

Relative P -Ideals of Hypergraphs and Ultrafilter Spectra

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Abstract

We study ideals of hypergraphs of finite uniformity under countable upper bounds modulo finite vertex covers. The canonical ideal generated by a free filter and finite weak colorings has this relative P -property exactly when the filter is the intersection of finitely many P -points. The classification is preserved by adjoining hypergraphs that are locally finite after a finite vertex deletion. The corresponding extension of a trace ideal induces capture ideals on increasing sequences. These are σ -ideals exactly for P -filters, and their uniformities are $\max\{\pi\chi(\mathcal{F}), \mathfrak{d}\}$. For an ultrafilter $\mathcal{U}$, the arity-two capture ideal equals the Louveau ideal exactly when $\mathcal{U}$ is a P -point. Successive capture ideals decrease with the arity, and equality characterizes Q -points. Every strict inclusion is witnessed by a closed set. Thus a non- Q P -point yields a decreasing sequence of σ -ideals, distinguished by closed sets, with equal uniformities. We also establish invariance under locally finite witness enlargements and analyze Stone kernels and countable block products. Consistently, nested ray-positive P -sets have the same singleton endpoint spectrum and different ray transversal numbers.

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1 Introduction

For an ideal of hypergraphs, the existence of countable upper bounds depends on the allowed differences between a hypergraph and its bound. We consider differences covered by finitely many vertices. Such differences meet any locally finite hypergraph in finitely many edges. An increasing path avoiding a relative upper bound therefore avoids each hypergraph it bounds. This connects the relative P -property of hypergraph ideals with countable closure of ideals on increasing sequences.

Fix an integer $k \geq 2$ and write $E_k = [\omega]^k$. A hypergraph is a subset of E_k . Write $A \subseteq^* B$ when $A \setminus B$ is finite. Let $\mathcal{V}_k$ consist of hypergraphs with a finite vertex cover, let $\mathcal{G}_{\text{fc}}^{(k)}$ consist of hypergraphs with a finite weak coloring, and let $\mathcal{L}_k$ consist of hypergraphs that become locally finite after finitely many vertices are deleted. A hypergraph ideal $\mathcal{J}$ is a $\mathcal{V}_k$ - P -ideal if every sequence $H_n \in \mathcal{J}$ has an upper bound $H \in \mathcal{J}$ satisfying $H_n \setminus H \in \mathcal{V}_k$ for all n . For $R = \{r_0 < r_1 < \dots\} \in [\omega]^\omega$, the consecutive hyperedges are

$$P_R^{(k)} = \{\{r_n, \dots, r_{n+k-1}\} : n < \omega\}.$$

The restriction of $\mathcal{V}_k$ to $P_R^{(k)}$ is the finite ideal. This permits upper bounds modulo $\mathcal{V}_k$ to control all sufficiently late consecutive hyperedges of a witness.

For a free filter $\mathcal{F}$ on ω , put

$$A_X^{(k)} = E_k \setminus [X]^k, \quad T_k(\mathcal{F}) = \{H \subseteq E_k : (\exists X \in \mathcal{F}) H \cap [X]^k \in \mathcal{V}_k\}.$$

We consider the two extensions

$$\mathcal{C}_k(\mathcal{F}) = \text{Id}(\mathcal{G}_{\text{fc}}^{(k)} \cup \{A_X^{(k)} : X \in \mathcal{F}\}), \quad \mathcal{J}_{\mathcal{F},k} = T_k(\mathcal{F}) + \mathcal{L}_k.$$

Here Id denotes the generated ideal, and $+$ denotes the ideal generated by a pair of ideals. The first extension incorporates finite colorings; the second incorporates local finiteness. Their relative P -properties have different filter-theoretic descriptions.

Theorem A. *Let $\mathcal{F}$ be a free filter and $2 \leq k < \omega$.*

- (i) *Both $\mathcal{C}_k(\mathcal{F})$ and $\mathcal{C}_k(\mathcal{F}) + \mathcal{L}_k$ have the relative P -property precisely when $\mathcal{F}$ is the intersection of a nonempty finite family of P -points.*
- (ii) *The ideal $\mathcal{J}_{\mathcal{F},k}$ has the relative P -property precisely when $\mathcal{F}$ is a P -filter.*
- (iii) *If $\mathcal{U}$ is an ultrafilter, then $\mathcal{J}_{\mathcal{U},k} = T_k(\mathcal{U})$ precisely when $\mathcal{U}$ is a Q -point.*

The proofs use finite partitions with coordinatewise eventual constancy, together with a row-tail construction for relative upper bounds. Theorem 3.7 proves the first assertion, Theorem 4.2 the second, and Theorem 5.1 the third. The row-tail construction also preserves the minimum size of an avoiding family when fewer than $\mathfrak{b}$ locally finite hypergraphs are adjoined to a trace ideal (Theorem 4.6).

The hypergraphs in $\mathcal{J}_{\mathcal{F},k}$ induce an ideal on $[\omega]^\omega$. Define

$$\begin{aligned}\Gamma_k(H) &= \{R \in [\omega]^\omega : |P_R^{(k)} \cap H| = \omega\}, \\ \mathcal{I}_k(\mathcal{F}) &= \{\mathcal{A} \subseteq [\omega]^\omega : (\exists H \in \mathcal{J}_{\mathcal{F},k}) \mathcal{A} \subseteq \Gamma_k(H)\}.\end{aligned}$$

An explicit cofinal family for this ideal is given by

$$(1.1) \quad \begin{aligned} N_{X,f}^{(k)} &= \{R : R \not\subseteq^* X\} \\ &\cup \{R \in [\omega]^\omega : r_{n+k-1} \leq f(r_n) \text{ for infinitely many } n\}, \end{aligned}$$

where $X \in \mathcal{F}$ and $f : \omega \rightarrow \omega$ is nondecreasing. This description determines both countable closure and uniformity. Here $\pi\chi(\mathcal{F})$ is the least size of a family of infinite sets that refines every member of $\mathcal{F}$ under almost inclusion, and $\mathfrak{d}$ is the dominating number. The uniformity $\text{non}(\mathcal{I})$ is the least size of a set outside $\mathcal{I}$.

Theorem B. *For every free filter $\mathcal{F}$ and $2 \leq k < \omega$,*

$$\begin{aligned}\mathcal{I}_k(\mathcal{F}) \text{ is a } \sigma\text{-ideal} &\iff \mathcal{F} \text{ is a } P\text{-filter}, \\ \text{non}(\mathcal{I}_k(\mathcal{F})) &= \max\{\pi\chi(\mathcal{F}), \mathfrak{d}\}.\end{aligned}$$

For a free ultrafilter $\mathcal{U}$, the ideals satisfy

$$\mathcal{I}_{k+1}(\mathcal{U}) \subseteq \mathcal{I}_k(\mathcal{U}), \quad \mathcal{I}_{k+1}(\mathcal{U}) = \mathcal{I}_k(\mathcal{U}) \iff \mathcal{U} \text{ is a } Q\text{-point}.$$

If $\mathcal{U}$ is not a Q -point, each strict inclusion is witnessed by a closed subset of $[\omega]^\omega$ belonging to $\mathcal{I}_k(\mathcal{U})$ and not to $\mathcal{I}_{k+1}(\mathcal{U})$. Moreover,

$$\mathcal{I}_2(\mathcal{U}) = \ell_{\mathcal{U}}^0 \iff \mathcal{U} \text{ is a } P\text{-point},$$

where $\ell_{\mathcal{U}}^0$ is the Louveau ideal. For every $k \geq 3$, $\mathcal{I}_k(\mathcal{U}) = \ell_{\mathcal{U}}^0$ holds precisely when $\mathcal{U}$ is both a P -point and a Q -point.

Consequently, a non- Q P -point yields

$$\ell_{\mathcal{U}}^0 = \mathcal{I}_2(\mathcal{U}) \supsetneq \mathcal{I}_3(\mathcal{U}) \supsetneq \mathcal{I}_4(\mathcal{U}) \supsetneq \cdots,$$

a strictly decreasing sequence of σ -ideals with the same uniformity. The separating closed sets are defined by finite blocks from a partition witnessing failure of the Q -point property. At a fixed arity, the ideal is unchanged if $P_R^{(k)}$ is enlarged to any locally finite subhypergraph of $[R]^k$. Thus the arity and the choice of locally finite witness enlargement have distinct effects.

Theorem 4.3 proves the basis description and countable-closure criterion, and Theorem 4.5 computes the uniformity. The Louveau identification and the arity classification are proved in Theorems 5.4 and 5.5.

The Stone space $X_k = \text{Ult}(\mathcal{P}(E_k)/\mathcal{V}_k)$ gives a topological interpretation. Its points are ultrafilters on E_k disjoint from $\mathcal{V}_k$. The k coordinate maps of an increasing hyperedge induce continuous maps $\pi_j : X_k \rightarrow \omega^*$, where ω^* is the space of free ultrafilters on ω . Let Z_k be their equalizer and write π for the common map on Z_k . For a proper ideal $\mathcal{J} \supseteq \mathcal{G}_{\text{fc}}^{(k)}$, its Stone kernel $L_{\mathcal{J}} = \{p \in X_k : p \cap \mathcal{J} = \emptyset\}$ lies in Z_k , and its endpoint spectrum is $\pi[L_{\mathcal{J}}]$.

A closed set is a P -set if every countable intersection of its neighborhoods is again a neighborhood. We call a nonempty closed $L \subseteq X_k$ ray-positive if every clopen neighborhood of L contains $\widehat{P_R^{(k)}}$ for some $R \in [\omega]^\omega$; here $\widehat{H}$ is the clopen set of ultrafilters containing H . Its ray transversal number $\nu_k(L)$ is the least size of a family of rays that supplies such a witness in every clopen neighborhood. A ray R avoids H if $|P_R^{(k)} \cap H| < \omega$, and an ideal is ray-proper if each of its members is avoided by some ray. We write $\beta\omega$ for the space of all ultrafilters on ω , including the principal ones; a function $e : \omega \rightarrow \omega$ induces the map $\beta e(\mathcal{U}) = \{A \subseteq \omega : e^{-1}[A] \in \mathcal{U}\}$.

Theorem C. (i) For a proper ideal $\mathcal{J} \supseteq \mathcal{G}_{\text{fc}}^{(k)}$, $L_{\mathcal{J}}$ is a P -set in X_k if and only if $\mathcal{J}$ is a $\mathcal{V}_k$ - P -ideal. If a nonempty $C \subseteq Z_k$ is a P -set in X_k and $\pi[C] = \{\mathcal{U}\}$, then $\mathcal{U}$ is a P -point.

(ii) Let $\omega = \dot{\bigcup}_n B_n$ be a partition into infinite sets, and on each $[B_n]^k$ let $\mathcal{J}_n$ be a proper ray-proper relative P -ideal containing the finite-chromatic ideal. Then

$$\mathcal{J} = \{H \subseteq E_k : H \cap [B_n]^k \in \mathcal{J}_n \text{ for every } n\}$$

has the same properties. On its kernel $L_{\mathcal{J}}$, the block-index function $e(i) = n$ for $i \in B_n$ induces a continuous surjection $\rho = \beta e \circ \pi : L_{\mathcal{J}} \rightarrow \beta\omega$. Every fiber is ray-positive. Every endpoint of a point in a nonprincipal fiber fails to be a P -point. Every nonempty P -set in the relative topology of $L_{\mathcal{J}}$ meets a principal fiber.

- (iii) If $\mathcal{U}$ is a P -point with $\pi\chi(\mathcal{U}) < \mathfrak{d}$, the kernels $L_{\mathcal{J}_{\mathcal{U},k}}$ and $L_{T_k(\mathcal{U})}$ are ray-positive P -sets in X_k with endpoint spectrum $\{\mathcal{U}\}$ and respective ray transversal numbers $\mathfrak{d}$ and $\pi\chi(\mathcal{U})$. They satisfy

$$L_{\mathcal{J}_{\mathcal{U},k}} \subsetneq L_{T_k(\mathcal{U})}.$$

The first and third assertions follow from the Stone correspondence and the filter classifications in Section 6. The product and fiber assertions are proved in Section 7.

The relevant antecedents are the finite-chromatic ideal and pair-reaping studied by Hrušák, Meza-Alcántara, and Minami [3], and the ultrafilter ideals studied by Brendle and Shelah [2]. The former work treats unbounded families of finite tuples, including higher finite arities, and the associated splitting invariants. The present paper uses overlapping consecutive hyperedges, finite vertex incidence, and upper bounds modulo finite vertex covers. Brendle and Shelah proved

$$\text{non}(\ell_{\mathcal{U}}^0) = \max\{\pi\chi_{\sigma}(\mathcal{U}), \mathfrak{d}\}.$$

For P -points, the uniformity formula in Theorem B recovers this value through an explicit hypergraph representation. We identify when that representation coincides with the Louveau ideal and classify the higher-arity capture ideals by strict inclusion. The latter inclusions are already separated by closed sets. The Louveau ideal itself originates in [5]; its successor-set description is recalled and proved in Section 5.

Countable decision on ω also connects these ideals to reaping. The dual of the countably complete decision filter is an instance of Schilhan's ideal associated with the refining relation [8, Definitions 2.9 and 7.6]. Theorem 2.7 characterizes finite-chromatic hypergraphs by the existence of a countable family of subsets of ω whose simultaneous decision ensures avoidance. Section 8 uses this characterization to represent $\mathfrak{r}$ and $\mathfrak{r}_{\sigma}$ by rays and to formulate ideal-extension questions.

2 Hypergraphs and increasing sequences

Finite vertex covers give an eventual inclusion order on the rows of a hypergraph. We establish this description, relate finite colorings to decisions of subsets of ω , and construct countable upper bounds by truncating rows.

2.1 Filters and cardinal characteristics

We work in ZFC. The set ω consists of the nonnegative integers. For bounds on finite subsets of ω , we use $\max \emptyset = -1$. For a set X , write $[X]^k$, $[X]^{<\omega}$, and $[X]^\omega$ for its k -element, finite, and countably infinite subsets. We write $A \subseteq^* B$ when $A \setminus B$ is finite. A filter is proper, closed under finite intersections and supersets. Unless stated otherwise, filters on ω are free, so they contain all cofinite sets and have no finite member. An ultrafilter is a maximal proper filter; equivalently, it contains exactly one of X and $\omega \setminus X$ for every $X \subseteq \omega$. Every proper filter extends to an ultrafilter by Zorn's lemma.

A *pseudointersection* of a family $\mathcal{A} \subseteq [\omega]^\omega$ is an infinite set B such that $B \subseteq^* A$ for every $A \in \mathcal{A}$. A free filter $\mathcal{F}$ is a *P-filter* if each countable subfamily has a pseudointersection belonging to $\mathcal{F}$. A *P-point* is a free ultrafilter that is a *P-filter*. A free ultrafilter $\mathcal{U}$ is a *Q-point* if, for every partition $\omega = \dot{\bigcup}_n C_n$ into finite sets, some $X \in \mathcal{U}$ satisfies $|X \cap C_n| \leq 1$ for all n .

