues are $\pm i\sqrt{V''}/2$, an oscillation; for $3 < r_0 < 6$ they are $\pm\sqrt{-V''}/2$, growth and decay. At $r_0 = 6$, the innermost stable circular orbit, the generator is nonzero with square zero: a Jordan block. This is the linearisation; the nonlinear plunge, and the inner edge of any particular disk, are separate questions. The unfolding coordinate $\sqrt{V''}/2$ is real outside and imaginary inside, and the gap divided by $\sqrt{|r_0 - 6|}$ tends to $2/\sqrt{648} = 0.0785674$ from both sides.

Proof. Differentiate V twice and substitute $L^2 = r^2/(r - 3)$; the numerator of $\frac{1}{2}V''$ is $r - 6$. The generator squares to $-\frac{1}{2}V'' I$. Near $r_0 = 6$, $\frac{1}{2}V'' \approx (r_0 - 6)/648$. $\square$

BH-ISCO-10 confirms each part: eigenvalues $\pm 0.02700i$ at $r_0 = 7$ and ± 0.06325 at $r_0 = 5$, and gap/ $\sqrt{|r_0 - 6|} = 0.078567$ outside and 0.078568 inside at $|r_0 - 6| = 10^{-5}$ [MEASURED]. The generator (43) has exactly the form of the optical threshold $M(D)$ above, with $\frac{1}{2}V''$ in the place of D .

The same generator gives the ISCO a classical reading. For a Newtonian central force $F \propto r^{-p}$, a nudged circular orbit has $\omega_r^2/\Omega^2 = 3 - p$, so circular orbits are stable exactly when $p < 3$; at $p = 3$, the inverse cube, a nudged orbit spirals. Defining an effective exponent p by the same

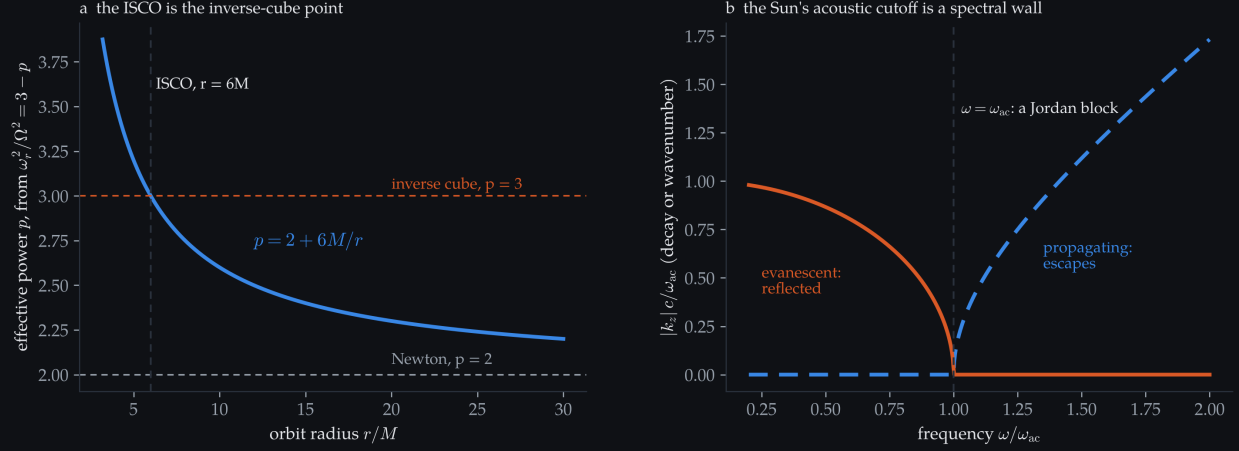


Figure 11: Two more spectral walls. (a) The effective power of gravity on circular Schwarzschild orbits, defined by $\omega_r^2/\Omega^2 = 3 - p$: $p = 2 + 6M/r$ runs from Newton's 2 far away to the inverse cube at the ISCO. (b) Lamb's acoustic cutoff: below ω_{ac} the vertical wave is evanescent and reflected, above it the wave propagates out; at the cutoff the generator is a Jordan block.

ratio for Schwarzschild, which is a reading of the epicyclic ratio and not a new force law, (43) with $\Omega = d\phi/d\tau = L/r^2$ gives

$$\frac{\omega_r^2}{\Omega^2} = 1 - \frac{6M}{r}, \quad p = 2 + \frac{6M}{r} \quad (44)$$

[REPRODUCTION, BH-ISCO-CUBE-17]: Newton's value far away, and the inverse-cube value of the effective exponent exactly at the ISCO. The author's essay on Walter Russell reads this as the radius at which Einstein's orbit becomes Russell's spiral [74]. Levin's whirling orbits live just inside it, where $p > 3$ (Figure 11).

The same generator read for light. For a null orbit the potential is $V = (1 - 2M/r)L^2/r^2$, and its one circular orbit, the photon ring at $r = 3M$, is unstable: the wobble generator has a real pair $\pm\lambda$. Measured in coordinate time, the growth rate equals the orbital rate, $\lambda = \Omega = 1/(3\sqrt{3}M)$, so in the orbit equation $u'' + u = 3Mu^2$ ($u = 1/r$, primes in azimuth) a nudge grows by exactly e per radian. Each extra half orbit that light makes before escaping therefore dims it by $e^{-\pi} = 0.0432$ and delays it by $\pi/\Omega = 3\sqrt{3}\pi M$ [REPRODUCTION, PHOTON-RING-WALL, V10-PUSH]. This is the known photon-ring result [99], and in the eikonal limit the same two rates give the quasinormal frequencies [100]. Integrating the nonlinear orbit gives 0.99998 per radian. The opposite run shows what is special: for a charged hole the rate per radian is $\sqrt{6Mu_0 - 6Q^2u_0^2 - 1}$ at the photon sphere u_0 , and the nonlinear orbits give 0.968 at $Q = 0.5M$ and 0.832 at $Q = 0.9M$, so the dimming per half orbit is $e^{-\pi}$ only when $Q = 0$ (Figure 12) [MEASURED, V10-PUSH FOR $Q = 0.5M$, BH-FIGS-V10 FOR $Q = 0.3M$ TO $0.9M$]. The photon ring is a single unstable orbit, not a wall in the sense of this section: no physical parameter carries it from oscillation to growth.

Levin and Perez-Giz built their periodic table of black-hole orbits and their homoclinic orbits on the unstable circular orbits inside this wall [46, 47]. A bound orbit with energy just below that of an unstable circular orbit whirls around it before zooming out, and the azimuth it sweeps per radial period grows like a logarithm,

$$\Delta\phi \simeq \frac{\Omega}{\lambda} \ln \frac{1}{\delta E} + \text{const}, \quad (45)$$

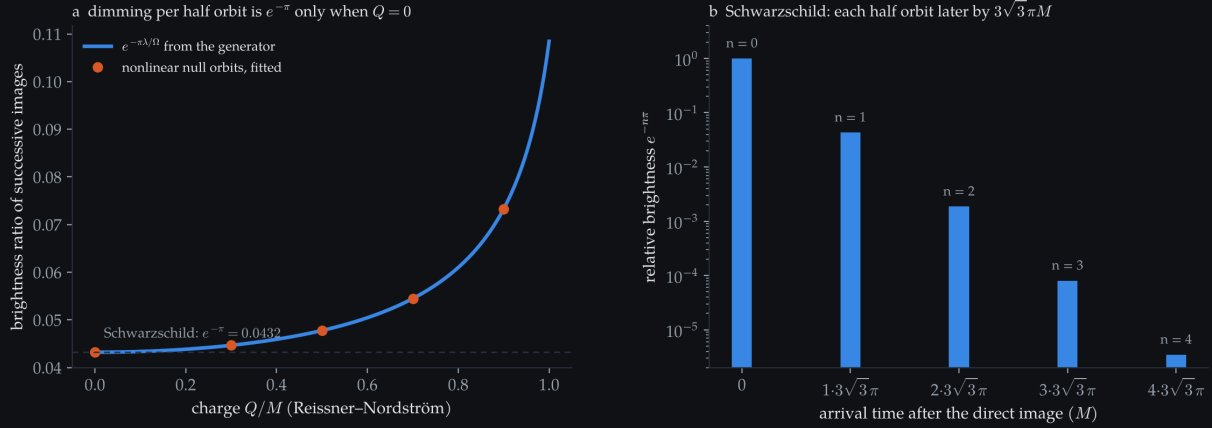


Figure 12: Light on the photon ring, read by the wobble generator. (a) Brightness ratio of successive images, $e^{-\pi\lambda/\Omega}$, against charge: the generator’s rate (line) and nonlinear null orbits (dots) agree, and the ratio is $e^{-\pi} = 0.0432$ only for Schwarzschild. (b) For Schwarzschild, each half orbit dims the image by $e^{-\pi}$ and delays it by $3\sqrt{3}\pi M$.

with Ω the angular rate and λ the growth rate of the unstable orbit. At $L^2 = 13$ ($r_u = 4.697224$) the measured slope is 1.898773 against $\Omega/\lambda = 1.898829$, as the orbits go from 2.55 to 6.02 whirls per period over $\delta E = 10^{-3}$ to 10^{-8} [REPRODUCTION, BH-ISCO-10]. The logarithm has the same source as the logarithm in static time near a horizon, $-\frac{1}{2\kappa} \log N$: passage close to a point that is left at a fixed exponential rate [STRUCTURAL].

