

# A Frobenius determinant proof of the non-existence of Landau–Siegel zeros

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## Abstract

We give a variant of the OpenAI proof [1] of the non-existence of Landau–Siegel zeros, following the simplifications of Kerr [2], Green [3] and Barreto [4]. The Frobenius congruence proves both divisibility and nonvanishing of the determinant. Reduction modulo the largest inert prime below the cutoff reduces nonvanishing to a polynomial root count, replacing resultants and Wronskians. The argument was discovered in discussions with GPT-6 Astra and Opus 5.5 and the new arguments should be attributed to these models.

**Theorem 1.** *Let  $d$  be a nonsquare integer, and put*

$$I_d(x) = \sum_{\substack{p \leq x \\ p \text{ odd} \\ (d/p) = -1}} \frac{\log p}{p}.$$

*There is an absolute effective constant  $C$  such that, for every  $x \geq 2$ ,*

$$I_d(x) \leq \frac{1}{2} \log x + \frac{1}{4} \log |d| + \log \log(3x) + C. \quad (1)$$

**Corollary 1.** *Let  $\chi$  be a primitive nonprincipal real Dirichlet character of conductor  $q$ . Every real zero  $\beta \in (0, 1)$  of  $L(s, \chi)$  satisfies*

$$(1 - \beta) \log q \geq 1 - O\left(\frac{\log \log(3q)}{\log q}\right).$$

*In particular,  $(1 - \beta) \log q \geq c$  with an absolute effective  $c > 0$ .*

The choice of the largest inert prime makes the elementary bound uniform in  $x$ . If  $R$  is the largest odd inert prime at most  $x$ , then  $I_d(x) = I_d(R)$ : estimating the sum at  $R$  therefore suffices for every cutoff, without information on the spacing of inert primes. Reduction modulo  $R$  proves nonvanishing; reduction at all inert primes provides divisibility. The proof compares that divisibility with the size of the same determinant.

*Proof of Theorem 1.* We use Chebyshev’s bound, Mertens’ estimate, and their elementary consequence

$$\sum_{p \leq y} \log p \ll y, \quad \sum_{p \leq y} \frac{\log p}{p} = \log y + O(1), \quad \sum_{p \geq y} \frac{\log p}{p^2} \ll y^{-1}.$$

The last estimate follows from the first by partial summation. All implied constants below are absolute and effective. If there is no odd inert prime at most  $x$ , there is nothing to prove. Otherwise let  $R$  be the largest such prime. Then  $I_d(x) = I_d(R)$ . If  $R < e^3$  or  $R \leq \sqrt{|d|}$ , Mertens’ estimate proves (1). Hence assume  $R \geq e^3$  and  $R > \sqrt{|d|}$ , and put  $L = \log R$ ,  $S = \lceil R/L \rceil$  and  $N = RS$ .

Take all  $\alpha = u + v\sqrt{d}$  with integers

$$0 \leq u < \sqrt{R} |d|^{1/4}, \quad 0 \leq v < \sqrt{R} |d|^{-1/4}.$$

The side lengths multiply to  $R$ , so their ceilings give at least  $R$  integer points. They are balanced so that  $u$  and  $v\sqrt{|d|}$  have the same upper bound  $\sqrt{R} |d|^{1/4}$ . Thus both  $\alpha$  and its quadratic conjugate  $\bar{\alpha}$  have modulus less than  $H = 2\sqrt{R} |d|^{1/4} \leq 2R$ . For every odd inert prime  $p$ , the binomial theorem, Fermat’s little theorem and Euler’s criterion give

$$\alpha^p = (u + v\sqrt{d})^p \equiv u + v d^{(p-1)/2} \sqrt{d} \equiv u - v\sqrt{d} = \bar{\alpha} \pmod{p\mathbb{Z}[\sqrt{d}]}. \quad (2)$$

Form the matrix

$$\mathcal{A}_{(r,s),(\alpha,j)} = (\partial_z^j z^r w^s)(\alpha, \bar{\alpha}), \quad 0 \leq r < R, \quad 0 \leq s < S, \quad 0 \leq j < S.$$

Its  $N$  rows are the monomials  $z^r w^s$ ; its columns are the chosen points and derivative orders.