A family $\mathcal{B} \subseteq [\omega]^\omega$ is a π -*base* of $\mathcal{F}$ if for every $X \in \mathcal{F}$ there is $B \in \mathcal{B}$ with $B \subseteq^* X$. It is a π_σ -*base* if, for every sequence $X_n \in \mathcal{F}$, one $B \in \mathcal{B}$ satisfies $B \subseteq^* X_n$ for every n . The least respective sizes are $\pi\chi(\mathcal{F})$ and $\pi\chi_\sigma(\mathcal{F})$. The members of these bases are arbitrary infinite subsets of ω . A filter base is a family $\mathcal{B} \subseteq \mathcal{F}$ such that every $X \in \mathcal{F}$ almost contains some $B \in \mathcal{B}$. Its least size is the *character* $\chi(\mathcal{F})$. Closing a base under finite deletions turns it into a base under actual inclusion without changing an infinite cardinality. These conventions agree with [2].

Every countable subfamily of a free filter has an infinite pseudointersection. Choose its n th point above the preceding point in the intersection of its first $n + 1$ members. Thus $\pi\chi_\sigma(\mathcal{F})$ is always defined. If $\mathcal{F}$ is a *P-filter*, a π -base refines the pseudointersection in the filter, so

$$(2.1) \quad \pi\chi_\sigma(\mathcal{F}) = \pi\chi(\mathcal{F}).$$

A free ultrafilter has no countable π -base. Indeed, a countable family of infinite sets can be split by one set, by successively choosing fresh points on both sides from each member. Whichever side belongs to the ultrafilter then contains no member of the alleged π -base modulo finite.

For functions $f, g \in \omega^\omega$, write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n . The bounding number $\mathfrak{b}$ is the least size of a $\leq^*$ -unbounded family, and the dominating number $\mathfrak{d}$ is the least size of a family cofinal under $\leq^*$. We use nondecreasing majorants freely. A dominating family may be chosen

to consist of functions h with $h(n) > n$. For background on these cardinals, see [1].

For a proper ideal $\mathcal{I}$ on a set S , its *uniformity* is

$$\text{non}(\mathcal{I}) = \min\{|A| : A \subseteq S, A \notin \mathcal{I}\}.$$

For a proper filter $\mathcal{F}$ on S , its transversal number is

$$\text{tr}(\mathcal{F}) = \min\{|A| : A \subseteq S, A \cap F \neq \emptyset \text{ for all } F \in \mathcal{F}\}.$$

The complement-dual ideal $\{A \subseteq S : S \setminus A \in \mathcal{F}\}$ has uniformity $\text{tr}(\mathcal{F})$, since A fails to meet a filter member exactly when its complement belongs to the filter.

2.2 Finite vertex covers and local finiteness

Fix $2 \leq k < \omega$. A k -uniform hypergraph on ω is a subset $H \subseteq E_k = [\omega]^k$. Its elements are its hyperedges. A set B is *independent* if $H \cap [B]^k = \emptyset$. A weak coloring is a map $c : \omega \rightarrow q$ such that no hyperedge is monochromatic. A finite weak coloring partitions the vertex set into finitely many independent sets. We write $\chi(H)$ for the least number of colors and put

$$\mathcal{G}_{\text{fc}}^{(k)} = \{H \subseteq E_k : \chi(H) < \omega\}.$$

This is an ideal. For a union of two finitely colorable hypergraphs, the ordered pair of their colorings is a finite weak coloring. Throughout, hypergraph coloring has this weak meaning.

For $F \subseteq \omega$, put

$$\text{Star}_k(F) = \{e \in E_k : e \cap F \neq \emptyset\}.$$

The finite-vertex-cover ideal is

$$\mathcal{V}_k = \{H \subseteq E_k : (\exists F \in [\omega]^{<\omega}) H \subseteq \text{Star}_k(F)\}.$$

It is contained in $\mathcal{G}_{\text{fc}}^{(k)}$. Assign distinct colors to the finitely many covering vertices and one further color to all remaining vertices. For $i \in \omega$, define the i th row by $H(i) = \{e \in H : \min e = i\}$.

Lemma 2.1 (Row description). *For $H, K \subseteq E_k$, the following are equivalent:*

- (i) $H \setminus K \in \mathcal{V}_k$;
- (ii) $H(i) \subseteq K(i)$ for all but finitely many i ;

(iii) for some N , every $e \in H$ with $\min e \geq N$ belongs to K .

Proof. If a finite set F covers $H \setminus K$, no error hyperedge can have minimum larger than every member of F . This proves (i) $\Rightarrow$ (iii). The equivalence of (ii) and (iii) is immediate. Under (iii), every error hyperedge meets $[0, N)$, proving (i). $\square$

A hypergraph is *locally finite* if every vertex belongs to finitely many hyperedges. Let $\mathcal{L}_k$ consist of hypergraphs H for which $H \cap [\omega \setminus F]^k$ is locally finite for some finite F .

Lemma 2.2. *A hypergraph belongs to $\mathcal{L}_k$ exactly when all but finitely many of its rows are finite. Consequently, $\mathcal{L}_k$ is an ideal containing $\mathcal{V}_k$. Every infinite vertex set contains an infinite independent subset for any member of $\mathcal{L}_k$.*

Proof. If deleting F gives local finiteness, every row with minimum above $\max F$ is finite. Conversely, suppose all rows with minimum at least N are finite. In the induced hypergraph on $[N, \infty)$, the hyperedges incident with a fixed vertex v lie in the finite union of rows indexed by $N, \dots, v$, so there are finitely many of them. The ideal assertions follow from this row description.

For the final assertion, first delete the finite exceptional vertices. Given finitely many already selected vertices, their incident hyperedges involve only finitely many vertices. Choose the next vertex in the prescribed infinite set outside that finite union and above the previous choice. No hyperedge can be completed by a later choice. The resulting infinite set is independent. $\square$

After identifying each row with ω , the ideal $\mathcal{L}_k$ becomes $\text{Fin} \otimes \text{Fin} = \{A \subseteq \omega \times \omega : \{n : A_n \text{ is infinite}\} \text{ is finite}\}$. Our upper bounds use the error ideal $\mathcal{V}_k$, whose members have finitely many nonempty rows.

Write $\mathcal{M}_k$ for the nondecreasing functions $f \in \omega^\omega$ with $f(i) \geq i + k$, and set

$$(2.2) \quad B_f^{(k)} = \{e \in E_k : \max e \leq f(\min e)\}.$$

Lemma 2.3 (Bounded-row hypergraphs). *For every $H \in \mathcal{L}_k$ there is $f \in \mathcal{M}_k$ such that $H \setminus B_f^{(k)} \in \mathcal{V}_k$. Moreover,*

$$B_f^{(k)} \setminus B_g^{(k)} \in \mathcal{V}_k \quad \Longleftrightarrow \quad f \leq^* g \quad (f, g \in \mathcal{M}_k).$$

Proof. Above the finitely many exceptional rows, take the maximum of all vertices in each finite row and then a nondecreasing majorant. This gives

the first assertion by Lemma 2.1. If $f \leq^* g$, row inclusion holds eventually. If $f(i) > g(i)$, the set

$$\{i, i+1, \dots, i+k-2, g(i)+1\}$$

is a k -element hyperedge in the i th row of the difference. Infinitely many such indices preclude a finite vertex cover, again by Lemma 2.1. $\square$

2.3 Witnesses and absolute avoidance

We regard $[\omega]^\omega$ as the space of its increasing enumerations, with the subspace topology inherited from ω^ω . The first finitely many entries specify a clopen cylinder. For each $R = \{r_0 < r_1 < \dots\}$, let

$$P_R^{(k)} = \{\{r_n, \dots, r_{n+k-1}\} : n < \omega\}.$$

A witness assignment W used below satisfies

$$(2.3) \quad P_R^{(k)} \subseteq W(R) \subseteq [R]^k, \quad W(R) \text{ is locally finite} \quad (R \in [\omega]^\omega).$$

Set

$$D_H^W = \{R : |W(R) \cap H| < \omega\}, \quad \Gamma_W(H) = [\omega]^\omega \setminus D_H^W.$$

When $W(R) = P_R^{(k)}$, we write $D_H^{(k)}$ and $\Gamma_k(H)$. We abbreviate $D_H^{(k)}$ to D_H when the arity is fixed. A hypergraph ideal is *W-ray-proper* if $D_H^W \neq \emptyset$ for every member H . The term *ray-proper* refers to consecutive hyperedges. A family $\mathcal{A} \subseteq [\omega]^\omega$ *avoids* an ideal $\mathcal{J}$ if for every $H \in \mathcal{J}$ some $R \in \mathcal{A}$ belongs to D_H^W . Thus the avoiding ray may depend on H .

Lemma 2.4 (The witness null ideal). *For every assignment satisfying (2.3),*

$$\Gamma_W(H) = \emptyset \iff H \in \mathcal{V}_k.$$

Furthermore,

$$\Gamma_W(H \cup K) = \Gamma_W(H) \cup \Gamma_W(K), \quad D_{H \cup K}^W = D_H^W \cap D_K^W,$$

and $H \triangle K \in \mathcal{V}_k$ implies $\Gamma_W(H) = \Gamma_W(K)$.

Proof. A finite vertex cover meets finitely many hyperedges of the locally finite witness. Thus every member of $\mathcal{V}_k$ has empty capture set. If $H \notin \mathcal{V}_k$, recursively choose $e_n \in H$ with $\max e_n < \min e_{n+1}$. A new choice is possible because an initial finite interval does not cover H . In the increasing enumeration of $R = \bigcup_n e_n$, each e_n is a consecutive block of length k . Hence $W(R) \cap H$ is infinite. The remaining identities follow from finite union and from the first assertion. $\square$

Example 2.5. For any fixed $d \geq k-1$, taking all k -subsets of $\{r_n, \dots, r_{n+d}\}$, for every n , gives an assignment satisfying (2.3). For $k = 2$, the assignment

$$W(R) = P_R^{(2)} \cup \{\{r_n, r_{2n+1}\}, \{r_n, r_{2n+2}\} : n < \omega\}$$

also satisfies (2.3). Every index has at most three incidences in the added binary-tree graph and at most two in the consecutive path. The complete hypergraph $[R]^k$ has infinite vertex degrees and lies outside (2.3).

For $X \subseteq \omega$, put

$$\Delta_k X = \{e \in E_k : 0 < |e \cap X| < k\}.$$

This hypergraph has a two-color weak coloring. A set $R \in [\omega]^\omega$ *decides* X if $R \subseteq^* X$ or $R \subseteq^* X^c$, where $X^c = \omega \setminus X$.

Lemma 2.6 (Cuts and decisions). *For every $R \in [\omega]^\omega$ and every witness assignment satisfying (2.3),*

$$R \in D_{\Delta_k X}^W \iff R \subseteq^* X \text{ or } R \subseteq^* X^c.$$

Also, $D_{A_X^{(k)}}^W = \{R : R \subseteq^* X\}$. Every member of $\mathcal{G}_{\text{fc}}^{(k)}$ is contained in a finite union of hypergraphs $\Delta_k X$.

Proof. If all sufficiently late consecutive k -blocks are monochromatic for $\mathbf{1}_X$, their overlaps force all sufficiently late entries to have one common value. Conversely, an eventually monochromatic sequence has mixed witness hyperedges only through its finitely many exceptional vertices, which have finite incidence. If $R \setminus X$ is infinite, infinitely many consecutive blocks meet $R \setminus X$; this proves the assertion for $A_X^{(k)}$.

For the final assertion, encode the finitely many colors of a weak coloring by distinct binary strings. Each hyperedge has two different colors, and some binary coordinate separates them. The cuts associated with those coordinates cover the hypergraph. $\square$

Theorem 2.7 (Absolute avoidance and finite weak colorings). *For $H \subseteq E_k$ and W satisfying (2.3), the following are equivalent:*

- (i) $H \in \mathcal{G}_{\text{fc}}^{(k)}$;
- (ii) *there are sets $X_n \subseteq \omega$ such that every R deciding all X_n belongs to D_H^W .*

Proof. If H has a finite weak coloring, a set deciding every color class is almost contained in one class. All hyperedges of $W(R) \cap H$ then meet finitely many exceptional vertices, so there are only finitely many.

Conversely, suppose $\chi(H) = \omega$ and fix $X_n \subseteq \omega$. Construct $C_0 \supseteq C_1 \supseteq \dots$ with $C_0 = \omega$, $\chi(H \cap [C_n]^k) = \omega$, and C_{n+1} contained in one side of X_n . If both sides had finite weak colorings, assigning disjoint finite palettes to them would give a finite weak coloring of $H \cap [C_n]^k$. Deleting finitely many vertices also preserves infinite weak chromatic number, since deleted vertices could otherwise receive separate colors. We can therefore choose $e_n \in H \cap [C_n]^k$ with $\max e_n < \min e_{n+1}$. The set $R = \bigcup_n e_n$ decides every X_n , because its tail is contained in C_{n+1} . Each e_n is a consecutive k -block of R , so $W(R) \cap H$ is infinite. This contradicts (ii). $\square$

For the consecutive assignment, $\Gamma_k(H)$ is a G_δ subset of $[\omega]^\omega$. It is the intersection over m of the union over $n \geq m$ of the clopen conditions $\{r_n, \dots, r_{n+k-1}\} \in H$.

2.4 Row-tail bounds and local finiteness

For hypergraphs $G, H \subseteq E_k$, call H a relative upper bound of G when $G \setminus H \in \mathcal{V}_k$. An ideal has the relative P -property if every countable subfamily has such an upper bound in the ideal. The next lemma reduces arbitrary relative upper bounds to explicit truncations of rows.

Let $G_0 \subseteq G_1 \subseteq \dots$ be hypergraphs on E_k . An arbitrary sequence in an ideal is reduced to this form by taking its successive finite unions. For $R \in \bigcap_n D_{G_n}$ define

$$\lambda_R(n) = \begin{cases} 0, & P_R^{(k)} \cap G_n = \emptyset, \\ 1 + \max\{\min e : e \in P_R^{(k)} \cap G_n\}, & P_R^{(k)} \cap G_n \neq \emptyset. \end{cases}$$

For a strictly increasing $h : \omega \rightarrow \omega$, set

$$(2.4) \quad H_h = \bigcup_n \{e \in G_n : \min e \geq h(n)\}.$$

Lemma 2.8. *Every H_h in (2.4) is a common $\mathcal{V}_k$ -upper bound of the G_n , and*

$$(2.5) \quad D_{H_h} = \{R \in \bigcap_n D_{G_n} : \lambda_R \leq^* h\}.$$

Every common $\mathcal{V}_k$ -upper bound H contains some H_h .

Proof. The difference $G_n \setminus H_h$ is covered by $[0, h(n))$. Thus any ray avoiding H_h also avoids every G_n . If $\lambda_R \leq^* h$, then for all but finitely many n no ray edge in G_n has minimum at least $h(n)$. The remaining finitely many G_n each meet the ray finitely, so R avoids H_h . If $\lambda_R(n) > h(n)$ for infinitely many n , choose corresponding ray edges in G_n with minimum at least $h(n)$. Since $h(n) \rightarrow \infty$, infinitely many distinct edges are obtained, all belonging to H_h . This proves (2.5). Finally, if $G_n \setminus H \in \mathcal{V}_k$, Lemma 2.1 gives a threshold a_n above which every row of G_n is contained in H . Choose a strictly increasing h with $h(n) \geq a_n$. Then $H_h \subseteq H$. $\square$

Lemma 2.9 (Locally finite pseudounions). *The ideal $\mathcal{L}_k$ is a $\mathcal{V}_k$ - P -ideal. More precisely, for any sequence $H_n \in \mathcal{L}_k$ there is a locally finite H with $H_n \setminus H \in \mathcal{V}_k$ for every n .*

Proof. Choose a_n such that the i th row of H_n is finite whenever $i \geq a_n$. Take a strictly increasing function h with $h(n) \geq a_n$, and define

$$(2.6) \quad H = \bigcup_{n < \omega} \{e \in H_n : \min e \geq h(n)\}.$$

For fixed i , only finitely many n satisfy $h(n) \leq i$. For each of these n , the row $H_n(i)$ is finite. Hence every row of H is finite, which implies local finiteness by the proof of Lemma 2.2. Moreover, $H_n \setminus H$ is covered by $[0, h(n))$. $\square$

The ideal $\mathcal{L}_k$ fails the ordinary P -ideal property modulo finite sets of hyperedges. Each full row belongs to $\mathcal{L}_k$, whereas an upper bound modulo finite hyperedges of all full rows would have infinitely many infinite rows.