Ringdown has spectral walls too, located and looped around. In massive-field perturbations of Kerr–Newman black holes, carrying the quasinormal spectrum around closed loops in parameter space permutes the mode frequencies [69]. In near-extremal Kerr the exceptional points form a cascade, with numerical evidence that it is infinite, and a path that encloses the first n of them returns the fundamental mode as the n th overtone [70]: one turn exchanges a pair, and the overtone reached counts the exceptional points enclosed. An exceptional point with hysteresis was reported earlier for Kerr [71]. That is the dynamo’s braid behaviour, and the count column of the ledger, in a black hole’s own spectrum [CITED RESULT; STRUCTURAL COMPARISON].

The Sun’s own ladder answers the same question the other way. The solar paper’s dynamo operator has five pair-birth points in $0 < \alpha_0 \leq 13$ [38]. BH-SUNLADDER-16b rebuilds that operator exactly and carries its spectrum around loops based at $\alpha_0 = 3$, below the first birth point, where every tracked eigenvalue is real and can be labelled by size. Each birth point joins its own pair: counting the least-damped mode as 0, the five points exchange eigenvalues (1,2), (3,4), (5,6), (7,8) and (9,10). A loop around the first n points gives n disjoint swaps. The discriminant winds exactly n times, two loop heights give the same permutation, the reversed loop gives the inverse, and every loop around several points equals the product of the single-point loops; the least-damped mode never moves [MEASURED]. So two families of spectral walls with the same local law, a square-root branch and one exchange per point, differ in how their points connect. The near-extremal Kerr cascade links its exceptional points through shared modes, so a loop climbs the overtones. The solar ladder is a stack of independent pairs, so a loop only swaps. Climb or swap is a checkable label for a family of exceptional points (Figure 13) [STRUCTURAL COMPARISON].

What these share is the defining property of a spectral wall. The square-root coordinate that

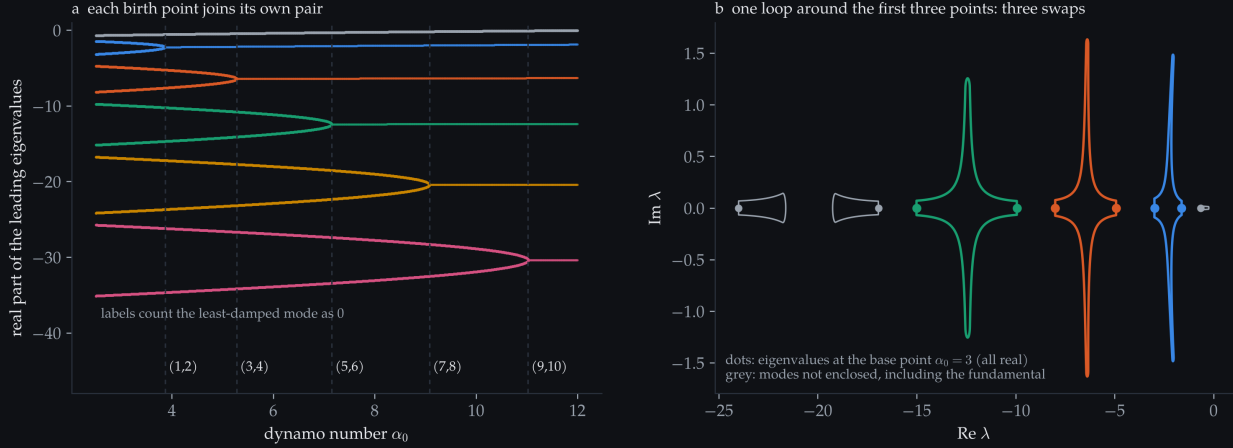


Figure 13: The Sun’s dynamo ladder swaps (BH-SUNLADDER-16b). (a) Real parts of the leading eigenvalues of the solar paper’s dynamo operator: at each birth point a different pair of real eigenvalues merges and leaves as a complex pair. (b) One loop in the complex α_0 plane around the first three birth points, starting from $\alpha_0 = 3$ where all eigenvalues are real (dots): each enclosed pair is exchanged, and nothing climbs.

unfolds the branch point is real on one side of the wall and imaginary on the other. A real eigenvalue pair turns into a complex pair along a physical path. The Jordan block at the wall is a property of a fixed operator on a fixed space, so no change of frame removes it. A loop of the physical parameter exchanges the pair.

8.2 Geometric walls

The coplanar fold. The unfolding coordinate $\sigma_3 = \text{sign}(\det X) \sqrt{w_3}$ is real on both sides, because the other side is the mirror image (Theorem 2). Along every physical path the spectrum is real on both sides (Theorem 7(iii)). The operator is self-adjoint in a metric that is regular on the cover, and the wall point is a cone (Theorem 8). The Jordan block of the Gram frame comes from the singular differential of the covering map, and the complex-loop sign comes with it. The chart-free sign is the conical one, carried by a real loop through the mirror (Theorem 10). The same fold appears one dimension down: an observer who records three bodies only through their distances sees the collinear circle as a fold, with a triangle and its mirror image as the two sheets that meet there [41] (Example 8.8 there).

The static base of a horizon. The static base of a Schwarzschild exterior has the same fold normal form, smooth and even in the signed lapse $\lambda = \pm\sqrt{N}$ (Section 9). The Penrose-inequality manuscript doubles the base across the horizon by $\lambda \mapsto -\lambda$, and calls the divergence of the coordinate expression at the horizon a coordinate singularity [7] [CITED, NOT CHECKED]. The Weyl operator of Kerr is type D, diagonalisable (Section 4.4), and the principal eigenvalue of the marginally trapped surface operator is simple [8].

	spectral wall	geometric wall
examples	Sun's dynamo-wave birth; optical threshold; $\sigma_x + i\gamma\sigma_z$; the ISCO; ringdown exceptional points	coplanar fold of Σ_4 ; static base of a horizon
unfolding coordinate	real on one side, imaginary on the other	real on both sides (the mirror, the doubled base)
spectrum along physical paths	real pair turns complex	real on both sides
Jordan block	in every frame	only in the folded chart
what a loop does	parameter loop exchanges the pair, winding +1	real loop through the mirror reverses the eigenline
what survives	pair projector, ledger, braid winding	pair projector, ledger, orientation, conical sign
term linear in time	$e^{\lambda t}(I + tN)$ in every norm: the ISCO drift, a double-pole ringdown in the two-mode model	removable: none in the kinetic metric (Section 4)

A black hole carries both kinds: the static base of its horizon is a geometric wall (the event horizon itself is a null causal boundary, Section 9) and the innermost stable orbit is a spectral one (Figure 14). In both kinds the pair projector and the ledger survive exactly. That is the common law, and it is the reason the ledger, rather than an eigenvector, is the right record to compare across the two (Figure 15). One caution keeps the ledger in its place: a quantity that varies smoothly through a wall cannot by itself detect the wall. The ledger records what survives; the orientation, the conical sign and the change of spectral type are what locate the wall.

Transient growth with no block present is a separate effect. The Sun's dynamo, with every eigenvalue damped at 0.9 of its steady-onset value and far from any exceptional point, still amplifies by 3.5, with an instantaneous growth rate of +5.84 against a modal rate of -0.063 [38]. That is non-normal growth, not the linear-in-time term of a Jordan block.

Codimension follows the symmetry class. How rare a degeneracy is depends on the class of the operator, not on whether the wall is spectral or geometric:

class of the family	degeneracy	codimension
real, non-symmetric	exceptional point (real pair turns complex)	1
complex, non-Hermitian	exceptional point	2
real symmetric	diabolic point (cone)	2
complex Hermitian	diabolic point	3

The first two rows are Proposition 12 of the gravity edition, with its caution that symmetry can lower the count [40]; the last two are von Neumann and Wigner's count [78, 79]. The class to use is the operator's hidden class, not the look of its matrix. In the Gram frame our wall block is real and non-symmetric, which by appearance would give codimension 1. It lands at codimension

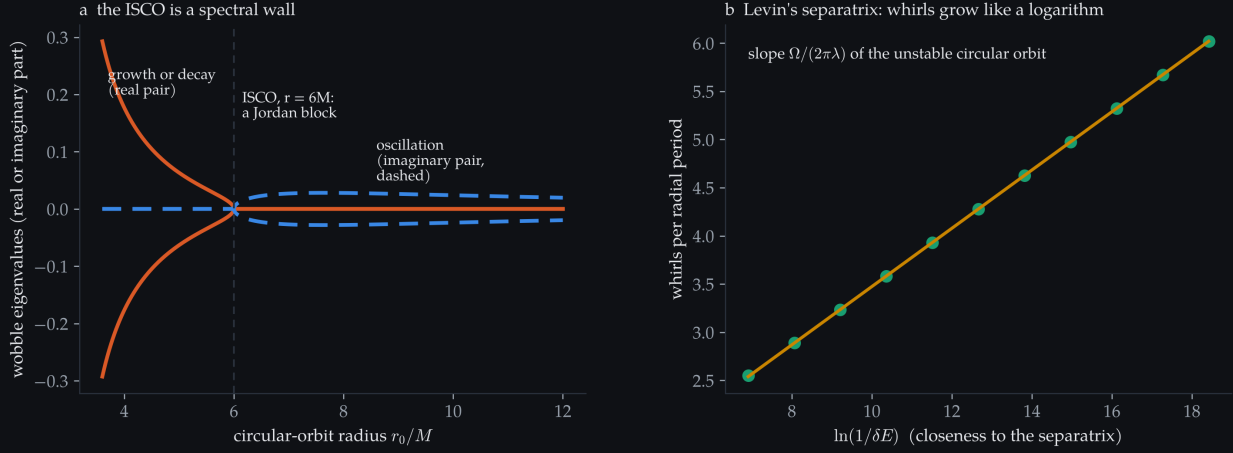


Figure 14: The black hole's spectral wall. (a) Eigenvalues of the wobble generator (43) of circular Schwarzschild orbits: an imaginary pair (oscillation, dashed) outside $r_0 = 6M$, a real pair (growth and decay, solid) inside, meeting at a Jordan block at the innermost stable circular orbit. (b) Levin's separatrix: whirls per radial period against $\ln(1/\delta E)$ at $L^2 = 13$; the line has the slope $\Omega/(2\pi\lambda)$ of the unstable circular orbit.