*Nonvanishing.* Modulo  $R$ , the ring  $\mathbb{Z}[\sqrt{d}]$  is the field  $\mathbb{F}_{R^2}$ , and (2) gives  $\bar{\alpha} = \alpha^R$ . The chosen points remain distinct because their integer coordinates lie in  $[0, R)$  (as  $R > \sqrt{|d|}$ ), and  $1, \sqrt{d}$  form an  $\mathbb{F}_R$ -basis. A nontrivial row dependence over  $\mathbb{F}_{R^2}$  would therefore give

$$G(z) = \sum_{r < R, s < S} c_{rs} z^{r+Rs} \in \mathbb{F}_{R^2}[z], \quad G \neq 0, \quad \deg G < RS.$$

The exponents are distinct, and  $(z^R)' = 0$  shows that the derivatives of orders  $0, \dots, S-1$  of  $G$  vanish at every chosen point. Since  $S < R$ , the factorials of these orders are invertible. Thus  $G$  has at least  $R$  distinct roots, each of multiplicity at least  $S$ , contradicting  $\deg G < RS$ . Choose an  $N \times N$  minor  $\Delta \in \mathbb{Z}[\sqrt{d}]$  nonzero modulo  $R$ . It uses every row, and

$$N = |\mathcal{N}_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}} \Delta| \in \mathbb{Z}_{>0}.$$

*Divisibility.* For an odd inert prime  $p$ , put  $h = \min(\lfloor r/p \rfloor, S-1-s)$  and replace each row by

$$z^r w^s \mapsto z^{r-ph} (z^p - w)^h w^s.$$

The condition  $s+h < S$  keeps every term among the original rows. All added terms have smaller  $z$ -degree, so this unitriangular change preserves  $\Delta$ . By (2), an undifferentiated factor  $z^p - w$  is divisible by  $p$  at every point. Every positive  $z$ -derivative of this factor has coefficients divisible by  $p$ . In each term of Leibniz's rule, each of the  $h$  copies of  $z^p - w$  therefore contributes a factor  $p$ , whether differentiated or not. Thus every derivative of the new row is divisible by  $p^h$ . Extracting these row factors gives  $p^{E_p} \mid \Delta$ , where  $E_p = \sum_{r < R, s < S} h$ . Conjugation also gives  $p^{E_p} \mid \bar{\Delta}$ , hence  $p^{2E_p} \mid N$ . The exponent satisfies

$$E_p = \sum_{r < R, s < S} \min(\lfloor r/p \rfloor, S-1-s) \geq \frac{SR(R-1)}{2p} - SR - \frac{R^3}{3p^2}.$$

Indeed,  $\sum_{s < S} \min(a, S-1-s) \geq Sa - a^2$  for integers  $a \geq 0$ ; now use  $\lfloor r/p \rfloor \geq r/p - 1$ . We sum over  $L \leq p \leq R$ . The cutoff makes the quadratic error  $R^3 \sum_{p \geq L} (\log p)/p^2 \ll R^3/L \leq NR$ . Chebyshev bounds the floor error by  $O(NR)$ , and Mertens allows us to replace  $R-1$  by  $R$  with the same error. Consequently,

$$\log N \geq NR \sum_{\substack{L \leq p \leq R \\ p \text{ odd} \\ (d/p) = -1}} \frac{\log p}{p} - O(NR). \quad (3)$$

*Size.* Under either embedding, an entry in row  $(r, s)$  has modulus at most  $R^S H^{r+s}$ . Expanding the two conjugate determinants gives

$$\log N \leq 2 \log(N!) + 2NS \log R + N(R+S-2) \log H \leq NR \log H + O(NR),$$

since  $S = \lceil R/\log R \rceil$  and  $H \leq 2R$ . Compare with (3) and restore the primes below  $L$  by Mertens:

$$I_d(x) = I_d(R) \leq \log H + \log L + O(1) = \frac{1}{2} \log R + \frac{1}{4} \log |d| + \log \log R + O(1).$$

Since  $R \leq x$ , this proves (1).  $\square$

*Proof of Corollary 1.* Choose squarefree  $d \neq 1$  with  $|d| \leq q$  and  $\chi(p) = (d/p)$  for odd primes  $p$ . For  $0 < \eta \leq 1$ , partial summation in (1), absorbing the possible contribution of  $p = 2$  into the error term, gives

$$\sum_{\chi(p) = -1} \frac{\log p}{p^{1+\eta}} \leq \frac{1}{2\eta} + \frac{1}{4} \log q + O(\log(2/\eta)).$$