3 Canonical ideals generated by filters

The trace ideal records the sets in a free filter as independent vertex sets. Adjoining the finite-chromatic ideal imposes an additional requirement on countable upper bounds. We characterize this requirement by finite partitions and then by the ultrafilters extending the filter.

3.1 Trace ideals and canonical filters

An ideal on E_k is downward closed and closed under finite unions. All hypergraph ideals considered below contain $\mathcal{V}_k$. Properness means $E_k \notin \mathcal{I}$. We write

$$\mathcal{I} + \mathcal{J} = \{H \subseteq E_k : (\exists I \in \mathcal{I})(\exists J \in \mathcal{J}) H \subseteq I \cup J\}.$$

The sum is the ideal generated by $\mathcal{I} \cup \mathcal{J}$.

Definition 3.1. For a free filter $\mathcal{F}$, its trace ideal is

$$T_k(\mathcal{F}) = \{H \subseteq E_k : (\exists X \in \mathcal{F}) H \cap [X]^k \in \mathcal{V}_k\}.$$

For a proper ideal $\mathcal{J}$ containing $\mathcal{V}_k$, its canonical filter is

$$\mathcal{F}_{\mathcal{J}} = \{X \subseteq \omega : A_X^{(k)} \in \mathcal{J}\}, \quad A_X^{(k)} = E_k \setminus [X]^k.$$

Proposition 3.2 (Trace ideals). *Let $\mathcal{F}$ be a free filter. Then $T_k(\mathcal{F})$ is a proper W -ray-proper ideal for every W satisfying (2.3), and*

$$T_k(\mathcal{F}) = \{H : (\exists X \in \mathcal{F}) H \subseteq A_X^{(k)}\}, \quad \mathcal{F}_{T_k(\mathcal{F})} = \mathcal{F}.$$

Furthermore,

- (i) $T_k(\mathcal{F})$ is a $\mathcal{V}_k$ - P -ideal if and only if $\mathcal{F}$ is a P -filter;
- (ii) $\mathcal{G}_{\text{fc}}^{(k)} \subseteq T_k(\mathcal{F})$ if and only if $\mathcal{F}$ is an ultrafilter.

Proof. Suppose $H \cap [X]^k$ is covered by a finite set F . Then $Y = X \setminus F \in \mathcal{F}$ and $H \cap [Y]^k = \emptyset$, so $H \subseteq A_Y^{(k)}$. The reverse inclusion is immediate. The identity

$$(3.1) \quad A_{X \cap Y}^{(k)} = A_X^{(k)} \cup A_Y^{(k)}$$

shows that $T_k(\mathcal{F})$ is an ideal. An infinite subset of Y is independent for H , proving properness and W -ray-properness. If $A_Z^{(k)} \subseteq A_Y^{(k)}$ for some infinite $Y \in \mathcal{F}$, then $Y \subseteq Z$. A point of $Y \setminus Z$, together with $k - 1$ other points of Y , would contradict the inclusion. Thus the canonical filter is exactly $\mathcal{F}$.

Assume $\mathcal{F}$ is a P -filter and $H_n \subseteq A_{X_n}^{(k)}$ with $X_n \in \mathcal{F}$. Choose $X \in \mathcal{F}$ with $X \subseteq^* X_n$ for all n . Then

$$H_n \setminus A_X^{(k)} \subseteq A_{X_n}^{(k)} \cap [X]^k \subseteq \text{Star}_k(X \setminus X_n) \in \mathcal{V}_k.$$

Conversely, let $X_n \in \mathcal{F}$ and take a common relative upper bound $H \in T_k(\mathcal{F})$ for the $A_{X_n}^{(k)}$. Choose $Y \in \mathcal{F}$ with $H \cap [Y]^k = \emptyset$. For a finite cover F_n of $A_{X_n}^{(k)} \setminus H$, an infinite $Y \setminus X_n$ would contain k points outside F_n . Their hyperedge belongs to the difference and misses its cover. Hence $Y \subseteq^* X_n$ for all n .

If $\mathcal{F}$ is an ultrafilter, it contains one cell of every finite coloring, so it contains an independent set for every member of $\mathcal{G}_{\text{fc}}^{(k)}$. Conversely, if every $\Delta_k X$ belongs to the trace ideal, there is $Y \in \mathcal{F}$ with $\Delta_k X \cap [Y]^k = \emptyset$. Since Y is infinite, it is contained in X or in X^c . Thus $\mathcal{F}$ decides every subset of ω and is an ultrafilter. $\square$

Lemma 3.3 (Canonical filter). *If $\mathcal{J}$ is a proper ideal containing $\mathcal{V}_k$, then $\mathcal{F}_{\mathcal{J}}$ is a free filter. If in addition $\mathcal{G}_{\text{fc}}^{(k)} \subseteq \mathcal{J}$ and $\mathcal{J}$ is a $\mathcal{V}_k$ - P -ideal, then $\mathcal{F}_{\mathcal{J}}$ is a P -filter.*

Proof. We have $A_{\omega}^{(k)} = \emptyset$. If a finite X belonged to the canonical filter, both $A_X^{(k)}$ and $[X]^k$ would belong to $\mathcal{J}$, contradicting properness. Equation (3.1) gives finite intersection closure. If $X \subseteq^* Y$, then

$$A_Y^{(k)} \setminus A_X^{(k)} \subseteq \text{Star}_k(X \setminus Y),$$

which proves upward closure modulo finite and includes all cofinite sets.

For the P -property, replace a given sequence by its successive finite intersections and write it as $X_0 \supseteq X_1 \supseteq \dots$. Choose $H \in \mathcal{J}$ and finite F_n such that

$$A_{X_n}^{(k)} \setminus H \subseteq \text{Star}_k(F_n).$$

For a vertex v , define its exit level

$$a(v) = \min\{n : v \notin X_n\} \in \omega \cup \{\infty\},$$

where the minimum of the empty set is ∞ , and put

$$(3.2) \quad Y = \{v : a(v) = \infty\} \cup \bigcup_{n < \omega} \{v \in F_n : a(v) \geq n\}.$$

If $v \in Y \setminus X_m$, its exit level is at most m , and therefore it belongs to some F_n with $n \leq m$. Hence

$$Y \setminus X_m \subseteq \bigcup_{n \leq m} F_n, \quad Y \subseteq^* X_m.$$

We claim $[Y^c]^k \subseteq H$. Otherwise take $e \in [Y^c]^k \setminus H$ and let $n = \min\{a(v) : v \in e\}$. All exit levels are finite, and $e \in A_{X_n}^{(k)} \setminus H$. Some $w \in e \cap F_n$ then satisfies $a(w) \geq n$, so (3.2) puts w in Y , a contradiction. Finally,

$$A_Y^{(k)} = [Y^c]^k \cup \Delta_k Y \in \mathcal{J}.$$

Thus $Y \in \mathcal{F}_{\mathcal{J}}$, and properness ensures that Y is infinite. $\square$

Proposition 3.4 (Infinite complete subhypergraphs). *Suppose $\mathcal{J} \supseteq \mathcal{G}_{\text{fc}}^{(k)}$ is a $\mathcal{V}_k$ - P -ideal. Every infinite $X \subseteq \omega$ contains an infinite B with $[B]^k \in \mathcal{J}$. In particular, $\mathcal{G}_{\text{fc}}^{(k)}$ is not a $\mathcal{V}_k$ - P -ideal.*

Proof. Let $T = \{s \in \omega^{<\omega} : |s| \geq 1\}$ and choose distinct labels $x_s \in X$. For $n < \omega$, let G_n consist of the hyperedges

$$\{x_{s_0}, \dots, x_{s_{k-1}}\}, \quad s_0 \subsetneq \dots \subsetneq s_{k-1}, \quad |s_0| = n + 1.$$

Color the labels at level $n + 1$ with one color and all other vertices with another. Each edge of G_n has exactly one vertex of the first color, so $G_n \in \mathcal{G}_{\text{fc}}^{(k)}$. Choose $H \in \mathcal{J}$ and finite F_n covering $G_n \setminus H$. Select a branch $s_0 \subsetneq s_1 \subsetneq \dots$, with $|s_n| = n + 1$, so that $x_{s_n} \notin \bigcup_{j \leq n} F_j$. At the first step there are infinitely many roots; at each later step there are infinitely many immediate successors. Thus every choice avoids the finite forbidden set. If $n_0 < \dots < n_{k-1}$, the corresponding edge lies in G_{n_0} and misses F_{n_0} , so it lies in H . Therefore $B = \{x_{s_n} : n < \omega\}$ satisfies $[B]^k \subseteq H$. An infinite complete k -uniform hypergraph has no finite weak coloring, proving the last assertion. $\square$

3.2 Finite ultrafilter spectra

Compactness turns the condition that every extending ultrafilter is a P -point into finiteness of the family of extensions. Identify $\beta\omega$ with the ultrafilters on ω , with clopen sets $X^\beta = \{\mathcal{U} : X \in \mathcal{U}\}$. The subspace ω^* consists of free ultrafilters, and $X^* = X^\beta \cap \omega^*$. These spaces are compact Hausdorff and zero-dimensional. For completeness, ultrafilters can be viewed as the Boolean-valued homomorphisms $\mathcal{P}(\omega) \rightarrow \{0, 1\}$; their defining equations give a closed subspace of the compact product $2^{\mathcal{P}(\omega)}$. The equations excluding singletons define the closed subspace ω^* . The clopen coordinate conditions separate points. Compactness implies that every clopen set is represented by a finite Boolean combination of coordinates, hence by a single subset of ω .

For a free filter $\mathcal{F}$, let

$$\text{Ext}(\mathcal{F}) = \{\mathcal{U} \in \omega^* : \mathcal{F} \subseteq \mathcal{U}\}.$$

It is nonempty and compact, and

$$(3.3) \quad \mathcal{F} = \bigcap \text{Ext}(\mathcal{F}).$$

Indeed, if $X \notin \mathcal{F}$, adjoining X^c to $\mathcal{F}$ gives a proper filter, which extends to an ultrafilter omitting X .

Lemma 3.5. *If every ultrafilter extending $\mathcal{F}$ is a P -point, then $\text{Ext}(\mathcal{F})$ is finite.*

Proof. For a P -point $\mathcal{U}$, any sequence of basic neighborhoods X_n^* of $\mathcal{U}$ contains a common basic neighborhood Y^* . Choose $Y \in \mathcal{U}$ with $Y \subseteq^* X_n$ for all n . Therefore, in the compact space $K = \text{Ext}(\mathcal{F})$, any countable intersection of open sets is open. Every countable subset D of such a T_1 space is closed, since its complement is the intersection of the open complements of its singletons. It is also discrete, since each $d \in D$ has the open neighborhood $K \setminus (D \setminus \{d\})$. If K were infinite, it would contain a countably infinite D , which would be a compact infinite discrete space. The cover of D by its singletons contradicts compactness. $\square$

Define the canonical ideal

$$\mathcal{C}_k(\mathcal{F}) = \text{Id}(\mathcal{G}_{\text{fc}}^{(k)} \cup \{A_X^{(k)} : X \in \mathcal{F}\}).$$

Its members are precisely the hypergraphs contained in $A_X^{(k)} \cup K$, with $X \in \mathcal{F}$ and $K \in \mathcal{G}_{\text{fc}}^{(k)}$.

Lemma 3.6. *The ideals $\mathcal{C}_k(\mathcal{F})$ and $\mathcal{C}_k(\mathcal{F}) + \mathcal{L}_k$ are W -ray-proper for every W satisfying (2.3). Both have canonical filter $\mathcal{F}$.*

Proof. It suffices to treat the larger ideal. A member is contained in $A_X^{(k)} \cup K \cup L$, where $X \in \mathcal{F}$, K has a finite weak coloring, and $L \in \mathcal{L}_k$. Some color class meets X infinitely. By Lemma 2.2, this intersection contains an infinite L -independent set. It is independent for the whole union.

The canonical filter contains $\mathcal{F}$. Suppose $A_Y^{(k)} \subseteq A_X^{(k)} \cup K \cup L$. If $X \setminus Y$ were infinite, choose an infinite L -independent $B \subseteq X \setminus Y$. Then $[B]^k \subseteq K$, contradicting the finite weak chromatic number of K . Thus $X \subseteq^* Y$, whence $Y \in \mathcal{F}$. $\square$

Theorem 3.7 (Canonical ideal classification). *For a free filter $\mathcal{F}$ and $2 \leq k < \omega$, the following are equivalent:*

- (i) $\mathcal{C}_k(\mathcal{F}) + \mathcal{L}_k$ is a $\mathcal{V}_k$ - P -ideal;
- (ii) $\mathcal{C}_k(\mathcal{F})$ is a $\mathcal{V}_k$ - P -ideal;
- (iii) for every sequence $c_n : \omega \rightarrow q_n$, with each q_n a positive integer, there are $X \in \mathcal{F}$ and a finite partition $X = X_0 \dot{\cup} \dots \dot{\cup} X_{m-1}$ such that $c_n \restriction X_i$ is eventually constant for all n, i ;
- (iv) every ultrafilter extending $\mathcal{F}$ is a P -point;
- (v) $\text{Ext}(\mathcal{F})$ is a nonempty finite set of P -points;

(vi) $\mathcal{F}$ is the intersection of a nonempty finite family of P -points.

Eventual constancy on a finite cell in (iii) is understood vacuously.

Proof. We first prove the equivalence of (i)–(iii), using a finite coloring to partition a set in the filter into cells. We then relate (iii) to ultrafilter extensions.

(i) $\Rightarrow$ (iii). For the given maps, put

$$D_n = \{e \in E_k : c_n \text{ is nonconstant on } e\}.$$

Then c_n weakly colors D_n . Choose a common relative upper bound $H \subseteq A_X^{(k)} \cup K \cup L$, with $X \in \mathcal{F}$, $K \in \mathcal{G}_{\text{fc}}^{(k)}$, $L \in \mathcal{L}_k$, and finite F_n covering $D_n \setminus H$. A finite weak coloring of K partitions X into independent cells X_i . Inside a cell,

$$D_n \cap [X_i]^k \subseteq L \cup \text{Star}_k(F_n).$$

Delete F_n and a finite exceptional set for local finiteness of L . If two c_n -values still occurred infinitely often, fix a vertex with the first value and choose arbitrarily many $(k-1)$ -subsets of the second infinite color class. This gives infinitely many hyperedges of L incident with the fixed vertex, a contradiction. Since c_n has finite range, precisely one value occurs on a cofinite subset of each infinite cell.