2, the joint condition (10), because the operator is self-adjoint in the kinetic metric (Theorem 4). A geometric wall is a real operator secretly in the symmetric class. The Sun's dynamo is a real operator in the non-symmetric class, and its pair births sit on codimension-one walls, as the solar paper says [38]. Ringdown operators are complex, and the Kerr–Newman exceptional lines lie in a three-parameter space (spin, charge and field mass), so they too are codimension two [69]. For complex operators codimension cannot tell spectral from geometric, which is why the divider in the table above is a Jordan block in every frame, not a count [CITED RESULTS; DERIVED READING].

8.3 Reflection and refraction on the pullback metric

The optical threshold above reappears inside the projector geometry. Take a two-parameter family of projectors whose pullback metric has a symmetry direction x ,

$$g = a(y) dx^2 + b(y) dy^2, \quad a = g_{\parallel}, \quad b = g_{\perp}. \quad (46)$$

A geodesic crossing the lines $y = \text{const}$ at angle θ from their normal conserves the Clairaut constant $C = a \sin^2 \theta$ [41] (Proposition 8.1 there), so it refracts by $a_1 \sin^2 \theta_1 = a_2 \sin^2 \theta_2$, Snell's law with index $n = \sqrt{g_{\parallel}}$, and it turns back where a falls to C , grazing the interface: total internal reflection of rays [41] (Proposition 8.3 there). The wave on the same metric is the Laplace–Beltrami equation $\Delta_g \psi + \omega^2 \psi = 0$. With $\psi = e^{ipx} u(y)$,

$$\left(\sqrt{a/b} u' \right)' + \sqrt{ab} \left(\omega^2 - \frac{p^2}{a} \right) u = 0, \quad p^2 = \omega^2 C, \quad (47)$$

and conservation of the x -momentum is the conservation of C . Where a and b are constant, $u = e^{\pm iky}$ with $k^2 = b\omega^2(a - C)/a$. The conserved flux is carried by $F = \sqrt{a/b} u'$, so u and F are continuous across a step, and the ratio $F/u = \pm iY$ defines the admittance

$$Y = \sqrt{a/b} k = \omega \sqrt{a - C} = \omega \sqrt{g_{\parallel}} \cos \theta. \quad (48)$$

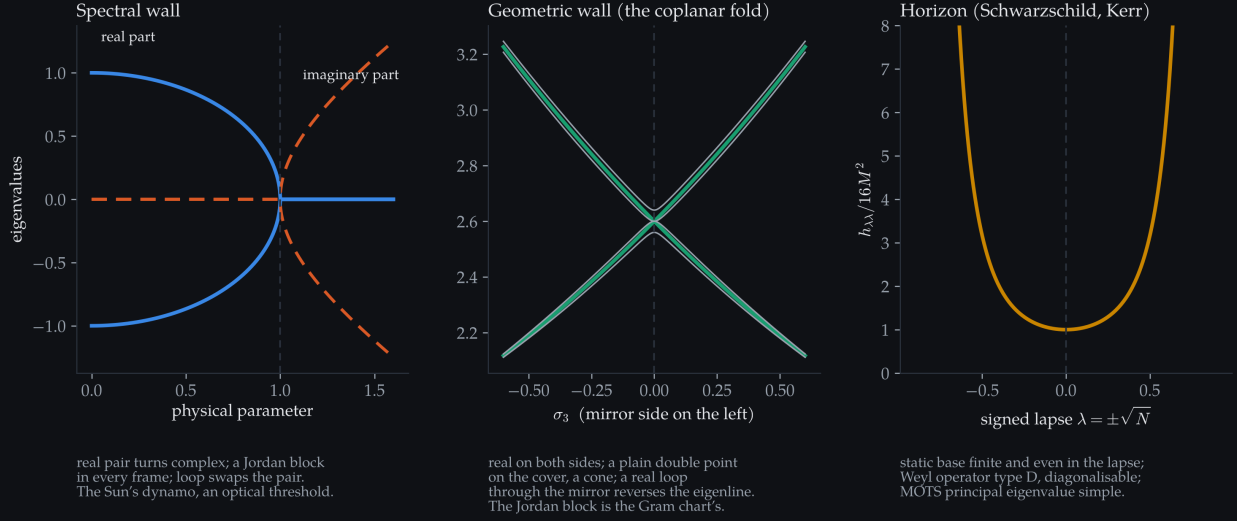


Figure 15: Spectral and geometric walls. Left: a spectral wall, $\sigma_x + i\gamma\sigma_z$, where a real pair (solid) becomes a complex pair (dashed imaginary parts). Middle: the coplanar wall on the cover, eigenvalues of the lifted transport against σ_3 on the joint condition (coloured) and beside it (grey); real on both sides, meeting at a cone. Right: the Schwarzschild static base in the signed lapse, finite and even across the doubled horizon.

The factor $\sqrt{b}$ in k cancels against the $1/\sqrt{b}$ in F . Matching u and F at a sharp step from (a_1, b_1) to (a_2, b_2) gives

$$\Gamma = \frac{\sqrt{a_1 - C} - \sqrt{a_2 - C}}{\sqrt{a_1 - C} + \sqrt{a_2 - C}}, \quad \sqrt{a_2 - C} \rightarrow i\sqrt{C - a_2} \text{ when } C > a_2 \quad (49)$$

[DERIVED]. Three things follow. First, $g_\perp$ drops out: the step reflects through $g_\parallel$ on both sides and the Clairaut constant alone. Second, $|\Gamma| = 1$ exactly when $C > a_2$, which is exactly when the geodesic turns, so total reflection begins at the Clairaut turning point. Near it, $\Gamma = 1 - 2\sqrt{a_2 - C}/\sqrt{a_1 - C} + O(a_2 - C)$: the square-root threshold of the optical case above, with $D = a_2 - C$. The optical threshold is the turning point of a geodesic, and its generator $M(D)$ is a Jordan block there. Third, at normal incidence $\Gamma = \tanh\left(\frac{1}{2} \ln(\sqrt{a_1}/\sqrt{a_2})\right)$, and where the index is set by a spectral gap at fixed coupling, $\sqrt{g_\parallel} \propto 1/\Delta$ [41] (Section 8.6 there, where $g_\parallel \Delta^2 = 9$), this is

$$\Gamma = \tanh\left(\frac{1}{2} \ln \frac{\Delta_2}{\Delta_1}\right), \quad (50)$$

the Smith coordinate of the peeling paper [76], now as the reflection of a wave at a change of gap [DERIVED]. Formula (49) is the transverse-electric Fresnel formula with admittance $\sqrt{a - C}$ [101], and interface conditions for media built from a metric are worked out in transformation optics [102]. What is specific here is the reading: the index is $\sqrt{g_\parallel}$ of a projector family, the incidence is fixed by C , and $g_\perp$ is absent at the step.

Tests, run several ways. Each test's kill condition was written before it ran (RS-03). One pre-registered gate failed: a power-law (Richardson) extrapolation of the smooth steps to zero width left 7.4×10^{-4} against a limit of 10^{-5} , because the error is a complex number whose phase turns as w shrinks, so a scalar power law misfits it; the first-order limit was then tested directly, with its own condition set before that run (RS-03c). Solving the wave equation on each side and matching

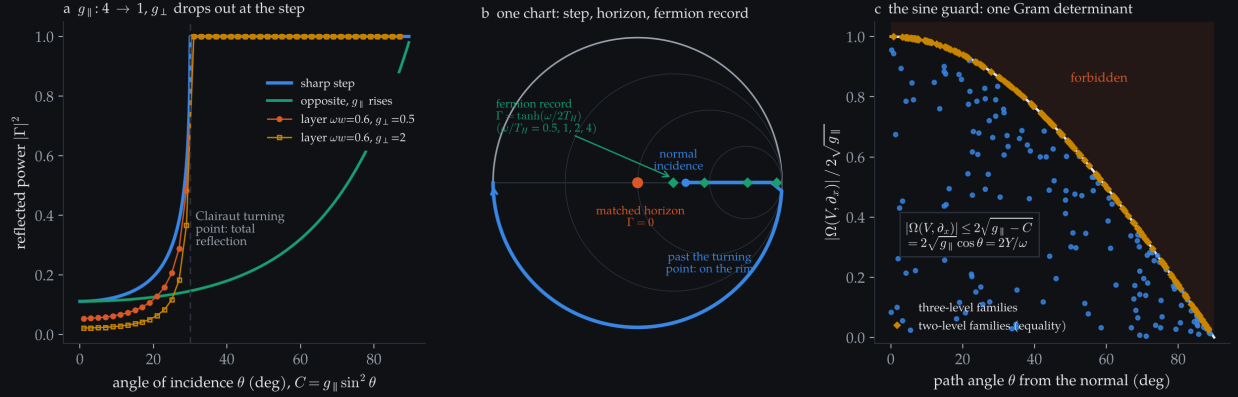


Figure 16: Reflection and refraction on the pullback metric. (a) A step in $g_{\parallel}$ from 4 to 1: at the sharp step $|\Gamma|^2$ is the same for every $g_{\perp}$ and reaches 1 at the Clairaut turning point $\theta_c = 30^\circ$; in a layer of finite width $g_{\perp}$ matters; with $g_{\parallel}$ rising there is no total reflection. (b) One Smith chart: as the angle grows the step's Γ runs from its normal-incidence value to 1 and then around the rim; the matched horizon sits at the centre, and the fermion record $\Gamma = \tanh(\omega/2T_H)$ lies on the real axis, reaching the centre only at $\omega = 0$. The three are different normalised impedances on one chart; their relative positions are not a physical identification. (c) The sine guard: the Berry curvature between a path and the symmetry direction stays below $2\sqrt{g_{\parallel}} \cos \theta$; two-level families sit on the ceiling.

