

Classical regularity and attainment in portfolio choice with nonhomothetic utility

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Abstract

We study classical regularity and attainment in terminal-utility portfolio choice with unspanned factors, nonnegative wealth, and exposures that are square-integrable in expectation. A joint estimate for logarithmic marginal utility controls relative risk aversion and factor sensitivity. In several dimensions, a risk-price component uniformly separated from the factor-hedging span gives classical solvability and a unique strong optimal portfolio on every finite horizon. A scalar comparison equation treats correlated single-factor markets. For state-independent C^4 utility, the preference assumptions reduce to positive lower and upper bounds on relative risk aversion. In an orthogonal single-factor market, a uniform finite-order condition allows the risk price to vanish and change sign. Subelliptic gradient estimates and a parabolic equation in the factor variable yield a classical value; verification and strict concavity give strong existence for its continuous feedback. In both settings, the optimal dual coefficient has a deterministic bound uniform over initial states and multipliers. Attainment has a separate integrability boundary. For deterministic opportunities with a finite static optimum, the prescribed strategy class attains the value exactly when the optimal payoff lies in L^2 under the physical probability. An explicit nonhomothetic utility gives a horizon and wealth classification with smooth finite values and nonattainment. Further results identify wealth-dependent purchasing-power hedges and a strict increase in the dual value under factor-information restrictions.

Keywords. Nonhomothetic utility; stochastic factors; classical regularity; degenerate Hamilton–Jacobi–Bellman equations; convex duality; square-integrable strategies.

1 Introduction

A changing investment opportunity set affects risky-asset demand through both current compensation for risk and future changes in investment opportunities. Merton’s continuous-time portfolio theory distinguishes myopic demand from intertemporal hedging demand [1, 2]. When relative risk aversion is constant, homogeneity often eliminates the wealth variable. Under general preferences, risk aversion varies with wealth, marginal wealth value interacts with factor states, and classical feedback requires joint wealth and factor derivatives. Terminal utility may also depend on the economic state. For example,

$$U(x, y) = U_0(xe^{-g(y)})$$

values terminal purchasing power against a factor-dependent price index. The role of this risk in dynamic asset allocation is studied by Brennan and Xia [3]; a martingale formulation of state-dependent preferences is given by Londoño [4]. Our factor is observable and exogenous, and the objective is ordinary expected terminal utility.

The regularity and verification question for stochastic-factor investment is formulated explicitly by Zariphopoulou [5, Problem 3]. We study classical solvability and attainment under the nonnegative-wealth constraint and square integrability in expectation of traded exposure. Two effects distinguish these questions. Factor noise can preserve classical regularity when the market

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price of risk vanishes. Conversely, the inverse marginal utility may produce a payoff outside $L^2(\mathbb{P})$ even when the value is smooth and the dual density has bounded coefficients. Our results give sufficient conditions for joint regularity and attainment, together with exact attainment and regularity criteria in specified submodels.

Let $A_V = -xV_{xx}/V_x$ denote indirect relative risk aversion, and let $s = \nabla_y \log V_x$ denote the factor sensitivity of marginal wealth value. In exposure coordinates, the optimal proportional position decomposes as

$$\frac{\vartheta^*}{x} = \frac{\lambda}{A_V} + \frac{C^\top s}{A_V}. \quad (1.1)$$

The matrix C records instantaneous covariances between factor innovations and traded risks. The second term lies in $\text{Ran } C^\top$. If the component of the market price of risk λ in the orthogonal complement of this space is uniformly separated from zero, hedging demand cannot offset the entire risky exposure. We prove that this primitive condition preserves parabolicity of the marginal equation and gives a classical solution on every finite horizon for several assets, several factors, and state-dependent utility. Correlated single-asset markets are treated by a separate scalar coefficient condition.

In an orthogonal single-factor market, we also consider smooth risk prices satisfying

$$\inf_y \sum_{j=0}^m |\lambda^{(j)}(y)|^2 > 0.$$

This condition permits λ to change sign and have zeros of any prescribed finite order. Starting from nondegenerate auxiliary markets, uniform Harnack estimates under commutator rescaling give compactness of the marginal gradients. An ordinary parabolic equation in the factor variable then recovers the classical derivatives of the value function. For the resulting continuous feedback, verification and strict concavity imply pathwise uniqueness, followed by strong existence on the original Brownian probability space.

All solvability theorems retain the original admissible class. Optimal wealth is strictly positive, proportional exposure is bounded, and the dual density generated by marginal utility has a bounded orthogonal Brownian coefficient. A bound determined by the data and horizon applies simultaneously to all initial wealth levels, factor states, and dual multipliers. In contrast, an exact criterion for deterministic opportunities shows that attainment in the original square-integrable control class is equivalent to the static optimal payoff belonging to $L^2(\mathbb{P})$. We construct a nonhomothetic utility family for which attainment depends on the horizon and wealth within a region of finite smooth values.

The financial implications concern two wealth effects. Purchasing-power preferences give the short-horizon hedging component

$$(1 - A_{V_0}^{-1}) C^\top \nabla g.$$

A utility mixture whose relative risk aversion crosses one can give this component opposite directions at low and high wealth. A second effect concerns the information used by the optimal dual control. Even with orthogonal factor and traded innovations, factor information alone cannot generate the optimal dual control for a finite mixture of distinct powers. The corresponding restricted dual value is strictly larger.

1.1 Relation to existing research

Utility maximization and convex duality in general semimartingale markets were developed by Kramkov and Schachermayer [6, 7]. Under their assumptions, including bounded relative risk aversion and σ -boundedness under the optimal numéraire, Kramkov and Sîrbu [8, Theorem 1] prove twice differentiability of primal and dual value functions with respect to initial values, inheritance of risk-aversion bounds, and initial-value derivatives of the optimizers. Building on

this wealth-regularity background, we study joint classical derivatives in the additional factor directions and derive a deterministic bound on the orthogonal coefficient of the optimal density from Markov coefficients.

Nadtochiy and Zariphopoulou [9] use a closely related marginal-HJB approach. In their complete author version, Assumptions 1–2 and Theorem 6.1 give convergence of a splitting approximation to the viscosity solution of the marginal equation. Theorems 6.2–6.3 treat classical solutions with positive regularization, and Theorem 6.6 gives performance estimates for the corresponding approximate portfolios. Their utility assumptions include comparison of marginal utility with a single power on the whole positive wealth axis. Our power mixtures may have different limiting power exponents, whereas our market assumptions impose stronger coefficient smoothness and additional risk-price geometry. Neither set of data conditions simply contains the other.

Bian, Miao, and Zheng [10, Theorems 3.8, 4.1, and 5.5] construct classical values for nonsmooth utility with deterministic time-dependent coefficients and cone constraints, and establish verification under their moment and continuity conditions. In the positive-wealth setting, Horst et al. [11, Theorems 4.1–4.2] respectively characterize an existing optimum by a forward–backward system and verify an adapted solution under integrability conditions. Mania and Tevzadze [12, Theorems 3.1–3.2] establish a backward stochastic partial differential equation and verification under semimartingale regularity assumptions on the value field. We construct a joint classical value directly for specified stochastic-factor markets, retaining square integrability in expectation of traded noise exposure as the admissibility requirement throughout.

The analytic tools are Lunardi’s local theorem for fully nonlinear evolution equations and Schauder estimates [13], together with the scalar divergence-form interior regularity of Aronson and Serrin [14, 15]. The finite-order degenerate case uses the parameter-uniform metric-measure estimates and Harnack inequality of Capogna, Citti, and Rea [16]. Intrinsic Schauder theory for nondivergence operators of Hörmander type is developed by Bramanti and Brandolini [17]. At the degenerate set, we obtain ordinary classical derivatives from first derivatives of the marginal function and the factor equation for the value. The passage from weak to strong existence uses the Yamada–Watanabe theorem [18]. The hypotheses of these results are verified in the corresponding proofs.

Section 2 states the model and main theorems. Sections 3–4 establish the uniform estimates and nondegenerate classical construction. Section 5 proves verification, convex duality, and approximation of utility, and Section 6 treats finite-order degeneracy. Section 7 studies dual information and purchasing-power hedges. Section 8 gives the classification of square-integrable attainment. The appendices contain preference examples, relative-curvature estimates, deterministic-factor regularity, and extensions in time and factor coordinates.

2 The market, preferences, and main results

2.1 Securities and the admissible class

Let W and B be independent standard Brownian motions of dimensions n and d , respectively, on their augmented natural filtration. Fix a constant matrix

$$C \in \mathbb{R}^{d \times n}, \quad c = \|C\|_{\text{op}} < 1, \quad R = I_d - CC^\top, \quad D = R^{1/2}. \quad (2.1)$$

The factor and discounted wealth satisfy

$$\begin{aligned} dY_t &= \beta(Y_t) dt + C dW_t + D dB_t, & Y_0 &= y, \\ dX_t &= \vartheta_t^\top \{\lambda(Y_t) dt + dW_t\}, & X_0 &= x > 0. \end{aligned} \quad (2.2)$$

The total instantaneous factor covariance is I_d , and R is the factor covariance left unspanned by traded risks. If securities have excess drift $\mu - r\mathbf{1}$ and nonsingular volatility matrix Σ , then

$$\lambda = \Sigma^{-1}(\mu - r\mathbf{1}), \quad \vartheta = \Sigma^\top \pi,$$

where π is the vector of discounted amounts held in risky assets. The integrability condition is therefore imposed on $\Sigma^\top \pi$.

Definition 2.1. For given (x, y, T) , let $\mathcal{A}_T(x, y)$ consist of all $\mathbb{R}^n$ -valued processes ϑ that are predictable on the full filtration and satisfy

$$\mathbb{E} \int_0^T |\vartheta_t|^2 dt < \infty, \quad X_t \geq 0 \quad (0 \leq t \leq T) \quad \mathbb{P}\text{-a.s.} \quad (2.3)$$

For terminal preferences $U(x, y)$ and remaining horizon τ , define

$$V(x, y, \tau) = \sup_{\vartheta \in \mathcal{A}_\tau(x, y)} \mathbb{E}_{x, y} U(X_\tau, Y_\tau). \quad (2.4)$$

Utility at $x = 0$ is defined by its right limit, which may equal $-\infty$.

For the preference classes below, utility has an integrable positive part under every admissible strategy; its negative part may have infinite expectation. In this paper, nonhomothetic utility means that relative risk aversion depends on wealth. State dependence of utility is an additional, separate feature. In calendar time, $V(x, y, t; T) = V(x, y, T - t)$. Utility is applied to discounted wealth; at a fixed horizon, a nominal-wealth objective is obtained by replacing $U(x, y)$ with $U(e^{rT}x, y)$.

2.2 Two classes of preferences

For noninteger $j + \alpha$, the space $C_b^{j+\alpha}$ consists of functions with bounded derivatives through order j and uniformly α -Hölder derivatives of order j .

Assumption 2.2 (Smooth state-dependent preferences). Fix $0 < \alpha < 1$. The market coefficients β, λ belong to the appropriate vector-valued spaces $C_b^{4+\alpha}$. Utility satisfies $U \in C^6((0, \infty) \times \mathbb{R}^d)$, $U_x > 0$, and $U_{xx} < 0$. Set

$$h_0(\ell, y) = \log U_x(e^\ell, y).$$

Assume $\nabla_{\ell, y} h_0 \in C_b^{4+\alpha}$, $h_0(0, \cdot)$ is bounded, and $U(1, \cdot) \in C_b^{4+\alpha}$. There are constants $0 < a_- \leq a_+ < \infty$ such that

$$-a_+ \leq \partial_\ell h_0 \leq -a_- < 0. \quad (2.5)$$

The function h_0 itself may grow linearly in ℓ . Terminal relative risk aversion is

$$A_U(x, y) = -\frac{x U_{xx}(x, y)}{U_x(x, y)},$$

and condition (2.5) gives it positive lower and upper bounds uniformly over wealth and the factor state.

Assumption 2.3 (State-independent C^4 preferences). Utility depends only on wealth and satisfies $U \in C^4(0, \infty)$, $U' > 0$, $U'' < 0$, and

$$0 < a_- \leq -\frac{x U''(x)}{U'(x)} \leq a_+ < \infty. \quad (2.6)$$

Both preference classes satisfy the Inada limits. Assumption 2.3 imposes local smoothness on higher derivatives. Market coefficient conditions are stated separately when that assumption is used. We write C_b^∞ for smooth functions whose derivatives of every order, including order zero, are bounded.

Let Π be the orthogonal projection onto $\ker C$. The principal market separation condition is

$$\delta = \inf_{y \in \mathbb{R}^d} |\Pi \lambda(y)| > 0. \quad (2.7)$$

Since $\text{Ran } C^\top$ is the orthogonal complement of $\ker C$, this condition requires a uniformly positive distance between λ and the factor-hedging span.

The following quantities depend only on primitive data. Denote spatial Jacobians by J_λ and J_β , and set

$$M = \|\lambda\|_\infty, \quad L = \|J_\lambda\|_{\infty, \text{op}}, \quad B_1 = \|J_\beta\|_{\infty, \text{op}}, \quad S_0 = \|\nabla_y h_0\|_\infty, \\ m_a = \max_{a \in [a_-, a_+]} |1 - a^{-1}|, \quad \Gamma = 1 + 2L^2/a_-^2 + 2B_1 + 2cLm_a, \quad E_0 = (MLm_a)^2, \quad (2.8)$$

$$K(T)^2 = (a_+^2 + S_0^2)e^{\Gamma T} + \frac{E_0}{\Gamma}(e^{\Gamma T} - 1). \quad (2.9)$$

For state-independent preferences, $S_0 = 0$.

2.3 Two classical solvability theorems

Write $\tilde{U}(z, y) = \sup_{x>0} \{U(x, y) - zx\}$. For a bounded predictable $\mathbb{R}^d$ -valued process ν , define

$$L_t^\nu = \mathcal{E} \left(- \int_0^t \lambda(Y_u)^\top dW_u - \int_0^t \nu_u^\top dB_u \right). \quad (2.10)$$

Here $\mathcal{E}$ denotes the stochastic exponential. For a deterministic $N < \infty$, set

$$v_N(z, y, T) = \inf_{|\nu| \leq N} \mathbb{E}_y \tilde{U}(zL_T^\nu, Y_T), \quad v = \inf_{N < \infty} v_N, \\ \mathcal{D}_N(x, y, T) = \inf_{z>0} \{v_N(z, y, T) + zx\}, \quad \mathcal{D} = \inf_{N < \infty} \mathcal{D}_N. \quad (2.11)$$

The bound on the coefficient is an essential bound with respect to $dt \otimes d\mathbb{P}$.

Theorem 2.4 (Nondegenerate markets). *Under Assumption 2.2 and condition (2.7), the value function is, on every finite horizon, a classical solution of*

$$V_\tau = \frac{1}{2} \Delta_y V + \beta \cdot \nabla_y V - \frac{|\lambda V_x + C^\top \nabla_y V_x|^2}{2V_{xx}}, \quad V(x, y, 0) = U(x, y), \quad (2.12)$$

with $V_x > 0$ and $V_{xx} < 0$. Define

$$h(\ell, y, \tau) = \log V_x(e^\ell, y, \tau), \quad p = h_\ell, \quad s = \nabla_y h, \quad k = \lambda + C^\top s. \quad (2.13)$$

Then $-a_+ \leq p \leq -a_-$, $p^2 + |s|^2 \leq K(T)^2$, and $|k| \geq \delta$. For some $\eta > 0$,

$$h - h_0 \in C([0, T]; bc^{4+\eta}), \quad h_\tau \in C([0, T]; bc^{2+\eta}). \quad (2.14)$$

The feedback

$$\vartheta^*(x, y, \tau) = - \frac{x k(\log x, y, \tau)}{p(\log x, y, \tau)}$$

generates the unique optimal wealth process on the given Brownian probability space. This process is strictly positive, and its exposure satisfies (2.3). The fixed-multiplier and outer dual problems are uniquely attained, with optimal coefficient

$$\nu_t^* = -D^\top s(\log X_t^*, Y_t, T - t), \quad |\nu_t^*| \leq \|D\|_{\text{op}} K(T). \quad (2.15)$$

If $N \geq \|D\|_{\text{op}} K(T)$, then $v_N = v$ for all multipliers and initial factor states, and $\mathcal{D}_N = \mathcal{D} = V$.

If preferences instead satisfy Assumption 2.3, while $\beta, \lambda \in C_b^{4+\alpha}$ and (2.7) are retained, classical regularity holds at positive remaining times and the terminal condition is attained locally uniformly. The gradient bounds, strong optimal portfolio, and dual conclusions continue to hold.

Classical regularity means continuity of V_τ , the first state derivatives, and the full state Hessian on the positive-wealth, positive-remaining-time domain. The equation holds there pointwise. The spaces bc^r are defined in Section 4. Uniqueness of controls means equality $dt \otimes d\mathbb{P}$ -almost everywhere, and uniqueness of wealth means indistinguishability. The high-order trace (2.14) applies to smooth preferences; the state-independent C^4 case has positive-time regularity and the original terminal trace. Section 4.4 gives alternative conditions for correlated single-factor markets.

Theorem 2.5 (Finite-order risk-price degeneracy). *Suppose $n = d = 1$, $C = 0$, and $\beta, \lambda \in C_b^\infty(\mathbb{R})$. For some integer $m \geq 1$, assume*

$$\inf_y \sum_{j=0}^m |\lambda^{(j)}(y)|^2 > 0. \quad (2.16)$$

For every utility satisfying Assumption 2.3, the value function has continuous derivatives

$$V_x, \quad V_{xx}, \quad V_y, \quad V_{xy}, \quad V_{yy}, \quad V_\tau,$$

on $(0, \infty) \times \mathbb{R} \times (0, \infty)$, satisfies (2.12) classically, and converges locally uniformly to U at the terminal time. Its marginal quantities h, p, s are continuous, with $p = h_\ell$ and $s = h_y$, and satisfy the same gradient bounds. The closed-loop system with feedback $\vartheta^* = -X^* \lambda(Y)/p$ has a unique strong solution on the given probability space. Wealth is strictly positive and the strategy belongs to (2.3). The optimal dual coefficient is $\nu^* = -s$, with $|\nu^*| \leq K(T)$. Consequently, every $N \geq K(T)$ gives equality of truncated and untruncated dual values and uniqueness of the optimizers.

Condition (2.16) allows the risk price to change sign. Its zeros concern the risk price; factor volatility remains equal to one. The estimates common to the two main theorems are established in Section 3. Theorem 2.4 is proved in Sections 4–5, and Theorem 2.5 in Section 6.