(iii) $\Rightarrow$ (ii). First we show that $\mathcal{F}$ is a P -filter. For a decreasing sequence $X_n \in \mathcal{F}$, apply (iii) to $\mathbf{1}_{X_n}$. Obtain $X \in \mathcal{F}$ and finitely many cells deciding all X_n . Call a cell bad if it is almost contained in X_n^c for some n ; this includes any finite cell. Let B be the union of the bad cells. If there are bad cells, choose N at least as large as all their witnessing indices. Then $B \cap X_N$ is finite, so $X \setminus B \in \mathcal{F}$. If there are none, put $B = \emptyset$. Every remaining cell is almost contained in every X_n , hence $X \setminus B$ is the required pseudointersection.

Now take $G_n \subseteq A_{X_n}^{(k)} \cup H_n$ in $\mathcal{C}_k(\mathcal{F})$, and choose $X \in \mathcal{F}$ with $X \subseteq^* X_n$ for all n . Apply (iii) to finite weak colorings c_n of the H_n . This gives $Y \in \mathcal{F}$ and cells Y_i on which all c_n are eventually constant. Set $Z = X \cap Y$. Let K consist of the k -subsets of Y that meet at least two cells. Color each Y_i separately and give $\omega \setminus Y$ an extra color. Thus $K \in \mathcal{G}_{\text{fc}}^{(k)}$ and $G = A_Z^{(k)} \cup K \in \mathcal{C}_k(\mathcal{F})$. The difference $A_{X_n}^{(k)} \setminus A_Z^{(k)}$ is covered by the finite set $Z \setminus X_n$. Hyperedges of H_n outside $A_Z^{(k)} \cup K$ lie inside one cell. After deleting the finitely many exceptions to the constant c_n -value in each of the finitely many cells, no such edge remains. Hence $G_n \setminus G \in \mathcal{V}_k$.

(ii) $\Rightarrow$ (i). By Lemma 2.9, $\mathcal{L}_k$ is a $\mathcal{V}_k$ - P -ideal. Given a sequence in the sum, bound its canonical parts and its locally finite parts separately. Their

union is a relative upper bound in the sum. Properness has already been proved in Lemma 3.6.

(iii) $\Rightarrow$ (iv). Let $\mathcal{U} \supseteq \mathcal{F}$ and let $X_n \in \mathcal{U}$. Apply (iii) to their characteristic functions. The resulting set $X \in \mathcal{F}$ belongs to $\mathcal{U}$, so exactly one cell C of the finite partition belongs to $\mathcal{U}$. It is infinite. Since both C and X_n lie in $\mathcal{U}$, the alternative $C \subseteq^* X_n^c$ is impossible. Thus $C \subseteq^* X_n$ for every n .

(iv) $\Rightarrow$ (v) $\Rightarrow$ (vi). Apply Lemma 3.5 and then (3.3).

(vi) $\Rightarrow$ (iii). Write $\mathcal{F} = \bigcap_{i < m} \mathcal{U}_i$, with $m \geq 1$ and all $\mathcal{U}_i$ P -points. For each i, n , one c_n -color class belongs to $\mathcal{U}_i$. Take a pseudointersection $Y_i \in \mathcal{U}_i$ of those classes. Then $Y = \bigcup_{i < m} Y_i$ belongs to every $\mathcal{U}_i$, hence to $\mathcal{F}$. Disjointify the sets Y_i by deleting from each one the preceding sets. Each resulting cell is contained in its original Y_i , so every c_n is eventually constant on it. $\square$

For an ultrafilter $\mathcal{U}$, Proposition 3.2 gives

$$(3.4) \quad \mathcal{C}_k(\mathcal{U}) = T_k(\mathcal{U}).$$

Thus $\mathcal{C}_k(\mathcal{U})$ has the relative P -property exactly when $\mathcal{U}$ is a P -point.

4 Locally finite extensions and capture ideals

Adding $\mathcal{L}_k$ to a trace ideal preserves its canonical filter and its relative P -property. The row bounds for $\mathcal{L}_k$ then give explicit bases for the induced ideals on increasing sequences and determine their uniformities.

4.1 Trace ideals with locally finite restrictions

The relative upper bounds from Lemma 2.9 combine with pseudointersections in the filter to give upper bounds in the sum ideal.

Definition 4.1. For a free filter $\mathcal{F}$, let

$$\mathcal{J}_{\mathcal{F},k} = T_k(\mathcal{F}) + \mathcal{L}_k.$$

Theorem 4.2 (Locally finite extensions). *For every free filter $\mathcal{F}$,*

$$(4.1) \quad H \in \mathcal{J}_{\mathcal{F},k} \iff (\exists X \in \mathcal{F}) H \cap [X]^k \in \mathcal{L}_k.$$

The ideal $\mathcal{J}_{\mathcal{F},k}$ is W -ray-proper for every W satisfying (2.3), its canonical filter is $\mathcal{F}$, and

$$(4.2) \quad \mathcal{J}_{\mathcal{F},k} \text{ is a } \mathcal{V}_k\text{-}P\text{-ideal} \iff \mathcal{F} \text{ is a } P\text{-filter}.$$

Also, $\mathcal{G}_{\text{fc}}^{(k)} \subseteq \mathcal{J}_{\mathcal{F},k}$ holds exactly when $\mathcal{F}$ is an ultrafilter.

Proof. If $H \subseteq G \cup L$ with $G \in T_k(\mathcal{F})$ and $L \in \mathcal{L}_k$, choose $X \in \mathcal{F}$ with $G \cap [X]^k \in \mathcal{V}_k$. Then $H \cap [X]^k \in \mathcal{L}_k$. Conversely, under the right-hand side of (4.1),

$$H \subseteq A_X^{(k)} \cup (H \cap [X]^k)$$

is a decomposition in the sum. Within the witnessing infinite X , delete finitely many exceptional vertices and choose an infinite independent set by Lemma 2.2. This proves W -ray-properness. If $A_Y^{(k)} \cap [X]^k \in \mathcal{L}_k$, an infinite $X \setminus Y$ would yield an infinite complete subhypergraph of a hypergraph locally finite after finite deletion. Hence $X \subseteq^* Y$, proving that the canonical filter is $\mathcal{F}$.

If $\mathcal{F}$ is a P -filter, both summands have the relative P -property by Proposition 3.2 and Lemma 2.9; bounding each part of a sequence separately proves the same property for their proper sum. Conversely, take $X_n \in \mathcal{F}$ and a common relative upper bound $H \in \mathcal{J}_{\mathcal{F},k}$ for the $A_{X_n}^{(k)}$. Choose $Y \in \mathcal{F}$ with $H \cap [Y]^k \in \mathcal{L}_k$. If $Y \setminus X_n$ were infinite, deleting a finite cover of $A_{X_n}^{(k)} \setminus H$ and the finitely many local-finiteness exceptions would leave an infinite set $B \subseteq Y \setminus X_n$ with $[B]^k \subseteq H$. This contradicts local finiteness. Thus $Y \subseteq^* X_n$ for all n , proving (4.2).

An ultrafilter already has $\mathcal{G}_{\text{fc}}^{(k)} \subseteq T_k(\mathcal{F})$. Conversely, if $\Delta_k X \in \mathcal{J}_{\mathcal{F},k}$, take a witnessing $Y \in \mathcal{F}$ in (4.1). If both $Y \cap X$ and $Y \setminus X$ were infinite, fixing one vertex in one side and varying $k-1$ vertices in the other would contradict local finiteness after finite deletion. Thus Y almost decides X , and $\mathcal{F}$ is an ultrafilter. $\square$

4.2 Explicit bases and countable closure

For a W -ray-proper ideal $\mathcal{J}$, set

$$\mathcal{J}_W(\mathcal{J}) = \{\mathcal{A} \subseteq [\omega]^\omega : (\exists H \in \mathcal{J}) \mathcal{A} \subseteq \Gamma_W(H)\}.$$

It is a proper ideal by Lemma 2.4. If $\mathcal{J}$ has the relative P -property, then it is a σ -ideal. Indeed, $H_n \setminus H \in \mathcal{V}_k$ implies $\bigcup_n \Gamma_W(H_n) \subseteq \Gamma_W(H)$.

Theorem 4.3 (Capture bases and witness invariance). *Let $\mathcal{F}$ be a free filter. For every W satisfying (2.3), the ideal $\mathcal{J}_W(\mathcal{J}_{\mathcal{F},k})$ is the same ideal $\mathcal{J}_k(\mathcal{F})$, with cofinal family*

$$(4.3) \quad \begin{aligned} N_{X,f}^{(k)} &= \{R : R \not\subseteq^* X\} \\ &\cup \{R \in [\omega]^\omega : r_{n+k-1} \leq f(r_n) \text{ for infinitely many } n\}, \end{aligned}$$

where $X \in \mathcal{F}$ and $f \in \mathcal{M}_k$. Every set in this cofinal family is G_δ , and

$$(4.4) \quad \mathcal{I}_k(\mathcal{F}) \text{ is a } \sigma\text{-ideal} \iff \mathcal{F} \text{ is a } P\text{-filter}.$$

Proof. By (4.1) and Lemma 2.3, every $H \in \mathcal{J}_{\mathcal{F},k}$ is contained modulo $\mathcal{V}_k$ in $A_X^{(k)} \cup B_f^{(k)}$ for some $X \in \mathcal{F}$ and $f \in \mathcal{M}_k$. Conversely, every such union belongs to $\mathcal{J}_{\mathcal{F},k}$. Lemma 2.6 identifies $\Gamma_W(A_X^{(k)})$ with $\{R : R \not\subseteq^* X\}$. We also have

$$(4.5) \quad \Gamma_W(B_f^{(k)}) = \Gamma_k(B_f^{(k)}).$$

For the nontrivial inclusion, infinitely many hyperedges of $W(R) \cap B_f^{(k)}$ have infinitely many different minimum vertices, because the witness is locally finite. If one minimum is r_n , then $r_{n+k-1} \leq \max e \leq f(r_n)$, so the consecutive block starting at r_n belongs to $B_f^{(k)}$. The opposite inclusion follows from $P_R^{(k)} \subseteq W(R)$. The finite-union identity for capture sets now gives (4.3) and witness invariance.

Both alternatives in (4.3) are infinitely-often conditions on clopen cylinders, and a finite union of G_δ sets is G_δ . The ideal contains all singletons. For a fixed R , the locally finite hypergraph $P_R^{(k)}$ captures R .

If $\mathcal{F}$ is a P -filter, Theorem 4.2 and the preceding observation give countable closure. Conversely, assume countable closure and take $X_n \in \mathcal{F}$. Each set $\{R : R \not\subseteq^* X_n\}$ belongs to the capture ideal, so their union is captured by one $H \in \mathcal{J}_{\mathcal{F},k}$. Choose $Y \in \mathcal{F}$ with $H \cap [Y]^k \in \mathcal{L}_k$. If $Y \setminus X_n$ were infinite, it would contain an infinite independent set R for H . This R belongs to the union but is not captured by H , a contradiction. Hence $Y \subseteq^* X_n$ for all n . $\square$

4.3 Uniformities and small extensions

For a W -ray-proper ideal, define

$$\nu_W(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega, (\forall H \in \mathcal{J}) \mathcal{A} \cap D_H^W \neq \emptyset\}.$$

If every countable subfamily of $\mathcal{J}$ has a common avoiding ray, define

$$\nu_{W,\omega}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega, (\forall (H_n)_{n<\omega} \in \mathcal{J}^\omega) \mathcal{A} \cap \bigcap_n D_{H_n}^W \neq \emptyset\}.$$

For consecutive witnesses use ν_k and $\nu_{k,\omega}$. By definition,

$$\nu_W(\mathcal{J}) = \text{non}(\mathcal{I}_W(\mathcal{J})).$$

Proposition 4.4 (Trace uniformities). *For every free filter $\mathcal{F}$ and W satisfying (2.3),*

$$\nu_W(T_k(\mathcal{F})) = \pi\chi(\mathcal{F}), \quad \nu_{W,\omega}(T_k(\mathcal{F})) = \pi\chi_\sigma(\mathcal{F}).$$

Proof. The hypergraphs $A_X^{(k)}$, $X \in \mathcal{F}$, are cofinal in $T_k(\mathcal{F})$, and their avoidance sets are exactly $\{R : R \subseteq^* X\}$. Thus the first assertion is the definition of a π -base. For a sequence of trace hypergraphs choose corresponding sets X_n and apply a π_σ -base. Conversely, the sequence $A_{X_n}^{(k)}$ forces a common almost subset of all X_n . $\square$

Theorem 4.5 (Uniformities of locally finite extensions). *For every free filter $\mathcal{F}$, every $k \geq 2$, and every W satisfying (2.3),*

$$(4.6) \quad \nu_W(\mathcal{J}_{\mathcal{F},k}) = \text{non}(\mathcal{J}_k(\mathcal{F})) = \max\{\pi\chi(\mathcal{F}), \mathfrak{d}\},$$

$$(4.7) \quad \nu_{W,\omega}(\mathcal{J}_{\mathcal{F},k}) = \max\{\pi\chi_\sigma(\mathcal{F}), \mathfrak{d}\}.$$

If $\mathcal{F}$ is a P -filter, these two values are equal.

Proof. The lower bounds by the two π -characters follow from Proposition 4.4. For the lower bound by $\mathfrak{d}$, let $\mathcal{A}$ avoid every member of $\mathcal{L}_k$. For $R \in \mathcal{A}$, let $t_R(m)$ be the k th element of $R \cap [m, \infty)$. Given a nondecreasing $f \in \mathcal{M}_k$, choose $R \in \mathcal{A} \cap D_{B_f}^W$. Writing $R = \{r_n : n < \omega\}$, we have eventually $r_{n+k-1} > f(r_n)$. If r_n is the first element of R at least m , then

$$t_R(m) = r_{n+k-1} > f(r_n) \geq f(m)$$

for all sufficiently large m . Thus $\{t_R : R \in \mathcal{A}\}$ is a dominating family, and $|\mathcal{A}| \geq \mathfrak{d}$.

For the upper bound in (4.6), choose a π -base $\mathcal{B}$ of size $\pi\chi(\mathcal{F})$ and a dominating family $\mathcal{D}$ of size $\mathfrak{d}$, consisting of increasing functions h with $h(i) > i$. Let $e_B : \omega \rightarrow B$ be the increasing enumeration. For each pair (B, h) , define

$$i_0 = 0, \quad i_{n+1} = h(i_n), \quad R_{B,h} = \{e_B(i_n) : n < \omega\}.$$

Fix $G \in \mathcal{J}_{\mathcal{F},k}$, and choose $X \in \mathcal{F}$ with $G \cap [X]^k \in \mathcal{L}_k$. Select $B \in \mathcal{B}$ with $B \subseteq^* X$. The pullback of $G \cap [B]^k$ under e_B has finite rows above some index. Choose $g(i)$ strictly greater than every index occurring in its i th row for all sufficiently large i . Take $h \in \mathcal{D}$ with $g \leq^* h$. For sufficiently large n , the next selected index i_{n+1} exceeds every vertex index in the row with minimum i_n .

Consequently the tail of $R_{B,h}$ is independent for G . Hyperedges of $W(R_{B,h})$ using earlier vertices form a finite set by local finiteness. Thus $R_{B,h} \in D_G^W$. The resulting family has size at most $\pi\chi(\mathcal{F}) \cdot \mathfrak{d} = \max\{\pi\chi(\mathcal{F}), \mathfrak{d}\}$.