at a sharp step, with no smoothing, reproduces (49) to 1.7×10^{-11} on 400 random steps, 89 of them beyond the turning point. Smooth steps of width w , with $\tanh$, error-function and linear-ramp profiles in both a and b , converge to Γ at first order in w ; for the $\tanh$ profile $|r - \Gamma|/(\omega w)$ settles to 0.0746, 0.105, 0.183 and 0.106 at $\theta = 10^\circ, 25^\circ, 31^\circ$ and 60° , followed down to $\omega w = 0.00125$. Sending the wave from the other side gives $r_{21} = -r_{12}$ at the step to 9×10^{-15} ; the flux balance $|r|^2 + (Y_2/Y_1)|t|^2 = 1$ holds to 6×10^{-14} for every profile and width; beyond the turning point $|r| = 1$ to 2×10^{-16} for every profile; and the gap form (50) holds to 3×10^{-13} on 200 random gap pairs [MEASURED, RS-03]. The opposite runs say where the law stops. In a layer of finite thickness $g_{\perp}$ is back: at $\omega w = 3$ quadrupling b_2 changes $|r|$ by up to 0.16, because b sets how many wavelengths the layer holds. When $g_{\parallel}$ rises across the step there is no total reflection below grazing. And as a layer thickens the onset of total reflection sharpens onto the turning point: half the power is reflected at $29.9960^\circ, 29.9995^\circ$ and 29.9999° for $\omega w = 10, 30$ and 100 , against $\theta_c = 30^\circ$ [MEASURED, RS-03] (Figure 16a).

The sine guard. One Gram determinant sets the admittance, the turning point and a curvature ceiling. For a unit path direction V at angle θ and the symmetry direction ∂_x ,

$$\det \begin{pmatrix} g(V, V) & g(V, \partial_x) \\ g(V, \partial_x) & g(\partial_x, \partial_x) \end{pmatrix} = g_{\parallel} - C = g_{\parallel} \cos^2 \theta, \quad |\Omega(V, \partial_x)| \leq 2\sqrt{g_{\parallel} - C} = \frac{2Y}{\omega}, \quad (51)$$

the second by the positivity bound (35) on the pair (V, ∂_x) [DERIVED]. This is the angle safeguard of [41] (equation (6.2) there), $|\Omega(V, W)| \leq 2\sqrt{g(V, V)g(W, W)} |\sin \theta_G|$, with $W = \partial_x$: the angle between the path and the symmetry direction is $\frac{\pi}{2} - \theta$, and its sine is $\cos \theta$. The Berry curvature that a path can feel against the symmetry direction is capped by the wave admittance, and both close at the turning point, where the path runs along ∂_x . On 600 random families the identity holds to 7×10^{-15} ; two-level families sit on the ceiling and three-level families below it (Figure 16c)

[MEASURED, V10-PUSH].

A polar stratum is a mirror. Near a polar stratum the parallel metric vanishes along the locus: in the family of [41] (Section 8.4 there), $a = \frac{9}{4} \sin^2 y$ and $b = \frac{1}{4}$, so every wave with $C > 0$ turns before it reaches $y = 0$. Computed from (47) with a regular solution at the pole, $|r| = 1$ to 2×10^{-16} , the reflection phase matches the ray phase $2 \int k dy - \pi/2$ to 0.02 rad at $\omega = 320$, and the wave's lateral shift $-\partial_p \arg r$ matches the geodesic's displacement between leaving and re-entering the lead to 2×10^{-4} at $\omega = 640$ for $\theta = 20^\circ$ to 65° , falling with ω , when the curved region joins the lead smoothly; with the lead joined by a kink, as first pre-registered, the shift oscillated and that gate failed, because the kink sends back a small echo [MEASURED, RS-01, RS-01B]. Section 9 puts the Killing horizon in the same class of degenerate forms, the polar one, yet the horizon is a matched load. The type of degeneracy of a form does not decide what a wave does there; the wave operator does [STRUCTURAL].

9 Black holes beside the wall

9.1 Three geometries, kept apart

Three different metrics appear in this section, and they are never mixed:

- (a) the Lorentzian spacetime metric of general relativity;
- (b) the Riemannian static base $h = g + N^{-1} X^b \otimes X_b$ of a stationary spacetime, the quotient by the Killing flow, with Killing norm $N = u^2 - |X|^2$ for lapse u and shift X ; it is used below for Schwarzschild only, since in Kerr the stationary Killing field turns spacelike in the ergoregion and the horizon cross-sections are not round;
- (c) the kinetic (Bures) metric of Σ_4 , and the projector metric $g_{ij}^P = \frac{1}{2} \text{Tr}(\partial_i P \partial_j P)$ of spectral projectors.

Comparisons below are between (b) and (c), or between operators, and are graded structural.

9.2 A horizon need not be a projector wall

Proposition 16 (local-symbol control). [DERIVED] Let $h(r, k) = N(r) (k \sigma_x + m \sigma_z)$ with $N = 1 - 2M/r$, a model local symbol of a massive two-component field in a static Schwarzschild frame (lower-order terms dropped). For $r > 2M$ its band projectors are independent of N , and the projector metric in k is $g_{kk}^P = m^2 / (4(k^2 + m^2)^2)$ at every such r . At $r = 2M$ the symbol vanishes and has no distinguished projectors; the projectors and the metric above are its limits as $r \rightarrow 2M$. The gap in local proper time, $2\sqrt{k^2 + m^2}$, stays open, and only the coordinate gap $2N\sqrt{k^2 + m^2}$ closes.

Proof. A positive scalar multiple does not change eigenvectors. $P = \frac{1}{2}(I + \hat{n} \cdot \sigma)$ with $\hat{n} = (k, 0, m) / \sqrt{k^2 + m^2}$, and $\frac{1}{2} \text{Tr}(\partial_k P)^2 = \frac{1}{4} |\partial_k \hat{n}|^2 = m^2 / (4(k^2 + m^2)^2)$. $\square$

A horizon can therefore coexist with an open local gap and a positive projector metric (Figure 17). This is the general fact that projector data are blind to positive rescaling, $H \mapsto \alpha H$, evaluated pointwise: projector geometry cannot set an absolute scale, so it cannot supply a mass. Every identification of a horizon with a projector wall must fail this control or explain why it does not. The offset paper proves the same fact for occupied projectors, as its Proposition 13, and shows that the boundary offset is unchanged under the rescaling too, so the offset does not supply the missing scale either [57].

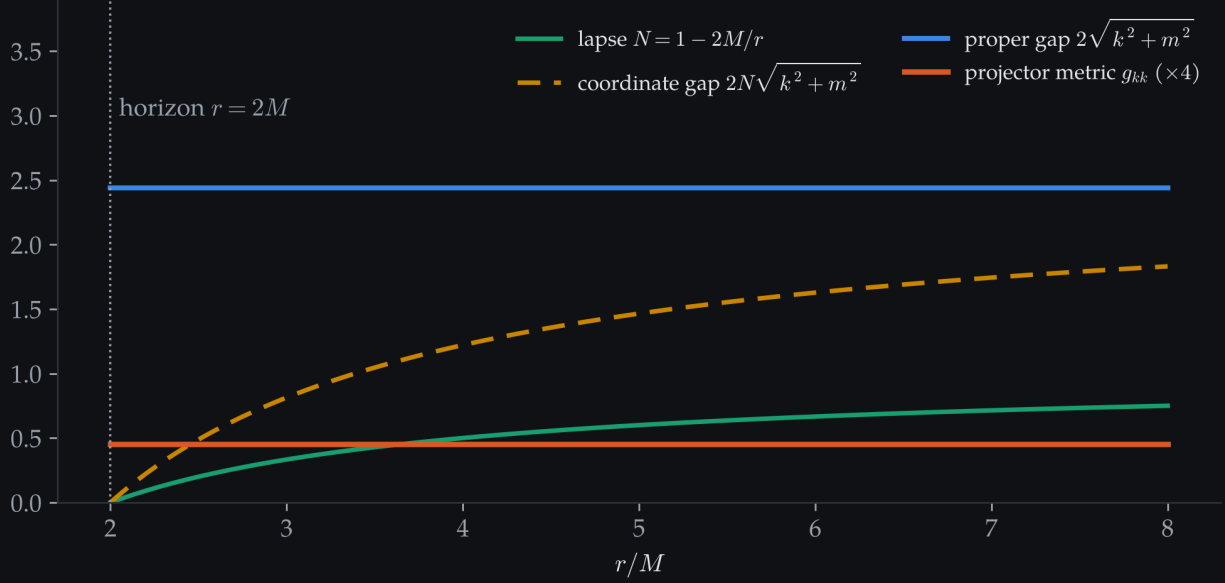


Figure 17: Local-symbol horizon control. The lapse and the coordinate gap vanish at $r = 2M$. The proper gap and the projector metric do not notice the horizon.