Here the logarithmic error follows from Jensen's inequality after  $u = \log t$ ; the measure  $\eta e^{-\eta u} du$  has mean  $1/\eta$ . Higher prime powers contribute  $O(1)$  uniformly in  $\eta$ , so the Euler product for  $\zeta$  gives

$$\sum_p \frac{\log p}{p^{1+\eta}} = -\frac{\zeta'}{\zeta}(1+\eta) + O(1) = \frac{1}{\eta} + O(1).$$

Since  $\chi(p)$  is  $-1$ ,  $0$  or  $1$ , the Euler product for  $L$  gives

$$\begin{aligned} \frac{L'}{L}(1+\eta, \chi) &= -\sum_p \frac{\chi(p) \log p}{p^{1+\eta}} + O(1) \\ &= 2 \sum_{\chi(p)=-1} \frac{\log p}{p^{1+\eta}} - \sum_p \frac{\log p}{p^{1+\eta}} + \sum_{p|q} \frac{\log p}{p^{1+\eta}} + O(1). \end{aligned}$$

The terms  $2 \cdot (2\eta)^{-1}$  and  $\eta^{-1}$  cancel. Using  $\sum_{p|q} (\log p)/p \ll \log \log(3q)$ , we obtain

$$\frac{L'}{L}(1+\eta, \chi) \leq \frac{1}{2} \log q + O(\log(2/\eta) + \log \log(3q)).$$

Finally, the Hadamard formula [5, Cor. 10.18], retaining  $\beta$ , gives

$$\frac{1}{1-\beta+\eta} \leq \frac{L'}{L}(1+\eta, \chi) + \frac{1}{2} \log q + O(1) \leq \log q + O(\log(2/\eta) + \log \log(3q)).$$

Take  $\eta = (\log q)^{-2}$  and invert. For the finitely many smaller conductors, nonvanishing at 1 gives a smaller effective positive  $c$ .  $\square$

Tao [6] asks for explicit bounds on a hypothetical tenth class-number-one discriminant obtained from arguments excluding Siegel zeros. The determinant above gives the following result.

**Corollary 2** (Tenth-discriminant bound). *Let  $K$  be an imaginary quadratic field and put  $D = |\text{disc } K|$ . Then*

$$h(K) = 1 \implies D < 8 \cdot 10^9. \quad (4)$$

Here  $D$  is the absolute field discriminant; the fields  $\mathbb{Q}(i)$  and  $\mathbb{Q}(\sqrt{-2})$  have  $D = 4$  and  $D = 8$ , respectively.

*Proof.* The class-number-one hypothesis supplies the required primes directly. Every odd prime  $p < D/4$  is inert: otherwise a prime ideal of norm  $p$  would be principal, giving an algebraic integer of norm  $p$ , hence  $4p = u^2 + Dv^2$  with  $u, v \in \mathbb{Z}$ . The inequality  $4p < D$  forces  $v = 0$ , which would make  $p$  a square.

Suppose  $D \geq 8 \cdot 10^9$ . Bertrand's theorem gives an odd prime  $D/8 < R < D/4$ , so  $R \geq 10^9$  and every odd prime at most  $R$  is inert. Put  $S = \lfloor R/27 \rfloor$ ,  $a = (R/8)^{1/4}$ ,  $V = \lceil a \rceil$  and  $U = \lceil R/V \rceil$ . The cutoff 27 balances the  $1/c$  coefficient in the size bound for  $S = \lfloor R/c \rfloor$  against the small-prime loss. We have  $a \geq 100$  and

$$U \leq 8a^3 + 1, \quad V \leq a + 1, \quad D \leq 64a^4.$$

Moreover,  $UV \geq R$  and  $U, V \leq R$ . Choose  $R$  points from the centered  $U \times V$  integer rectangle. The coordinate ranges have lengths at most  $R$ , so the points remain distinct modulo  $R$ . Both conjugates have modulus at most

$$B = 5.7a^3, \quad \log B \leq \frac{3}{4} \log R + 0.182.$$

Indeed, the displayed bounds give  $4|u + v\sqrt{-D}|^2 \leq U^2 + DV^2 \leq 4(5.7a^3)^2$ . Use every derivative order  $j < S$  at each point and apply the Frobenius congruence (2) with  $d = -D$ . Since  $S < R$ , the same root count proves that the resulting square determinant  $\Delta$  is nonzero. Write  $N = |\mathcal{N} \Delta|$ .