For (4.7), use a π_σ -base $\mathcal{B}$. Given $G_n \in \mathcal{J}_{\mathcal{F},k}$, choose $X_n \in \mathcal{F}$ with $G_n \cap [X_n]^k \in \mathcal{L}_k$, and one $B \in \mathcal{B}$ with $B \subseteq^* X_n$ for every n . Pull back all the induced hypergraphs on B . Lemma 2.9 gives a locally finite relative upper bound L for them. Choose g bounding the rows of L , then $h \in \mathcal{D}$ as above. The resulting ray avoids L . Each difference between a pullback and L has a finite vertex cover, so this one ray avoids all the G_n . The lower bound by $\pi\chi_\sigma(\mathcal{F})$ uses a countable sequence of endpoint hypergraphs, and the lower bound by $\mathfrak{d}$ already follows from the one-hypergraph case. Finally use (2.1). $\square$

Theorem 4.6 (Small locally finite perturbations). *Suppose $\mathcal{F}$ is a free P -filter and $\mathcal{H} \subseteq \mathcal{L}_k$ has size less than $\mathfrak{b}$. There is a locally finite hypergraph L such that, for every W satisfying (2.3), the ideal*

$$\mathcal{K} = \text{Id}(T_k(\mathcal{F}) \cup \{L\})$$

contains $T_k(\mathcal{F}) \cup \mathcal{H}$, is W -ray-proper and a $\mathcal{V}_k$ - P -ideal, and satisfies

$$\nu_W(\mathcal{K}) = \nu_{W,\omega}(\mathcal{K}) = \pi\chi(\mathcal{F}).$$

For any π -base $\mathcal{B}$, infinite subsets $B' \subseteq B$, one for each $B \in \mathcal{B}$, can be chosen so that $\{B' : B \in \mathcal{B}\}$ avoids every member of $\mathcal{K}$.

Proof. For $H \in \mathcal{H}$, choose $f_H \in \mathcal{M}_k$ from Lemma 2.3. Since $|\mathcal{H}| < \mathfrak{b}$, there is $f \in \mathcal{M}_k$ eventually dominating all f_H . Set $L = B_f^{(k)}$. Then $H \setminus L \in \mathcal{V}_k \subseteq T_k(\mathcal{F})$, so $\mathcal{K}$ contains $\mathcal{H}$. In the countable case one can equivalently use the explicit row-tail bound (2.6). For each $B \in \mathcal{B}$, take an infinite L -independent $B' \subseteq B$. If $G \subseteq A_X^{(k)} \cup L$ with $X \in \mathcal{F}$, choose $B \subseteq^* X$. Then $B' \subseteq^* X$, and $W(B')$ meets $A_X^{(k)}$ only finitely while missing L . Thus the refined family avoids G .

For a sequence $G_n \subseteq K_n \cup L$ with $K_n \in T_k(\mathcal{F})$, choose a relative upper bound $K \in T_k(\mathcal{F})$ by Proposition 3.2. Then $K \cup L$ bounds the sequence in $\mathcal{K}$. Properness follows from the avoiding family. The lower bound $\pi\chi(\mathcal{F})$ follows from the endpoint hypergraphs. A family avoiding every single member of a relative P -ideal also avoids every countable sequence, by applying it to a common relative upper bound. $\square$

Starting with a π -base of minimum size in Theorem 4.6 gives an avoiding family of size $\pi\chi(\mathcal{F})$. Adjoining fewer than $\mathfrak{b}$ members of $\mathcal{L}_k$ therefore preserves this value. Adjoining all of $\mathcal{L}_k$ gives the value $\max\{\pi\chi(\mathcal{F}), \mathfrak{d}\}$ by Theorem 4.5.

5 The Louveau ideal and the arity hierarchy

We now specialize the filter to a free ultrafilter $\mathcal{U}$. The P -point property determines countable closure of the capture ideals. The Q -point property determines whether their dependence on the arity is strict.

5.1 Absorption of locally finite hypergraphs

An independent set belonging to $\mathcal{U}$ places a hypergraph in $T_k(\mathcal{U})$. The Q -point property provides such sets for every locally finite hypergraph by selecting well-separated vertices.

Theorem 5.1 (Q -points and local finiteness). *For a free ultrafilter $\mathcal{U}$ and $k \geq 2$, the following are equivalent:*

- (i) $\mathcal{U}$ is a Q -point;
- (ii) $\mathcal{L}_k \subseteq T_k(\mathcal{U})$;
- (iii) $\mathcal{J}_{\mathcal{U},k} = T_k(\mathcal{U})$.

Proof. The equivalence of (ii) and (iii) follows from the definition of the ideal sum. Assume (ii), and let $\omega = \bigcup_n C_n$ be a partition into finite sets. The hypergraph

$$H = \bigcup_n [C_n]^k$$

is locally finite. Choose $X \in \mathcal{U}$ and a finite vertex cover F of $H \cap [X]^k$. Then

$$|(X \setminus F) \cap C_n| \leq k - 1 \quad (n < \omega).$$

Partition $X \setminus F$ into $k - 1$ sets, each meeting every C_n in at most one point. Since their union belongs to $\mathcal{U}$, one of these sets belongs to $\mathcal{U}$. Thus $\mathcal{U}$ is a Q -point.

Conversely, assume (i) and let $H \in \mathcal{L}_k$. Remove a finite vertex set F so that the remaining hypergraph is locally finite. Its *two-shadow* is the graph joining two vertices whenever they occur together in some hyperedge.

Each vertex belongs to finitely many hyperedges, each of finite size, so the two-shadow is locally finite.

Choose $0 = a_0 < a_1 < a_2 < \dots$ with $a_1 > \max F$ and, recursively, with every neighbor of a vertex below a_{n+1} lying below a_{n+2} . Local finiteness makes each choice possible. Put $I_n = [a_n, a_{n+1})$. An edge of the two-shadow joins vertices in the same interval or in adjacent intervals. The Q -point property gives $X \in \mathcal{U}$ with $|X \cap I_n| \leq 1$ for all n . Exactly one of the unions of the even and the odd intervals belongs to $\mathcal{U}$. Intersect X with that union and remove the finite exceptional vertices, obtaining $Y \in \mathcal{U}$. No edge of the two-shadow has both endpoints in Y . Therefore $H \cap [Y]^k = \emptyset$, and $H \in T_k(\mathcal{U})$. $\square$

Petrenko and Protasov give a related characterization of Q -points by thin sets in locally finite metric spaces [6, Remark 6.2].

5.2 A successor-set basis for the Louveau ideal

Let $\omega^{\uparrow < \omega}$ be the set of finite strictly increasing sequences of natural numbers. We identify a member of $[\omega]^\omega$ with its increasing enumeration. If s is a nonempty finite sequence, $\max s$ denotes its last entry; set $\max \emptyset = -1$. For $i > \max s$, write $s \frown i$ for the sequence obtained by appending i .

A $\mathcal{U}$ -Laver tree is a nonempty tree $T \subseteq \omega^{\uparrow < \omega}$, closed under initial segments, with a stem $s_T \in T$ such that every node of T is comparable with s_T under extension, and

$$\text{succ}_T(t) = \{i > \max t : t \frown i \in T\} \in \mathcal{U} \quad (t \in T, s_T \subseteq t).$$

Its branch set $[T]$ consists of the infinite increasing sequences all of whose initial segments belong to T . A subtree in this definition is again a $\mathcal{U}$ -Laver tree, whose stem extends or equals the original stem.

Definition 5.2. The *Louveau ideal* $\ell_{\mathcal{U}}^0$ consists of all $A \subseteq [\omega]^\omega$ such that every $\mathcal{U}$ -Laver tree T has a $\mathcal{U}$ -Laver subtree $S \subseteq T$ with $[S] \cap A = \emptyset$.

We use the increasing-sequence presentation of the Louveau ideal from [2, Section 3, p. 2653], following Louveau's construction [5]. Successor sets give an explicit basis for this ideal.

For a family $\mathbf{B} = (B_s : s \in \omega^{\uparrow < \omega})$ with $B_s \in \mathcal{U}$, define

$$(5.1) \quad N(\mathbf{B}) = \{R : \text{for infinitely many } n, r_n \notin B_{R \restriction n}\}.$$

We may always replace B_s by $B_s \setminus [0, \max s]$.

Lemma 5.3 (Successor-set basis). *The sets $N(\mathbf{B})$ form a cofinal family in $\ell_{\mathcal{U}}^0$ under inclusion. In particular, $\ell_{\mathcal{U}}^0$ is a proper σ -ideal.*

Proof. Given T and $\mathbf{B}$, retain the stem of T and restrict the successor set at each later node s to $\text{succ}_T(s) \cap B_s$. These intersections belong to $\mathcal{U}$. Every branch of the resulting subtree obeys B_s at all stages beyond its stem, so it misses $N(\mathbf{B})$. Thus every set in (5.1) belongs to $\ell_{\mathcal{U}}^0$.

Conversely, let $A \in \ell_{\mathcal{U}}^0$. Call a finite increasing sequence s *good* if there is a $\mathcal{U}$ -Laver tree with stem s whose branch set misses A . Put $G_s = \{i > \max s : s \smallfrown i \text{ is good}\}$. If $G_s \in \mathcal{U}$, the corresponding trees can be joined at s to form such a tree with stem s . Hence s is good.

If some s were not good, then $\{i > \max s : i \notin G_s\} \in \mathcal{U}$. Repeating this observation at every nongood node produces a $\mathcal{U}$ -Laver tree with stem s all of whose nodes above the stem are nongood. The definition of $\ell_{\mathcal{U}}^0$ gives a subtree with branch set disjoint from A . Its stem is then a good node of the constructed tree, a contradiction. Thus every s is good.

For each s , choose a witnessing tree T_s with stem s . For a finite increasing sequence t , let

$$B_t = (\max t, \infty) \cap \bigcap \{\text{succ}_{T_s}(t) : s \subseteq t, t \in T_s\}.$$

There are only finitely many initial segments s of t , and the displayed intersection includes $s = t$. Each factor belongs to $\mathcal{U}$, so $B_t \in \mathcal{U}$. If R eventually obeys these successor sets, choose N after its last violation and put $s = R \upharpoonright N$. An induction shows that $R \upharpoonright n \in T_s$ for every $n \geq N$: at such a node, the definition of $B_{R \upharpoonright n}$ includes its successor set in T_s . Hence $R \in [T_s]$, and $R \notin A$. This proves $A \subseteq N(\mathbf{B})$.

For countable closure, suppose $A_j \subseteq N(\mathbf{B}^j)$. At a node s , put

$$C_s = \bigcap_{j \leq |s|} B_s^j \in \mathcal{U}.$$

A branch eventually obeying $\mathbf{C}$ eventually obeys each $\mathbf{B}^j$. Therefore $\bigcup_j A_j \subseteq N(\mathbf{C})$. The whole space is not null, since every $\mathcal{U}$ -Laver tree has an infinite branch. A singleton $\{R\}$ is contained in $N(\mathbf{B})$ by setting $B_{R \upharpoonright n} = \omega \setminus \{r_n\}$ and choosing arbitrary members of $\mathcal{U}$ at the remaining nodes. Downward closure is immediate. $\square$

The cofinal basis in Lemma 5.3 is the one used by Brendle and Shelah in their calculation of the cardinal characteristics of $\ell_{\mathcal{U}}^0$ [2, proof of Theorem 2, p. 2654].

5.3 The graph representation of the Louveau ideal

For a P -point, one set $X \in \mathcal{U}$ is almost contained in every successor set of a given basis element. A function f bounds the finite exceptions at nodes with bounded maximum. For rays eventually contained in X , the graph bound $r_{n+1} > f(r_n)$ then ensures eventual obedience to all successor conditions along the ray.

Theorem 5.4 (The Louveau representation). *For every free ultrafilter $\mathcal{U}$,*

$$\mathcal{I}_2(\mathcal{U}) \subseteq \ell_{\mathcal{U}}^0.$$

Moreover,

$$(5.2) \quad \mathcal{I}_2(\mathcal{U}) = \ell_{\mathcal{U}}^0 \iff \mathcal{U} \text{ is a } P\text{-point}.$$

Proof. Fix $X \in \mathcal{U}$ and $f \in \mathcal{M}_2$. In any $\mathcal{U}$ -Laver tree, restrict each successor set above a node s to $X \cap (f(\max s), \infty)$, using X at the empty node. Every branch of the resulting tree is eventually contained in X and eventually satisfies $r_{n+1} > f(r_n)$. Thus $N_{X,f}^{(2)} \in \ell_{\mathcal{U}}^0$. The basis theorem for $\mathcal{I}_2(\mathcal{U})$ gives the inclusion.

Suppose now that $\mathcal{U}$ is a P -point, and let $A \in \ell_{\mathcal{U}}^0$. By Lemma 5.3, choose $B_s \in \mathcal{U}$ with $A \subseteq N(\mathbf{B})$. There are countably many finite increasing sequences, so take $X \in \mathcal{U}$ such that $X \subseteq^* B_s$ for every s . For each m , only finitely many increasing sequences have maximum at most m . Choose a nondecreasing $f \in \mathcal{M}_2$ satisfying

$$(5.3) \quad f(m) > \max \left(\bigcup_{\max s \leq m} (X \setminus B_s) \right) \quad (m < \omega),$$

with the maximum of the empty set interpreted as -1 .

If $R \notin N_{X,f}^{(2)}$, then eventually $r_n \in X$ and $r_{n+1} > f(r_n)$. Apply (5.3) to $s = R \upharpoonright (n+1)$, whose maximum is r_n . It follows that $r_{n+1} \in B_s$ for all sufficiently large n . Hence $R \notin N(\mathbf{B})$, and therefore $R \notin A$. We have shown $A \subseteq N_{X,f}^{(2)}$, so $A \in \mathcal{I}_2(\mathcal{U})$.

Conversely, equality with $\ell_{\mathcal{U}}^0$ makes $\mathcal{I}_2(\mathcal{U})$ a σ -ideal. By Theorem 4.3, this implies that $\mathcal{U}$ is a P -point. $\square$

For comparison, Brendle and Shelah proved

$$(5.4) \quad \text{non}(\ell_{\mathcal{U}}^0) = \max\{\pi\chi_{\sigma}(\mathcal{U}), \mathfrak{d}\}$$

for every free ultrafilter [2, Theorem 2(c), p. 2654]. For P -points they obtained the formula with $\pi\chi(\mathcal{U})$ [2, Corollary 3.2(c), p. 2656]. Theorems 4.5 and 5.4 supply a hypergraph representation and a direct row-bound proof of that formula in the P -point case. Higher arities distinguish ideals even when their uniformities agree.

5.4 Strict dependence on the arity

Increasing the arity makes the condition $r_{n+k-1} \leq f(r_n)$ stronger. To prove strict inclusion when $\mathcal{U}$ is not a Q -point, we select successive groups of exactly k vertices from finite partition cells. Every $(k+1)$ -element hyperedge then meets at least two groups, which the local-finiteness construction keeps independent of one another.