The local thermal law of a flat horizon has been measured on the operator side. By the Bisognano–Wichmann theorem, the entanglement Hamiltonian of a half-space is 2π times the boost generator, which is the origin of the Unruh temperature. The offset paper compares the near-cut slope of a lattice entanglement Hamiltonian with the continuum value $2\pi/v_F$ and reports agreement at about two percent in its tested range [57]. That is a measured instance of the structure that makes a Rindler horizon thermal, on a fermion chain; it says nothing about a curved horizon [CITED MEASUREMENT; STRUCTURAL].

9.3 The shared fold normal form

For a Schwarzschild exterior the static base is $h = dr^2/N + r^2 d\Omega^2$, $N = 1 - 2M/r$. Writing $r = 2M/(1 - N)$,

$$h_{NN} = \frac{4M^2}{N(1 - N)^4} \sim \frac{1}{4\kappa^2 N}, \quad \kappa = \frac{1}{4M}, \quad (52)$$

and in the lapse $\lambda = \sqrt{N}$, $h_{\lambda\lambda} = 16M^2(1 - \lambda^2)^{-4}$, finite and even, with $h_{\lambda\lambda}(0) = \kappa^{-2}$. Proper distance from the horizon is $\approx \lambda/\kappa$, so the lapse is linear in proper distance. Since the curvature of $-N^2 dt^2 + dx^2$ is $R = -2N''/N$, a lapse linear in proper distance is flat: near the horizon the (t, ρ) plane is a Rindler wedge with $N = \kappa\rho$ [40] [DERIVED, BH-ISCO-CUBE-17]. The fold needs that linear lapse, so it needs $\kappa \neq 0$. At an extremal horizon proper distance diverges logarithmically and the lapse is exponential in it: there is no fold [DERIVED, BH-ISCO-CUBE-17]. In four space dimensions ($N = 1 - 2M/r^2$, $\kappa = 1/\sqrt{2M}$) the same computation gives $Nh_{NN} \rightarrow M/2 = 1/(4\kappa^2)$ and $h_{\lambda\lambda}(0) = 2M = \kappa^{-2}$ [DERIVED; CHECKED SYMBOLICALLY, BH-FOLD-PARALLEL-01]. On Σ_4 , $w g_{ww} \rightarrow \frac{1}{4}$ and $g_{\sigma\sigma} = 1$ at the wall. Both are the fold normal form

$$ds^2 = \frac{dX^2}{4c^2 X} = \frac{d\lambda^2}{c^2}, \quad X = \lambda^2, \quad \lambda \mapsto -\lambda \text{ an isometry}, \quad (53)$$

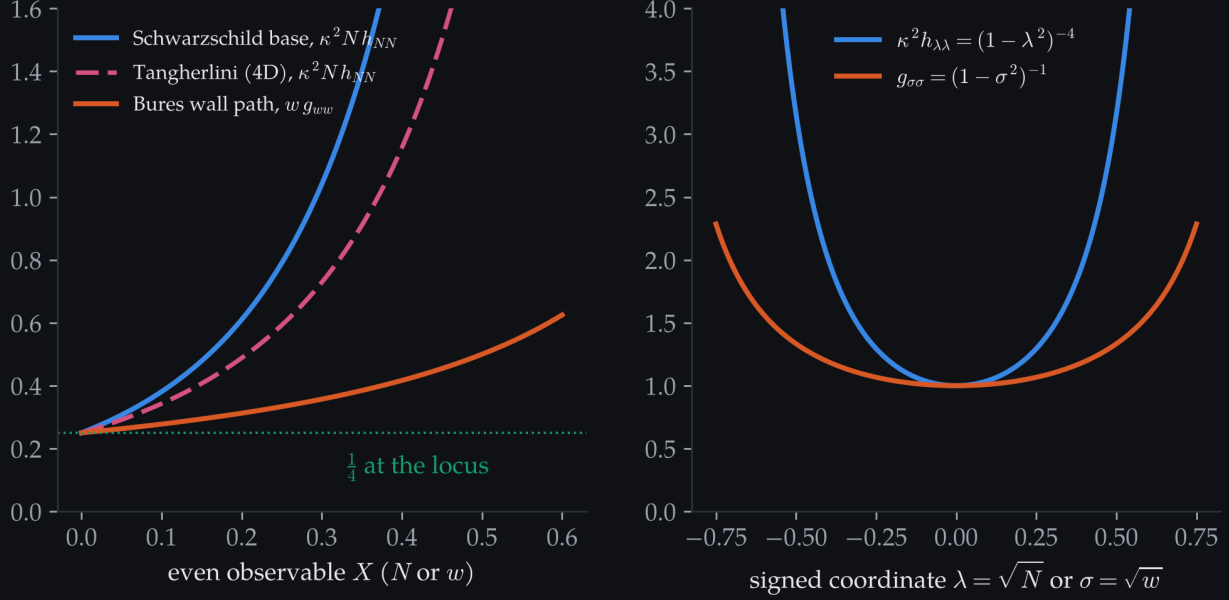


Figure 18: The shared fold normal form. Left: in the even observable (N or w), $c^2 X g_{XX} \rightarrow \frac{1}{4}$ for the Schwarzschild and Tangherlini static bases and for the Bures wall path. Right: in the signed coordinate ($\lambda = \sqrt{N}$ or $\sigma = \sqrt{w}$) both metrics are finite and even.

with $c = \kappa$ in general relativity and $c = 1$ here (Figure 18). The Penrose-inequality manuscript doubles the static base across the horizon with an odd lapse, $h_{\pm} = ((1 \pm \lambda)/2)^4 h$ [7]. It calls the divergence of the coordinate expression at the horizon “a coordinate singularity” and says the apparent degeneracy “is removed by the isotropic radius or the regular base collar” [CITED, NOT CHECKED]. Two cautions belong with the doubling. The second sheet of the doubled base is a second exterior, not the black-hole interior. And the event horizon is a null causal boundary of spacetime; the fold is a property of the Riemannian base, not of that boundary. That is the same lesson as Theorem 2, and it is the author’s own rule for degenerate strata: a stratum whose kernel points across it is a fold, silent at first order and resolved at the next, not a barrier [41] (Section 8.5 there).

The grade is structural, for three reasons. The constant does not match: κ is physical, our c is a fixed normalisation, and matching powers without a fixed parameter is insufficient. The base h is a quotient of a Lorentzian spacetime by a flow that goes null at the horizon, and nothing on our side goes null. And the fold normal form is generic: any smooth metric seen through the square of a coordinate has it, Whitney’s A_2 [77].

9.4 A non-self-adjoint horizon operator

The stability operator of a marginally outer trapped surface S is

$$Ls = -\Delta_S s + 2K(\nu, \nabla_S s) + (\text{tr}_S \nabla_\nu K - |II|^2 - \text{Ric}(\nu, \nu))s. \quad (54)$$

Here ν is the outward normal in the slice, K the extrinsic curvature of the slice, and Q denotes the zeroth-order coefficient in parentheses. The drift term $2K(\nu, \nabla_S s)$ makes L non-self-adjoint. Its principal eigenvalue is still real and simple, with positive right and left eigenfunctions φ, φ^* [8]. In

the equality case of the spacetime Penrose inequality, the lapse restricted to the horizon satisfies $2L^*u = cH$, with $H \geq 0$ the outward mean curvature and c a constant [7] [CITED, NOT CHECKED]: the lapse answers to the *adjoint* operator. When the drift is exact, $K(v, \cdot) = d\psi$, a weighted similarity removes it:

$$e^{-\psi}Le^\psi = -\Delta_S + (Q + |\nabla\psi|^2 - \Delta_S\psi), \quad (55)$$

which is self-adjoint, and a rotational drift cannot be removed this way [DERIVED; CHECKED SYMBOLICALLY, BH-DRIFT-SYM-02]. This parallels Theorem 5, where the transport operator is similar to a symmetric one through $S = \text{diag}\sqrt{w/t}$, and Theorem 4, where it is self-adjoint in the kinetic metric. On our side the symmetrising weight degenerates at the wall; on the horizon side a scalar weight e^ψ need not degenerate at the horizon. The principal eigenvalue of L is always simple, so it is never part of a Jordan block; that says nothing about the other eigenvalues. On our side, read on the cover where the metric is regular, the wall point is not a Jordan block either: it is a plain double point (Theorem 8). The Jordan block belongs to the folded Gram chart. Read this way the two sides agree, and the meeting point is the left/right (biorthogonal) structure together with the absence of a defect in the regular chart [STRUCTURAL].

9.5 A degenerate form with a kernel

On a Killing horizon the induced form is degenerate. The generator direction spans its kernel, and the pullback to any section is the same round metric [7] [CITED, NOT CHECKED]. The author's IQGT paper sorts degenerate strata in exactly these terms [41]. A stratum is *polar* if the kernel runs along it, and then nothing moves along it at any order (Lemma 8.5 there); it is a *fold* if the kernel points across it, and then the family is silent at first order and moves at second, with two sheets meeting, while an odd first order is only a parametrisation artefact (Proposition 8.6 there). At a fold the degenerate metric does not determine the motion across it: a direction in the kernel must be supplied, which is the structure basic to Carrollian geometry [96] (Remark 8.7 there). A Killing horizon is the polar case: its kernel is the generator and runs along the locus. The supplied kernel vector field, the Carrollian structure, is on the operator side the Riesz block. Our wall is the fold case. In the Gram picture the kernel of the degenerating coefficient points *across* the locus [STRUCTURAL].