2.4 Attainment under the exposure integrability constraint

The next result concerns deterministic investment opportunities. It uses a utility class without a positive lower bound on relative risk aversion.

Theorem 2.6 (Attainment for deterministic opportunities). *Consider one traded Brownian risk with a bounded deterministic market price of risk c_t on $[0, T]$. The filtration may also contain an independent factor Brownian motion. Let U be a smooth, strictly increasing, strictly concave utility on $(0, \infty)$ with the Inada limits, and put*

$$Z_t = \mathcal{E} \left(- \int_0^t c_u dW_u \right), \quad I = (U')^{-1}.$$

For the prescribed $x > 0$, suppose a multiplier $z > 0$ satisfies

$$H = I(zZ_T), \quad \mathbb{E}[Z_T H] = x, \quad \mathbb{E}|U(H)| + \mathbb{E}|\tilde{U}(zZ_T)| < \infty. \quad (2.17)$$

The value in the admissible class (2.3) equals $\mathbb{E}U(H)$ and is attained if and only if

$$H \in L^2(\mathbb{P}). \quad (2.18)$$

When attained, the optimal terminal wealth, wealth process, and exposure are unique. Otherwise, admissible portfolios replicating bounded strictly positive terminal payoffs approach the value.

Theorem 2.6 is proved in Section 8. Theorem 8.2 gives an explicit utility for which this criterion separates attainment from nonattainment within a region where the value is finite and smooth. Thus the exposure integrability condition remains relevant after the classical HJB equation has been solved.

2.5 Scope and portfolio implications

Condition (2.7) requires $\text{rank } C < n$. If C has full column rank, then $\ker C = \{0\}$ and the condition cannot hold. A concrete example has $n = d + 1$ and

$$C = (\text{diag}(\rho_1, \dots, \rho_d) \ 0), \quad \lambda(y) = (\lambda_1(y), \dots, \lambda_d(y), \epsilon)^\top, \quad \epsilon > 0.$$

If $\max_i |\rho_i| < 1$, then $|\Pi\lambda| \geq \epsilon$. The first d risk prices may change sign, and the drift may be chosen as $\beta_i = -b_i \tanh y_i$. The final traded direction is risky and earns a risk premium, while its instantaneous covariance with factor innovations is zero. The unspanned factor covariance remains positive definite.

Projecting (1.1) gives

$$\left| \Pi \frac{\vartheta^*}{x} \right| \geq \frac{\delta}{a_+}, \quad \frac{\vartheta^*}{x} \in \text{span}(\{\lambda(y)\} \cup \text{Ran } C^\top).$$

Thus at most $1 + \text{rank } C$ risky funds span the optimal position state by state. The funds themselves may depend on the factor state; their holdings also depend on preferences and wealth. The optimal ν^* specifies a supporting martingale measure for a particular investor and initial state.

The nondegeneracy assumptions are sufficient conditions. For example, if $C = 0$, $\lambda = 0$, and utility is state-independent, square-integrable wealth is a martingale. Jensen's inequality gives $V = U$, attained by zero exposure. The bounded smooth drift assumptions exclude unbounded Ornstein–Uhlenbeck drifts such as $\beta(y) = -by$. Multidimensional factor covariance is constant in the basic model; the coordinate extension in the appendix treats suitable one-dimensional nonconstant diffusions.

State-dependent preferences also have a specified range. If $U_x(x, y) = x^{-\gamma(y)}$ with nonconstant γ , then $\nabla_y h_0 = -\ell \nabla \gamma(y)$ is generally unbounded, so Assumption 2.2 fails. The purchasing-power transformation with bounded smooth g satisfies a different set of verifiable conditions, given in Section 7.3. The power mixtures allow risk aversion to remain above one, remain below one, or cross one. Horizons may be arbitrarily large and finite, although constants generally depend on the horizon. Horizon-independent bounds are stated separately for scalar markets.

3 The marginal equation and uniform a priori estimates

3.1 The logarithmic marginal transformation and principal matrix

We first derive the equation formally in the classical setting, then solve it as an independent Cauchy problem. Put $u = V_x$. Then

$$u_x = \frac{u}{x}p, \quad u_{xx} = \frac{u}{x^2}(p\ell + p^2 - p), \quad \nabla_y u = us, \quad \frac{|\lambda u + C^\top \nabla_y u|^2}{2u_x} = \frac{xu|k|^2}{2p}.$$

Differentiating (2.12) in x and dividing by u yields

$$h_\tau = \frac{|k|^2}{2p^2} h_{\ell\ell} - \frac{(Ck) \cdot \nabla_y h_\ell}{p} + \frac{1}{2} \Delta_y h + F(y, p, s), \quad h(0) = h_0, \quad (3.1)$$

where

$$A = \frac{1}{2} \begin{pmatrix} |k|^2/p^2 & -(Ck)^\top/p \\ -Ck/p & I_d \end{pmatrix}, \quad F = \beta \cdot s + \frac{1}{2}|s|^2 - \frac{1}{2}|k|^2(1 + p^{-1}). \quad (3.2)$$

Thus $h_\tau = A : D^2 h + F$. In the smooth-data construction, a classical branch is understood to have bounded $h - h_0$ and bounded spatial derivatives through the orders used below on each compact time interval of existence. The function h itself may grow linearly in ℓ . For $\xi = (\xi_0, \xi_y)$,

$$2\xi^\top A\xi = \left| \frac{k}{p}\xi_0 - C^\top \xi_y \right|^2 + |D^\top \xi_y|^2, \quad \det A = \frac{k^\top (I_n - C^\top C)k}{2^{d+1}p^2}. \quad (3.3)$$

The determinant follows from the Schur complement of the lower-right block. Since $\|C\|_{\text{op}} < 1$, for $p \neq 0$ the principal matrix is positive definite exactly when $k \neq 0$.

3.2 Exact equations for the gradients

Regard A, F as functions of (y, p, s) and define

$$\mathcal{L}_h = A : D^2 + b_0 \partial_\ell + b_y \cdot \nabla_y, \quad b_0 = A_p : D^2 h + F_p, \quad (b_y)_i = A_{s_i} : D^2 h + F_{s_i}.$$

The drifts are explicitly

$$\begin{aligned} b_0 &= -\frac{|k|^2}{p^3} p_\ell + \frac{(Ck) \cdot \nabla_y p}{p^2} + \frac{|k|^2}{2p^2}, \\ b_y &= \frac{Ck}{p^2} p_\ell - \frac{CC^\top}{p} \nabla_y p + \beta + s - (1 + p^{-1})Ck. \end{aligned} \quad (3.4)$$

Set $\lambda_i = \partial_{y_i} \lambda$ and $\beta_i = \partial_{y_i} \beta$. For a scalar function w , write

$$\mathcal{Q}_w = \frac{k}{p} w_\ell - C^\top \nabla_y w.$$

Lemma 3.1. *Every classical solution with $p < 0$ satisfies*

$$\begin{aligned} (\partial_\tau - \mathcal{L}_h)p &= 0, \\ (\partial_\tau - \mathcal{L}_h)s_i &= \frac{\lambda_i \cdot \mathcal{Q}_p}{p} + \beta_i \cdot s - (1 + p^{-1})k \cdot \lambda_i. \end{aligned} \quad (3.5)$$

Proof. The coefficients have no explicit dependence on ℓ , so all chain-rule terms from differentiation in ℓ enter the linearized drift. Differentiation in y_i also produces the explicit derivatives at fixed p, s ,

$$A_{y_i} : D^2 h = \frac{k \cdot \lambda_i}{p^2} p_\ell - \frac{(C\lambda_i) \cdot \nabla_y p}{p}, \quad F_{y_i} = \beta_i \cdot s - (1 + p^{-1})k \cdot \lambda_i.$$

Combining these expressions with $\partial_{y_i} p = \partial_\ell s_i$ proves both equations. $\square$

Proposition 3.2 (Risk-aversion interval and joint gradient bound). *On each compact time interval of a smooth-data classical branch,*

$$-a_+ \leq p \leq -a_-, \quad p^2 + |s|^2 \leq K(\tau)^2. \quad (3.6)$$

The constant in the second estimate uses neither a positive lower bound on $|k|$ nor higher-order derivative bounds for utility.

Proof. The first equation in (3.5) has no zeroth-order term, and the constants $-a_+$ and $-a_-$ are solutions. On an existing compact time strip the linearized coefficients are bounded. Add $\varepsilon e^{A\tau}(1 + \ell^2 + |y|^2)$, apply the maximum principle on expanding spatial balls, and let $\varepsilon \downarrow 0$. This preserves the interval. The barrier exponent A may depend on the drift bounds of that strip; the resulting interval bounds do not depend on A .

Set $\mathcal{R} = p^2 + |s|^2$. The product rule for the second-order operator and (3.3) give

$$\begin{aligned} (\partial_\tau - \mathcal{L}_h)\mathcal{R} = & -|\mathcal{Q}_p|^2 - |D^\top \nabla_y p|^2 - \sum_{i=1}^d (|\mathcal{Q}_{s_i}|^2 + |D^\top \nabla_y s_i|^2) \\ & + \frac{2}{p} (J_\lambda s) \cdot \mathcal{Q}_p + 2s^\top J_\beta^\top s - 2(1 + p^{-1})k \cdot (J_\lambda s). \end{aligned} \quad (3.7)$$

The Hessian term outside the negative squares satisfies

$$\frac{2}{|p|} |J_\lambda s| |\mathcal{Q}_p| \leq \frac{1}{2} |\mathcal{Q}_p|^2 + \frac{2L^2}{a_-^2} |s|^2.$$

Since $|k| \leq M + c|s|$, we also have

$$2|1 + p^{-1}| |k| |J_\lambda s| \leq 2MLm_a |s| + 2cLm_a |s|^2 \leq E_0 + (1 + 2cLm_a) |s|^2.$$

Discarding the remaining nonpositive terms gives

$$(\partial_\tau - \mathcal{L}_h)\mathcal{R} \leq \Gamma \mathcal{R} + E_0.$$

Initially $\mathcal{R} \leq a_+^2 + S_0^2$. Comparison with $r' = \Gamma r + E_0$ gives (2.9). Whole-space comparison uses the same localization as the interval argument. $\square$

3.3 Divergence form, function bounds, and a time modulus

Equation (3.1) has the exact divergence form

$$h_\tau = \operatorname{div} \mathcal{J} + F, \quad \mathcal{J} = \begin{pmatrix} -|k|^2/(2p) \\ s/2 \end{pmatrix}, \quad \mathbf{B} = D_{(p,s)} \mathcal{J} = \begin{pmatrix} |k|^2/(2p^2) & -(Ck)^\top/p \\ 0 & I_d/2 \end{pmatrix}. \quad (3.8)$$

The divergence acts on all (ℓ, y) coordinates. The matrix $\mathbf{B}$ is generally nonsymmetric; its symmetric part is $\mathbf{A}$.

Lemma 3.3. *The gradient components of the solution satisfy*

$$\begin{aligned} p_\tau - \operatorname{div}(\mathbf{B} \nabla p) &= \operatorname{div} \begin{pmatrix} F \\ 0 \end{pmatrix}, \\ (s_i)_\tau - \operatorname{div}(\mathbf{B} \nabla s_i) &= \operatorname{div} \begin{pmatrix} -k \cdot \lambda_i/p \\ F e_i \end{pmatrix}. \end{aligned} \quad (3.9)$$

Here e_i is the i th coordinate vector in $\mathbb{R}^d$.

Proof. Differentiate (3.8) in ℓ and y_i . Differentiation of the flux through its gradient variables gives $\mathbf{B} \nabla p$ or $\mathbf{B} \nabla s_i$. Its explicit y_i derivative at fixed gradient is $(-k \cdot \lambda_i/p, 0)^\top$. The remaining source terms are the displayed divergences. The calculation retains the full nonsymmetric flux Jacobian. $\square$

For smooth initial data, let

$$C_0 = \left\| \mathbf{A}(y, \nabla h_0) : D^2 h_0 + F(y, \nabla h_0) \right\|_\infty.$$

Since the equation does not contain h itself, $h_0 + C_0 \tau$ and $h_0 - C_0 \tau$ are a supersolution and a subsolution, respectively. Hence

$$|h - h_0| \leq C_0 \tau. \quad (3.10)$$

If two initial functions have a bounded difference, the same comparison gives

$$\sup_{0 \leq \tau \leq T} \|h^1(\tau) - h^2(\tau)\|_\infty \leq \|h_0^1 - h_0^2\|_\infty. \quad (3.11)$$

Comparison is applied to bounded differences. Along the segment between two gradients, p remains negative and the principal matrix remains nonnegative by (3.3). The localized maximum principle therefore applies.

Lemma 3.4 (A time modulus independent of higher initial derivatives). *Suppose a family of smooth solutions on $[0, T]$ has common gradient and coefficient bounds. There is a common constant A_T such that*

$$\sup_z |h(z, t) - h(z, s)| \leq A_T(\sqrt{|t - s|} + |t - s|), \quad z = (\ell, y). \quad (3.12)$$

Proof. By (3.8), the flux $\mathcal{J}$ and source F are uniformly bounded. Convolve in space with a compactly supported mollifier of radius r . The average changes between s and t by at most $A|t - s|(r^{-1} + 1)$. The gradient bound makes each difference between an average and its point value at most Kr . For $|t - s| \leq 1$, choose $r = \sqrt{|t - s|}$. Partitioning longer time intervals into pieces of length at most one proves the estimate after increasing the constant. $\square$

4 Classical construction in nondegenerate markets

4.1 Function spaces for the local Cauchy problem

For noninteger $r > 0$, let $bc^r(\mathbb{R}^{1+d})$ be the closure of C_b^r in the norm of C_b^r . Higher bounded Hölder regularity implies membership in every strictly lower-exponent little-Hölder space. We use the dense Banach pair

$$E_0^\eta = bc^{2+\eta}(\mathbb{R}^{1+d}), \quad E_1^\eta = bc^{4+\eta}(\mathbb{R}^{1+d}), \quad 0 < \eta < \alpha. \quad (4.1)$$

A strict solution is a function in $C([0, t_0]; E_1^\eta) \cap C^1([0, t_0]; E_0^\eta)$ that satisfies its equation in E_0^η .

We use the local existence theorem of Lunardi [13, Theorem 8.1.1, p. 290] in the following form. Let E_1 be densely and continuously embedded in E_0 , let $\mathcal{O} \subset E_1$ be open, and let $\mathcal{F} : [0, T] \times \mathcal{O} \rightarrow E_0$ be continuously Fréchet differentiable in f . Suppose $D_f \mathcal{F}(t, f)$ is sectorial on E_0 with domain E_1 and graph norm equivalent to the E_1 norm. Assume also that $\mathcal{F}, D_f \mathcal{F}$ are locally Lipschitz in f and locally Hölder continuous in t in their respective Banach norms. Every compatible initial value then gives a local strict solution. Density makes the compatibility condition $\mathcal{F}(0, f_0) \in \overline{E_1}^{E_0}$ automatic. The theorem also gives uniqueness in a weighted positive-time Hölder class; we additionally use the maximum principle for uniqueness in the classical class with bounded differences considered here.

Put $f = h - h_0$ and define

$$\mathcal{F}(f) = A(y, \nabla h_0 + \nabla f) : D^2(h_0 + f) + F(y, \nabla h_0 + \nabla f). \quad (4.2)$$

Consider the open subset of E_1^η specified by

$$\sup_{\ell, y} (h_0 + f)_\ell < 0, \quad \inf_{\ell, y} |\lambda + C^\top \nabla_y (h_0 + f)| > 0. \quad (4.3)$$

In a small neighborhood of each point, the gradients have a common upper bound and both separation conditions have a common positive margin. The principal matrix is therefore uniformly positive definite.

Lemma 4.1 (Local strict solution). *Under Assumption 2.2, suppose*

$$\inf_{\ell, y} |\lambda(y) + C^\top \nabla_y h_0(\ell, y)| > 0.$$

For sufficiently small $\eta > 0$, equation (3.1) then has a local strict solution $f = h - h_0$. Every strong E_1^η endpoint trace satisfying (4.3) can be used to restart the equation.

Proof. The Hölder product estimate and Taylor's formula for smooth compositions show that (4.2) maps E_1^η into E_0^η , is continuously Fréchet differentiable, and has a locally Lipschitz derivative on small bounded neighborhoods in the open set. Its derivative is

$$D\mathcal{F}(f)v = A : D^2v + b_0 v_\ell + b_y \cdot \nabla_y v,$$

with coefficients in $bc^{2+\eta}$. Although h_0 itself may be unbounded, the map uses only its bounded derivatives.

We verify the operator domain. The realization of the preceding elliptic operator P on bc^η is sectorial with domain $bc^{2+\eta}$ and equivalent graph norm; see Lunardi [13, Theorems 3.1.14–3.1.15 and Corollary 3.1.16, pp. 87–90]. The part of P on its domain, endowed with the graph norm, has the restricted resolvent. The identity $P(z - P)^{-1} = (z - P)^{-1}P$ preserves the sectorial bounds in that norm.

The domain of this part is

$$\{v \in bc^{2+\eta} : Pv \in bc^{2+\eta}\} = bc^{4+\eta}.$$

To prove the equality, take difference quotients twice in the elliptic equation and apply the same Schauder estimate successively. The commutators contain bounded coefficient derivatives through order two and lower-order derivatives of v . Interpolation absorbs the latter and yields

$$\|v\|_{C_b^{4+\eta}} \leq A(\|Pv\|_{C_b^{2+\eta}} + \|v\|_\infty). \quad (4.4)$$

Smooth approximation and resolvent difference estimates give the corresponding little-Hölder conclusion. The reverse inclusion follows from the product rule. Thus the domain of $D\mathcal{F}(f)$ on E_0^η is exactly E_1^η . The ellipticity constants and coefficient norms are locally uniform, so the resolvent and graph-norm constants can be chosen locally uniformly as well.

The time conditions are automatic for the autonomous equation, and the initial value $f_0 = 0$ lies in (4.3). The local theorem gives a strict solution. The same mapping and sectoriality arguments apply at every stated strong endpoint. Linearizing the difference of two solutions along the gradient and Hessian segments gives bounded coefficients and a nonnegative principal matrix. The maximum principle makes a bounded difference with zero initial value identically zero. $\square$

4.2 Scalar interior estimates and whole-space regularity

Write $Q_r(z_0, t_0) = B_r(z_0) \times (t_0 - r^2, t_0)$ and use the parabolic distance $|z - z'| + |t - t'|^{1/2}$ in Hölder seminorms.