Theorem 5.5 (Arity classification). *For every free ultrafilter $\mathcal{U}$ and every $k \geq 2$,*

$$(5.5) \quad \mathcal{I}_{k+1}(\mathcal{U}) \subseteq \mathcal{I}_k(\mathcal{U}).$$

Equality holds in (5.5) if and only if $\mathcal{U}$ is a Q -point. If $\mathcal{U}$ is not a Q -point, there is a closed set $\mathcal{A}_k \subseteq [\omega]^\omega$ such that

$$\mathcal{A}_k \in \mathcal{I}_k(\mathcal{U}) \setminus \mathcal{I}_{k+1}(\mathcal{U}).$$

Proof. Take $X \in \mathcal{U}$ and $f \in \mathcal{M}_{k+1}$. The implication

$$r_{n+k} \leq f(r_n) \implies r_{n+k-1} \leq f(r_n)$$

shows that $N_{X,f}^{(k+1)} \subseteq N_{X,f}^{(k)}$. This proves the inclusion of the ideals.

If $\mathcal{U}$ is a Q -point, Theorem 5.1 gives $\mathcal{I}_{\mathcal{U},k} = T_k(\mathcal{U})$ for every k . Every member of $T_k(\mathcal{U})$ is contained in $A_X^{(k)}$ for some $X \in \mathcal{U}$. Consequently

$$(5.6) \quad \mathcal{I}_k(\mathcal{U}) = \{\mathcal{A} : (\exists X \in \mathcal{U})(\forall R \in \mathcal{A}) R \not\subseteq^* X\},$$

which does not depend on k .

Suppose that $\mathcal{U}$ is not a Q -point. Fix a partition $\omega = \dot{\bigcup}_m C_m$ into finite sets having no selector in $\mathcal{U}$. For every $X \in \mathcal{U}$ and every positive integer j , there are infinitely many m with

$$(5.7) \quad |X \cap C_m| \geq j.$$

Indeed, if $j = 1$, the assertion follows because a finite union of finite blocks is finite. For $j \geq 2$, failure of (5.7) would, after deleting finitely many blocks,

bound all intersection sizes by $j - 1$. Partitioning the remaining $\mathcal{U}$ -set into $j - 1$ selectors would put one selector in $\mathcal{U}$, a contradiction.

Fix $k \geq 2$, and put $H_k = \bigcup_m [C_m]^k$. Define

$$(5.8) \quad \mathcal{A}_k = \{R = \{r_n : n < \omega\} : (\forall j < \omega)(\exists m < \omega) \{r_{kj}, \dots, r_{kj+k-1}\} \subseteq C_m\}.$$

For a fixed j , the condition in (5.8) depends only on the first $k(j + 1)$ entries of R . It therefore defines a clopen subset of $[\omega]^\omega$. Their countable intersection $\mathcal{A}_k$ is closed in $[\omega]^\omega$. Every $R \in \mathcal{A}_k$ has infinitely many consecutive k -windows in H_k , namely those starting at the indices kj . Thus

$$\mathcal{A}_k \subseteq \Gamma_k(H_k) \in \mathcal{I}_k(\mathcal{U}).$$

We prove that $\mathcal{A}_k \notin \mathcal{I}_{k+1}(\mathcal{U})$.

Let $H \in \mathcal{I}_{k+1}$. Choose $X \in \mathcal{U}$ such that $H \cap [X]^{k+1}$ is locally finite after a finite vertex deletion, and make that deletion from X . We recursively select k -element sets

$$b_i \subseteq X \cap C_{m_i}, \quad \max b_i < \min b_{i+1},$$

so that no new vertex belongs to an H -hyperedge meeting a previously chosen vertex. At a given stage the chosen vertex set is finite. Its incident hyperedges in $H \cap [X]^{k+1}$, and all vertices of those hyperedges, form a finite set. Exclude this set, all vertices below the previous maximum, and the previously used blocks. Only finitely many blocks are affected. Equation (5.7) supplies a remaining block containing at least k points of X , from which to select the next b_i .

Let $R = \bigcup_i b_i$. Since each b_i has exactly k elements and precedes b_{i+1} , it occupies the positions $ki, \dots, ki + k - 1$ in the increasing enumeration of R . Each b_i lies in one block C_{m_i} , so $R \in \mathcal{A}_k$. Every $(k + 1)$ -element subset of R meets at least two of the clusters b_i . If it were an H -hyperedge, a vertex in its latest cluster would belong to an H -hyperedge meeting an earlier chosen vertex, contrary to the construction. Thus

$$[R]^{k+1} \cap H = \emptyset.$$

In particular, $R \notin \Gamma_{k+1}(H)$. Since H was arbitrary, no single member of $\mathcal{I}_{k+1}$ captures $\mathcal{A}_k$. The closed set $\mathcal{A}_k$ therefore witnesses strict inclusion in (5.5). $\square$

Corollary 5.6. *If $\mathcal{U}$ is a P -point which is not a Q -point, then*

$$\ell_{\mathcal{U}}^0 = \mathcal{I}_2(\mathcal{U}) \supsetneq \mathcal{I}_3(\mathcal{U}) \supsetneq \mathcal{I}_4(\mathcal{U}) \supsetneq \cdots$$

is a strictly decreasing sequence of proper σ -ideals, with each strict inclusion witnessed by a closed subset of $[\omega]^\omega$. Every term has a cofinal family of G_δ sets and uniformity $\max\{\pi\chi(\mathcal{U}), \mathfrak{d}\}$.

Proof. Apply Theorems 4.3, 4.5, 5.4, and 5.5. □

Corollary 5.7. *For every free ultrafilter $\mathcal{U}$ and every $k \geq 3$,*

$$\mathcal{I}_k(\mathcal{U}) = \ell_{\mathcal{U}}^0 \iff \mathcal{U} \text{ is both a } P\text{-point and a } Q\text{-point.}$$

Proof. Equality with the Louveau ideal implies that $\mathcal{I}_k(\mathcal{U})$ is a σ -ideal, so Theorem 4.3 makes $\mathcal{U}$ a P -point. If it were not a Q -point, Theorem 5.5 would give $\mathcal{I}_k(\mathcal{U}) \subsetneq \mathcal{I}_2(\mathcal{U}) = \ell_{\mathcal{U}}^0$. Conversely, a P -point which is a Q -point satisfies the required equality by (5.6) and Theorem 5.4. □

6 Stone duality and endpoint spectra

The quotient by finite vertex covers translates relative pseudounions into countable neighborhood properties. We develop this correspondence for all finite arities. Throughout this section, the canonical witnesses are $P_R^{(k)}$.

6.1 The quotient space and its endpoint maps

Let

$$\mathbb{B}_k = \mathcal{P}(E_k)/\mathcal{V}_k, \quad X_k = \text{Ult}(\mathbb{B}_k).$$

Here $\text{Ult}(\mathbb{B}_k)$ is the Stone space of Boolean ultrafilters on $\mathbb{B}_k$. For $H \subseteq E_k$, its basic clopen set is

$$\hat{H} = \{p \in X_k : [H]_{\mathcal{V}_k} \in p\}.$$

Equivalently, points of X_k are ultrafilters on E_k which are disjoint from $\mathcal{V}_k$, and we write $H \in p$ in this representation. The ultrafilter extension argument shows that these clopen sets separate points and that every cover by basic clopen sets has a finite subcover. Thus X_k is compact, Hausdorff, and zero-dimensional. We use the usual Stone correspondence between Boolean algebras and their clopen algebras; see [4].

The passage to capture sets can identify distinct Boolean classes. For $k = 2$, let $M = \{\{2n, 2n + 1\} : n < \omega\}$. Then $M \notin \mathcal{V}_2$, but

$$\Gamma_2(E_2) = \Gamma_2(E_2 \setminus M) = [\omega]^\omega.$$

Indeed, two consecutive edges of an increasing ray cannot both belong to the matching M , so every ray has infinitely many edges in $E_2 \setminus M$.

For later use, Boolean order gives

$$(6.1) \quad H \setminus K \in \mathcal{V}_k \iff \widehat{H} \subseteq \widehat{K}.$$

To see the reverse implication directly, if $H \setminus K \notin \mathcal{V}_k$, its nonzero class extends to an ultrafilter, giving a point in $\widehat{H} \setminus \widehat{K}$.

For $e = \{i_0 < \dots < i_{k-1}\}$, put $s_j(e) = i_j$. The inverse image of a finite subset of ω under s_j has a finite vertex cover. Hence inverse image induces a Boolean homomorphism $\mathcal{P}(\omega)/\text{Fin} \rightarrow \mathbb{B}_k$, and a continuous map

$$\pi_j : X_k \longrightarrow \omega^*, \quad X \in \pi_j(p) \iff s_j^{-1}[X] \in p.$$

Define the closed equalizer

$$Z_k = \{p \in X_k : \pi_0(p) = \dots = \pi_{k-1}(p)\}.$$

On Z_k write π for the common endpoint map.

Proposition 6.1. *For $p \in X_k$,*

$$p \in Z_k \iff \Delta_k X \notin p \text{ for every } X \subseteq \omega \iff p \cap \mathcal{G}_{\text{fc}}^{(k)} = \emptyset.$$

Proof. For a fixed X , the set $\Delta_k X$ is the finite union of the symmetric differences $s_i^{-1}[X] \Delta s_j^{-1}[X]$, $i < j < k$. An ultrafilter avoids this union exactly when all the sets $s_j^{-1}[X]$ have the same membership value in it. Requiring this for every X is precisely equality of the endpoint ultrafilters. Every cut has finite weak chromatic number. Conversely, a finite weak coloring can be encoded by finitely many binary coordinates, so every member of $\mathcal{G}_{\text{fc}}^{(k)}$ is contained in a finite union of cuts by Lemma 2.6. Primeness of p completes the proof. $\square$

For a proper ideal $\mathcal{J} \supseteq \mathcal{V}_k$, set

$$(6.2) \quad L_{\mathcal{J}} = \{p \in X_k : p \cap \mathcal{J} = \emptyset\} = \bigcap_{H \in \mathcal{J}} (X_k \setminus \widehat{H}).$$

This is a nonempty closed set. Conversely, a nonempty closed $L \subseteq X_k$ defines

$$\mathcal{J}_L = \{H \subseteq E_k : \widehat{H} \cap L = \emptyset\}.$$

These operations are mutually inverse and reverse inclusion. For example, if $H \notin \mathcal{J}$, its class is nonzero in $\mathcal{P}(E_k)/\mathcal{J}$; an ultrafilter on that quotient supplies $p \in L_{\mathcal{J}} \cap \widehat{H}$. This proves $\mathcal{J}_{L_{\mathcal{J}}} = \mathcal{J}$. The converse follows by separating a point outside a closed set with a clopen set.

6.2 Relative pseudounions and closed sets

A *P-set* in a topological space is a closed set L such that every countable intersection of neighborhoods of L is again a neighborhood of L . This definition will always refer to a specified ambient space.

The restriction of $\mathcal{V}_k$ to $P_R^{(k)}$ is the finite ideal. Finite vertex covers meet it finitely, and every finite hypergraph has a finite vertex cover. Therefore the clopen set $\widehat{P_R^{(k)}} \subseteq X_k$ is homeomorphic to ω^* . A nonempty closed set $L \subseteq X_k$ will be called *ray-positive* if every clopen neighborhood of L contains some $\widehat{P_R^{(k)}}$.

Theorem 6.2 (Stone correspondence for relative ideals). *Let $\mathcal{J}$ be a proper ideal on E_k containing $\mathcal{V}_k$. Then*

- (i) $\mathcal{G}_{\text{fc}}^{(k)} \subseteq \mathcal{J}$ if and only if $L_{\mathcal{J}} \subseteq Z_k$;
- (ii) $\mathcal{J}$ is a $\mathcal{V}_k$ -*P-ideal* if and only if $L_{\mathcal{J}}$ is a *P-set* in X_k ;
- (iii) $\mathcal{J}$ is *ray-proper* if and only if $L_{\mathcal{J}}$ is *ray-positive*.

Proof. For (i), the forward direction follows from Proposition 6.1. If $H \in \mathcal{G}_{\text{fc}}^{(k)} \setminus \mathcal{J}$, the quotient-ultrafilter argument following (6.2) gives a point of $L_{\mathcal{J}}$ containing H . It cannot lie in Z_k , proving the converse.

Assume that $\mathcal{J}$ is a relative *P-ideal*. Given open neighborhoods O_n of $L_{\mathcal{J}}$, compactness and zero-dimensionality give clopen neighborhoods $X_k \setminus \widehat{G_n} \subseteq O_n$. By the correspondence between ideals and closed sets, $G_n \in \mathcal{J}$. Choose $G \in \mathcal{J}$ with $G_n \setminus G \in \mathcal{V}_k$. Equation (6.1) yields

$$L_{\mathcal{J}} \subseteq X_k \setminus \widehat{G} \subseteq \bigcap_n (X_k \setminus \widehat{G_n}) \subseteq \bigcap_n O_n.$$

Thus the kernel is a *P-set*. Conversely, for $G_n \in \mathcal{J}$, apply the *P-set* property to the clopen neighborhoods $X_k \setminus \widehat{G_n}$, then refine their common neighborhood

to a clopen neighborhood $X_k \setminus \widehat{G}$. Its containment of the kernel gives $G \in \mathcal{J}$, and its inclusion in every $X_k \setminus \widehat{G}_n$ gives $G_n \setminus G \in \mathcal{V}_k$.

Every clopen neighborhood of the kernel has the form $X_k \setminus \widehat{G}$ with $G \in \mathcal{J}$. It contains $\widehat{P_R^{(k)}}$ exactly when $P_R^{(k)} \cap G \in \mathcal{V}_k$, which is equivalent to finiteness of that intersection. This proves (iii). $\square$

For a proper ideal $\mathcal{J} \supseteq \mathcal{G}_{\text{fc}}^{(k)}$, the *endpoint spectrum* is the nonempty compact set

$$K_{\mathcal{J}} = \pi[L_{\mathcal{J}}] \subseteq \omega^*.$$

Proposition 6.3. *For every proper $\mathcal{J} \supseteq \mathcal{G}_{\text{fc}}^{(k)}$,*

$$(6.3) \quad \mathcal{F}_{\mathcal{J}} = \bigcap K_{\mathcal{J}}, \quad K_{\mathcal{J}} = \text{Ext}(\mathcal{F}_{\mathcal{J}}).$$

For a free ultrafilter $\mathcal{U}$,

$$(6.4) \quad L_{T_k(\mathcal{U})} = Z_k \cap \pi^{-1}(\{\mathcal{U}\}) =: Z_{k,\mathcal{U}}.$$

For a free filter $\mathcal{F}$,

$$L_{\mathcal{C}_k(\mathcal{F})} = Z_k \cap \pi^{-1}[\text{Ext}(\mathcal{F})].$$

Proof. If $p \in Z_k$ and $\pi(p) = \mathcal{U}$, then

$$[X]^k = \bigcap_{j < k} s_j^{-1}[X] \in p \iff X \in \mathcal{U}.$$

Taking complements shows that $A_X^{(k)}$ is avoided by every point of $L_{\mathcal{J}}$ exactly when X belongs to every member of $K_{\mathcal{J}}$. This is the first equality of (6.3).

For any nonempty compact $K \subseteq \omega^*$, one has $K = \text{Ext}(\bigcap K)$. Indeed, an ultrafilter $\mathcal{U} \notin K$ can be separated from K by a clopen set X^* containing K and missing $\mathcal{U}$. Then $X \in \bigcap K$ but $X \notin \mathcal{U}$. This proves the second equality.

