

A Frobenius determinant proof of the non-existence of Landau–Siegel zeros

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Abstract

We give a variant of the OpenAI proof [1] of the non-existence of Landau–Siegel zeros, following the simplifications of Kerr [2], Green [3] and Barreto [4]. The Frobenius congruence proves both divisibility and nonvanishing of the determinant. At a large inert prime supplied by the hypothetical zero, it reduces nonvanishing to a polynomial root count, replacing resultants and Wronskians. The argument was discovered in discussions with GPT-6 Astra and Opus 5.5 and the new arguments should be attributed to these models.

Theorem (OpenAI [1]). *There is an absolute $c > 0$ such that every real zero $\beta \in (0, 1)$ of a primitive nonprincipal real Dirichlet L -function of conductor $q \geq 3$ satisfies $(1 - \beta) \log q \geq c$.*

Proof. Suppose the theorem is false. For every integer $k \geq 4$, choose such a character and zero with $(1 - \beta) \log q \leq k^{-2}$. Choose a squarefree $d \neq 1$ with $|d| \leq q$ such that $\chi(p) = (d/p)$ for odd primes p . We call a prime inert when $\chi(p) = -1$. All these quantities may depend on k ; all limits are as $k \rightarrow \infty$.

Put $\delta = 1 - \beta$, $Y = q^k$ and $t = 1/\log Y$. Then $\log q = o(\log Y)$ and $\delta \log Y \leq 1/k$. The Euler products and the Hadamard formula [5, Cor. 10.18] give

$$\frac{2}{e} \sum_{\substack{p \leq Y \\ \chi(p)=1}} \frac{\log p}{p} \leq -\frac{\zeta'}{\zeta}(1+t) - \frac{L'}{L}(1+t, \chi) \leq \frac{1}{t} - \frac{1}{t+\delta} + \frac{1}{2} \log q + O(1) = o(\log Y).$$

Here $t^{-1} - (t+\delta)^{-1} \leq \delta t^{-2} = o(\log Y)$. Mertens' estimate and $\sum_{p|q} (\log p)/p \leq \log q$ now imply

$$\sum_{\substack{p \leq Y \\ \chi(p)=-1}} \frac{\log p}{p} \sim \log Y.$$

For sufficiently large k , let p_0 be the largest inert prime at most Y . The preceding sum is unchanged when Y is replaced by p_0 , so Mertens' estimate gives $\log p_0 \sim \log Y$. Set

$$m = \lfloor p_0/k^2 \rfloor, \quad R = k^2 m, \quad S = km, \quad T = S - m, \quad N = RS.$$

Since $\log p_0 \sim k \log q$ and $q \geq 3$, we have $k^2 = o(p_0)$, hence $R \sim p_0$ and $\log R \sim \log Y$. Since $p_0/S \sim k$, Mertens' estimate shows that omitting primes outside $[k^2, S]$ costs $O(\log k)$. Thus

$$\sum_{\substack{k^2 \leq p \leq S \\ \chi(p)=-1}} \frac{\log p}{p} \sim \log R.$$

For every odd inert prime p , the binomial theorem, Fermat's little theorem and Euler's criterion give

$$(u + v\sqrt{d})^p \equiv u + v d^{(p-1)/2} \sqrt{d} \equiv u - v\sqrt{d} \pmod{p\mathbb{Z}[\sqrt{d}]} \quad (u, v \in \mathbb{Z}). \quad (1)$$

Let $M = \lceil \sqrt{p_0} \rceil$, and form the matrix

$$\mathcal{A}_{(r,s),(u,v,j)} = (\partial_z^j z^r w^s)(u + v\sqrt{d}, u - v\sqrt{d}),$$

with rows indexed by $0 \leq r < R$, $0 \leq s < S$, and columns by $0 \leq u, v < M$, $0 \leq j < S$, all indices integral. We claim that $\mathcal{A}$ has row rank N modulo p_0 . In $K = \mathbb{Z}[\sqrt{d}]/p_0\mathbb{Z}[\sqrt{d}] \cong \mathbb{F}_{p_0^2}$, the $M^2 \geq p_0$ residues $u + v\sqrt{d}$ are distinct because $M \leq p_0$ and $1, \sqrt{d}$ form an $\mathbb{F}_{p_0}$ -basis.

By (1), a nontrivial row dependence would give a polynomial

$$G(z) = \sum_{r < R, s < S} c_{rs} z^{r+p_0s} \in K[z]$$

whose derivatives of orders $0, \dots, S-1$ vanish at all these residues: in characteristic p_0 , $(z^{p_0})' = 0$. Since $R \leq p_0$, the exponents $r + p_0s$ are distinct, so $G \neq 0$ and $\deg G \leq R-1 + p_0(S-1) < p_0S$. Since $S < p_0$, the factorials $j!$ for $j < S$ are invertible in K . Thus G has at least p_0 distinct roots of multiplicity at least S , contradicting its degree bound. Choose an $N \times N$ minor $\Delta \in \mathbb{Z}[\sqrt{d}]$ nonzero modulo p_0 ; it uses every row. Set $D = |\mathcal{N}_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}} \Delta| \in \mathbb{Z}_{>0}$.

For each odd inert prime $k^2 \leq p \leq S$, replace the rows with $s < T$ by

$$z^r w^s \mapsto z^{r-ph} (z^p - w)^h w^s, \quad h = \lfloor r/p \rfloor.$$

Here $h < m$, so $s + h < S$, and all added monomials have smaller z -degree. Ordered by increasing r , this is a unitriangular row transformation and therefore preserves Δ . By (1), $z^p - w \equiv 0 \pmod{p}$ at every grid point; its positive z -derivatives have coefficients divisible by p . By Leibniz's rule, every entry of the new row is divisible by p^h . Taking norms gives

$$p^{2T \sum_{r < R} \lfloor r/p \rfloor} \mid D.$$

The estimate $\sum_{r < R} \lfloor r/p \rfloor = R^2/(2p) + O(R)$ gives, uniformly for $p \leq S = R/k$,

$$\sum_{r < R} \lfloor r/p \rfloor \geq (1 - O(k^{-1})) \frac{R^2}{2p}.$$

Summing over these primes and using $T/S \rightarrow 1$ gives

$$\log D \geq (1 - O(k^{-1})) T R^2 \sum_{\substack{k^2 \leq p \leq S \\ \chi(p) = -1}} \frac{\log p}{p} = (1 - o(1)) N R \log R. \quad (2)$$

Under either embedding, the grid coordinates have modulus at most $B = 2M\sqrt{q}$, and

$$\log B = \frac{1}{2} \log p_0 + \frac{1}{2} \log q + O(1) = \left(\frac{1}{2} + o(1)\right) \log R.$$

Since $j < S$, each entry in row (r, s) has modulus at most $R^S B^{r+s}$. Using $\sum_{r < R, s < S} (r+s) = N(R+S-2)/2$ and expanding the two conjugate determinants gives

$$\begin{aligned} \log D &\leq 2 \log(N!) + 2SN \log R + N(R+S) \log B \\ &\leq \left(\frac{1}{2} + o(1)\right) N R \log R, \end{aligned}$$

since $S/R = 1/k \rightarrow 0$ and $\log(N!) = o(NR \log R)$. This contradicts (2). $\square$

References

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