Lemma 4.2 (Scalar estimate with a bounded divergence source). *Let $w \in L^\infty(Q_{2r}) \cap L_t^2 H_z^1(Q_{2r})$ be a weak solution of*

$$w_t - \operatorname{div}(B \nabla w) = \operatorname{div} J, \quad |B| \leq \Lambda, \quad \xi^\top B \xi \geq \kappa |\xi|^2, \quad J \in L^\infty.$$

The matrix B need only be measurable. There are $\gamma \in (0, 1)$ and $A < \infty$, depending only on dimension, κ , and Λ , such that

$$r^\gamma [w]_{C^{\gamma, \gamma/2}(Q_r)} \leq A(\|w\|_{L^\infty(Q_{2r})} + r\|J\|_{L^\infty(Q_{2r})}). \quad (4.5)$$

Proof. This is the bounded-source case of the scalar local Hölder theorem of Aronson and Serrin [14]; an explicit statement under the original structural assumptions is also given by Aronson [15, p. 7]. Choose divergence flux $\mathcal{A}(z, t, u, q) = Bq + J$ and zero nondifference source. Then

$$q \cdot \mathcal{A} \geq \frac{\kappa}{2}|q|^2 - \frac{|J|^2}{2\kappa}, \quad |\mathcal{A}| \leq \Lambda|q| + |J|.$$

The lower-order structural coefficients multiplying u can all be set to zero. The two inhomogeneous structural functions can be taken as $|J|/\sqrt{2\kappa}$ and $|J|$. They are bounded and satisfy the local mixed-integrability requirements of the original theorem. These inequalities use no symmetry of B . With truncation test function $(w - a)_+\zeta^2$, the source term is controlled by

$$|J \cdot \nabla(w - a)_+|\zeta^2 \leq \frac{\kappa}{2}|\nabla(w - a)_+|^2\zeta^2 + C_\kappa|J|^2\mathbf{1}_{\{w>a\}}\zeta^2.$$

Scalar iteration gives the estimate on a unit cylinder. Under $z = z_0 + r\hat{z}$ and $t = t_0 + r^2\hat{t}$, the vector source becomes rJ , giving (4.5). $\square$

Suppose the maximal classical branch has endpoint $T_* < \infty$ and $|k| \geq \delta_* > 0$ on $[0, T_*)$. By (3.3) and the gradient bound,

$$\det A \geq \frac{(1 - c^2)\delta_*^2}{2^{d+1}a_+^2}, \quad \text{tr } A \leq \frac{1}{2} \left(d + \frac{(M + cK(T_*))^2}{a_-^2} \right).$$

Dividing the determinant lower bound by the d th power of the trace upper bound gives a positive lower bound on the least eigenvalue. Hence B is bounded and uniformly coercive. The vector sources in (3.9) are uniformly bounded; their pointwise divergences need not be controlled. Applying Lemma 4.2 separately to $p, s_1, \dots, s_d$ yields uniform whole-space parabolic Hölder bounds on every $[\varepsilon, T_*)$. Each application is scalar, with the coefficients of the actual solution.

We next use the localized Schauder estimate of Lunardi [13, Theorem 5.1.10, p. 192]. If

$$w_t - a^{ij}w_{ij} - b^i w_i = f$$

is uniformly parabolic on Q_{2r} and its coefficients and f belong to $C^{\theta, \theta/2}$, then

$$\|w\|_{C^{2+\theta, 1+\theta/2}(Q_r)} \leq A_r(\|w\|_{L^\infty(Q_{2r})} + \|f\|_{C^{\theta, \theta/2}(Q_{2r})}). \quad (4.6)$$

This local form follows by nested space-time cutoffs, extension of the coefficients preserving ellipticity, and Hölder interpolation to absorb the first-order cutoff terms. The constant depends on the ellipticity bounds, coefficient Hölder norms, dimension, and r . It requires no data beyond the upper endpoint of the cylinder.

Choose $0 < \theta < \min\{\alpha, \gamma\}$. The gradient estimates give uniform parabolic Hölder norms for A, F . On a cylinder centered at (ℓ_0, y_0) , subtract the constant $h_0(\ell_0, y_0)$ from h . The height bound (3.10) and the global Lipschitz property of h_0 give a uniform local function bound. Applying (4.6) to (3.1) yields second spatial derivatives and the time derivative. The drifts in (3.4) and the gradient-equation sources are now Hölder continuous with uniform bounds. A further application gives third spatial derivatives of h . Differentiating the gradient equations once more produces commutators containing only controlled derivatives of order at most three, and a final application gives fourth spatial derivatives. The equation then controls two spatial derivatives of h_τ .

On nested positive-time strips, for some $\theta > 0$,

$$\sup_{\varepsilon \leq \tau < T_*} (\|f(\tau)\|_{C_b^{4+\theta}} + \|f_\tau(\tau)\|_{C_b^{2+\theta}}) < \infty, \quad f = h - h_0. \quad (4.7)$$

The whole-space norms follow by taking the supremum of fixed-radius local estimates, with distant point pairs controlled by derivative suprema. Constants may depend on higher-order primitive derivative bounds, but remain bounded as $\tau \uparrow T_*$.

4.3 Strong endpoint traces and continuation

Lemma 4.3. *If (4.7) holds, then for every $0 < \eta < \theta$ there is $f_* \in bc^{4+\eta}$ such that*

$$f(\tau) \rightarrow f_* \text{ in } bc^{4+\eta}, \quad f_\tau(\tau) \rightarrow \mathcal{F}(f_*) \text{ in } bc^{2+\eta}. \quad (4.8)$$

If $p \leq -a_-$ and $|k| \geq \delta_*$ hold before the endpoint, then the trace satisfies (4.3).

Proof. For $\varepsilon \leq s < t < T_*$, the time-derivative bound gives $\|f(t) - f(s)\|_{C_b^{2+\theta}} \leq A|t - s|$. The difference also has a uniform $C_b^{4+\theta}$ bound. Hölder interpolation [13, Proposition 1.1.3 and Section 1.2] yields

$$\|f(t) - f(s)\|_{C_b^{4+\eta}} \leq A_\eta |t - s|^{(\theta-\eta)/2}. \quad (4.9)$$

The interpolation weight is determined by $4 + \eta = a(2 + \theta) + (1 - a)(4 + \theta)$, so $a = (\theta - \eta)/2$. Each $f(t)$ has higher Hölder regularity and therefore belongs to $bc^{4+\eta}$. Completeness gives the strong limit. The map (4.2) is continuous from this space to $bc^{2+\eta}$, proving the second convergence. Uniform gradient convergence preserves the margins in the open-set conditions. $\square$

Proposition 4.4 (Continuation of the uniformly parabolic branch). *Under Assumption 2.2 and $\inf |k_0| > 0$, a finite maximal uniformly parabolic existence time $T_* < \infty$ implies*

$$\liminf_{\tau \uparrow T_*} \inf_{\ell, y} |k(\ell, y, \tau)| = 0. \quad (4.10)$$

Under (2.7), the classical branch exists on every finite horizon and satisfies (2.14).

Proof. If (4.10) fails, the earlier compact time strips and the positive lower bound near the endpoint give a positive margin throughout $[0, T_*)$. The preceding two subsections provide a strong trace, and Lemma 4.1 restarts the solution, a contradiction. The separation condition gives

$$\Pi k = \Pi \lambda, \quad |k| \geq |\Pi \lambda| \geq \delta,$$

which excludes a finite maximal time. The initial strict solution, positive-time estimates, and (4.9) give high-order strong continuity. The maximum principle makes constructions on different horizons agree on their common intervals. $\square$

The criterion allows the infimum to approach zero along states escaping to infinity. It concerns continuation of the uniformly parabolic branch. Finite-order degeneracy is treated separately in Section 6.

4.4 Coefficient comparison and uniform intervals for one correlated factor

Let $n = d = 1$, $0 < |\rho| < 1$, and $C = \rho$. Define

$$c_\rho = \beta' - \lambda'/\rho, \quad g_\rho = \lambda\lambda'/\rho - (\beta\lambda)' - \lambda''/2, \quad k_0 = \lambda + \rho h_{0y}. \quad (4.11)$$

Lemma 4.5. *Every classical branch satisfies*

$$(\partial_\tau - \mathcal{L}_h)k = c_\rho k + g_\rho. \quad (4.12)$$

Proof. Since $k = \lambda + \rho s$, the gradient equation gives

$$(\partial_\tau - \mathcal{L}_h)k = \frac{\rho\lambda'}{p} \left(\frac{k}{p} p_\ell - \rho p_y \right) + \rho\beta's - \rho k\lambda'(1 + p^{-1}) - \frac{1}{2}\lambda'' - b_y\lambda'.$$

Substituting $b_y = \rho k p_\ell / p^2 - \rho^2 p_y / p + \beta + s - \rho k(1 + p^{-1})$ cancels the Hessian terms and the terms involving $\rho k(1 + p^{-1})$, leaving $\rho\beta's - (\beta + s)\lambda' - \lambda''/2$. Substituting $\rho s = k - \lambda$ proves the claim. $\square$

Theorem 4.6 (Scalar nondegeneracy conditions). *Under Assumption 2.2, suppose there are $e \in \{-1, 1\}$ and $\delta_0 > 0$ such that*

$$ek_0 \geq \delta_0, \quad eg_\rho \geq 0.$$

The smooth-preference conclusions of Theorem 2.4 hold, with its lower bound on $|k|$ replaced by

$$ek \geq \delta_0 e^{-\|c_\rho^-\|_\infty \tau}. \quad (4.13)$$

Alternatively, if constants $0 < \underline{k} \leq \bar{k}$ satisfy

$$\underline{k} \leq k_0 \leq \bar{k}, \quad c_\rho \underline{k} + g_\rho \geq 0, \quad c_\rho \bar{k} + g_\rho \leq 0, \quad (4.14)$$

then $\underline{k} \leq k \leq \bar{k}$ on every horizon, and the optimal dual coefficient has the uniform bound

$$|\nu^*| \leq \frac{\sqrt{1-\rho^2}}{|\rho|} \sup_{y, a \in [\underline{k}, \bar{k}]} |a - \lambda(y)|. \quad (4.15)$$

Proof. After an exponential time change in (4.12), compare ek with the deterministic subsolution on the right of (4.13). The reaction coefficient is globally bounded and the drift is bounded on each existing compact strip, so whole-space comparison applies. Constant subsolutions and supersolutions prove the case (4.14). The positive margin and Proposition 4.4 give every finite horizon. The financial conclusions follow from the common verification in Section 5. Finally, $s = (k - \lambda)/\rho$ and (2.15) give (4.15). $\square$

The constant conditions also admit an exact necessity statement. In the following result, assume further that $\beta, \lambda \in C_b^\infty$. Fix $0 < \underline{k} < \bar{k}$ and suppose $\lambda(y) \in [\underline{k}, \bar{k}]$. Let $\mathcal{U}_{\underline{k}, \bar{k}}$ consist of all state-dependent utilities satisfying the smooth-preference conditions and having initial k_0 in this interval. Individual utilities may have different finite derivative bounds and risk-aversion bounds.

Theorem 4.7 (A necessary and sufficient uniform interval condition). *The interval $[\underline{k}, \bar{k}]$ is preserved by the classical branch for every $U \in \mathcal{U}_{\underline{k}, \bar{k}}$ if and only if*

$$c_\rho(y)\underline{k} + g_\rho(y) \geq 0, \quad c_\rho(y)\bar{k} + g_\rho(y) \leq 0 \quad (y \in \mathbb{R}). \quad (4.16)$$

When these conditions hold, the classical value and optimal portfolio exist on every finite horizon for each utility in the class, and (4.15) holds uniformly over horizons and the preference class.

Proof. Sufficiency follows from Theorem 4.6. Suppose $c_\rho(y_0)\underline{k} + g_\rho(y_0) < 0$ at some y_0 . Choose a smooth compactly supported cutoff χ , equal to one near y_0 , with $0 \leq \chi \leq 1$. Set

$$k_0(y) = \chi(y)\underline{k} + (1 - \chi(y))\lambda(y), \quad \psi'(y) = \frac{k_0(y) - \lambda(y)}{\rho}.$$

Since ψ' has compact support, it has a bounded smooth antiderivative. Choose a smooth nonhomothetic utility U_0 satisfying the required conditions, such as the two-power utility in Appendix A, and put $U(x, y) = e^{\psi(y)}U_0(x)$. This utility belongs to the specified class and realizes the chosen k_0 . Near y_0 , k_0 equals $\underline{k}$ and all its spatial derivatives vanish. The initial strict solution is differentiable in time, so

$$k_\tau(\ell, y_0, 0) = c_\rho(y_0)\underline{k} + g_\rho(y_0) < 0.$$

The solution immediately crosses the lower endpoint, a contradiction. If the upper inequality fails, replace $\underline{k}$ by $\bar{k}$ in the construction to obtain the same contradiction. $\square$

The necessity statement quantifies over state-dependent preferences. For a fixed state-independent utility, the initial quantity $k_0 = \lambda$ is prescribed, and (4.16) remains a sufficient condition.

An explicit market family is

$$\beta = 0, \quad \lambda(y) = a + b \tanh(\omega y), \quad \omega > 0, \quad \rho b > 0, \quad a > |b| + |\rho|\omega. \quad (4.17)$$

Writing $t_y = \tanh(\omega y)$ gives

$$c_\rho = -\lambda'/\rho, \quad g_\rho = \frac{\lambda'}{\rho} \{a + (b + \rho\omega)t_y\}, \quad \lambda'/\rho > 0.$$

Since $|b + \rho\omega| = |b| + |\rho|\omega$, we can choose

$$\underline{k} = a - |b| - |\rho|\omega, \quad \bar{k} = a + |b| + |\rho|\omega.$$

For state-independent utility, $k_0 = \lambda$ lies in this interval. Hence

$$\frac{a - |b| - |\rho|\omega}{a_+} \leq \frac{\vartheta^*}{x} \leq \frac{a + |b| + |\rho|\omega}{a_-}, \quad |\nu^*| \leq \sqrt{1 - \rho^2} \left(\frac{2|b|}{|\rho|} + \omega \right). \quad (4.18)$$

The same conclusion holds for state-dependent utility whose initial k_0 lies in the displayed interval. For state-independent preferences satisfying Assumption 2.3, Theorem 5.4 gives these conclusions under the same market conditions.

In arbitrary dimensions, on a given horizon T , suppose

$$\lambda_0 = \inf_y |\lambda(y)| > 0, \quad cK(T) < \lambda_0, \quad (4.19)$$

which gives $|k| \geq \lambda_0 - cK(T) > 0$. Proposition 4.4 and the subsequent verification yield all conclusions on that horizon. This condition provides a finite-horizon neighborhood of zero covariance.

5 Optimal wealth, convex duality, and utility approximation

5.1 Recovering the value from its marginal function

First impose Assumption 2.2 and let h be the solution constructed in the preceding section. Set

$$u(x, y, \tau) = e^{h(\log x, y, \tau)}, \quad \hat{p}(x, y, \tau) = p(\log x, y, \tau), \quad \hat{k}(x, y, \tau) = k(\log x, y, \tau).$$

Let $a(y, \tau)$ solve the linear Cauchy problem

$$a_\tau = \frac{1}{2} \Delta_y a + \beta \cdot \nabla_y a - \frac{e^{h(0, y, \tau)} |k(0, y, \tau)|^2}{2p(0, y, \tau)}, \quad a(y, 0) = U(1, y) \quad (5.1)$$

and define

$$\hat{V}(x, y, \tau) = a(y, \tau) + \int_1^x u(r, y, \tau) dr. \quad (5.2)$$

The source is bounded on every finite horizon and has the required Hölder regularity. Thus a is supplied by linear parabolic theory or the Feynman–Kac formula.

Lemma 5.1. *The function (5.2) satisfies (2.12) with initial value U , and*

$$\hat{V}_x = u > 0, \quad \hat{V}_{xx} = u\hat{p}/x < 0, \quad \nabla_y \hat{V}_x = u s(\log x, y, \tau).$$

For every finite T , there are $A, N_0 < \infty$ such that

$$\begin{aligned} |\hat{V}(x, y, \tau)| + u(x, y, \tau) &\leq A(1 + x^{N_0} + x^{-N_0}), \\ |\tilde{U}(z, y)| &\leq A(1 + z^{N_0} + z^{-N_0}), \quad U^+(x, y) \leq A(1 + x), \end{aligned} \quad (5.3)$$

uniformly over the factor state and $0 \leq \tau \leq T$.

Proof. Reversing the marginal transformation yields

$$u_\tau = \frac{1}{2}\Delta_y u + \beta \cdot \nabla_y u - \partial_x \left(\frac{|\lambda u + C^\top \nabla_y u|^2}{2u_x} \right).$$

Thus the HJB residual of $\widehat{V}$ is independent of x . At $x = 1$, equation (5.1) makes it zero. The fundamental theorem of calculus gives the initial value and the stated derivatives.

Boundedness of $h_0(0, y)$ gives positive uniform lower and upper bounds on $U_x(1, y)$. Integrating the risk-aversion interval gives, for $x \geq 1$,

$$U_x(1, y)x^{-a+} \leq U_x(x, y) \leq U_x(1, y)x^{-a-}.$$

For $0 < x \leq 1$, the endpoint exponents are exchanged. The height estimate (3.10) then bounds u , and integration in (5.2) gives two-sided polynomial growth bounds for $\widehat{V}$. Logarithmic terms are dominated by a slightly larger power. The inverse marginal $I(z, y) = (U_x(\cdot, y))^{-1}(z)$ also has finite power bounds at both endpoints. The identity $\widetilde{U}(z, y) = U(I(z, y), y) - zI(z, y)$ gives the conjugate bound. Finally, for $x \geq 1$, concavity gives $U(x, y) \leq U(1, y) + U_x(1, y)(x - 1)$. Monotonicity treats $x \leq 1$, proving the uniform linear bound on the positive part. $\square$

5.2 Optimal wealth and marginal density on the given probability space

Define the proportional exposure

$$\kappa(\ell, y, \tau) = -\frac{k(\ell, y, \tau)}{p(\ell, y, \tau)}. \quad (5.4)$$

It is bounded on every finite strip and, for smooth data, has the required local Lipschitz regularity. Couple the factor equation with

$$d\ell_t^* = \left(\kappa(\ell_t^*, Y_t, T - t) \cdot \lambda(Y_t) - \frac{1}{2} |\kappa(\ell_t^*, Y_t, T - t)|^2 \right) dt + \kappa(\ell_t^*, Y_t, T - t)^\top dW_t \quad (5.5)$$

and take $\ell_0^* = \log x$. Local Lipschitz regularity gives a pathwise unique local strong solution, and bounded drift and diffusion exclude finite-time explosion. Put $X_t^* = e^{\ell_t^*}$. Then $X_t^* > 0$, and $\vartheta_t^* = X_t^* \kappa_t$ satisfies the original wealth equation.