By Proposition 3.2, every member of $T_k(\mathcal{U})$ is contained in some $A_X^{(k)}$ with $X \in \mathcal{U}$. The ideal also contains $\mathcal{G}_{\text{fc}}^{(k)}$. Avoiding it is consequently equivalent to lying in Z_k and having endpoint $\mathcal{U}$. This proves (6.4). For $\mathcal{C}_k(\mathcal{F})$, avoiding its generators means lying in Z_k and having an endpoint containing every $X \in \mathcal{F}$. This gives the last equality. $\square$

6.3 Reflection from a single endpoint

Lemma 6.4 (Transitivity of P -sets). *Let X be compact Hausdorff, let L be a P -set in X , and let C be a P -set in the relative topology of L . Then C is a P -set in X .*

Proof. Let O_n be open neighborhoods of C in X . By the relative P -set property, choose an open $V \subseteq X$ with $C \subseteq L \cap V \subseteq L \cap \bigcap_n O_n$. By compactness and regularity, choose an open neighborhood U of C whose closure misses $L \setminus V$. The closed sets

$$F_n = \overline{U} \cap (X \setminus O_n)$$

are disjoint from L . The P -set property of L , applied to $X \setminus F_n$, gives a neighborhood W of L disjoint from $\bigcup_n F_n$. Then $U \cap W$ is a neighborhood of C contained in every O_n . $\square$

A point x of a space K is a *relative P -point* if $\{x\}$ is a P -set in K . For $\mathcal{U} \in \omega^*$, this is equivalent to the ultrafilter definition of a P -point. In fact, the basic neighborhoods are X^* , $X \in \mathcal{U}$, and a basic neighborhood Y^* refines all X_n^* exactly when $Y \subseteq^* X_n$ for all n .

Theorem 6.5 (Endpoint reflection). *Let $\emptyset \neq C \subseteq Z_k$ be a P -set in X_k . If $\pi[C] = \{\mathcal{U}\}$, then $\mathcal{U}$ is a P -point. More generally, if $\mathcal{J} \supseteq \mathcal{G}_{\text{fc}}^{(k)}$ is a proper $\mathcal{V}_k$ - P -ideal and $\mathcal{U}$ is a relative P -point of $K_{\mathcal{J}}$, then $\mathcal{U}$ is a P -point of ω^* .*

Proof. The ideal $\mathcal{J}_C$ contains $\mathcal{G}_{\text{fc}}^{(k)}$ and is a relative P -ideal by Theorem 6.2. Proposition 6.3 gives $\mathcal{F}_{\mathcal{J}_C} = \mathcal{U}$. Lemma 3.3 then makes $\mathcal{U}$ a P -filter. As it is an ultrafilter, it is a P -point.

For the second assertion, the continuous surjection $\pi : L_{\mathcal{J}} \rightarrow K_{\mathcal{J}}$ is closed. Given a relative open neighborhood O of its fiber over $\mathcal{U}$, the set $K_{\mathcal{J}} \setminus \pi[L_{\mathcal{J}} \setminus O]$ is an open neighborhood of $\mathcal{U}$ whose inverse image is contained in O . For countably many such O , the relative P -point property gives a common neighborhood of $\mathcal{U}$. Hence the fiber is a relative P -set in $L_{\mathcal{J}}$. The latter is an ambient P -set by Theorem 6.2. Lemma 6.4 makes the fiber an ambient P -set, and the first assertion applies. $\square$

In particular, $Z_{k,\mathcal{U}}$ is a P -set if and only if $\mathcal{U}$ is a P -point, by (6.4), Proposition 3.2, and Theorem 6.2.

6.4 Equal spectra with different transversal numbers

For a ray-positive closed set $L \subseteq X_k$, define its ray transversal number by $\nu_k(L) = \nu_k(\mathcal{J}_L)$. Thus it is the least size of a family $\mathcal{A} \subseteq [\omega]^\omega$ such that every clopen neighborhood of L contains $\widehat{P_R^{(k)}}$ for some $R \in \mathcal{A}$.

Theorem 6.6 (Equal spectra with distinct transversal numbers). *Suppose $\mathcal{U}$ is a P -point with $\pi\chi(\mathcal{U}) < \mathfrak{d}$. For every $k \geq 2$, the closed sets*

$$L_0 = L_{T_k(\mathcal{U})}, \quad L_1 = L_{\mathcal{J}_{\mathcal{U},k}}$$

are nonempty ray-positive P -sets satisfying

$$L_1 \subsetneq L_0, \quad \pi[L_0] = \pi[L_1] = \{\mathcal{U}\},$$

whereas

$$\nu_k(L_0) = \pi\chi(\mathcal{U}) < \mathfrak{d} = \nu_k(L_1).$$

Proof. Proposition 3.2 and Theorem 4.2 show that both ideals are proper, ray-proper, and relative P -ideals, with common canonical filter $\mathcal{U}$. Their spectra are consequently $\{\mathcal{U}\}$ by Proposition 6.3. Their transversal numbers are given by Proposition 4.4 and Theorem 4.5. The different numbers force strict inclusion of the ideals, and therefore strict reverse inclusion of their Stone kernels. $\square$

We record how the hypothesis is realized in a familiar model. The *ultrafilter number* is $\mathfrak{u} = \min\{\chi(\mathcal{U}) : \mathcal{U} \text{ is a free ultrafilter}\}$, where $\chi(\mathcal{U})$ is the least size of a base contained in $\mathcal{U}$.

Lemma 6.7. *If a free ultrafilter $\mathcal{U}$ satisfies $\chi(\mathcal{U}) < \mathfrak{d}$, then it is a P -point.*

Proof. Suppose that $X_0 \supseteq X_1 \supseteq \dots$ belongs to $\mathcal{U}$ and has no pseudointersection in $\mathcal{U}$. We may arrange $X_0 = \omega$. The sets $X_n \setminus X_{n+1}$ together with $\bigcap_n X_n$ form a countable partition into sets outside $\mathcal{U}$, after empty cells are omitted. There must be infinitely many nonempty cells. Enumerate them as C_n . No set meeting every C_n finitely belongs to $\mathcal{U}$: such a set would be almost contained in every X_n .

Take a base $\mathcal{B} \subseteq \mathcal{U}$ of size $\chi(\mathcal{U})$ for actual inclusion. A base for almost inclusion can be made one for actual inclusion by adding its finite deletions, with no change of this infinite cardinality. Define

$$g_B(n) = \min \left(B \cap \bigcup_{j \geq n} C_j \right) \quad (B \in \mathcal{B}).$$

The minimum exists because the tail union belongs to $\mathcal{U}$. Given a nondecreasing $f \in \omega^\omega$, the set

$$Y_f = \bigcup_j (C_j \cap [0, f(j)))$$

meets each cell finitely, so it does not belong to $\mathcal{U}$. Choose $B \in \mathcal{B}$ contained in its complement. If $g_B(n) \in C_j$, then $j \geq n$ and $g_B(n) \geq f(j) \geq f(n)$. Hence the family $(g_B : B \in \mathcal{B})$ is dominating, contradicting $\chi(\mathcal{U}) < \mathfrak{d}$. $\square$

Corollary 6.8. *If $\mathfrak{u} = \aleph_1 < \mathfrak{d} = \aleph_2$, there is a P -point $\mathcal{U}$ which is not a Q -point and for which, for every $k \geq 2$,*

$$\nu_k(T_k(\mathcal{U})) = \aleph_1, \quad \text{non}(\mathcal{I}_k(\mathcal{U})) = \aleph_2.$$

For this ultrafilter, the strict σ -ideal sequence in Corollary 5.6 and the spectral separation in Theorem 6.6 both hold.

Proof. Choose $\mathcal{U}$ with $\chi(\mathcal{U}) = \aleph_1$. Lemma 6.7 makes it a P -point. By Section 2, a free ultrafilter has no countable π -base. Since $\pi\chi(\mathcal{U}) \leq \chi(\mathcal{U}) = \aleph_1$, we obtain $\pi\chi(\mathcal{U}) = \aleph_1$. If $\mathcal{U}$ were a Q -point, Theorem 5.1 would identify $T_k(\mathcal{U})$ with $\mathcal{I}_{\mathcal{U},k}$, contradicting their different transversal numbers. The other conclusions now follow from the preceding theorems. $\square$

The hypothesis of Corollary 6.8 holds in the Miller model [7, p. 326].

7 Countable block products

On a countable partition into infinite blocks, local relative upper bounds can be combined by including the first $n + 1$ members of a given sequence on the n th block. The resulting Boolean quotient is a product, and its Stone space has a natural map to $\beta\omega$ given by the block indices. We use the consecutive witnesses $P_R^{(k)}$ throughout. Let

$$\omega = \bigcup_{n < \omega} B_n, \quad |B_n| = \aleph_0, \quad E_n = [B_n]^k.$$

All local definitions use the order inherited from ω . Write $\mathcal{V}_{k,n}$ and $\mathcal{G}_{\text{fc}}^{(k)}(B_n)$ for the local finite-cover and finite-chromatic ideals. Given ideals $\mathcal{J}_n$ on E_n , define

$$(7.1) \quad \mathcal{J} = \{H \subseteq E_k : H \cap E_n \in \mathcal{J}_n \text{ for all } n < \omega\}.$$

The set of all hyperedges meeting more than one block belongs to $\mathcal{J}$; we call these the cross-block hyperedges.

7.1 The relative P -property and the Boolean product

Theorem 7.1 (Countable block products). *Suppose each $\mathcal{J}_n$ is a proper ray-proper $\mathcal{V}_{k,n}$ - P -ideal containing $\mathcal{G}_{\text{fc}}^{(k)}(B_n)$. Then the ideal in (7.1) is proper, ray-proper, contains $\mathcal{G}_{\text{fc}}^{(k)}$, and is a $\mathcal{V}_k$ - P -ideal. Moreover,*

$$(7.2) \quad \mathcal{P}(E_k)/\mathcal{J} \cong \prod_{n < \omega} \mathcal{P}(E_n)/\mathcal{J}_n.$$

Proof. A finite weak coloring restricts to one on each block, proving the stated containment. If $H \in \mathcal{J}$, choose a block B_n and use local ray-properness to find $R \subseteq B_n$ with $|P_R^{(k)} \cap H| < \omega$. This proves ray-properness and properness.

Let $G_i \in \mathcal{J}$ for $i < \omega$. On each block, choose $H_n \in \mathcal{J}_n$ such that

$$(G_i \cap E_n) \setminus H_n \in \mathcal{V}_{k,n} \quad (i < \omega).$$

Set

$$H'_n = H_n \cup \bigcup_{i \leq n} (G_i \cap E_n) \in \mathcal{J}_n,$$

and let H contain all cross-block hyperedges and satisfy $H \cap E_n = H'_n$. Then $H \in \mathcal{J}$. For a fixed i , the difference $G_i \setminus H$ lies in the finitely many blocks $n < i$. Its intersection with each of these blocks has a finite vertex cover, and the union of those finite covers is a global finite vertex cover. Thus $G_i \setminus H \in \mathcal{V}_k$.

The Boolean homomorphism

$$H \mapsto \langle [H \cap E_n]_{\mathcal{J}_n} : n < \omega \rangle$$

is onto. Choose a representative on each disjoint block and take their union. Its kernel is exactly $\mathcal{J}$, which proves (7.2). $\square$

7.2 Principal and nonprincipal fibers

Put $L = L_{\mathcal{J}} \subseteq Z_k$. Define the block-index function $e : \omega \rightarrow \omega$ by $e(i) = n$ for $i \in B_n$. For an ultrafilter $\mathcal{U}$ on ω , its image under e is

$$e(\mathcal{U}) = \{A \subseteq \omega : e^{-1}[A] \in \mathcal{U}\} \in \beta\omega.$$

This continuous extension is denoted by βe . The common endpoint map defines

$$(7.3) \quad \rho = \beta e \circ \pi : L \longrightarrow \beta\omega.$$

For $q \in \beta\omega$, let $L_q = \rho^{-1}(\{q\})$. For $A \subseteq \omega$, write $A^\beta = \{q \in \beta\omega : A \in q\}$; principal points are included in this clopen set.

Theorem 7.2 (Block fibers). *Under the hypotheses of Theorem 7.1, the map (7.3) is onto and every fiber L_q is ray-positive. For a principal point n , the fiber L_n is naturally homeomorphic to $L_{\mathcal{J}_n}$ in the local Stone space, and $\bigcup_n L_n$ is dense in L . If $q \in \omega^*$ and $p \in L_q$, then $\pi(p)$ is not a P -point.*

Proof. Choose $p_n \in L_{\mathcal{J}_n}$ for every n . For $q \in \beta\omega$, define an ultrafilter on E_k by

$$H \in p_q \iff \{n : H \cap E_n \in p_n\} \in q.$$

The ultrafilter axioms follow coordinatewise from those of the p_n and q . The ultrafilter p_q avoids $\mathcal{J}$, since every member of $\mathcal{J}$ is avoided on every coordinate. Thus $p_q \in L$. For $A \subseteq \omega$, all endpoint preimages of $e^{-1}[A]$ have full intersection with E_n if $n \in A$, and empty intersection if $n \notin A$. Consequently $\rho(p_q) = q$, proving surjectivity. For principal $q = n$, this construction identifies the fiber with the Stone space of the n th quotient in (7.2).

More explicitly, $\widehat{E_n} \subseteq X_k$ is homeomorphic to the local Stone space of $\mathcal{P}(E_n)/\mathcal{V}_{k,n}$. A point $p \in L$ lies in $\widehat{E_n}$ exactly when all its endpoints contain B_n , and hence exactly when $\rho(p) = n$. Therefore

$$(7.4) \quad L_n = L \cap \widehat{E_n}.$$

The product representation shows density of their union. Indeed, every nonempty relatively clopen set in L is represented by a nonzero tuple in (7.2); some coordinate is nonzero, and an ultrafilter on that coordinate gives a point of the clopen set in L_n .

Fix $q \in \beta\omega$, and let $O = X_k \setminus \widehat{G}$ be a clopen neighborhood of L_q . The compact set $K = L \cap \widehat{G}$ has closed image $\rho[K]$ not containing q . Choose a clopen A^β with $q \in A^\beta$ and $A^\beta \cap \rho[K] = \emptyset$. For every $n \in A$, the principal fiber L_n misses $\widehat{G}$, whence $G \cap E_n \in \mathcal{J}_n$ by the local correspondence between ideals and closed sets. Choose one such n and a local ray $R \subseteq B_n$ avoiding $G \cap E_n$. Then $\widehat{P_R^{(k)}} \subseteq O$. This proves ray-positivity of every fiber.