9.6 Two type D objects, and where the exceptional points are

Kerr is type D at every point (Section 4.4): its Weyl operator has a doubled eigenvalue and no Jordan block. The exceptional points of black-hole physics are elsewhere, in the quasinormal spectrum. Exceptional points and exceptional lines have recently been located there, with double-pole excitation and pseudospectral structure [9, 10, 11]. In the language of Section 8 these are spectral walls of the ringdown operator, the kind the Sun's dynamo shows. The orbits around the hole carry one as well: the innermost stable circular orbit is a spectral wall of the circular-orbit family (Proposition 15). Theorem 8 suggests a concrete question for that setting: is there a natural signed coordinate, real on both sides, in which a ringdown exceptional point becomes a plain crossing? If so it is geometric; if the coordinate turns imaginary across it, it is spectral [PROPOSED].

The excitation of the two merging modes shows the pair rule of Section 4 on the gravitational side. In Kerr–de Sitter, where exceptional points are generic in the quasinormal spectrum, the excitation factors of two merging overtones grow large near an exceptional point and acquire nearly opposite phases [67]. In Kerr, the paired excitation factors are enhanced in almost opposite

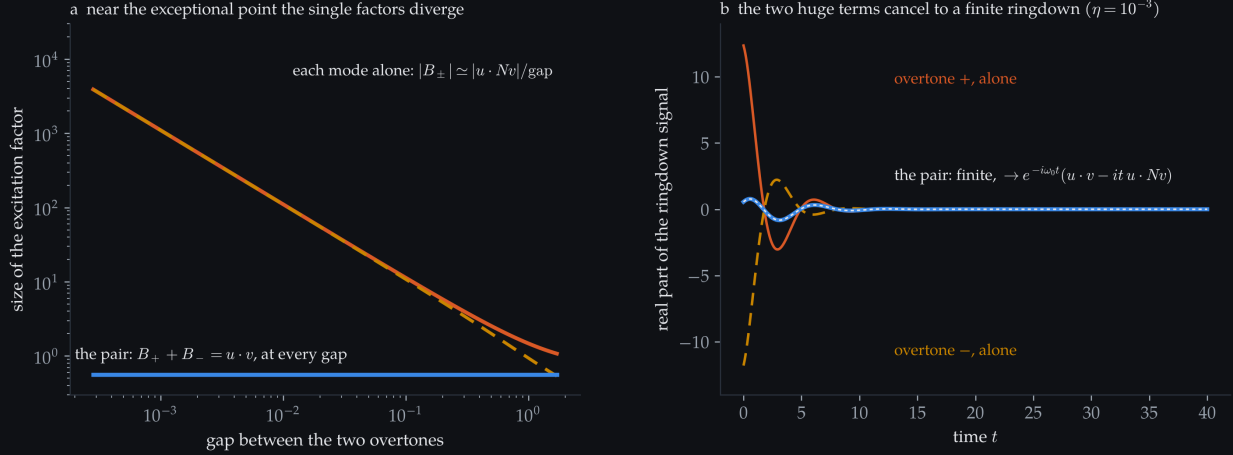


Figure 19: Two merging ringdown overtones (BH-QNM-13). (a) Each excitation factor diverges like the Jordan coupling divided by the gap between the overtones, while their sum is fixed by the pair projector. (b) Near the exceptional point the two single-mode terms are large and opposite; the ringdown they make together is finite and approaches the Jordan form with its term linear in time (dotted).

directions near an avoided crossing, with the product of the factor difference and the frequency difference nearly constant [68], and the Nariai case is treated analytically, including destructive excitation and transient linear growth [11]. In our terms, a single eigenprojector grows like $|A_{12}|/\text{gap}$ at a Jordan point while the pair projector stays exact. The author’s light paper shows the mechanism in the simplest case: for $\sigma_x + i\gamma\sigma_z$ the left–right overlap of a single mode collapses as $\sqrt{1 - \gamma^2} \approx \sqrt{2(1 - \gamma)}$, with Petermann factor $(1 - \gamma^2)^{-1}$, while the contour count stays 2 [37]. An excitation factor carries the inverse overlap, so it grows like the inverse square root of the distance to the exceptional point. The gap closes like that same square root, so this is the $1/\text{gap}$ law. BH-QNM-13 checks this in a two-mode model of a ringdown exceptional point, complex symmetric as quasinormal modes are under their bilinear product. Each excitation factor grows like $|u \cdot Nv|/\text{gap}$, with N the Jordan coupling, reaching 391 at gap 2.8×10^{-3} , and the two acquire opposite phases ($\arg B_+/B_- = 0.99986\pi$). Their sum is $u \cdot v$ at every gap, because $P_+ + P_-$ is the pair projector. The ringdown signal stays bounded through the exceptional point and becomes $e^{-i\omega_0 t}(u \cdot v - it u \cdot Nv)$ there, a term linear in time, with the two-mode sum approaching it like the square of the gap. A contour around both modes gives the same signal to 3×10^{-13} at and beside the point, and one loop around it exchanges the two modes and their factors while the ledger returns single-valued (Figure 19) [MEASURED FOR THE MODEL]. The model is synthetic: it is not a Teukolsky computation or detector data, and for a particular source the coefficient of the linear term can vanish. The black-hole results it is set beside are for different fields and backgrounds (a massive scalar on Kerr–Newman, gravitational perturbations of Kerr–de Sitter, the Nariai model), which are not interchangeable; what carries across is the structure. Destructive excitation and transient linear growth near an exceptional line are already studied in that literature [11]. What this paper adds is the reading: the contour and the ledger are what survive the exceptional point, and the single modes are not [STRUCTURAL COMPARISON FOR KERR–DE SITTER].

9.7 A square that vanishes is not equality

In the anti-de Sitter part of the Penrose manuscript the second-order deficit is a lapse-weighted square whose kernel is exactly four-dimensional. Equality is decided only after the kernel directions are carried to fourth order, where a second weighted square and a degree-one obstruction appear [7] [CITED, NOT CHECKED]. Three results here share the discipline. The fixed-family trace-log ledger is even to all orders, $Z(t) - Z(0) = \sum_j \log(1 + t^2 \omega_j^2)$, so its first-order term vanishes while the ledger is not constant [37]. The handed part Ω cancels across the mirror while each side is handed (Section 4). And Ramanujan's f and b cancel to a finite sum while each diverges (Section 4.5). In each case a vanishing combination says nothing about the parts. The parts have to be followed further, and the author's own rule says how far: to the first order that does not vanish. That order decides whether a silent direction folds or passes through [41] (Proposition 8.6 there), and whether a nonnegative branch touches zero or crosses it, an even first order being a touch and an odd one a crossing [40] (Proposition 17 there) [STRUCTURAL, DISCIPLINE].

9.8 What exterior data recover

At equality the Penrose manuscript recovers the full initial data (g, K) of a Schwarzschild slice, through a Lagrange multiplier turned into Killing initial data, a static base and its doubling [7] [CITED, NOT CHECKED]. Its bordered map at the zero eigenvalue of the trapped-surface operator is a Schur complement, the move of Section 5. Our hidden-block identity $P(X + QYQ)P = PXP$ says that an exterior compression alone cannot recover a hidden block (Section 5.5). The two statements are consistent. Recovery needs structure beyond the exterior record, and in their theorem that structure is supplied by the Einstein constraint equations and the dominant energy condition. Nothing on our side plays that role [STRUCTURAL].

9.9 What an identification would need

A chart wall can be identified with a horizon only after each of the following is in place: (1) one operator $K[g, \Phi]$ coupling the projector data to a physical metric; (2) a fixed pairing (the inner product is structural); (3) a source response with a probe; (4) an areal spectral measure; (5) a four-dimensional conserved field equation, with lensing; (6) a wave sector. None is supplied here. In particular there is nothing on our side corresponding to mass, the dominant energy condition, the area inequality, the value of the surface gravity, charge, or uniqueness. Proposition 16 explains why projector data alone cannot provide the first of these.

Coplanar wall of Σ_4	Horizon	Grade
fold normal form, $c = 1$	static base, $c = \kappa$	shared form, verified
apparent exceptional point is a chart effect (Gram frame)	apparent singularity is a coordinate singularity	same lesson
transport self-adjoint in the kinetic metric; regular on the cover	MOTS operator similar to self-adjoint when the drift is exact	same object type
plain double point on the cover; Jordan block only in the Gram frame	principal eigenvalue always simple (no Jordan block there)	agreement, read in the regular chart
read on the cover, diabolic (a cone)	Weyl operator type D, diabolic	same kind; different unfolding exponent
kernel across the locus (fold)	kernel along the generator	same object type, opposite case
conical sign -1 for a real loop through the mirror	—	ours only
matched load, $Z_b = I$, $S_b = 0$	horizon surface resistance equal to the vacuum impedance	structural, sourced
spectral walls: the Sun's dynamo, the optical threshold	the innermost stable circular orbit, a Jordan block between oscillation and growth	same object type
Bell: outside record thermal and blind to the pair; one-way books need the flux	Hawking pair beats CHSH; anomaly fixed by Hawking's flux	reproduction, structural
pair projector and ledger survive	—	ours only
exterior compression cannot recover the hidden block	equality recovers (g, K) with constraints and energy condition	consistent
—	mass, energy condition, area bound, κ , uniqueness	theirs only

No isomorphism is claimed.