For every $q > 0$, the bounded logarithmic drift and martingale quadratic variation imply

$$\mathbb{E} \sup_{0 \leq t \leq T} ((X_t^*)^q + (X_t^*)^{-q}) < \infty. \quad (5.6)$$

To obtain this bound, apply the exponential martingale and Doob inequalities to positive and negative multiples of the martingale part. Taking $q = 2$ yields $\mathbb{E} \int |\vartheta^*|^2 < \infty$.

Apply the localized Itô formula to $\widehat{V}(X_t^*, Y_t, T - t)$ along the optimal feedback. Completing the square in the Hamiltonian makes the drift zero. The bounds (5.3) and (5.6) dominate its running absolute value by an integrable random variable. Hence

$$\widehat{V}(x, y, T) = \mathbb{E} U(X_T^*, Y_T). \quad (5.7)$$

Let $M_t = u(X_t^*, Y_t, T - t)$ and $s_t = s(\ell_t^*, Y_t, T - t)$. Equation (3.1) and Itô's formula give

$$\frac{dM_t}{M_t} = -\lambda(Y_t)^\top dW_t + (D^\top s_t)^\top dB_t. \quad (5.8)$$

Indeed, the traded coefficient of $d \log M$ is

$$p\kappa + C^\top s = -k + C^\top s = -\lambda,$$

the orthogonal coefficient is $D^\top s$, and the drift is $-(|\lambda|^2 + |D^\top s|^2)/2$. Applying Itô's formula to the exponential gives (5.8).

Set $z = M_0$ and $L^* = M/z$. Then L^* has the form (2.10), with $\nu^* = -D^\top s$. Bounded coefficients make L^* a strictly positive true martingale with all finite positive and negative moments and the required running maximum moments. The product L^*X^* has zero drift and bounded proportional diffusion coefficients $(\kappa - \lambda, D^\top s)$. Consequently,

$$\mathbb{E}[L_T^* X_T^*] = x, \quad U_x(X_T^*, Y_T) = zL_T^*. \quad (5.9)$$

Proposition 5.2 (Verification and uniqueness in the admissible class). *The function $\hat{V}$ equals the value function (2.4). In the original admissible class, optimal terminal wealth is unique almost surely, the optimal wealth process is unique up to indistinguishability, and the traded exposure is unique with respect to $dt \otimes d\mathbb{P}$.*

Proof. For any admissible wealth X^ϑ , the product L^*X^ϑ is a nonnegative local martingale. Hence

$$\mathbb{E}[L_T^* X_T^\vartheta] \leq x.$$

The local calculation is first stopped where the density, wealth, and integrated squared exposure are bounded, and the stopping times are then removed. Bounded λ and the square-integrability condition make X_T^ϑ integrable, so the positive part of utility is integrable. The statewise Fenchel inequality gives

$$\begin{aligned} \mathbb{E}U(X_T^\vartheta, Y_T) &\leq \mathbb{E}\tilde{U}(zL_T^*, Y_T) + zx \\ &= \mathbb{E}U(X_T^*, Y_T) = \hat{V}(x, y, T). \end{aligned}$$

The conjugate term is integrable by (5.3) and the density moment bounds; equality uses (5.9). The argument also covers competitors whose utility has infinite negative-part expectation.

Admissible wealth and exposure pairs form a convex set. If two optimal terminal wealths differ with positive probability, strict concavity makes their average yield strictly larger expected utility, a contradiction. For any optimal wealth X , its terminal value equals X_T^* . Thus the nonnegative supermartingale L^*X has initial and terminal expectations both equal to x . It is a true martingale closed by its terminal value, and

$$L_t^* X_t = \mathbb{E}[L_T^* X_T^* \mid \mathcal{F}_t] = L_t^* X_t^*.$$

Positivity of L^* gives equality of the wealth paths. Uniqueness of continuous semimartingale decomposition then gives $\int_0^T |\vartheta_t - \vartheta_t^*|^2 dt = 0$ almost surely. $\square$

5.3 Dual attainment and uniform control bounds

The following proposition isolates the duality argument used below.

Proposition 5.3. *Suppose $U(1, \cdot)$ is bounded, utility and its conjugate satisfy (5.3), the finite value function satisfies $V_x > 0$ and $V_{xx} < 0$, and, for every fixed (y, T) , its marginal maps $(0, \infty)$ onto $(0, \infty)$. For each initial wealth, assume an admissible optimal wealth with finite expected utility and a density with bounded coefficients satisfy (5.9). Assume the coefficient bound is independent of initial wealth and the factor state. Then*

$$v(z, y, T) = \sup_{x>0} \{V(x, y, T) - zx\}, \quad V(x, y, T) = \inf_{z>0} \{v(z, y, T) + zx\}. \quad (5.10)$$

The fixed-multiplier and outer dual problems are uniquely attained. Whenever a deterministic control bound contains all the constructed optimal coefficients, the truncated and untruncated values agree for every initial state and multiplier.

Proof. For $z > 0$, surjectivity and strict monotonicity of the marginal select the unique x with $V_x(x, y, T) = z$. Take the corresponding X^*, L^* . For every other density L^ν with bounded coefficients, $L^\nu X^*$ is a nonnegative local martingale, and therefore

$$\begin{aligned}\mathbb{E}\tilde{U}(zL_T^\nu, Y_T) &\geq \mathbb{E}U(X_T^*, Y_T) - z\mathbb{E}[L_T^\nu X_T^*] \\ &\geq V(x, y, T) - zx.\end{aligned}$$

The density L^* gives equality in both steps. Strict concavity makes the same x the unique maximizer in the Legendre transform; the outer multiplier $z = V_x(x, y, T)$ also gives equality. This proves conjugacy and attainment.

If another density L attains the same fixed-multiplier value, let its terminal Fenchel gap against X_T^* be

$$\Delta = \tilde{U}(zL_T, Y_T) + zL_T X_T^* - U(X_T^*, Y_T) \geq 0.$$

The budget inequality gives $0 \leq \mathbb{E}\Delta \leq 0$. Thus $zL_T = U_x(X_T^*, Y_T) = zL_T^*$. The two true martingales have the same terminal value, so their entire paths and Brownian coefficients agree. If (z, L) is another optimal outer pair, set $z^* = V_x(x, y, T)$. Then

$$0 = \mathbb{E}\Delta + z\{x - \mathbb{E}[L_T X_T^*]\}.$$

Both terms are nonnegative, giving $zL_T = z^*L_T^*$ and the budget equality. Substitution yields $(z^*/z)x = x$, so $z = z^*$. Finally, a truncation containing the actual optimal coefficient leaves each minimum unchanged. $\square$

The proposition also applies to the enlarged density class $\mathfrak{L}_T$. This class consists of (2.10) generated by predictable coefficients satisfying $\int_0^T |\nu_t|^2 dt < \infty$ almost surely. The densities are strictly positive on the entire interval, including the terminal time, and may be strict local martingales. Since

$$\tilde{U}(zL_T, Y_T) \geq U(1, Y_T) - zL_T, \quad \mathbb{E}L_T \leq 1,$$

the negative part of the conjugate is integrable, and the preceding lower bound remains valid in the extended expectation sense. If such a density attains the value, Fenchel equality forces $L_T = L_T^*$. Its terminal expectation is one, so it is itself a closed true martingale and the same uniqueness proof applies.

The risk-aversion interval makes $\ell \mapsto h(\ell, y, T)$ strictly decreasing and onto $\mathbb{R}$. The smooth construction therefore satisfies the proposition's marginal-surjectivity condition. Formula (2.15) completes the portfolio and duality conclusions of Theorems 2.4 and 4.6.

A deterministic coefficient bound also implies that $\int \nu^* dB$ is BMO, namely

$$\sup_{\sigma \leq T} \left\| \mathbb{E} \left[\int_{\sigma}^T |\nu_t^*|^2 dt \middle| \mathcal{F}_{\sigma} \right] \right\|_{\infty} < \infty,$$

where the supremum is over stopping times. The converse fails. The deterministic function $\nu_t = (T - t)^{-1/4}$ is unbounded for $t < T$, while every remaining conditional energy is at most $2\sqrt{T}$. Exact deterministic truncation uses the essential bound on the coefficient itself.

5.4 Approximation of state-independent C^4 utility

Theorem 5.4. *Let $\beta, \lambda \in C_b^{4+\alpha}$. Suppose the market satisfies (2.7), or a condition of Theorem 4.6 applicable to $k_0 = \lambda$. For every utility satisfying Assumption 2.3, the C^4 -utility conclusions of Theorem 2.4 hold. The fixed-horizon condition (4.19) is also sufficient on the corresponding horizon.*

Proof. Let η_ε be a nonnegative compactly supported mollifier on $\mathbb{R}$, and set $h_0^\varepsilon = h_0 * \eta_\varepsilon$. Global Lipschitz continuity gives

$$-a_+ \leq (h_0^\varepsilon)' \leq -a_-, \quad \|h_0^\varepsilon - h_0\|_\infty \leq a_+ \varepsilon \int |r| \eta(r) dr. \quad (5.11)$$

Define $(U^\varepsilon)'(x) = \exp h_0^\varepsilon(\log x)$ and $U^\varepsilon(1) = U(1)$. Each mollified utility satisfies the smooth assumptions. The preceding classical construction and verification give its actual solutions $h^\varepsilon, V^\varepsilon$.

By (3.11),

$$\sup_{0 \leq \tau \leq T} \|h^\varepsilon(\tau) - h^\delta(\tau)\|_\infty \leq \|h_0^\varepsilon - h_0^\delta\|_\infty.$$

After subtracting any fixed member, the family is therefore uniformly Cauchy on the whole space-time strip. The gradient estimate is uniform in the mollification parameter, and the initial factor gradient is zero. Lemma 3.4 gives a uniform time modulus. The limit has the original initial trace and satisfies

$$\sup_{\ell, y} |h(\ell, y, \tau) - h_0(\ell)| \leq A_T(\sqrt{\tau} + \tau). \quad (5.12)$$

The market conditions give a positive lower bound on $|k^\varepsilon|$ uniform in ε . On any positive remaining-time strip, apply Lemma 4.2 and the subsequent Schauder estimates to the divergence equations for the gradients. The local estimates are applied to h^ε after subtracting its center constant. The function bound follows from the uniform gradient bound, time modulus, and (5.11). Thus the required higher derivatives are uniformly controlled on positive-time compact sets, without using higher derivatives of h_0^ε . Extract a subsequence and identify its derivative limits through the uniform function limit. This gives a classical marginal solution at positive times.

The source in the recovery equation (5.1) is bounded on every finite horizon and has local parabolic regularity at positive times. Its semigroup integral defines a , with $a \rightarrow U(1)$ at time zero. Formula (5.2) gives a classical candidate value and the HJB equation, with locally uniform terminal trace U . Estimate (5.12) and the risk-aversion bounds give the uniform wealth-growth bounds needed for verification.

The feedback coefficients are locally Lipschitz on every $[0, T - \delta]$ and bounded on $[0, T]$. The local strong solutions agree on overlaps. Bounded drift and quadratic variation give a finite continuous terminal limit for log-wealth, with all the moments in (5.6). Perform classical verification and the marginal-density Itô calculation up to $T - \delta$, then let $\delta \downarrow 0$. Estimate (5.12) and the wealth moments identify the terminal marginal $U'(X_T^*)$, while bounded density coefficients give true martingales. The arguments of Propositions 5.2 and 5.3 apply, yielding admissibility, optimality, uniqueness, and all dual conclusions. $\square$

5.5 Legendre geometry

Let $x = x(q, y, \tau)$ be the unique solution of $V_x = e^q$, and define

$$\mathcal{W}(q, y, \tau) = V(x(q, y, \tau), y, \tau) - e^q x(q, y, \tau).$$

Envelope and implicit differentiation give

$$\begin{aligned} P &= -\mathcal{W}_q = e^q x, & Q &= \mathcal{W}_{qq} - \mathcal{W}_q = -\frac{e^{2q}}{V_{xx}} > 0, \\ H &= Q/P = -1/p, & G &= \nabla_y \mathcal{W}_q / Q = -s, & k &= \lambda - C^\top G. \end{aligned} \quad (5.13)$$

All primal and dual quantities are evaluated at corresponding states. If $P_0 = e^q I(e^q, y)$ and $Q_0 = H_0 P_0$ are the terminal quantities, and $\|h - h_0\|_\infty \leq A_T$, then

$$e^{-A_T/a_-} \leq \frac{P}{P_0} \leq e^{A_T/a_-}, \quad \frac{a_-}{a_+} e^{-A_T/a_-} \leq \frac{Q}{Q_0} \leq \frac{a_+}{a_-} e^{A_T/a_-}. \quad (5.14)$$

Indeed, the marginal equations $h(\ell, y, \tau) = q$ and $h_0(\ell_0, y) = q$, together with $|h_{0\ell}| \geq a_-$, imply $|\ell - \ell_0| \leq A_T/a_-$. Combine this with $P = e^q e^\ell$ and $H \in [1/a_+, 1/a_-]$.

The dual equation is

$$\mathcal{W}_\tau = \frac{1}{2}\Delta_y \mathcal{W} + \beta \cdot \nabla_y \mathcal{W} + \frac{1}{2}|\lambda|^2 Q - (C\lambda) \cdot \nabla_y \mathcal{W}_q - \frac{(\nabla_y \mathcal{W}_q)^\top R \nabla_y \mathcal{W}_q}{2Q}. \quad (5.15)$$

Its second-order linearization has matrix

$$A^d = \frac{1}{2} \begin{pmatrix} |\lambda|^2 + G^\top R G & -(C\lambda + R G)^\top \\ -(C\lambda + R G) & I_d \end{pmatrix}, \quad \det A^d = \frac{k^\top (I_n - C^\top C) k}{2^{d+1}}.$$

At a state with the required classical derivatives, degeneracy of this principal matrix is therefore equivalent to vanishing of the entire optimal traded exposure. The uniform lower bound in (5.14) concerns normalized curvature; unnormalized Q is strictly positive pointwise.

6 Finite-order degeneracy and strong optimal portfolios

6.1 Commutator rescaling and uniform local estimates

This section proves Theorem 2.5. In coordinates $(\ell, y) \in \mathbb{R}^2$, define

$$X_0 = \lambda(y) \partial_\ell, \quad Y = \partial_y.$$

The nonzero iterated commutators are, up to sign,

$$\text{ad}_Y^j X_0 = \lambda^{(j)}(y) \partial_\ell. \quad (6.1)$$

Condition (2.16) therefore gives Hörmander's spanning condition of step at most $m + 1$. For $0 < \varepsilon \leq 1$, set

$$v_j^\varepsilon(y) = \varepsilon^j \lambda^{(j)}(y), \quad X_j^\varepsilon = v_j^\varepsilon \partial_\ell \quad (0 \leq j \leq m), \quad Y = \partial_y. \quad (6.2)$$

All these vector fields have zero Euclidean divergence. Write $Z^\varepsilon = (X_0^\varepsilon, \dots, X_m^\varepsilon, Y)$ and let d_ε be their control distance. An absolutely continuous curve with controls a_i has velocity and length

$$\dot{\gamma} = \sum_i a_i Z_i^\varepsilon(\gamma), \quad \int \left(\sum_i |a_i|^2 \right)^{1/2} dt,$$

respectively. The distance is the infimum of these lengths over curves joining the two prescribed points. The distance d_0 is generated by X_0, Y .

Lemma 6.1 (A parameter-uniform local modulus). *On a fixed compact spatial neighborhood, let w^ε be bounded local-energy weak solutions of*

$$\partial_t w^\varepsilon - \sum_{i,j} Z_i^\varepsilon(a_{ij}^\varepsilon Z_j^\varepsilon w^\varepsilon) = \sum_i b_i^\varepsilon Z_i^\varepsilon w^\varepsilon + \sum_i Z_i^\varepsilon J_i^\varepsilon + f^\varepsilon. \quad (6.3)$$

The local energy condition requires w^ε and its indicated horizontal derivatives to be square-integrable on compact cylinders. Suppose a^ε is uniformly bounded and uniformly positive definite in the vector-field index space, and $b^\varepsilon, J^\varepsilon, f^\varepsilon, w^\varepsilon$ have bounds independent of ε . On smaller compact cylinders separated from the initial time, there are $A < \infty$ and $\gamma \in (0, 1)$, independent of ε , such that

$$|w^\varepsilon(z, t) - w^\varepsilon(z', t')| \leq A(d_\varepsilon(z, z') + |t - t'|^{1/2})^\gamma. \quad (6.4)$$

The family is locally equicontinuous in the ordinary topology. At fixed ℓ , the distance on the right can be replaced by $|y - y'| + |t - t'|^{1/2}$.

Proof. Fix compact sets $K_0 \Subset K_1$ and a positive distance from the initial time. Write

$$\delta_m^2 = \inf_y \sum_{j=0}^m |\lambda^{(j)}(y)|^2 > 0.$$

At every point of K_1 , one of the determinants formed by Y and $\text{ad}_Y^j X_0$ has magnitude at least $\delta_m/\sqrt{m+1}$. The required finite-order coefficient norms are bounded on a neighborhood of K_1 . The scaling in (6.2) is the commutator scaling of Capogna et al. [16, Section 4]. Their Propositions 4.4 and 4.6 supply constants C_D, C_P and a radius $r_0 > 0$, independent of ε , such that

$$\begin{aligned} |B_\varepsilon(z, 2r)| &\leq C_D |B_\varepsilon(z, r)|, \\ \int_{B_\varepsilon(z, r)} |f - f_{B_\varepsilon(z, r)}|^2 &\leq C_P r^2 \int_{B_\varepsilon(z, 2r)} |Z^\varepsilon f|^2, \end{aligned} \quad (6.5)$$

for $z \in K_0$ and $0 < r < r_0$. Here balls and averages use d_ε and Lebesgue measure. Redundant signed copies in the commutator list change the control norm by a factor bounded in terms of m . These estimates therefore hold for the displayed fields. Since their Euclidean speeds are uniformly bounded, r_0 can also be chosen so that all these balls lie in K_1 . A common doubling dimension larger than two can be used throughout.