Finally, suppose $q \in \omega^*$, $p \in L_q$, and $\pi(p) = \mathcal{U}$. For every m ,

$$Y_m = \omega \setminus \bigcup_{n \leq m} B_n \in \mathcal{U}.$$

If $\mathcal{U}$ were a P -point, choose $A \in \mathcal{U}$ with $A \subseteq^* Y_m$ for all m . Then $A \cap B_n$ is finite for every n . Hence $[A]^k \cap E_n = [A \cap B_n]^k \in \mathcal{J}_n$, so $[A]^k \in \mathcal{J}$. On the other hand, $A \in \pi_j(p)$ for each $j < k$, and therefore $[A]^k = \bigcap_{j < k} s_j^{-1}[A] \in p$. This contradicts $p \in L$. $\square$

7.3 Minimal closed sets and singleton spectra

Theorem 7.3. *Every ray-positive closed $L \subseteq Z_k$ contains an inclusion-minimal ray-positive closed subset C . Every such C has $\pi[C] = \{\mathcal{U}\}$ for a unique free ultrafilter $\mathcal{U}$.*

Proof. Consider a decreasing chain of nonempty ray-positive closed subsets of L . Compactness makes its intersection C nonempty. If O is a clopen neighborhood of C , some member of the chain is contained in O . Otherwise the closed sets obtained by removing O would have the finite intersection property, yielding a point of $C \setminus O$. The contained member supplies a ray clopen set in O , so C remains ray-positive. Zorn's lemma now gives an inclusion-minimal member.

Fix $X \subseteq \omega$, and put $Q_X = \widehat{[X]^k}$. Since $C \subseteq Z_k$,

$$C = (C \cap Q_X) \dot{\cup} (C \cap Q_{X^c}).$$

At least one piece is ray-positive. If both failed to be ray-positive, choose clopen neighborhoods $O_0 \subseteq Q_X$ and $O_1 \subseteq Q_{X^c}$ of the respective pieces, neither containing a ray clopen set. For an empty piece use the empty neighborhood. Their union contains C , and hence contains $\widehat{P_R^{(k)}}$ for some R . Since

$$Q_X \cup Q_{X^c} = X_k \setminus \widehat{\Delta_k X},$$

the ray avoids $\Delta_k X$. Lemma 2.6 gives $R \subseteq^* X$ or $R \subseteq^* X^c$. The entire clopen set $\widehat{P_R^{(k)}}$ then lies in one of Q_X, Q_{X^c} and hence in the corresponding O_i , a contradiction.

Minimality implies $C \subseteq Q_X$ or $C \subseteq Q_{X^c}$ for every X . The choices determine an ultrafilter $\mathcal{U} = \{X : C \subseteq Q_X\}$; finite intersections are preserved, and finite sets have empty Q_X . Each point of C has common endpoint $\mathcal{U}$, proving the assertion and uniqueness. $\square$

Corollary 7.4. *If a ray-positive P -set $L_0 \subseteq Z_k$ exists, then there is a ray-positive P -set $L \subseteq Z_k$ containing a minimal ray-positive closed set whose endpoint is not a P -point.*

Proof. Take the ideal dual to L_0 , copy it onto each infinite block, and apply Theorem 7.1. The resulting kernel L is a ray-positive P -set by Theorem 6.2. Choose $q \in \omega^*$. By Theorem 7.2, L_q is ray-positive and every endpoint in it is non- P . Theorem 7.3 supplies a minimal ray-positive closed $C \subseteq L_q$ with one endpoint. That endpoint is non- P . $\square$

Theorem 7.5 (Localization of relative P -sets). *Let L be the block-product kernel. Every nonempty relative P -set $M \subseteq L$ meets some principal fiber L_n , and $M \cap L_n$ is a nonempty relative P -set in L . In particular, $\rho^{-1}[\omega^*]$ contains no nonempty relative P -set of L .*

Proof. If M missed all L_n , each $L \setminus L_n$ would be a clopen neighborhood of M . Their intersection would contain a nonempty relatively open neighborhood of M , disjoint from the dense set $\bigcup_n L_n$, a contradiction. Since L_n is relatively clopen, $M \cap L_n$ is again a relative P -set. $\square$

Theorem 7.6 (Singleton spectra in block products). *The block-product kernel L contains a nonempty P -set C of X_k with $|\pi[C]| = 1$ if and only if some local kernel $L_{\mathcal{J}_n}$ contains a nonempty P -set of its local Stone space with a single endpoint.*

Proof. The local Stone space embeds as the clopen set $\widehat{E}_n \subseteq X_k$. A P -set of that clopen space is a P -set of X_k , so a local witness gives a global one. Conversely, a global witness $C \subseteq L$ is a relative P -set in L . By Theorem 7.5, $C \cap L_n$ is nonempty for some n . Equation (7.4) gives $C \cap L_n = C \cap \widehat{E}_n$, a clopen part of the ambient P -set C . It remains an ambient P -set and has the same unique endpoint. Under the local homeomorphism it gives the required witness. $\square$

7.4 The reduced block product

Define

$$\mathcal{J}_{\text{red}} = \{H \subseteq E_k : H \cap E_n \in \mathcal{J}_n \text{ for all but finitely many } n\}.$$

It contains $\mathcal{G}_{\text{fc}}^{(k)}$ and is ray-proper, since any member can be avoided in a block where its restriction belongs to $\mathcal{J}_n$. Allowing finitely many exceptional blocks destroys the relative P -property.

Proposition 7.7. *Under the hypotheses of Theorem 7.1, $\mathcal{J}_{\text{red}}$ is not a $\mathcal{V}_k$ - P -ideal. More generally, a ray-proper $\mathcal{V}_k$ - P -ideal cannot contain both every E_n and all cross-block hyperedges.*

Proof. Suppose a ray-proper relative P -ideal $\mathcal{K}$ contains these sets. Take a common relative upper bound $H \in \mathcal{K}$ of the E_n . Choose finite $F_n \subseteq B_n$ covering $E_n \setminus H$, so that

$$[B_n \setminus F_n]^k \subseteq H.$$

The union H' of H and all cross-block hyperedges belongs to $\mathcal{K}$. Let $R = \{r_i : i < \omega\}$ be any increasing sequence. If the block indices of consecutive vertices change infinitely often, infinitely many consecutive k -windows meet more than one block, and hence belong to H' . If the indices change only finitely often, the sequence is eventually contained in some B_n . Its late windows then avoid the finite set F_n and belong to H . Thus every ray meets H' infinitely, contradicting ray-properness. The reduced product contains all the stipulated sets, so the general assertion applies to it. $\square$

8 Countable decision and extension problems

The preceding classification results also give an exact formulation of several extension questions motivated by reaping. We first specify the cardinal parameters and then give the row criterion for adjoining a pseudounion while preserving a fixed avoiding family.

8.1 The finite and countable reaping numbers

A set $X \subseteq \omega$ *splits* an infinite $R \subseteq \omega$ if both $R \cap X$ and $R \setminus X$ are infinite. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *reaping* if no single X splits all its members. The least size of a reaping family is $\mathfrak{r}$. A family is σ -*reaping* if for every sequence $X_n \subseteq \omega$ some $R \in \mathcal{A}$ decides all the X_n , where deciding means $R \subseteq^* X_n$ or $R \subseteq^* X_n^c$. Its least possible size is $\mathfrak{r}_\sigma$. Vojtáš's question asks whether these two cardinals are equal in ZFC [9, 1].

Lemma 8.1. *There is a family of size $\mathfrak{r}$ which simultaneously decides every finite family of subsets of ω .*

Proof. Fix a reaping family $\mathcal{A}$ of size $\mathfrak{r}$. Set $\mathcal{A}_1 = \mathcal{A}$. Given $\mathcal{A}_m$, choose a bijection $e_B : \omega \rightarrow B$ for each $B \in \mathcal{A}_m$ and put

$$\mathcal{A}_{m+1} = \{e_B[C] : B \in \mathcal{A}_m, C \in \mathcal{A}\}.$$

Given $X_0, \dots, X_m$, first choose $B \in \mathcal{A}_m$ deciding the first m sets. Then choose $C \in \mathcal{A}$ deciding the inverse image of $X_m \cap B$ under e_B . The subset $e_B[C] \subseteq B$ decides all $m+1$ sets. Every $\mathcal{A}_m$ has size at most $\mathfrak{r}$, and $\bigcup_{m \geq 1} \mathcal{A}_m$ has size $\mathfrak{r}$ and the desired property. We used that $\mathfrak{r}$ is uncountable, which follows by splitting any countable family of infinite sets with a recursive choice of two fresh points from each set at infinitely many stages. $\square$

Proposition 8.2. *For every $k \geq 2$,*

$$\nu_k(\mathcal{G}_{\text{fc}}^{(k)}) = \mathfrak{r}, \quad \nu_{k,\omega}(\mathcal{G}_{\text{fc}}^{(k)}) = \mathfrak{r}_\sigma.$$

Proof. An avoiding family applied to the cuts $\Delta_k X$ is a reaping family by Lemma 2.6, proving the first lower bound. Conversely, take the family from Lemma 8.1. A finite weak coloring of a hypergraph gives finitely many color classes; an infinite set deciding all these classes is almost contained in one of them. Its sufficiently late consecutive windows therefore avoid the hypergraph.

For a countable family of finite-chromatic hypergraphs, the color classes of fixed finite colorings form a countable family. A σ -reaping witness deciding all those classes avoids every hypergraph. The converse again uses the countable sequence of cuts associated with a given sequence X_n . $\square$

Let $D_X = \{R : R \subseteq^* X \text{ or } R \subseteq^* X^c\}$. Write $\mathcal{F}_{\text{fin}}$ for the filter on $[\omega]^\omega$ generated by all D_X , and write $\mathcal{F}_{\text{fin}}^\sigma$ for the family of supersets of sets of the form $\bigcap_n D_{X_n}$.

Proposition 8.3. *The family $\mathcal{F}_{\text{fin}}^\sigma$ is the least countably complete proper filter containing $\mathcal{F}_{\text{fin}}$, and*

$$\text{tr}(\mathcal{F}_{\text{fin}}) = \mathfrak{r}, \quad \text{tr}(\mathcal{F}_{\text{fin}}^\sigma) = \mathfrak{r}_\sigma.$$

For every $k \geq 2$,

$$\{H \subseteq E_k : D_H \in \mathcal{F}_{\text{fin}}^\sigma\} = \mathcal{G}_{\text{fc}}^{(k)}.$$

Proof. For any sequence X_n , recursively choose decreasing infinite sets C_n deciding the first n sets by taking an infinite side of each successive partition. A diagonal increasing choice $r_n \in C_n$ gives $R \in \bigcap_n D_{X_n}$. Thus the basic sets are nonempty. Countable intersections of basic sets are again basic by enumerating the sets in a double sequence. This proves properness and countable completeness. Any countably complete filter containing every D_X contains these basic intersections, proving minimality. The transversal formulas follow from Lemma 8.1 and the definition of σ -reaping. The final equality is Theorem 2.7. $\square$

For the refining relation on $[\omega]^\omega$

$$X \text{ ref } R \iff R \subseteq^* X \text{ or } R \subseteq^* X^c,$$

the ideal dual to $\mathcal{F}_{\text{fin}}^\sigma$ is the ideal of families which are not ref- ω -dominating, in the terminology of [8, Definitions 2.9 and 7.6]. Explicitly, it consists of all families contained in a countable union of splitting sets $\{R : X \text{ splits } R\}$.

If $\mathcal{J} \supseteq \mathcal{G}_{\text{fc}}^{(k)}$ is ray-proper and a relative P -ideal, then

$$(8.1) \quad \mathfrak{r}_\sigma \leq \nu_k(\mathcal{J}).$$

Indeed, for any sequence of cuts, take a common relative upper bound $H \in \mathcal{J}$. A ray avoiding H decides every set defining those cuts. Thus an avoiding family for $\mathcal{J}$ is σ -reaping, proving (8.1) directly.

8.2 Row bounds preserving an avoiding family

The row-tail bounds and the functions λ_R are defined in Section 2.4. We apply their normal form to preservation of a fixed avoiding family.

Let $\mathcal{A} \subseteq [\omega]^\omega$ avoid every member of a ray-proper ideal $\mathcal{J}$. The sets $\mathcal{A} \cap D_K$, $K \in \mathcal{J}$, form a base of a proper filter $\mathcal{F}_{\mathcal{J}, \mathcal{A}}$ on $\mathcal{A}$. A subset of $\mathcal{A}$ is *positive* for this filter if it meets every one of its members.

Proposition 8.4 (One-step preservation). *Let $G_0 \subseteq G_1 \subseteq \dots$ belong to $\mathcal{J}$. There is a common $\mathcal{V}_k$ -upper bound H such that $\mathcal{A}$ avoids every member of $\text{Id}(\mathcal{J} \cup \{H\})$ if and only if some strictly increasing $h \in \omega^\omega$ makes*

$$\{R \in \mathcal{A} \cap \bigcap_n D_{G_n} : \lambda_R \leq^* h\}$$

positive for $\mathcal{F}_{\mathcal{J}, \mathcal{A}}$.

Proof. The family $\mathcal{A}$ avoids every member of $\text{Id}(\mathcal{J} \cup \{H\})$ exactly when

$$\mathcal{A} \cap D_K \cap D_H \neq \emptyset \quad (K \in \mathcal{J}).$$

Thus the condition is positivity of $\mathcal{A} \cap D_H$. If an upper bound H has this property, Lemma 2.8 gives $H_h \subseteq H$, so $D_H \subseteq D_{H_h}$ and positivity is preserved. Equation (2.5) gives the displayed set. Conversely, a positive set of the displayed form permits adjoining H_h , which is a relative upper bound by the same lemma. $\square$

Corollary 8.5. *Suppose $C = \mathcal{A} \cap \bigcap_n D_{G_n}$ is positive for $\mathcal{F}_{\mathcal{J}, \mathcal{A}}$, and this filter has a base of size less than $\mathfrak{b}$. Then the one-step extension in Proposition 8.4 exists.*

Proof. For each member B of a base of size less than $\mathfrak{b}$, choose $R_B \in B \cap C$. The family of functions λ_{R_B} has a common $\leq^*$ -upper bound. Increase that bound to a strictly increasing h . The corresponding set in Proposition 8.4 meets every basic set B , and is therefore positive. $\square$

8.3 Further questions

Question 8.6. Does ZFC prove the existence of a ray-proper $\mathcal{V}_2$ - P -ideal $\mathcal{J} \supseteq \mathcal{G}_{\text{fc}}^{(2)}$ with $\nu_2(\mathcal{J}) = \mathfrak{r}$?

By (8.1), an affirmative answer gives $\mathfrak{r} = \mathfrak{r}_\sigma$. If there is a P -point $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \mathfrak{r}$, then $T_2(\mathcal{U})$ already supplies the required ideal. Theorem 6.6 also shows that enlargements with the same endpoint spectrum can increase the relevant transversal number.

Proposition 8.4 gives two requirements for an extension construction. The set $C = \mathcal{A} \cap \bigcap_n D_{G_n}$ must be positive, and for some h the subset $\{R \in C : \lambda_R \leq^* h\}$ must also be positive. Theorem 4.6 verifies both requirements for small families of locally finite perturbations of trace ideals, after refining a minimum-size π -base without changing its cardinality.

Question 8.7. For $k \geq 2$, does every ray-positive P -set L of X_k with $L \subseteq Z_k$ contain a nonempty P -set of X_k concentrated over one endpoint?

Theorem 6.5 makes such an endpoint a P -point. Corollary 7.4 shows that minimality of a ray-positive closed subset does not ensure this conclusion. Theorem 7.6 reduces the existence question for a countable block product to the corresponding question in one of its components.

For a P -point $\mathcal{U}$ which is not a Q -point, the ideals $\mathcal{J}_k(\mathcal{U})$ form a strictly decreasing sequence with equal uniformities and closed sets separating successive terms. It remains to determine their other cardinal characteristics and the reducibility relations between distinct terms.

Declaration of AI use

ChatGPT was used for proofreading and language polishing.

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