10 Colour extension

Internal non-Abelian charge is incorporated at the standard grade. A charge $q \in \mathfrak{g}$ coupled along the trajectory through $\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + g_s\text{Tr}(qA_\mu)\dot{x}^\mu$ obeys, together with dynamics for the charge on its coadjoint orbit (the internal degrees of freedom the Wong system needs beyond the minimal coupling), the Wong equations [12]

$$\dot{q} = -g_s[A_\mu\dot{x}^\mu, q], \quad (56)$$

$$\frac{d}{dt}(g_{\mu\nu}\dot{x}^\nu) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = g_s\text{Tr}(qF_{\mu\nu})\dot{x}^\nu, \quad (57)$$

with $F = dA + g_s[A, A]$: the charge is dragged and rotated by transport, and its current back-reacts as a Lorentz-type force. A colour singlet decouples.

Proposition 17 (conditional persistence of non-normality). [DERIVED, CONDITIONAL] *Let $K = K_{\text{geom}} + K_{\text{gauge}}$ and suppose $\|[K_{\text{gauge}}, K_{\text{gauge}}^\top]\| > \|R_{\text{mixed}}\|$, where*

$$R_{\text{mixed}} = [K_{\text{geom}}, K_{\text{gauge}}^\top] + [K_{\text{gauge}}, K_{\text{geom}}^\top].$$

Then $\|[K, K^\top]\| > 0$ even at perfect isotropy $t_1 = t_2 = t_3$, where $[K_{\text{geom}}, K_{\text{geom}}^\top] = 0$.

The algebra is a reverse triangle inequality. Its content lies entirely in the hypothesis, whose verification for a concrete field configuration is open.

11 Proposed extensions

Nothing in this section is derived. Each item is an ansatz, recorded because it points somewhere concrete.

Superradiant extraction. Motivated by transient superradiance in modally stable regimes [6], an extraction rate for the gauge-extended shape system is proposed, combining an alignment term and an advective non-normality term:

$$\frac{dE}{dt} \stackrel{?}{=} \alpha_s \int \text{Tr}(J_{\text{colour}}^\mu F_{\mu\nu} \dot{W}^\nu) \|J\|^2 + \gamma \int \text{Tr}([D_t K, K^\top] J J^\dagger).$$

Its derivation from an energy functional with the kinetic metric of Theorem 1 has not been carried out.

Junction matching. A Darmon–Israel-type matching [13] of discrete wall data, such as the turning angle $\Theta_n = \arccos\langle v_{\text{max}}(n), v_{\text{max}}(n+1) \rangle$ of the leading direction, to a continuum exterior is a programme, not a result.

Interface analogues. Polariton and other non-Hermitian interfaces plausibly realise the same one-way and Jordan structure near their exceptional points. A quantitative trapping law awaits a dimensionally consistent derivation.

12 Open problems

1. Derive the block structure of a concrete hyperboloidal ringdown operator near an exceptional line, and decide whether it is spectral or geometric in the sense of Section 8: does a signed coordinate, real on both sides, unfold it? The exceptional points located in near-extremal Kerr and in Kerr–Newman, with their loops in a physical parameter plane [70, 69], are the first concrete places to ask.
2. Do the $\hat{Z}$ invariants of $\Sigma(3, 4, 11)$ and $\Sigma(3, 4, 13)$, the ± 1 surgeries on 8_{19} , turn from false theta into mock theta functions under $q \rightarrow q^{-1}$, as $\Sigma(2, 3, 7)$ is conjectured to, and do their radial limits show the pair cancellation of Section 4.5? Recent work on $\hat{Z}$ and surgery [65, 66] may already decide the first part.
3. Find the frame-free analogue of the Petrov classification for the transport operator on the square-case ladder $N = d + 1$, $d = 4, 5$, where more than two eigenvalues can meet at a

wall, and compare its unfolding exponents with types II, III, D and N. The interior half is settled: for every d the eigenvalue block obeys a secular equation with interlacing real roots, so exceptional points occur only on the boundary $\det G = 0$, where an eigenvalue is pinned at $2t_k$ [75]. The boundary classification is what remains. It now reduces to one computation that can fail: build the eigenvalue block of the transport for $d = 4$ on the stratum where one eigenvalue of G vanishes and at the corner where two vanish, and decide whether a size-3 Jordan block, the analogue of type III, occurs at the corner. The labels to compare are frame-free: the leading Puiseux exponent and the monodromy permutation on each stratum.

4. *Answered in Section 4.8.* Give the physical content of the qutrit dictionary: what do fidelity and quantum Fisher information measure mechanically on Σ_4 ? Section 4.8: the Fisher information is eight times the shape kinetic energy, the wall distance is a Bures angle, and fidelity is the Gram observer, blind to the mirror (BH-QUTRIT-18).
5. *Partly answered in Section 4.7.* Find a physically motivated potential that stiffens toward coplanarity, and compute its transverse frequency and the resulting displacement suppression. A thin disk gives the frequency ν exactly; every distance potential has stiffness $2\partial V/\partial w_3$; equal-mass gravity stiffens the square with $\omega = \sqrt{2(2\sqrt{2} - 1)} = 1.912290$, small-energy amplitude $\sqrt{2E}/\omega$ and a softening quartic term (28), and its stiffness at any flat shape is (26), but the rhombus turns over at $k_* = 0.144557$ (BH-QUTRIT-18, BH-STIFF-19). What remains is the region of flat shapes on which gravity stiffens, and its boundary.
6. Establish or refute the hypothesis of the conditional colour proposition for a concrete non-Abelian background.
7. State a Bell-type bound for operator records across the wall, with the hidden-block identity as the model of a local record, and decide whether the one-way wall of Section 5 can exceed it.
8. Follow the innermost stable orbit (a spectral wall) and the horizon (a geometric fold) together as the Kerr spin grows, and find where, if anywhere, the two kinds of wall meet. A candidate is extremality: there the fold loses its surface gravity (Section 9) and the ringdown exceptional points pile up [70, 69]. That the prograde ISCO reaches the horizon radius in Boyer–Lindquist coordinates as the spin reaches its maximum, while the two stay at different proper locations in the extremal throat, is established [84]; the open task is an invariant comparison that gives “meet” a coordinate-free meaning.
9. The accelerated-detector reading of horizon temperature (Bell and Leinaas) needs an absolute scale, which projector data cannot supply; find the extra structure that would let it join this comparison.
10. Supply any one of the six items of the identification list in Section 9.

A Proofs of the results used

Theorem 1. At $X = \text{diag}(\sqrt{w_i})$, the horizontal lift H of a base direction E solves $X^\top H + H^\top X = E$ together with horizontality, $XH^\top$ symmetric. The metric is $g(E_a, E_b) = \langle H_a, H_b \rangle_F$. For a diagonal E the lift is diagonal with $h_i = E_{ii}/(2\sqrt{w_i})$, giving $\sum_i E_{ii}E'_{ii}/(4w_i)$. For $E_{(ij)}$ the lift has $H_{ij} = \sqrt{w_i}/(\sqrt{2}(w_i + w_j))$ and $H_{ji} = \sqrt{w_j}/(\sqrt{2}(w_i + w_j))$: it satisfies $\sqrt{w_i}H_{ij} + \sqrt{w_j}H_{ji} = 1/\sqrt{2}$ and $\sqrt{w_i}H_{ji} = \sqrt{w_j}H_{ij}$, so $\|H\|^2 = 1/(2(w_i + w_j))$. Distinct directions have lifts with disjoint supports, hence the orthogonality. The resulting metric is the Bures metric of density matrices at W [14, 15].

Persistence bounds. Let Γ isolate part of the spectrum of A with margin $\delta = \text{dist}(\Gamma, \sigma(A))$, let $C_\Gamma = \delta \sup_\Gamma \|(z - A)^{-1}\|$, and let $A' = A + E$ with $\|E\| \leq \epsilon$.

Lemma 18. *If $\epsilon \sup_\Gamma \|(z - A)^{-1}\| < 1$, then $(z - A')^{-1}$ exists on Γ and $\|(z - A')^{-1}\| \leq \|(z - A)^{-1}\| / (1 - \epsilon \|(z - A)^{-1}\|)$.*

Proof. $z - A' = (z - A)(I - (z - A)^{-1}E)$, and the Neumann series of the second factor converges geometrically. $\square$

Theorem 19. *For $\epsilon < \delta / C_\Gamma$, the Riesz projectors over Γ satisfy $\|P' - P\| \leq \frac{\ell(\Gamma) C_\Gamma^2 \epsilon}{2\pi \delta (\delta - \epsilon C_\Gamma)}$.*

Proof. $P' - P = \frac{1}{2\pi i} \oint_\Gamma (z - A')^{-1} E (z - A)^{-1} dz$ by the resolvent identity. Bound the integrand by Lemma 18, with $\|(z - A)^{-1}\| \leq C_\Gamma / \delta$. $\square$

Projector norm near a Jordan block. For a real $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real distinct eigenvalues $\lambda_\pm$ and gap $g = \lambda_+ - \lambda_-$, $P_+ = (B - \lambda_-)/g$ is a rank-one idempotent, so its singular values are $\|P_+\|$ and 0 and $\|P_+\|^2 = \|P_+\|_F^2 = ((a - \lambda_-)^2 + b^2 + c^2 + (d - \lambda_-)^2) / g^2$. With $a - \lambda_- = \frac{1}{2}(a - d + g)$, $d - \lambda_- = \frac{1}{2}(d - a + g)$ and $g^2 = (a - d)^2 + 4bc$ this is $\|P_+\|^2 = 1 + (b - c)^2 / g^2$ exactly, and the same for P_- [DERIVED; CHECKED ON 500 RANDOM MATRICES TO 4×10^{-14} , V10-CORRECTIONS]. As $c \rightarrow 0$, $\|P_+\| g \rightarrow |b|$.