For $\varepsilon > 0$, the control distance satisfies $|Z^\varepsilon d_\varepsilon(\cdot, z_0)| \leq 1$ almost everywhere in the horizontal dual norm. Metric cutoffs therefore have uniform horizontal derivative bounds. The divergence flux $a^\varepsilon \xi + J^\varepsilon$ in the weak equation satisfies

$$\xi \cdot (a^\varepsilon \xi + J^\varepsilon) \geq \frac{\kappa}{2} |\xi|^2 - \frac{|J^\varepsilon|^2}{2\kappa}, \quad |a^\varepsilon \xi + J^\varepsilon| \leq \Lambda |\xi| + |J^\varepsilon|.$$

The nondiffegence term is bounded by $|b^\varepsilon| |\xi| + |f^\varepsilon|$. All structural functions in Capogna et al. [16, (3.2)] can therefore be chosen bounded. Fixed mixed-integrability exponents satisfying their strict inequalities can be selected for the common doubling dimension. These choices are independent of ε .

There is also a uniform local energy bound. Let η be a space-time cutoff, compactly supported in space and vanishing at the lower time boundary. Testing (6.3) with $w^\varepsilon \eta^2$ and using Young's inequality gives

$$\begin{aligned} \sup_t \int |w^\varepsilon|^2 \eta^2 + \iint \eta^2 |Z^\varepsilon w^\varepsilon|^2 \\ \leq C \iint |w^\varepsilon|^2 (|\eta \eta_t| + |Z^\varepsilon \eta|^2 + |b^\varepsilon|^2 \eta^2) + C \iint (|J^\varepsilon|^2 + |f^\varepsilon| |w^\varepsilon|) \eta^2. \end{aligned} \quad (6.6)$$

The integrals are over the cutoff cylinder, and the supremum is over its time interval. The constant depends on the lower and upper ellipticity bounds in the field indices. Steklov averages justify this test for local-energy weak solutions.

For $Q_r^\varepsilon(z, t) = B_\varepsilon(z, r) \times (t - r^2, t)$ contained in the given cylinder, the inhomogeneous Harnack inequality in Capogna et al. [16, Theorem 1.2], applied to positive oscillations, gives constants $0 < \vartheta < 1$ and $\sigma > 0$ for which

$$\text{osc}_{Q_{r/8}^\varepsilon} w^\varepsilon \leq \vartheta \text{osc}_{Q_r^\varepsilon} w^\varepsilon + C r^\sigma.$$

All constants are uniform for $0 < r < r_0$. The bounded source norms enter the last constant; the drift has a uniformly bounded rescaled norm. Iterating on nested backward cylinders gives (6.4).

Adding the rescaled fields can only shorten distance, so $d_\varepsilon \leq d_0$. The local ball-box estimate for the fixed-step system gives

$$d_0(z, z') \leq A |z - z'|^{1/(m+1)},$$

for nearby points, with constants determined by finite-order coefficient bounds and the spanning margin on the compact neighborhood. At fixed ℓ , direct motion along Y gives $d_0((\ell, y), (\ell, y')) \leq |y - y'|$. These inequalities prove both continuity conclusions. $\square$

6.2 Auxiliary markets and their gradient equations

Take $m + 1$ traded Brownian motions $W^0, \dots, W^m$, independent of the factor Brownian motion B , with risk-price vector

$$v^\varepsilon = (\lambda, \varepsilon \lambda', \dots, \varepsilon^m \lambda^{(m)}), \quad \Lambda_\varepsilon^2 = \sum_{j=0}^m (v_j^\varepsilon)^2. \quad (6.7)$$

For every $\varepsilon > 0$, $\inf \Lambda_\varepsilon > 0$. Theorem 5.4 therefore gives the value function and strong optimal portfolio for this orthogonal auxiliary market. Denote its marginal quantities by $h^\varepsilon, p^\varepsilon, s^\varepsilon$. Since

$$\sup_{\varepsilon \leq 1} \|v^\varepsilon\|_\infty < \infty, \quad \sup_{\varepsilon \leq 1} \|(v^\varepsilon)'\|_\infty < \infty,$$

the gradient bounds are uniform in ε . These two norms converge to $\|\lambda\|_\infty$ and $\|\lambda'\|_\infty$, respectively.

Put $a(p) = 1/(2p^2)$ and define

$$\begin{aligned} \mathcal{P}_\varepsilon w &= w_\tau - \sum_{j=0}^m X_j^\varepsilon (a(p^\varepsilon) X_j^\varepsilon w) - \frac{1}{2} Y^2 w \\ &\quad - \sum_{j=0}^m \frac{v_j^\varepsilon}{2(p^\varepsilon)^2} X_j^\varepsilon w - (\beta + s^\varepsilon) Y w. \end{aligned} \quad (6.8)$$

The gradients satisfy

$$\begin{aligned} \mathcal{P}_\varepsilon p^\varepsilon &= 0, \\ \mathcal{P}_\varepsilon s^\varepsilon &= \sum_{j=0}^m X_j^\varepsilon \left(-\frac{(v_j^\varepsilon)'}{p^\varepsilon} \right) + \beta' s^\varepsilon - \sum_{j=0}^m v_j^\varepsilon (v_j^\varepsilon)' \left(1 + \frac{1}{p^\varepsilon} \right). \end{aligned} \quad (6.9)$$

To verify these identities, write the auxiliary marginal equation as

$$h_\tau^\varepsilon = \partial_\ell \left(-\frac{\Lambda_\varepsilon^2}{2p^\varepsilon} \right) + \frac{1}{2} h_{yy}^\varepsilon + \beta s^\varepsilon + \frac{1}{2} (s^\varepsilon)^2 - \frac{1}{2} \Lambda_\varepsilon^2 (1 + (p^\varepsilon)^{-1}).$$

Differentiate in ℓ and y , using $p_y^\varepsilon = s_\ell^\varepsilon$ and $\partial_y \Lambda_\varepsilon^2 = 2 \sum_j v_j^\varepsilon (v_j^\varepsilon)'$. Differentiation of the source through the gradients gives the horizontal drifts in (6.8). Explicit differentiation of the flux gives the horizontal divergence in (6.9). The calculation divides only by p^ε , never by a risk-price component.

The principal matrix in the vector-field indices is diagonal, with its first $m + 1$ entries equal to $a(p^\varepsilon)$ and its last entry equal to $1/2$. The risk-aversion interval makes it uniformly positive definite. The drifts, vector sources, and scalar sources have bounds independent of ε . Each approximation is classical at positive times and satisfies the local energy requirement; estimate (6.6) bounds that energy uniformly on smaller cylinders. Lemma 6.1 gives a uniform Hölder modulus for $p^\varepsilon, s^\varepsilon$ on positive-time compact sets. Lemma 3.4 also applies to h^ε . A diagonal subsequence therefore satisfies

$$h^\varepsilon \rightarrow h, \quad p^\varepsilon \rightarrow p, \quad s^\varepsilon \rightarrow s, \quad (6.10)$$

locally uniformly on positive-time compact sets. Integration along spatial line segments gives $p = h_\ell$ and $s = h_y$. Local uniform convergence of the bounded fluxes and sources passes the marginal equation to the limit in distributional form and gives

$$\sup_{\ell, y} |h(\ell, y, \tau) - h_0(\ell)| \leq A_T(\sqrt{\tau} + \tau). \quad (6.11)$$

Passing to the limit in the auxiliary gradient bounds, and using convergence of the coefficient norms, yields the precise bound $K(T)$ for the original risk price.

6.3 Recovery of ordinary classical derivatives from the factor equation

The preceding compactness gives ordinary first derivatives of h . Second factor derivatives of the value are obtained from a separate equation. For fixed $x > 0$, the auxiliary value satisfies

$$V_\tau^\varepsilon - \frac{1}{2}V_{yy}^\varepsilon - \beta V_y^\varepsilon = q^\varepsilon(x, y, \tau), \quad V^\varepsilon(x, y, 0) = U(x), \quad (6.12)$$

where

$$q^\varepsilon(x, y, \tau) = -\frac{\Lambda_\varepsilon(y)^2 x e^{h^\varepsilon(\log x, y, \tau)}}{2p^\varepsilon(\log x, y, \tau)}. \quad (6.13)$$

On every compact wealth interval, the source is uniformly bounded over all factor states and finite horizons. This follows from the risk-aversion interval, uniform time modulus, and boundedness of h_0 on that wealth interval.

Let P_t be the factor semigroup. The Feynman–Kac formula gives

$$V^\varepsilon(x, y, \tau) = U(x) + \int_0^\tau P_{\tau-r} q^\varepsilon(x, \cdot, r)(y) dr. \quad (6.14)$$

The sources converge locally uniformly to

$$q(x, y, \tau) = -\lambda(y)^2 x e^{h(\log x, y, \tau)} / (2p(\log x, y, \tau)).$$

For initial wealth and factor states in compact sets, the contribution of $r \in [0, \delta]$ is at most $C_T \delta$. On $[\delta, T]$, first restrict the factor path to a large compact set and use local uniform convergence of the source there. The complementary contribution is bounded by a constant times the exit probability of the factor, which tends to zero uniformly over the compact set of initial states. Indeed, $Y_t = y + B_t + \int_0^t \beta(Y_u) du$ and β is bounded. Letting $\varepsilon \downarrow 0$, then enlarging the compact set and taking $\delta \downarrow 0$, proves local uniform convergence of (6.14). Its limit defines V .

Fix a compact wealth interval $I \Subset (0, \infty)$ and nested factor cylinders $Q_r \Subset Q_{2r}$ at positive remaining times. At each fixed ℓ , the direct Y path in Lemma 6.1 gives

$$\sup_{0 < \varepsilon \leq 1, x \in I} \|q^\varepsilon(x, \cdot, \cdot)\|_{C^{\gamma, \gamma/2}(Q_{2r})} < \infty.$$

Local uniform convergence, combined with Hölder interpolation, gives convergence of the sources in every smaller exponent $\eta \in (0, \gamma)$. For example,

$$\|q^\varepsilon - q^{\varepsilon'}\|_{C^{\eta, \eta/2}} \leq C \|q^\varepsilon - q^{\varepsilon'}\|_\infty^{1-\eta/\gamma},$$

on the fixed cylinder, uniformly in $x \in I$. The factor Schauder estimate applied to the difference of (6.12) then gives

$$\begin{aligned} & \|V^\varepsilon(x) - V^{\varepsilon'}(x)\|_{C^{2+\eta, 1+\eta/2}(Q_r)} \\ & \leq C (\|V^\varepsilon(x) - V^{\varepsilon'}(x)\|_{L^\infty(Q_{2r})} + \|q^\varepsilon(x) - q^{\varepsilon'}(x)\|_{C^{\eta, \eta/2}(Q_{2r})}). \end{aligned} \quad (6.15)$$

Both terms tend to zero uniformly on I . To check the wealth parameter, apply the same estimate to $V^\varepsilon(x_j, \cdot, \cdot) - V^\varepsilon(x, \cdot, \cdot)$ for $x_j \rightarrow x$. Joint local equicontinuity of the sources gives uniform convergence of their differences, and interpolation gives convergence in each smaller Hölder exponent. The value differences are uniformly small because $V_x^\varepsilon = e^{h^\varepsilon}$ is uniformly bounded on compact wealth intervals. Passing to the limit proves joint continuity of V_y, V_{yy}, V_τ .

Wealth derivatives follow from the marginal integral identity. For example, pass to the limit in

$$V^\varepsilon(x_2, y, \tau) - V^\varepsilon(x_1, y, \tau) = \int_{x_1}^{x_2} e^{h^\varepsilon(\log r, y, \tau)} dr$$

and then differentiate to obtain

$$V_x = e^h, \quad V_{xx} = e^h p/x < 0, \quad V_{xy} = e^h s. \quad (6.16)$$

Together with the factor equation, this proves the classical HJB equation

$$V_\tau = \frac{1}{2}V_{yy} + \beta V_y - \frac{\lambda^2 V_x^2}{2V_{xx}}. \quad (6.17)$$

All marginal quantities here are evaluated at $\ell = \log x$.

Zero exposure in each auxiliary market gives $V^\varepsilon \geq U(x)$, hence $V \geq U(x)$. The uniform marginal bounds and the representation at $x = 1$ give, for all y and $\tau \leq T$,

$$|V(x, y, \tau)| \leq A_T(1 + x^N + x^{-N}), \quad V^+(x, y, \tau) \leq A_T(1 + x). \quad (6.18)$$

The second bound also follows from concavity and the uniform bounds on $V(1, y, \tau)$ and $V_x(1, y, \tau)$. Formula (6.14) gives the locally uniform terminal trace U .

6.4 Pathwise uniqueness from optimality

Lemma 6.2. *In the orthogonal market with bounded λ and bounded globally Lipschitz β , let v solve (6.17) classically at positive remaining times, have locally uniform terminal trace U , and satisfy $v_x > 0$, $v_{xx} < 0$, $v \geq U(x)$, and (6.18). Suppose its maximizing proportional exposure*

$$\kappa = -\lambda v_x / (x v_{xx})$$

is bounded on each finite horizon and continuous at positive remaining times, and the feedback system has a weak solution. Assign any bounded value to the feedback at calendar time T ; this leaves its stochastic integrals unchanged. Every weak feedback solution then attains v in the original admissible class on its Brownian filtration. The feedback system is pathwise unique and has a strong solution on the original Brownian probability space.

Proof. Every weak feedback solution has bounded logarithmic drift and diffusion, hence all finite positive and negative running wealth moments. Apply Itô's formula to $v(X_t, Y_t, T - t)$ before the terminal time. The maximizing feedback gives zero drift. The growth bound dominates its running absolute value by an integrable function of the positive and negative running wealth powers. It is a uniformly integrable martingale. Continuity of the state path and the local uniform terminal trace identify its terminal value as $U(X_T)$, so its expectation is $v(x, y, T)$.

For an arbitrary admissible competitor, fix a cash shift $a > 0$. The wealth $X + a$ starts from $x + a$ and is self-financing under the same exposure. The HJB inequality makes $v(X_t + a, Y_t, T - t)$ a local supermartingale bounded below by $U(a)$. Localization and Fatou's lemma at the terminal time give

$$\mathbb{E}U(X_T + a) \leq v(x + a, y, T).$$

Since $U(X_T) \leq U(X_T + a)$ and the positive part is integrable, letting $a \downarrow 0$ gives $\mathbb{E}U(X_T) \leq v(x, y, T)$. This argument applies to any filtration in which the driving coordinates remain Brownian motions.

Now place two weak feedback solutions with the same initial state on one Brownian space. The factor equation is Lipschitz, so the factor paths agree. Both wealth processes are admissible and optimal, and their average is admissible. Strict concavity forces equality of their terminal wealths. Let the minimal martingale density be

$$Z = \mathcal{E} \left(- \int \lambda(Y_t) dW_t \right).$$

For either feedback wealth, ZX has bounded proportional diffusion coefficient $\kappa - \lambda$ and is a true martingale. Hence

$$Z_t X_t^1 = \mathbb{E}[Z_T X_T^1 | \mathcal{F}_t] = \mathbb{E}[Z_T X_T^2 | \mathcal{F}_t] = Z_t X_t^2.$$

Positivity of Z gives equality of the wealth paths, proving pathwise uniqueness. Weak existence and pathwise uniqueness yield strong existence on the given probability space by the Yamada–Watanabe theorem [18]. Any other optimal strategy has the same terminal wealth. Its nonnegative supermartingale ZX has equal initial and terminal expectations, so the terminal value likewise determines its entire wealth path and exposure. $\square$

6.5 Weak limits of auxiliary controls and dual densities

For the strong auxiliary-market optimizers, the proportional exposures and orthogonal density coefficients are

$$\kappa_j^\varepsilon = -v_j^\varepsilon/p^\varepsilon, \quad \nu^\varepsilon = -s^\varepsilon.$$

They are uniformly bounded in ε and the initial state. Consider jointly optimal log-wealth, the factor, the logarithm of the normalized marginal density, and all driving Brownian coordinates. This joint process has bounded drift and diffusion. The Burkholder–Davis–Gundy inequality gives fourth-moment increment bounds of order $|t - s|^2$, hence tightness of the continuous-path laws.

Let Ξ^ε collect these coordinates, and write $\mathcal{G}_t^\varepsilon$ for their joint generator. For each $\phi \in C_c^\infty$, the process

$$\phi(\Xi_t^\varepsilon) - \phi(\Xi_0^\varepsilon) - \int_0^t \mathcal{G}_u^\varepsilon \phi(\Xi_u^\varepsilon) du$$

is a martingale. On $[0, T - \delta]$, the generators converge locally uniformly by (6.10). Their bounded coefficients and tightness control excursions outside compact sets; the integral over the last interval of length δ is bounded by $C_\phi \delta$. Testing against bounded continuous functions of finitely many earlier coordinates and passing to the limit preserves the martingale identities. Localized linear and quadratic tests preserve all cross-variations. In particular, the retained driving coordinates are continuous martingales with covariance matrix tI . Lévy’s characterization makes them a vector Brownian motion in the joint limiting filtration. Its coordinates are independent. The limit satisfies

$$\begin{aligned} d\ell_t &= (\kappa\lambda - \kappa^2/2) dt + \kappa dW_t^0, & \kappa &= -\lambda/p, \\ dY_t &= \beta(Y_t) dt + dB_t, & \frac{dL_t}{L_t} &= -\lambda(Y_t) dW_t^0 + s(\ell_t, Y_t, T - t) dB_t. \end{aligned} \quad (6.19)$$

For $j \geq 1$, the auxiliary traded coefficients vanish in the limit. More precisely, the uniform second wealth moment gives

$$\mathbb{E} \int_0^T |\vartheta_j^\varepsilon|^2 dt \leq C_T \varepsilon^{2j}.$$

This constructs the weak feedback solution required by Lemma 6.2. That lemma identifies V as the value of the original market and gives pathwise uniqueness and strong existence on the given probability space. All possible subsequential value limits consequently agree, and the marginal derivatives in (6.16) are uniquely determined.

The auxiliary normalized marginal identity is

$$L_t^\varepsilon = \frac{\exp h^\varepsilon(\ell_t^\varepsilon, Y_t, T - t)}{\exp h^\varepsilon(\log x, y, T)}.$$

On $[0, T - \delta]$, local uniform convergence of h^ε and weak convergence of the joint continuous paths pass this identity to the limit, first at rational times and then by continuity. As $t \uparrow T$, the deterministic estimate (6.11) gives

$$|h(\ell_t, Y_t, T - t) - h_0(\ell_t)| \leq A_T(\sqrt{T - t} + T - t).$$

The right-hand side tends to zero. The bounded coefficients of log-wealth give a finite continuous terminal limit. Since $h_0(\ell) = \log U'(e^\ell)$, the terminal identity is

$$L_t = \frac{\exp h(\ell_t, Y_t, T - t)}{z}, \quad U'(X_T) = zL_T, \quad z = \exp h(\log x, y, T). \quad (6.20)$$

This identity is obtained through the joint martingale-law limit. Classical Itô verification is applied to the constructed V . Given the strong feedback state and Brownian coordinates, the linear density equation has a unique solution, so the identity transfers to the given probability space.