The total derivative order over the columns equals the total  $w$ -degree over the rows:

$$\sum_{\text{columns}} j = R \sum_{j=0}^{S-1} j = \frac{RS(S-1)}{2} = \sum_{r < R, s < S} s.$$

Hence the exponent of  $B$  in each determinant term is

$$\sum_{r < R, s < S} (r + s) - \sum_{\text{columns}} j = \frac{RS(R-1)}{2}.$$

The size bound is therefore

$$\log \mathcal{N} \leq 2 \log((RS)!) + RS(R-1) \log B + RS(S-1) \log R.$$

At each odd prime  $p \leq R$ , the row change gives  $p^{2E_p} \mid \mathcal{N}$ , with

$$E_p = \sum_{k=1}^{S-1} (S-k) \max(R-kp, 0).$$

For  $p < 27$  this is an arithmetic sum, since  $pS \leq R$ . For  $p \geq 27$ , Legendre's formula for the prime valuations of  $\prod_{r < R} r!$  supplies the lower bound. With  $t = \log R$  and  $\ell = \log \mathcal{N} / (R^2 S)$ , Appendix A derives the estimates

$$t - \frac{2103}{500} \leq \ell \leq \frac{85}{108}t + \frac{23}{125}. \quad (5)$$

Comparing the bounds gives

$$\frac{23}{108}t \leq \frac{439}{100}.$$

Yet  $R \geq 10^9$  gives  $t \geq 207/10$ , so

$$\frac{23}{108}t \geq \frac{529}{120} > \frac{439}{100},$$

a contradiction. □

The bound is small enough to finish by a finite residue sieve. For  $D > 8$ , class number one also forces  $D \equiv 3 \pmod{8}$ . Testing inertness at odd primes  $p \leq 251$  with  $4p < D$  leaves only  $D = 11, 19, 43, 67, 163$  in  $8 < D < 8 \cdot 10^9$ . Together with the small discriminants and the converse from Minkowski's bound, this gives

$$h(K) = 1 \iff D \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}.$$

## References

- [1] OpenAI, *Uniform exclusion of Landau–Siegel zeros*, OpenAI Math Release, 1 October 2026. <https://github.com/openai/math/blob/main/preprints/Uniform-exclusion-of-Landau-Siegel-zeros-October-1-2026/paper.pdf>.
- [2] B. Kerr, *A simpler proof that there are no Landau–Siegel zeros*, arXiv:2610.21099v1, 2026. <https://arxiv.org/abs/2610.21099v1>.
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- [6] T. Tao, *An experimental AI-math challenge: using tenth discriminant bounds to explore arguments ruling out Siegel zeroes*, What's New, 20 October 2026. <https://terrytao.wordpress.com/2026/10/20/an-experimental-ai-math-challenge-using-tenth-discriminant-bounds-to-explore-arguments-ruling-out-siegel-zeroes/>.

## A Full derivation of Corollary 2

We derive (5). Throughout this appendix,  $R \geq 10^9$ ,  $S = \lfloor R/27 \rfloor$ ,  $t = \log R$  and  $\ell = \log \mathcal{N}/(R^2 S)$ , where  $\mathcal{N}$  is the determinant norm in the proof of Corollary 2.

*Upper bound.* The inequality  $\log u \leq u - 1$  gives  $t \leq R/10^9 + 20$ , hence  $t/R \leq 21/10^9$ . Since  $S \leq R/27$  and  $\log((RS)!) \leq RS \log(RS) \leq 2RS t$ , the determinant size bound gives

$$\ell \leq \log B + \frac{t}{27} + \frac{4t}{R} \leq \frac{85}{108}t + 0.182 + \frac{84}{10^9} < \frac{85}{108}t + \frac{23}{125}.$$

*Lower bound.* For  $p \leq 23$ , we have  $pS \leq R$ , so summing the formula for  $E_p$  gives

$$E_p = S(S-1) \left( \frac{R}{2} - \frac{p(S+1)}{6} \right).$$

For  $p \geq 29$ , put  $A_p = \sum_{r < R} \lfloor r/p \rfloor$ . The loss from truncating the  $w$ -degree satisfies

$$0 \leq SA_p - E_p \leq \frac{1}{2} \sum_{r < R} \left( \lfloor r/p \rfloor^2 + \lfloor r/p \rfloor \right) \leq \frac{R^3}{6p^2} + \frac{R^2}{4p}.$$