Jordan semigroup norm. $e^{-Jt} = e^{-bt} \begin{pmatrix} 1 & -\mu t \\ 0 & 1 \end{pmatrix}$, and the largest singular value of $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ is $(\sqrt{4 + s^2} + |s|) / 2 = e^{\text{arsinh}(|s|/2)}$. So $\log \|e^{-Jt}\| = \text{arsinh}(|\mu|t/2) - bt$. The first term is concave in $t \geq 0$ with slope $|\mu|/2$ at $t = 0$, so the difference is positive for some $t > 0$ if and only if $|\mu|/2 > b$.

Interface Schur complement (31). Solve the first and third block rows of $(K - z)x = y$ for the side variables with y supported on the boundary: $x_A = -(A - z)^{-1} B x_\Sigma$ and $x_E = -(E - z)^{-1} G x_\Sigma$. Substitute into the middle row.

Proposition 13. The second block row of LW is $L_{ib} - L_{ii} L_{ii}^{-1} L_{ib} = 0$, so $LW = \begin{pmatrix} Z_b \\ 0 \end{pmatrix}$ and $W^* LW = Z_b$, hence $\text{Herm } Z_b = W^* \text{Herm } L W$. Then $(Z + I)^*(Z + I) - (Z - I)^*(Z - I) = 2(Z + Z^*)$, and multiplying on both sides by $(Z + I)^{-1}$ gives the identity.

Equation (33). With $\mathcal{G} = (I, S_b)^\top$ and $\mathcal{J} = \text{diag}(I, -I)$, $\mathcal{G}^* \mathcal{J} \mathcal{G} = I \cdot I - S_b^* S_b$, and the second equality is (32).

The identities of Section 5.4. (a) The upper-left block of P^2 is $C^2 + BB^\dagger$; equating it with C gives $C - C^2 = BB^\dagger$. (b) $K(t) = B e^{-iDt} B^\dagger$ at $t = 0$. (c) $PH^2P = PH(P + Q)HP$, so $PH^2P - (PHP)^2 = PHQHP$, whose kept block is $BB^\dagger$. (d) As (c), with D_T , P_W and Q_W ; Q_W has rank one, so the difference has rank at most one, and $g_W(H, P_W D_T Q_W D_T P_W H) = g_W(Q_W D_T P_W H, Q_W D_T P_W H) \geq 0$ because D_T is g_W -self-adjoint and Q_W is a g_W -orthogonal projection.

Retention model. The Laplace transform of the outward flux κx after an impulse E_0 is $\hat{F}(s)/E_0 = \kappa(s+b)/(s^2 + (\kappa+a+b)s + \kappa b)$. Its value at $s = 0$ is 1 for $b > 0$, and $-\hat{F}'(0)/E_0 = (1+a/b)/\kappa$. At $b = 0$, $x = E_0 e^{-(\kappa+a)t}$ and $y \rightarrow aE_0/(\kappa+a)$.

B Checks

Every derived statement above was also checked by computation, and every measured number comes from these scripts. Each script states its kill condition before it runs. They are included in the code archive.

- **BH-WALL-CHECK-01:** the Kendall control (4.0000000 at three points); unit coefficients of the shape block in σ ; length $\arcsin \sqrt{\varepsilon}$ to 10^{-6} ; metric-normalised curvature finite at the wall.
- **BH-CORE-02:** symbolic block structure (Theorem 3); interior discriminant positive; joint condition; braid gate BH-N-01; the ledger table; volume-structure self-adjointness; the Jordan semigroup formula.
- **BH-WALL-03:** closed form of $D(w_3)$ and the location of its roots; the Schur–Cayley identities; the local-symbol horizon control.
- **BH-NONNORMAL-04:** $\|[A, A^\top]\|_F$ in the Frobenius frame, anisotropic and isotropic cases.
- **BH-FOLD-PARALLEL-01** and **BH-DRIFT-SYM-02:** the static-base normal forms in three and four dimensions, and the drift symmetrisation identity.
- **BH-EP-SPINOR-04:** the Gram-frame pairing (19) and its residue $\frac{1}{2}$ symbolically; the monodromy after one, two and four loops of w_3 ; the Majorana exchange relations; the handed part Ω and its cancellation across the mirror.
- **BH-LIFT-05:** kinetic self-adjointness (symbolic, 5×5); $A = P_W D_T P_W$; the lifted operator (14) and $B(0) = \lambda_* I$; complex loops in both frames; the folded share $\frac{1}{2}$; kinetic and Gram-frame projector norms; real loops through the mirror in both frames; the cone; real spectrum on both sides.
- **BH-PETROV-06:** the map (21) and its determinant; the Jordan structure of all five Petrov types; the unfolding laws toward II and D; Kerr type D at three points.
- **BH-SHADOW-07:** Ramanujan’s radial limits at -1 , i and $e^{i\pi/3}$, with the finite sum (23).
- **BH-KNOT-08:** the volume conjecture for 4_1 to $N = 800$; the trefoil series; mirror conjugation; signatures.
- **BH-SPINE-09:** the $BB^\dagger$ identities, the transport compression, the Cayley and graph identities, the geometric-tensor inequality, the one-way readout, the hidden-block identity and the retention model.
- **BH-ISCO-10:** the wobble generator of circular Schwarzschild orbits, its sign change and Jordan block at $r_0 = 6$, the square-root law on both sides, and Levin’s whirl law at $L^2 = 13$.
- **BH-BELL-11:** the thermal outside record and its blindness to the pair phase; the CHSH value of a Hawking pair; same record, opposite futures; the anomaly flux against the thermal flux; the associator and reduced-square identities; the one-way balance.
- **BH-MATCHED-12:** the horizon surface resistivity against the vacuum impedance, and the Cayley map at and beside the matched value.
- **BH-QNM-13:** two merging ringdown overtones in a complex-symmetric two-mode model; single excitation factors against the gap; the pair sum; the bounded signal and its Jordan form; the contour integral; one and two loops around the exceptional point.
- **BH-KNOT-14:** the Rosso–Jones invariant of torus knots, checked against the unknot, two Jones polynomials and the trefoil series; growth of the 8_{19} invariant and its period-16 constant;

mirror conjugation; Alexander polynomial, genus and Moser surgeries.

- **BH-OFFSET-15**: the outside half of a fermionic Hawking pair by partial trace; its offset against ω/T_H for every phase; four modes; the rescaling.
- **BH-SUNLADDER-16b**: the solar dynamo operator rebuilt from the solar paper's census script; its birth points; loops around the first n , at two heights and reversed; discriminant winding; the product of single-point loops.
- **BH-ISCO-CUBE-17**: the inverse-cube reading $p = 2 + 6M/r$ of the ISCO; $R = -2N''/N$ and the Rindler lapse; the extremal lapse; Lamb's acoustic cutoff and its nilpotent generator.
- **BH-QUTRIT-18**: quantum Fisher information against the kinetic metric on random motions; the wall distance against the Bures angle; the mirror pair's fidelity; the thin-disk minimum over orientations; the sign of the transverse stiffness for gravity and like charges on random flat configurations, with the square as an exact control; the closed-form cone slope. Its first kill condition, that gravity does not stiffen, failed, and the run note records the correction.
- **BH-STIFF-19**: the gravitational stiffness at a flat shape in closed form, against the lift by finite differences; the square; the rhombus sign change and its exact root k_* ; the stiffness minimised over flat shapes away from collisions; the fraction of random flat shapes that stiffen.
- **RS-01** and **RS-01b**: the polar stratum as a wave mirror; reflection magnitude, phase against the ray phase, lateral shift against the geodesic displacement, with a kinked and a smooth lead.
- **RS-02**, **RS-03**, **RS-03b** and **RS-03c**: reflection at a step of the pullback metric; the sharp step by direct matching on random parameters; three smooth profiles and their first-order convergence; reciprocity; flux balance; $g_\perp$ in a thick layer; total reflection beyond the turning point and its adiabatic onset; the gap form.
- **PHOTON-RING-WALL**: the wobble generator of the null circular orbit, its Lyapunov rate, the half-orbit factor $e^{-\pi}$ and delay $3\sqrt{3}\pi M$, symbolically.
- **V10-CORRECTIONS**: the common-frame hypothesis of Theorem 4 and its counterexample; the Jacobian of the normalised flow; the curvature of the eigenvalue block; the real-qubit hemisphere; the rank of the hidden block; the frame-change term of the connection; the exact projector norm; the Fisher information at a rank change; the exact potential of the lifted square; the local symbol at $r = 2M$.
- **V10-PUSH**: the photon ring from nonlinear null orbits and its charged opposite; the sine guard on two- and three-level families; the fermion offset as a Smith coordinate by partial trace; the compression $\mathcal{D}$ for Hermitian and oblique projectors; Davis–Kahan persistence of a cluster against the turning of its vectors.
- **BH-FIGS-05**, **BH-FIGS-NEW-05**, **BH-FIGS-V6**, **BH-FIGS-V7**, **BH-FIGS-V8**, **BH-FIGS-V8-4**, **BH-FIGS-V8-5** and **BH-FIGS-V10**: every figure, computed from the same formulas.

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