The proportional diffusion coefficients of L and LX are bounded. Both are therefore true martingales, and $\mathbb{E}[L_T X_T] = x$. Proposition 5.3 now gives unique attainment in the fixed-multiplier and outer dual problems. Since $-a_+ \leq h_\ell \leq -a_-$, each multiplier selects a unique initial wealth. The bound $|\nu^*| \leq K(T)$ is uniform over this choice and gives exact deterministic truncation. This completes the proof of Theorem 2.5.

6.6 Examples and the scope of degeneracy

For example, $\lambda(y) = \tanh y$ satisfies

$$\lambda^2 + (\lambda')^2 = \tanh^2 y + \operatorname{sech}^4 y \geq \frac{3}{4},$$

whereas $\lambda(y) = \sin y$ makes the same sum identically one. For any fixed positive integer r , the first nonzero derivative of $(\tanh y)^r$ and $(\sin y)^r$ at a zero has order r . Their behavior at infinity or periodicity then verifies (2.16).

For an orthogonal factor satisfying $dY = b(Y) dt + a_F(Y) dB$ with $0 < a_0 \leq a_F \leq a_1$, set

$$\Phi(y) = \int_0^y a_F(v)^{-1} dv, \quad \bar{\beta} = (b/a_F - a'_F/2) \circ \Phi^{-1}, \quad \bar{\lambda} = \lambda \circ \Phi^{-1}.$$

If the normalized coefficients belong to C_b^∞ and

$$\inf_y \sum_{j=0}^m |(a_F \partial_y)^j \lambda(y)|^2 > 0, \quad (6.21)$$

then Theorem 2.5 transfers to the original factor coordinate through the Lamperti transformation. The optimal coefficient in that coordinate is $\nu^* = -a_F \partial_y \log V_x$. Admissibility under this transformation is checked in Appendix C.

The finite-order condition is uniform. Infinitely flat zeros, an open interval on which the risk price vanishes, simultaneous decay at infinity of all the indicated derivatives, and endogenous zero-participation sets in general correlated markets lie outside its hypotheses. Appendix B gives another class of factor degeneracies with an exact regularity classification.

7 Wealth-dependent hedging and dual information

7.1 The exact dual value under factor information

Consider one traded risk and one orthogonal factor, $dY = \beta(Y) dt + dB$. Assume $\beta, \lambda \in C_b^{4+\alpha}$, $0 < \lambda_0 \leq \lambda \leq M$, and λ is nonconstant. Terminal utility is state-independent, with conjugate

$$\tilde{U}(z) = \sum_{j=1}^J c_j z^{-r_j}, \quad c_j > 0, \quad r_j > 0, \quad (7.1)$$

where at least two exponents are distinct and repeated exponents have been combined. This utility satisfies the smooth-preference assumptions; see Appendix A.

Set

$$Z_t = \mathcal{E} \left(- \int_0^t \lambda(Y_u) dW_u \right), \quad N_t^\nu = \mathcal{E} \left(- \int_0^t \nu_u dB_u \right), \quad A_t = \int_0^t \lambda(Y_u)^2 du.$$

The factorization $L^\nu = ZN^\nu$ holds even when ν uses the full filtration. Let $\mathcal{F}_t^B$ denote the factor Brownian filtration and define

$$\mathcal{D}_B(x, y, T) = \inf_{z > 0, \nu \text{ bounded and } \mathcal{F}^B\text{-predictable}} \{ \mathbb{E} \tilde{U}(z Z_T N_T^\nu) + zx \}. \quad (7.2)$$

This quantity restricts the information available to the dual coefficient.

Set $\kappa_r = r(r+1)/2$. For $A \geq 0$, define

$$\mathfrak{F}(x, A) = \min_{z > 0} \left\{ \sum_j c_j e^{\kappa_{r_j} A} z^{-r_j} + xz \right\}. \quad (7.3)$$

The objective is strictly convex and tends to infinity at both endpoints. Its unique minimizer $\zeta_x(A)$ satisfies

$$x = \sum_j r_j c_j e^{\kappa_{r_j} A} \zeta_x(A)^{-r_j-1}. \quad (7.4)$$

The implicit function theorem gives smoothness, positive lower and upper bounds, and bounded derivatives on compact A intervals.

Proposition 7.1. *The restricted dual problem is attained by a bounded factor-predictable coefficient, and*

$$\mathcal{D}_B(x, y, T) = \mathbb{E}_y \mathfrak{F}(x, A_T).$$

Proof. For a factor-predictable coefficient, condition on the entire B path. The terminal factor density N_T^ν is then known and the traded integral is Gaussian, so

$$\mathbb{E}[Z_T^{-r} \mid \mathcal{F}_T^B] = e^{\kappa_r A_T}.$$

Using $\mathbb{E} N_T^\nu = 1$, the restricted objective equals

$$\mathbb{E} \left[\sum_j c_j e^{\kappa_{r_j} A_T} (z N_T^\nu)^{-r_j} + x z N_T^\nu \right] \geq \mathbb{E} \mathfrak{F}(x, A_T).$$

Set $z_B = \mathbb{E} \zeta_x(A_T)$ and prescribe terminal factor density $N_T = \zeta_x(A_T)/z_B$. It has deterministic positive lower and upper bounds and expectation one.

Let Y^v be the factor started at v and define

$$H(t, a, v) = \frac{1}{z_B} \mathbb{E} \zeta_x \left(a + \int_0^{T-t} \lambda(Y_u^v)^2 du \right).$$

For $0 \leq a \leq M^2 T$, the argument of ζ_x lies in $[0, M^2 T]$, so H has a uniform positive lower bound. The initial-state derivative of the factor is $J_u = \exp(\int_0^u \beta'(Y_r^v) dr)$. Hence

$$|H_v| \leq \frac{T e^{\|\beta'\|_\infty T} \|(\lambda^2)'\|_\infty}{z_B} \sup_{0 \leq a \leq M^2 T} |\zeta'_x(a)|.$$

Differentiating under the expectation and using the backward equation justify Itô's formula for $N_t = H(t, A_t, Y_t)$, giving $dN_t = H_v(t, A_t, Y_t) dB_t$. Thus $\nu_t = -H_v/H$ is bounded and factor-predictable, realizes the prescribed terminal density, and attains the pointwise minimum in (7.3). $\square$

7.2 Strict improvement with the full filtration

Theorem 7.2. *For every $x > 0$, $y \in \mathbb{R}$, and $T > 0$,*

$$V(x, y, T) = \mathcal{D}(x, y, T) < \mathcal{D}_B(x, y, T) = \mathbb{E}_y \mathfrak{F}(x, A_T). \quad (7.5)$$

The unique bounded optimal dual coefficient is not factor-predictable. In particular, it cannot be a deterministic function of time and the current factor alone.

Proof. Suppose a bounded factor-predictable coefficient ν is locally optimal, at a fixed multiplier $z > 0$, against every bounded predictable perturbation on the full filtration. For such a perturbation a , put

$$J_a = \int_0^T a_t dB_t + \int_0^T a_t \nu_t dt.$$

The exact density ratio is

$$\log \frac{N_T^{\nu+\varepsilon a}}{N_T^\nu} = -\varepsilon J_a - \frac{1}{2} \varepsilon^2 \int_0^T a_t^2 dt.$$

Exponential moments of stochastic integrals with bounded integrands justify differentiating the objective under expectation. Define

$$R_T = \sum_j r_j c_j z^{-r_j} L_T^{-r_j}, \quad R_t = \mathbb{E}[R_T | \mathcal{F}_t], \quad dR_t = H_t^B dB_t + H_t^W dW_t.$$

The first variation is $\mathbb{E}[R_T J_a]$. Martingale representation, Itô isometry, and conditional expectation yield

$$0 = \mathbb{E} \int_0^T a_t (H_t^B + \nu_t R_t) dt, \quad H^B + \nu R = 0 \quad dt \otimes d\mathbb{P}\text{-a.e.} \quad (7.6)$$

For each exponent r , let

$$S_t^r = \mathbb{E}[e^{\kappa_r A_T} N_T^{-r} | \mathcal{F}_t^B], \quad dS_t^r = h_t^r dB_t.$$

Conditional Gaussian integration over future traded noise gives

$$\mathbb{E}[L_T^{-r} | \mathcal{F}_t] = Z_t^{-r} e^{-\kappa_r A_t} S_t^r.$$

Substituting into (7.6) yields

$$\sum_j r_j c_j z^{-r_j} Z_t^{-r_j} e^{-\kappa_{r_j} A_t} (h_t^{r_j} + \nu_t S_t^{r_j}) = 0. \quad (7.7)$$

For almost every $t > 0$, condition on $\mathcal{F}_t^B$. The parenthesized coefficients are known, while Z_t^{-1} is a nondegenerate lognormal random variable with support $(0, \infty)$, since $A_t \geq \lambda_0^2 t$. Distinct real powers are linearly independent on this interval. Indeed, divide a vanishing finite sum by its largest power and let the argument tend to infinity, then proceed by induction. Thus $h_t^r = -\nu_t S_t^r$ for every exponent.

Uniqueness of the linear stochastic equation gives $S_t^r = S_0^r N_t$. At the terminal time,

$$N_T = (S_0^r)^{-1/(r+1)} e^{r A_T/2}. \quad (7.8)$$

Two distinct exponents force A_T to be deterministic.

The function λ^2 is nonconstant. Choose two small neighborhoods on which its ranges are separated. Construct two continuous factor paths from the same initial state y , each reaching one of these neighborhoods in an arbitrarily short initial period and remaining there until the terminal time. Their opportunity clocks differ when the initial period is sufficiently short. Gronwall's inequality makes the solution map of the additive-noise factor equation continuous with respect

to the driving path. Each factor path γ corresponds to the driving path $\gamma_t - y - \int_0^t \beta(\gamma_u) du$. Brownian full support and continuity of the opportunity-clock functional give two disjoint clock intervals with positive probability, contradicting (7.8).

Proposition 7.1 supplies a bounded optimizing pair for the restricted problem. If its value equaled the full-information value, its coefficient would also be locally optimal against all full-filtration perturbations at the same multiplier, giving the preceding contradiction. The inequality is therefore strict. Theorem 2.4 supplies attainment of the full dual problem, and the same argument excludes factor predictability of its optimizer. $\square$

The optimal coefficient has the wealth–factor Markov representation (2.15). The theorem shows that the wealth coordinate generally cannot be removed. Inequality (7.5) concerns a restriction on dual information. It does not define or compare the investor’s welfare under a different information constraint on the primal portfolio.

7.3 Purchasing-power preferences and short-horizon hedging demand

Let U_0 be a state-independent utility satisfying Assumption 2.2, and let $g \in C_b^{6+\alpha}(\mathbb{R}^d)$. Define

$$U(x, y) = U_0(xe^{-g(y)}). \quad (7.9)$$

Here $e^{g(Y_T)}$ is the terminal price index relative to the unit in which wealth is measured, with deterministic positive lower and upper bounds. If $h_{00}(\ell) = \log U'_0(e^\ell)$, then

$$\begin{aligned} h_0(\ell, y) &= h_{00}(\ell - g(y)) - g(y), \\ p_0 &= h'_{00}(\ell - g(y)), \quad s_0 = (A_{U_0}(e^{\ell - g(y)}) - 1)\nabla g(y), \\ \tilde{U}(z, y) &= \tilde{U}_0(ze^{g(y)}). \end{aligned} \quad (7.10)$$

These identities follow from the chain rule and a change of variables in the definition of the conjugate. Boundedness of g places $e^{-g(y)}$ in a compact subset of the positive axis, and all the required preference conditions follow.

Write

$$A_0(\ell, y) = A_{U_0}(e^{\ell - g(y)}), \quad v_g(y) = C^\top \nabla g(y), \quad \mathcal{H}(\ell, y, \tau) = \frac{C^\top s(\ell, y, \tau)}{-p(\ell, y, \tau)}.$$

The vector $\mathcal{H}$ is the factor-hedging component of proportional exposure.

Proposition 7.3 (Short-horizon purchasing-power demand). *In a market covered by the smooth nondegenerate results, for every fixed finite T there is $A_T < \infty$ such that*

$$\begin{aligned} \left| \mathcal{H}(\ell, y, \tau) - \left(1 - \frac{1}{A_0(\ell, y)}\right) v_g(y) \right| &\leq A_T \tau, \\ \left| \frac{\vartheta^*(e^\ell, y, \tau)}{e^\ell} - \frac{\lambda(y)}{A_0(\ell, y)} - \left(1 - \frac{1}{A_0(\ell, y)}\right) v_g(y) \right| &\leq A_T \tau, \end{aligned} \quad (7.11)$$

for all states and $0 \leq \tau \leq T$.

Proof. By (2.14), $\|h_\tau\|_{C_b^1}$ is bounded on $[0, T]$. Hence $\|p(\tau) - p_0\|_\infty + \|s(\tau) - s_0\|_\infty \leq A_T \tau$. Both denominators $-p$ and $-p_0$ are at least a_- . Substitute (7.10) into the feedback (5.4) and estimate the differences of the quotients to obtain both inequalities. $\square$

Consider the normalized power utilities

$$u_\gamma(x) = \begin{cases} (x^{1-\gamma} - 1)/(1 - \gamma), & \gamma \neq 1, \\ \log x, & \gamma = 1 \end{cases},$$

and set $U_0 = c_1 u_{\gamma_1} + c_2 u_{\gamma_2}$, where $c_i > 0$ and $0 < \gamma_1 < 1 < \gamma_2$. Its relative risk aversion

$$A_{U_0}(x) = \frac{c_1 \gamma_1 x^{-\gamma_1} + c_2 \gamma_2 x^{-\gamma_2}}{c_1 x^{-\gamma_1} + c_2 x^{-\gamma_2}}$$

tends to $\gamma_2 > 1$ at low wealth and to $\gamma_1 < 1$ at high wealth.

Corollary 7.4. *Under these purchasing-power preferences and the smooth nondegenerate market assumptions, fix y_0 with $v_g(y_0) \neq 0$. There exist a low-wealth compact interval, a high-wealth compact interval, a neighborhood of y_0 , and $\tau_0 > 0$ such that, for $0 \leq \tau \leq \tau_0$,*

$$\mathcal{H}(\log x, y, \tau) \cdot v_g(y) > 0 \quad \text{on the low-wealth interval,}$$

$$\mathcal{H}(\log x, y, \tau) \cdot v_g(y) < 0 \quad \text{on the high-wealth interval.}$$

Proof. The two endpoint limits of risk aversion and boundedness of g allow us to choose compact wealth intervals on which $A_0 - 1$ has respectively a strictly positive and a strictly negative margin. Shrink the factor neighborhood so that $|v_g|$ has a positive lower bound. The terminal inner product is $(1 - A_0^{-1})|v_g|^2$, with opposite signs and uniform margins on the two compact state sets. Estimate (7.11) preserves these signs for sufficiently short horizons. $\square$

This corollary describes a change in the direction of the factor-hedging component. For $U_0(x) = \log x$, one has $s_0 = 0$ and the price index contributes only an additive utility term. Direct calculation gives

$$V(x, y, \tau) = \log x + \mathbb{E}_y \left[\frac{1}{2} \int_0^\tau |\lambda(Y_t)|^2 dt - g(Y_\tau) \right], \quad \vartheta^*/x = \lambda, \quad \nu^* = 0.$$

General power mixtures and bounded logarithmic modulations are developed in Appendix A.

8 Attainment under square-integrable strategies

8.1 Proof of the terminal-payoff criterion

Throughout this subsection, c , U , Z , z , and H satisfy the hypotheses of Theorem 2.6. In particular, c is deterministic and bounded, and the budget and integrability conditions are (2.17). Write

$$A = \int_0^T c_t^2 dt$$

for the accumulated squared market price of risk.

Proof of Theorem 2.6. For any competing wealth process, ZX is a nonnegative local martingale. The budget inequality and Fenchel's inequality therefore give

$$\mathbb{E}U(X_T) \leq \mathbb{E}\tilde{U}(zZ_T) + zx = \mathbb{E}U(H). \quad (8.1)$$

Concavity bounds the positive part of utility by a linear function, while the square-integrability condition makes X_T integrable. The expectation is consequently well defined in the extended sense.

Let

$$H_n = \frac{x}{\mathbb{E}[Z_T(H \wedge n)]} (H \wedge n).$$

The payoff is bounded, strictly positive, and measurable with respect to the traded Brownian motion, because c is deterministic. Under $d\mathbb{Q} = Z_T d\mathbb{P}$, the process $W_t^{\mathbb{Q}} = W_t + \int_0^t c_u du$ is Brownian and remains independent of the additional factor Brownian motion. Hence conditioning H_n on the full filtration is the same as conditioning it on the traded filtration. Martingale

representation in that filtration gives a replication with initial wealth x and bounded nonnegative wealth.

We verify square-integrable admissibility under $\mathbb{P}$. With $M = \|c\|_\infty$,

$$|\vartheta|^2 + 2cX\vartheta \geq \frac{1}{2}|\vartheta|^2 - 2M^2X^2.$$

For a localizing sequence $\tau_j \uparrow T$ and $\kappa > 2M^2$, the stopped Itô identity gives

$$\frac{1}{2}\mathbb{E} \int_0^{\tau_j} e^{\kappa t} |\vartheta_t|^2 dt \leq \mathbb{E}[e^{\kappa \tau_j} X_{\tau_j}^2] - X_0^2.$$

The bounded replication wealth is continuous through T . Dominated convergence on the right and monotone convergence on the left prove $\mathbb{E} \int_0^T |\vartheta_t|^2 dt < \infty$.

The normalization constants decrease to one and are at least one, so $U(H_n) \geq U(H \wedge n)$. The negative part of $U(H \wedge 1)$ is integrable by (2.17). Subtracting that variable and applying monotone convergence yields $\mathbb{E}U(H \wedge n) \rightarrow \mathbb{E}U(H)$. Together with (8.1), this proves the value assertion and the existence of the approximating sequence.

If an admissible portfolio attains the value, its terminal Fenchel gap and budget slack are nonnegative and have zero expected sum. Strict concavity forces $X_T = H$. Every admissible portfolio, however, satisfies

$$\mathbb{E}|X_T|^2 \leq 3x^2 + 3(1 + TM^2)\mathbb{E} \int_0^T |\vartheta_t|^2 dt < \infty. \quad (8.2)$$

This proves necessity.