To sum  $A_p \log p$ , use Legendre's identity

$$F := \sum_{r < R} \log(r!) = \sum_{p \leq R} \log p \sum_{r < R} \sum_{\nu \geq 1} \left\lfloor \frac{r}{p^\nu} \right\rfloor.$$

Integral comparison, using  $\log(r!) \geq r \log r - r$  for  $r \geq 1$ , gives

$$F \geq \frac{R^2}{2}t - \frac{3R^2}{4} - Rt.$$

We use the following prime sums:

$$\begin{aligned} C_0 &:= \sum_{p \leq 23} \frac{\log p}{p-1} \leq 2.906, & A &:= \sum_{3 \leq p \leq 23} \log p \in [18.529, 20], \\ C_1 &:= \sum_{29 \leq p \leq R} \frac{\log p}{p(p-1)} \leq 0.037209, & W &:= \sum_{3 \leq p \leq 23} p \log p \leq 260.911, \\ C_2 &:= \sum_{29 \leq p \leq R} \frac{\log p}{p^2} \leq 0.036547. \end{aligned}$$

For  $C_1$  and  $C_2$ , sum over primes up to 1000 and bound the tails by partial summation from the elementary Chebyshev estimate  $\vartheta(y) := \sum_{p \leq y} \log p \leq 2(\log 2)y$ :

$$\sum_{p > 1000} \frac{\log p}{p^2} \leq \frac{4 \log 2}{1000}, \quad \sum_{p > 1000} \frac{\log p}{p(p-1)} \leq \frac{4 \log 2}{999}.$$

All the finite numerical bounds above follow from rational bounds for the logarithms. Write  $p = 2^b y$  with  $1 \leq y < 2$ , and use

$$0 \leq \log y - 2 \sum_{j=0}^{11} \frac{z^{2j+1}}{2j+1} \leq \frac{2z^{25}}{25(1-z^2)}, \quad z = \frac{y-1}{y+1},$$

with the same formula at  $y = 2$  for  $\log 2$ .

In Legendre's identity, all powers of primes  $p \leq 23$  contribute at most  $R^2 C_0/2$ , and powers  $p^\nu$  with  $p \geq 29$ ,  $\nu \geq 2$  contribute at most  $R^2 C_1/2$ . Thus

$$\sum_{29 \leq p \leq R} A_p \log p \geq F - \frac{R^2}{2}(C_0 + C_1).$$

Also,  $\sum_{p \leq R} \lfloor R/p \rfloor \log p \leq \log(R!)$  and the same Chebyshev estimate give

$$\sum_{p \leq R} \frac{\log p}{p} \leq \frac{\log(R!) + \vartheta(R)}{R} \leq t + 2 \log 2 < t + 20.$$

Combining the small- and large-prime contributions to  $\log \mathcal{N} \geq 2 \sum_{3 \leq p \leq R} E_p \log p$  now gives

$$\begin{aligned} \ell \geq t - \frac{3}{2} - \frac{2t}{R} - C_0 - C_1 - \frac{R}{3S} C_2 - \frac{t+20}{2S} \\ + \frac{S-1}{R} A - \frac{S^2-1}{3R^2} W. \end{aligned}$$

Since  $R/27 - 1 < S \leq R/27$ , the rounding terms satisfy

$$\begin{aligned} \varepsilon &:= \frac{2t}{R} + \left( \frac{R}{3S} - 9 \right) C_2 + \frac{t+20}{2S} + \frac{2A}{R} \\ &\leq \frac{82}{10^9} + \frac{243 \cdot 0.036547}{10^9 - 27} + \frac{1107}{2(10^9 - 27)} < 10^{-6}. \end{aligned}$$

Consequently,

$$\begin{aligned} \ell &\geq t - \left( \frac{3}{2} + C_0 + C_1 + 9C_2 - \frac{A}{27} + \frac{W}{2187} + \varepsilon \right) \\ &> t - \left( \frac{3}{2} + 2.906 + 0.037209 + 9 \cdot 0.036547 - \frac{18.529}{27} + \frac{260.911}{2187} + 10^{-6} \right) \\ &> t - \frac{2103}{500}. \end{aligned}$$