Conversely, suppose $H \in L^2(\mathbb{P})$. Replicate the unnormalized payoffs $H \wedge n$, each at its own initial price. For differences of two such replications, the exponential-square Itô estimate gives

$$\mathbb{E} \int_0^T |\Delta \vartheta_t|^2 dt \leq 2e^{\kappa T} \mathbb{E}|\Delta H|^2.$$

The initial-price difference satisfies $|\Delta x|^2 \leq \mathbb{E}Z_T^2 \mathbb{E}|\Delta H|^2$. The wealth equation, Cauchy–Schwarz, and the Burkholder–Davis–Gundy inequality then imply

$$\mathbb{E} \sup_{t \leq T} |\Delta X_t|^2 + \mathbb{E} \int_0^T |\Delta \vartheta_t|^2 dt \leq C_T \mathbb{E}|\Delta H|^2. \quad (8.3)$$

The terminal truncations are Cauchy in $L^2(\mathbb{P})$, so the wealth processes and exposures converge in the displayed spaces. Their initial prices converge to x . The limiting wealth is nonnegative, has terminal value H , and satisfies the original wealth equation and admissibility conditions. This is also a particular instance of a linear L^2 backward stochastic equation [19].

Strict concavity determines the terminal wealth of every optimizer. For any optimal wealth, the nonnegative supermartingale ZX has initial and terminal expectation x , and is therefore a martingale closed by $Z_T H$. This determines its wealth path. Uniqueness of the semimartingale decomposition then determines the exposure. $\square$

8.2 A smooth utility with vanishing relative risk aversion

Fix $\zeta > 0$ and $b_* > 1$, and prescribe the inverse marginal utility

$$I_\zeta(z) = z^{-b_*} \exp \left\{ \zeta [\log(1 + z^{-1})]^2 \right\}. \quad (8.4)$$

Define

$$\tilde{U}_\zeta(z) = \int_z^\infty I_\zeta(r) dr, \quad U_\zeta(x) = \inf_{z > 0} \{ \tilde{U}_\zeta(z) + xz \}. \quad (8.5)$$

The condition $b_* > 1$ makes this integral finite at every $z > 0$. The function I_ζ is smooth, strictly decreasing, and onto $(0, \infty)$. Consequently U_ζ is a C^∞ , strictly increasing, strictly concave utility with the Inada limits and $U_\zeta(0) = 0$. Its relative risk aversion is

$$A_{U_\zeta}(I_\zeta(z)) = \left[b_* + \frac{2\zeta \log(1 + z^{-1})}{1 + z} \right]^{-1}. \quad (8.6)$$

It varies with wealth and tends to zero as wealth tends to infinity.

For $A > 0$, let $N \sim N(0, 1)$ and set

$$Z_A = e^{-\sqrt{A}N - A/2}, \quad H_{z,A} = I_\zeta(zZ_A).$$

Lemma 8.1 (Exact moment boundary). *For every $q > 0$ and $z > 0$,*

$$\mathbb{E}H_{z,A}^q < \infty \iff \begin{cases} q\zeta A < 1/2, & \text{or} \\ q\zeta A = 1/2 \text{ and } \log z > A/2 + b_*/(2\zeta) \end{cases}. \quad (8.7)$$

Proof. Set $u = \sqrt{A}N + c_0$, where $c_0 = A/2 - \log z$. As $N \rightarrow +\infty$,

$$\log(1 + e^u) = u + O(e^{-u}), \quad \log I_\zeta(e^{-u}) = b_*u + \zeta u^2 + o(1).$$

After multiplication by the standard normal density, the logarithm of the integrand is

$$(q\zeta A - 1/2)N^2 + q\sqrt{A}\{b_* + 2\zeta(A/2 - \log z)\}N + O(1). \quad (8.8)$$

The $O(1)$ remainder has a finite limit. A negative quadratic coefficient gives integrability, and a positive one gives divergence. When the quadratic coefficient is zero, integrability on the positive half-line holds exactly when the linear coefficient is strictly negative. If that coefficient also vanishes, the integrand tends to a positive constant and its integral diverges. On the negative half-line, $I_\zeta(e^{-u}) = e^{b_*u}(1 + o(1))$, whose every positive power is integrable. This proves (8.7). $\square$

8.3 The horizon and wealth boundary for attainment

Theorem 8.2. *For utility (8.5) and deterministic investment opportunities, suppose*

$$0 < A < \frac{1}{2\zeta}. \quad (8.9)$$

For every $x > 0$, the value in the original admissible class is finite, equals the static conjugate value, and is smooth as a function of (x, A) on this open region. Attainment in (2.3) is classified as follows.

1. *If $0 < A < 1/(4\zeta)$, the optimum is attained at every positive initial wealth.*
2. *If $A = 1/(4\zeta)$, define*

$$z_c = \exp\{A/2 + b_*/(2\zeta)\}, \quad x_c = \mathbb{E}[Z_A I_\zeta(z_c Z_A)].$$

The optimum is attained exactly when $0 < x < x_c$. It is not attained at $x = x_c$ or at any larger wealth.

3. *If $1/(4\zeta) < A < 1/(2\zeta)$, the supremum is not attained at any positive initial wealth.*

In all three cases, the orthogonal dual coefficient $\nu = 0$ attains the dual value.

Proof. On every compact subset of (8.9), the budget integrand $Z_A I_\zeta(z Z_A)$ and its parameter derivatives admit Gaussian integrable bounds with a strict margin in the quadratic exponent. The conjugate term has the same type of bound. Specifically, for large positive u , the substitution $r = e^{-v}$ yields

$$\tilde{U}_\zeta(e^{-u}) = \tilde{U}_\zeta(1) + \int_0^u e^{(b_*-1)v + \zeta[\log(1+e^v)]^2} dv.$$

This is bounded by a polynomial in $1 + u$ times $e^{\zeta u^2 + (b_*-1)u + O(1)}$; for negative u it has order $e^{(b_*-1)u}$. Parameter differentiation of any fixed order introduces only polynomial factors. All required expectations are therefore finite and smooth.

The budget map

$$x(z, A) = \mathbb{E}[Z_A I_\zeta(z Z_A)]$$

is strictly decreasing, with strictly negative z -derivative $\mathbb{E}[Z_A^2 I'_\zeta(z Z_A)]$. Monotone convergence gives the limit infinity as $z \downarrow 0$, while dominated convergence gives zero as $z \uparrow \infty$. Every $x > 0$ consequently selects a unique smooth multiplier $z(x, A)$. The terminal Fenchel equality makes $U_\zeta(H) = \tilde{U}_\zeta(z Z_A) + z Z_A H$ nonnegative and integrable. Theorem 2.6 applies, and (8.7) with $q = 2$ gives the attainment classification. At the critical point, strict decrease of the budget map makes $z > z_c$ equivalent to $x < x_c$.

Put

$$v(z, A) = \mathbb{E}\tilde{U}_\zeta(z Z_A), \quad F_\zeta(x, A) = \inf_{z>0} \{v(z, A) + zx\}.$$

Differentiating these integrable Gaussian expressions gives

$$v_A = \frac{1}{2} z^2 v_{zz}, \quad v_{zz} > 0, \quad (F_\zeta)_x = z, \quad (F_\zeta)_{xx} = -1/v_{zz} < 0.$$

The implicit function theorem therefore gives smoothness of F_ζ and

$$(F_\zeta)_A = -\frac{(F_\zeta)_x^2}{2(F_\zeta)_{xx}}. \tag{8.10}$$

For a constant risk price c , the function $V(x, \tau) = F_\zeta(x, c^2 \tau)$ solves the portfolio HJB equation classically and has terminal trace equal to the original utility.

For each fixed z , choose $x = x(z, A)$. The density Z_A has zero orthogonal coefficient and realizes the finite static conjugate value. Every other dual density with bounded coefficients gives a Fenchel upper bound for the bounded replicating approximations in the proof of Theorem 2.6. Letting their expected utilities converge to $F_\zeta(x, A)$ shows that every competing fixed-multiplier objective is at least $v(z, A)$. Hence the zero orthogonal coefficient attains both the fixed-multiplier and outer dual values. $\square$

For example, if $b_* = 2$, $\zeta = 3/8$, and the constant risk price is one, the critical horizon for square-integrable attainment is $T = 2/3$, and the value is finite and smooth for $0 < T < 4/3$. The critical multiplier is $z_c = e^3$. In particular, at $T = 1$ every positive initial wealth has a finite classical value, while the supremum is unattained in the original square-integrable strategy class. A nondegenerate Brownian factor with any fixed correlation may be added, with coefficients independent of it. The original asset and factor coefficients can all be smooth and bounded, and both volatilities can be bounded away from zero. This example separates the moment requirement on the optimal payoff from classical regularity of the value.

9 Conclusion

The risk-aversion interval and joint gradient estimate for logarithmic marginal utility provide common a priori bounds for nonhomothetic portfolio choice. With several assets and state-dependent preferences, a risk-price component outside the factor-hedging span ensures uniform

parabolicity. A scalar reaction equation supplies another class of primitive conditions in correlated single-factor markets. For state-independent utility, mollification reduces the assumptions to C^4 regularity and positive lower and upper bounds on relative risk aversion. In an orthogonal single-factor market, finite-order zeros of the risk price remain compatible with a classical value and a strong optimal portfolio on the given probability space. Bounded orthogonal coefficients of the optimal density yield exact deterministic dual truncation, uniformly over initial states and multipliers.

The moment condition for attainment plays a separate role. With deterministic opportunities and an integrable static problem, attainment in the original class of exposures square-integrable in expectation is equivalent to membership of the static optimal payoff in $L^2(\mathbb{P})$. The utility family studied above gives a strict horizon and wealth boundary within a region of finite smooth values. Short-horizon purchasing-power demand and the strict dual information result show how nonhomothetic preferences make risk demand and the supporting measure depend on wealth. The deterministic-factor classification in the appendix reduces value regularity to regularity of the actual opportunity clock and realizes every finite order of derivative loss. These results concern risk-price geometry, factor dynamics, and terminal-payoff integrability, under the assumptions stated in the respective theorems.

A Power mixtures and relative curvature

A.1 Two classes of nonhomothetic utility mixtures

Let μ be a finite nonzero positive measure supported on $[r_-, r_+] \subset (0, \infty)$. Define

$$\tilde{U}(z) = \int z^{-r} \mu(dr), \quad U(x) = \inf_{z>0} \{\tilde{U}(z) + xz\}, \quad I(z) = \int rz^{-r-1} \mu(dr). \quad (\text{A.1})$$

Compact support permits differentiation under the integral to every order. The inverse marginal I is strictly decreasing and maps $(0, \infty)$ onto itself. The unique minimizer is therefore smooth, with $U' = z$ and $U'' = 1/I'(z) < 0$, and the Inada limits hold. As $z \uparrow \infty$, both $\tilde{U}(z)$ and $zI(z)$ tend to zero, so $U(0) = 0$.

Terminal relative risk tolerance is

$$H_0(q) = \frac{\int r(r+1)e^{-rq} \mu(dr)}{\int re^{-rq} \mu(dr)} \in [1+r_-, 1+r_+]. \quad (\text{A.2})$$

This is the mean of $r+1$ under the probability measure $re^{-rq} \mu(dr) / \int re^{-rq} \mu(dr)$. Its normalized moments satisfy $m'_j = -m_{j+1} + m_j m_1$. All their derivatives are therefore polynomials in moments of a random variable in a fixed compact interval, and are uniformly bounded. Since $\partial_\ell = -H_0^{-1} \partial_q$, all required logarithmic wealth derivatives of relative risk aversion are bounded. A bound on the next derivative supplies the corresponding Hölder regularity, proving Assumption 2.2. If μ is not concentrated at one point, H'_0 is minus the variance of r under the tilted measure and is strictly negative, so relative risk aversion is nonconstant. In every market covered by the paper,

$$1+r_- \leq H \leq 1+r_+.$$

A second class is defined directly through primal utility. Let χ be a finite nonzero positive measure supported on $[\gamma_-, \gamma_+] \subset (0, \infty)$, and set

$$U(x) = \int u_\gamma(x) \chi(d\gamma), \quad u_\gamma(x) = \begin{cases} (x^{1-\gamma} - 1)/(1-\gamma), & \gamma \neq 1, \\ \log x, & \gamma = 1 \end{cases}. \quad (\text{A.3})$$

The integral representation $u_\gamma(x) = \int_1^x r^{-\gamma} dr$ removes the apparent singularity at $\gamma = 1$. Consequently,

$$U'(x) = \int x^{-\gamma} \chi(d\gamma), \quad A_U(x) = \frac{\int \gamma x^{-\gamma} \chi(d\gamma)}{\int x^{-\gamma} \chi(d\gamma)} \in [\gamma_-, \gamma_+].$$

The same normalized-moment argument applies directly in logarithmic wealth coordinates. Moreover, $U(1) = 0$ and $h_0(0) = \log \chi([\gamma_-, \gamma_+])$. All smooth preference conditions hold, and the exponent interval may lie on either side of one or cross one. If the measure is not concentrated at a single point, $\partial_\ell A_U$ is a strictly negative tilted variance, making the utility nonhomothetic. Both constructions describe a single investor's utility and a single common portfolio choice.

A useful two-power conjugate is

$$\tilde{U}(z) = z^{-1} + \frac{1}{2}z^{-2}, \quad I(z) = z^{-2} + z^{-3}, \quad A_U(I(z)) = \frac{z+1}{2z+3}. \quad (\text{A.4})$$

Its true indirect relative risk tolerance remains in the interval $2 \leq H \leq 3$ in every applicable market.

A.2 Bounded logarithmic modulation and horizon-uniform curvature

Take $a, r > 0$ and $f \in C_b^{6+\alpha}(\mathbb{R})$. Suppose that, for some $0 < e < 1$,

$$1 - e \leq f \leq 1 + e, \quad g = rf - f', \quad \inf g > 0, \quad \delta_f = \|g'/g\|_\infty < r. \quad (\text{A.5})$$

Define $\tilde{U}_f(z) = az^{-r}f(\log z)$ and let U_f be the associated utility obtained by the concave Fenchel transform. Its inverse marginal is

$$I(z) = az^{-r-1}g(\log z), \quad \frac{d \log I(e^q)}{dq} = -(r+1) + \frac{g'(q)}{g(q)}.$$

Thus $I' < 0$, the range of I is the positive axis, and $H_0 = r+1-g'/g$ belongs to $[r+1-\delta_f, r+1+\delta_f]$. The positive lower and upper bounds, composition estimates, and inverse-coordinate formula show that U_f satisfies the smooth preference assumptions, with $U_f(0) = 0$. The function f need not be periodic.

Proposition A.1. *Suppose U_f and the reference conjugate $\tilde{U}_0(z) = az^{-r}$ are covered by the same market theorem in this paper, and denote their true logarithmic dual values by $\mathcal{W}_f, \mathcal{W}_0$. Then*

$$(1-e)\mathcal{W}_0 \leq \mathcal{W}_f \leq (1+e)\mathcal{W}_0, \quad r+1-\delta_f \leq H_f \leq r+1+\delta_f, \\ (1-e) \left(1 - \frac{\delta_f}{r}\right) \left(1 - \frac{\delta_f}{r+1}\right) \leq \frac{Q_f}{Q_0} \leq (1+e) \left(1 + \frac{\delta_f}{r}\right) \left(1 + \frac{\delta_f}{r+1}\right). \quad (\text{A.6})$$

The comparison constants are independent of the investment horizon.

Proof. For the same fixed multiplier and each admissible density, the terminal conjugates satisfy $(1-e)\tilde{U}_0 \leq \tilde{U}_f \leq (1+e)\tilde{U}_0$. Taking expectations and then infima gives the value comparison by the proved dual representation. Preservation of the risk-aversion interval gives the bounds on H_f . Homogeneity of the reference problem gives $\mathcal{W}_0(q, y, \tau) = e^{-r\tau}A(y, \tau)$, so $P_0 = r\mathcal{W}_0$ and $Q_0 = r(r+1)\mathcal{W}_0$.

For the modulated value, $P_q = (1-H_f)P$, and hence for every $u \geq 0$,

$$P(q)e^{-(r+\delta_f)u} \leq P(q+u) \leq P(q)e^{-(r-\delta_f)u}.$$

The value comparison implies $\mathcal{W}_f(q) \rightarrow 0$ as $q \rightarrow \infty$. Therefore $\mathcal{W}_f(q) = \int_0^\infty P(q+u) du$. Integration yields $(r-\delta_f)\mathcal{W}_f \leq P \leq (r+\delta_f)\mathcal{W}_f$. Using $Q = H_f P$ and dividing by the reference curvature proves the claim. $\square$

For example, if $r = 1$, $f(q) = 1 + \varepsilon \sin q$, and $0 < |\varepsilon| \leq 1/10$, then

$$g(q) = 1 + \varepsilon(\sin q - \cos q), \quad \delta_f \leq \frac{\sqrt{2}|\varepsilon|}{1 - \sqrt{2}|\varepsilon|} < 1.$$

Each such fixed nonzero amplitude is admissible on every finite horizon in the corresponding global market classes.

A.3 The power-utility representation for one correlated factor

If $\tilde{U}_0(z) = az^{-r}$, then

$$U_0(x) = (r+1)a \left(\frac{x}{ar} \right)^{r/(r+1)}.$$

In the one-asset, one-factor model, let

$$\phi_\tau = \frac{1}{2}\phi_{yy} + (\beta + r\rho\lambda)\phi_y + \frac{r(1+r\rho^2)}{2}\lambda^2\phi, \quad \phi(y, 0) = 1. \quad (\text{A.7})$$

The Feynman–Kac representation makes ϕ strictly positive and smooth on each finite horizon. The corresponding candidates are

$$V_0(x, y, \tau) = U_0(x)\phi^{1/(1+r\rho^2)}, \quad \mathcal{W}_0(q, y, \tau) = ae^{-rq}\phi^{(r+1)/(1+r\rho^2)}. \quad (\text{A.8})$$

Writing $V_0 = U_0F$ reduces the HJB equation to

$$F_\tau = \frac{1}{2}F_{yy} + (\beta + r\rho\lambda)F_y + \frac{r}{2}\lambda^2F + \frac{r\rho^2}{2}\frac{F_y^2}{F}.$$

For $F = \phi^\gamma$, the coefficient of the quadratic gradient term is proportional to $\gamma((1+r\rho^2)\gamma - 1)$. Thus $\gamma = 1/(1+r\rho^2)$ gives (A.7). This is the power transformation of Zariphopoulou [20] in the present parameters. In the market classes covered here, the verification in Section 5 identifies it with the true value.

B Exact regularity in deterministic factor regions

B.1 The opportunity clock for a compact mixture of powers

Take the conjugate mixture (A.1) from Appendix A, and set $\kappa_r = r(r+1)/2$. For $A \in \mathbb{R}$, define

$$\mathfrak{F}_\mu(x, A) = \min_{z>0} \left\{ \int e^{\kappa_r A} z^{-r} \mu(dr) + xz \right\}. \quad (\text{B.1})$$

The unique minimizer $z = z(x, A)$ satisfies

$$x = \int r e^{\kappa_r A} z^{-r-1} \mu(dr).$$

Strict convexity, coercivity at both endpoints, and the implicit function theorem show that $\mathfrak{F}_\mu$ is smooth on $(0, \infty) \times \mathbb{R}$, with

$$(\mathfrak{F}_\mu)_A(x, A) = \int \kappa_r e^{\kappa_r A} z(x, A)^{-r} \mu(dr) > 0. \quad (\text{B.2})$$

Negative values of A are used only to extend this deterministic function smoothly. Actual opportunity clocks remain nonnegative.

Proposition B.1. *In a market with bounded deterministic risk price c_t , put $A = \int_0^T c_t^2 dt$. The true value for the conjugate mixture (A.1) in the original class (2.3) is $\mathfrak{F}_\mu(x, A)$, and is attained by a strictly positive wealth process. The filtration may contain additional independent Brownian information.*

Proof. Set $Z_t = \mathcal{E}(-\int_0^t c_u dW_u)$ and $z = z(x, A)$. Every admissible portfolio satisfies $\mathbb{E}[Z_T X_T] \leq x$. The identity $\mathbb{E}Z_T^{-r} = e^{\kappa_r A}$ and Fenchel's inequality make (B.1) a common upper bound.

Let $A_t = \int_t^T c_u^2 du$, and define

$$J_r(t) = rz^{-r-1}e^{\kappa_r A_t}Z_t^{-r-1}, \quad X_t^* = \int J_r(t)\mu(dr), \quad \vartheta_t^* = c_t \int (r+1)J_r(t)\mu(dr).$$

Itô's formula gives

$$dJ_r = (r+1)c_t J_r dW_t + (r+1)c_t^2 J_r dt.$$

The exponent set is compact and c is bounded, so all required moments are uniform in r , permitting interchange of the parameter integral and stochastic integrals. Thus X^* starts from x , is strictly positive, and satisfies the wealth equation. It has running moments of every positive order and square-integrable exposure. At the terminal time, $X_T^* = I(zZ_T)$ and $\mathbb{E}[Z_T X_T^*] = x$. The budget and Fenchel equalities prove optimality. $\square$

B.2 Local equivalence of regularity

Suppose the factor diffusion vanishes on a neighborhood of the reachable paths under consideration. Its deterministic flow satisfies

$$\dot{\varphi}_u(y) = b(\varphi_u(y)), \quad \varphi_0(y) = y.$$

Define

$$A(y, \tau) = \int_0^\tau \lambda(\varphi_u(y))^2 du. \quad (\text{B.3})$$

For each initial state the opportunity set is deterministic. Proposition B.1 gives $V(x, y, \tau) = \mathfrak{F}_\mu(x, A(y, \tau))$.

Theorem B.2. *Fix $x_0 > 0$. On an open set of factor and positive remaining-time states where the preceding representation holds,*

$$V(x_0, \cdot, \cdot) \in C_{\text{loc}}^{2,1} \iff A \in C_{\text{loc}}^{2,1}. \quad (\text{B.4})$$

At fixed horizon, the same equivalence holds for C_{loc}^j and $C_{\text{loc}}^{j,\gamma}$, for every integer $j \geq 0$ and $0 < \gamma \leq 1$. If the stated regularity holds at one positive wealth level, it holds at all positive wealth levels.

Proof. Composition with the smooth function $\mathfrak{F}_\mu$ proves sufficiency. By (B.2), for fixed x_0 the map $A \mapsto \mathfrak{F}_\mu(x_0, A)$ has a smooth inverse near every actual clock value, including $A = 0$. Hence

$$A(y, \tau) = \mathfrak{F}_\mu(x_0, \cdot)^{-1}(V(x_0, y, \tau)),$$

locally. Smooth composition preserves the stated regularity classes on compact subsets, proving necessity. Wealth derivatives and the required mixed derivatives follow from the chain rule for the smooth outer function. $\square$

B.3 Every finite order of derivative loss

Proposition B.3. *For every integer $j \geq 1$, there exist globally Lipschitz market and factor coefficients of at most linear growth, a constant strictly positive asset volatility, a smooth bounded strictly positive risk price, and a nonhomothetic C^∞ utility such that the true value is locally $C^{j-1,1}$ but not C^j in the factor at zero. The optimizing portfolios for initial states in this neighborhood satisfy (2.3). The construction applies to every fixed $\rho \in (-1, 1)$.*

Proof. Take the two-power utility (A.4), and set

$$b(y) = |y|, \quad a_F(y) = \begin{cases} 0, & |y| \leq 1, \\ \exp[-(y^2 - 1)^{-2}], & |y| > 1 \end{cases}.$$

At ± 1 all derivatives from outside the interval tend to zero, so the flat extension is smooth; its first derivative is globally bounded. Choose $\chi \in C_c^\infty$ equal to one near zero, and put $\lambda(y) = 1 + \varepsilon \chi(y) y^j$,

where $\varepsilon > 0$ is small enough that $\lambda \geq 1/2$. Take the asset volatility to be a constant $\sigma_0 > 0$ and the excess drift to be $\sigma_0 \lambda$. All coefficients satisfy the stated conditions.

For every fixed finite horizon, a sufficiently small initial y makes the entire factor path remain where $a_F = 0$ and $\chi = 1$. The unique path is ye^u for $y \geq 0$ and ye^{-u} for $y < 0$. Both Brownian terms vanish, independently of ρ . The two opportunity clocks are exactly

$$A_{\pm}(y, \tau) = \tau + 2\varepsilon y^j \int_0^\tau e^{\pm ju} du + \varepsilon^2 y^{2j} \int_0^\tau e^{\pm 2ju} du. \quad (\text{B.5})$$

The derivatives of orders below j agree at zero, while

$$\partial_y^j A_+(0, \tau) - \partial_y^j A_-(0, \tau) = 2\varepsilon(j-1)! \{e^{j\tau} + e^{-j\tau} - 2\} > 0.$$

The true-value representation and (B.2) give

$$\partial_y^j V_+(x, 0, \tau) - \partial_y^j V_-(x, 0, \tau) = 2\varepsilon(j-1)! \{e^{j\tau} + e^{-j\tau} - 2\} (\mathfrak{F}_\mu)_A(x, \tau) > 0. \quad (\text{B.6})$$

Piecewise smoothness and matching lower-order derivatives give $C^{j-1,1}$ regularity. Proposition B.1 supplies a strictly positive admissible optimizer for initial states in the neighborhood. Inverse density moments under a bounded risk price also give a finite dual upper bound at every initial factor state. The optimization problem is therefore finite globally. $\square$

In contrast, if $b = a_F = 0$ and $\lambda(y) = \tanh y$, then $A(y, \tau) = \tau \tanh^2 y$ is smooth. The true value for the same nonhomothetic utility remains smooth at $y = 0$, where the entire optimal risky exposure vanishes. In the deterministic-factor class, equivalence (B.4) identifies regularity of the actual opportunity clock.

C Time dependence, factor coordinates, and sensitivity

C.1 A Banach time topology for nonautonomous coefficients

For every finite calendar interval $[0, T]$, assume

$$\beta \in C^1([0, T]; C_b^{4+\alpha}(\mathbb{R}^d; \mathbb{R}^d)), \quad \lambda \in C^1([0, T]; C_b^{4+\alpha}(\mathbb{R}^d; \mathbb{R}^n)). \quad (\text{C.1})$$

Time differentiation is taken in the displayed Banach norms. Thus, for $a = \beta, \lambda$,

$$\sup_{t \leq T} (\|a(t)\|_{C_b^{4+\alpha}} + \|\partial_t a(t)\|_{C_b^{4+\alpha}}) < \infty, \quad \|a(t) - a(s)\|_{C_b^{4+\alpha}} \leq M_T |t - s|.$$

Keep C constant and require

$$\inf_{0 \leq t \leq T, y} |\Pi \lambda(t, y)| > 0. \quad (\text{C.2})$$

The terminal preference satisfies Assumption 2.2. Write the value as $V(x, y, t; T)$ and set $h(\ell, y, \tau) = \log V_x(e^\ell, y, T - \tau; T)$.

Proposition C.1. *Under (C.1)–(C.2), the smooth-preference conclusions of Theorem 2.4 hold on every fixed interval $[0, T]$. A single deterministic bound on the dual coefficient applies to all initial calendar times, initial states, and multipliers within that interval.*

Proof. Replace the autonomous coefficients in (4.2) by $\beta(T - \tau, y), \lambda(T - \tau, y)$, and denote the resulting map by $\mathcal{F}_T(\tau, f)$. On every bounded parabolic neighborhood, the Hölder product and composition estimates give

$$\begin{aligned} & \|\mathcal{F}_T(\tau, f) - \mathcal{F}_T(\sigma, f)\|_{E_0^\eta} \\ & + \|D_f \mathcal{F}_T(\tau, f) - D_f \mathcal{F}_T(\sigma, f)\|_{\mathcal{L}(E_1^\eta, E_0^\eta)} \leq A|\tau - \sigma|. \end{aligned}$$

The sectorial-domain verification in Lemma 4.1 is uniform on this neighborhood. Lunardi's nonautonomous local theorem [13, Theorem 8.1.1] therefore applies in the actual Banach norms.

The gradient estimate uses only spatial differentiation, so $K(T)$ is formed from the supremum of the coefficient norms over $[0, T]$. The height barriers hold with $C_0 = \sup_{\tau \leq T} \|\mathcal{F}_T(\tau, 0)\|_\infty$. Time-Lipschitz coefficients and the established gradient bounds provide the parabolic Hölder conditions in (4.6). The same spatial differentiation gives (4.7). Endpoint interpolation produces a strong trace in $bc^{4+\eta}$, and joint continuity of $\mathcal{F}_T$ gives convergence of the time derivative in $bc^{2+\eta}$. The nonautonomous local theorem restarts the solution at this trace. The reconstruction equation and Itô verification use coefficients at the current calendar time, and the budget and duality arguments remain valid. Norms over the full interval also control every shorter remaining horizon. $\square$

For the scalar comparison equation, calendar-time dependence contributes the source term $-\lambda_t$. All coefficients below are evaluated at $(T - \tau, y)$:

$$(\partial_\tau - \mathcal{L}_h)k = \left(\beta_y - \frac{\lambda_y}{\rho}\right)k + \frac{\lambda\lambda_y}{\rho} - (\beta\lambda)_y - \frac{1}{2}\lambda_{yy} - \lambda_t.$$

When the corresponding conditions on signed sources or deterministic subsolutions and supersolutions hold, the comparison argument of Theorem 4.6 continues to apply.

C.2 Nonconstant one-dimensional factor diffusion

Suppose the physical factor satisfies

$$dY = b(Y)dt + a_F(Y)(\rho dW + \sqrt{1 - \rho^2} dB), \quad 0 < a_0 \leq a_F \leq a_1 < \infty.$$

Assume the coefficients have the smoothness required for the transformations below; in particular, take $a_F \in C^6$. The Lamperti transformation

$$z = \Phi(y) = \int_0^y a_F(v)^{-1} dv$$

is a global smooth increasing diffeomorphism. The normalized data are

$$\bar{b} = (b/a_F - a'_F/2) \circ \Phi^{-1}, \quad \bar{\lambda} = \lambda \circ \Phi^{-1}, \quad \bar{U}(x, z) = U(x, \Phi^{-1}(z)). \quad (\text{C.3})$$

Itô's formula gives unit factor volatility in the normalized coordinate. Whenever (C.3) satisfies the hypotheses of the applicable theorem, its conclusions transfer to the original coordinate. A deterministic invertible change of factor coordinates preserves the filtration, the wealth equation, and the expected integral of squared traded exposure. The original-coordinate feedback formulas are

$$k = \lambda + \rho a_F \partial_y \log V_x, \quad \vartheta^* = -xk/p, \quad \nu^* = -\sqrt{1 - \rho^2} a_F \partial_y \log V_x.$$

The scalar source conditions are differentiated in the normalized coordinate. In the orthogonal finite-order theorem, $\partial_z = a_F \partial_y$, so (6.21) becomes exactly (2.16). The corresponding normalized coefficients must satisfy the C_b^∞ assumptions.

C.3 Initial-state sensitivity in smooth nondegenerate models

Proposition C.2. *Fix a finite horizon covered by the smooth nondegenerate results. The optimal orthogonal dual coefficient is continuously differentiable as a function of initial logarithmic wealth and factors, with values in $L^2(\Omega \times [0, T]; \mathbb{R}^d)$. The same conclusion holds with the initial logarithmic multiplier in place of initial logarithmic wealth.*

Proof. Write $Z_t = (\ell_t^*, Y_t)$. Its drift and diffusion columns from (2.2) and (5.5) are denoted by b^* and σ_j^* . The smooth nondegenerate estimates give uniform bounds on their first two state derivatives over the fixed horizon. The variational matrix with respect to the initial state $a = (\ell, y)$ satisfies

$$dJ_t = Db^*(Z_t, t)J_t dt + \sum_{j=1}^{n+d} D\sigma_j^*(Z_t, t)J_t d\mathcal{B}_t^j, \quad J_0 = I_{1+d}, \quad \mathcal{B} = (W, B).$$

The Burkholder–Davis–Gundy and Gronwall inequalities give all required running moments of finite order of J . Subtracting the equations for neighboring initial states yields $\mathbb{E} \sup_t |Z_t^{a+v} - Z_t^a|^r \leq A_r |v|^r$. Expand the coefficients to first order. Their bounded second derivatives bound the remainders by the squared state difference. Substitution into the integral equations shows that the difference quotients converge in $L^2(\Omega; C[0, T])$ to the displayed variational equation. The same estimate for neighboring variational equations gives continuity of the derivative.

Since $\nu_t^* = -D^\top s(Z_t, T - t)$,

$$D_a \nu_t^* = -D^\top \nabla_{\ell, y} s(Z_t, T - t) J_t$$

holds in the stated L^2 space. For an initial multiplier e^q , the initial wealth is determined by $h(\ell, y, T) = q$, with $\ell_q = 1/p$ and $\nabla_y \ell = -s/p$. The chain rule proves the second assertion. $\square$

Supplementary material and data availability

The supplementary file `anc/identities.py` contains exact symbolic calculations for the marginal and gradient equations, principal matrices, density identities, utility transformations, and moment boundaries. The calculations require no empirical dataset. The dependency specification and execution instructions are included with the supplementary files.

AI-assisted language editing

ChatGPT was used for proofreading and language editing. The authors retain responsibility for the content of the manuscript.

References

- [1] R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413, 1971. doi:10.1016/0022-0531(71)90038-X.
- [2] R. C. Merton. An intertemporal capital asset pricing model. *Econometrica*, 41(5):867–887, 1973. doi:10.2307/1913811.
- [3] M. J. Brennan and Y. Xia. Dynamic asset allocation under inflation. *Journal of Finance*, 57(3):1201–1238, 2002. doi:10.1111/1540-6261.00459.
- [4] J. A. Londoño. State-dependent utility. *Journal of Applied Probability*, 46(1):55–70, 2009. doi:10.1239/jap/1238592116.
- [5] T. Zariphopoulou. Optimal asset allocation in a stochastic factor model—an overview and open problems. In H. Albrecher, W. J. Runggaldier, and W. Schachermayer, editors, *Advanced Financial Modelling*, volume 8 of Radon Series on Computational and Applied Mathematics. Walter de Gruyter, Berlin, 2009. doi:10.1515/9783110213140.427. Author manuscript TZ-50.
- [6] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999. doi:10.1214/aoap/1029962818.
- [7] D. Kramkov and W. Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, 13(4):1504–1516, 2003. doi:10.1214/aoap/1069786508.

- [8] D. Kramkov and M. Sirbu. On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, 16(3):1352–1384, 2006. doi:10.1214/105051606000000259.
- [9] S. Nadtochiy and T. Zariphopoulou. An approximation scheme for solution to the optimal investment problem in incomplete markets. *SIAM Journal on Financial Mathematics*, 4(1):494–538, 2013. doi:10.1137/120869080. Complete author version TZ-58.
- [10] B. Bian, S. Miao, and H. Zheng. Smooth value functions for a class of nonsmooth utility maximization problems. *SIAM Journal on Financial Mathematics*, 2(1):727–747, 2011. doi:10.1137/100793396.
- [11] U. Horst, Y. Hu, P. Imkeller, A. Réveillac, and J. Zhang. Forward–backward systems for expected utility maximization. *Stochastic Processes and their Applications*, 124(5):1813–1848, 2014. doi:10.1016/j.spa.2014.01.004.
- [12] M. Mania and R. Tevzadze. Backward stochastic PDEs related to the utility maximization problem. *Georgian Mathematical Journal*, 17(4):705–740, 2010. doi:10.1515/gmj.2010.038.
- [13] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Progress in Nonlinear Differential Equations and Their Applications, Vol. 16. Birkhäuser, Basel, 1995. doi:10.1007/978-3-0348-9234-6.
- [14] D. G. Aronson and J. Serrin. Local behavior of solutions of quasilinear parabolic equations. *Archive for Rational Mechanics and Analysis*, 25:81–122, 1967. doi:10.1007/BF00281291.
- [15] D. G. Aronson. Harnack, Hölder, Gauss and Widder: Serrin’s parabolic legacy. *arXiv preprint*, arXiv:1611.09417, 2016. arXiv:1611.09417.
- [16] L. Capogna, G. Citti, and G. Rea. A subelliptic analogue of Aronson–Serrin’s Harnack inequality. *Mathematische Annalen*, 357(3):1175–1198, 2013. doi:10.1007/s00208-013-0937-y. Author manuscript arXiv:1109.4596v2.
- [17] M. Bramanti and L. Brandolini. Schauder estimates for parabolic nondivergence operators of Hörmander type. *Journal of Differential Equations*, 234(1):177–245, 2007. doi:10.1016/j.jde.2006.07.015.
- [18] T. Yamada and S. Watanabe. On the uniqueness of solutions of stochastic differential equations. *Journal of Mathematics of Kyoto University*, 11(1):155–167, 1971. doi:10.1215/kjm/1250523691.
- [19] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990. doi:10.1016/0167-6911(90)90082-6.
- [20] T. Zariphopoulou. A solution approach to valuation with unhedgeable risks. *Finance and Stochastics*, 5:61–82, 2001. doi:10.1007/PL00000040.