

Hall polynomials for representation-finite algebras

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Abstract

We prove the Hall polynomial existence conjecture for representation-finite algebras over finite fields.

1 Introduction

Let $k = \mathbb{F}_q$ and let A be a finite-dimensional k -algebra of finite representation type. For a finite-dimensional right A -module M and a finite extension E/k , write $M_E = M \otimes_k E$. For such modules X, Y, L , define

$$F_{X,Y}^L(E) = \#\{U \subseteq L_E : U \simeq Y_E, L_E/U \simeq X_E\}.$$

The first lower index is the quotient and the second is the submodule. A finite extension is *conservative for A* if every indecomposable A -module remains indecomposable after scalar extension. We say that A has Hall polynomials if, for every X, Y, L , there is a polynomial $\varphi_{X,Y}^L \in \mathbb{Z}[T]$ such that

$$F_{X,Y}^L(E) = \varphi_{X,Y}^L(|E|) \quad \text{for every conservative } E/k. \quad (1.1)$$

Ringel's Hall polynomial existence conjecture asserts that every representation-finite algebra over a finite field has this property; see [5, §1.1]. Hall polynomials for representation-finite hereditary algebras were established by Ringel [10]. Hall polynomials for nilpotent cyclic-quiver representations provide another basic counting input; see [6, Theorem 16].

Theorem 1.1. *Every representation-finite algebra over a finite field has Hall polynomials.*

The proof first treats geometrically standard algebras. A representation-finite algebra over an algebraically closed field is *mesh-standard* if its category of indecomposable modules is equivalent, with its object labels, to the mesh category of its Auslander–Reiten quiver. We call A *geometrically standard* if $A_{\bar{k}}$ is mesh-standard, where $\bar{k}$ is an algebraic closure of k . The mesh relations come from almost split sequences; their finite-field modulation is described in Section 5.

Theorem 1.2. *Every geometrically standard representation-finite algebra over a finite field has Hall polynomials.*

For Theorem 1.2, the contravariant defect of a short exact sequence determines its minimal endpoints and middle term through its dimension vector, top and socle. An orbit formula expresses Hall numbers through the masses of these defects. We compute the masses on the stable Auslander algebra by completing finite translation intervals, decomposing local data

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into strings and bands, and evaluating the quadratic character sums. Total masses on periodic Dynkin components determine the remaining band contribution.

The standard-form classification [3] reduces the additional characteristic-two cases to penny configurations. Their radical-to-socle-quotient maps differ by a root shear. For each graph map into a quotient module, we extend that shear modulo maps through projectives; pullback by a target automorphism then permutes extension classes while preserving the required middle-module labels. At isolated roots this gives all Hall numbers of the standard form. The three possible root attachments have two exceptional indecomposables; matrix-rank counts and the recursions in Section 8 determine their remaining Hall coefficients. Section 9 completes the proof over the original finite field by node splitting and central restriction of scalars.

Use of generative AI. ChatGPT and Codex were used to develop mathematical arguments, perform computational checks, and draft and revise the manuscript. The characteristic-two arguments were developed and cross-checked with Codex agents and revised with Codex.

2 Defects and Hall numbers

Write $\text{mod } A$ for the category of finite-dimensional right A -modules and add M for the full subcategory of direct summands of finite direct sums of M . The notation M^n denotes a direct sum of n copies of M , with $M^0 = 0$. For a finite extension E/k put $Q = |E|$. Fix representatives $I_1, \dots, I_m$ of the indecomposable A -modules, and write

$$K_i = \text{End}_A(I_i) / \text{rad } \text{End}_A(I_i) = \mathbb{F}_{q^{d_i}}, \quad D_A = \text{lcm}(d_1, \dots, d_m).$$

Finite division rings are fields. Since k is perfect, radicals commute with separable scalar extension. Thus $E = \mathbb{F}_{q^r}$ is conservative precisely when $\gcd(r, D_A) = 1$. Moreover, the modules $(I_i)_E$ form a complete, nonrepeating indecomposable list over every such E . Indeed, an A_E -module is a direct summand of the scalar extension of its restriction to A , because E/k is separable. The restriction is in $\text{add}(\bigoplus_i I_i)$. The semisimple quotient of the endomorphism algebra of this sum becomes $\prod_i (K_i \otimes_k E)$, a product of fields; this also proves that the extended indecomposables remain pairwise nonisomorphic.

Throughout the remainder of this section, E/k is conservative for A .

Write $h(M, N) = \dim_k \text{Hom}_A(M, N)$. For $M = \bigoplus_i I_i^{n_i}$, set

$$a_M(T) = T^{h(M, M) - \sum_i d_i n_i^2} \prod_i \prod_{j=0}^{n_i-1} (T^{d_i n_i} - T^{d_i j}). \quad (2.1)$$

Empty products equal one, so $a_0 = 1$. The formula for the units of a finite algebra, applied to $\text{End}_A(M)$, gives $a_M(|E|) = |\text{Aut}(M_E)|$. All these polynomials are monic.

We use the subring

$$\mathcal{R} = \mathbb{Z}[T, T^{-1}, (T^j - 1)^{-1} : j \geq 1] \subseteq \mathbb{Q}(T).$$

The inverses of the polynomials in (2.1) belong to $\mathcal{R}$.

Lemma 2.1. *Let $R \in \mathcal{R}$. If $R(Q)$ is an integer for an unbounded set of positive integers Q , at which it is defined, then $R \in \mathbb{Z}[T]$.*

Proof. Write $R = P/B$ with $P, B \in \mathbb{Z}[T]$ and B monic. Division in $\mathbb{Z}[T]$ gives $P = BH + U$ with $\deg U < \deg B$. Along the given integers, $U(Q)/B(Q)$ is an integer tending to zero. Hence U has infinitely many zeros and vanishes. $\square$

2.1 Minimal exact sequences

Put $G = \bigoplus_i I_i$ and let

$$\Lambda = \text{End}_A(G)/[\text{proj } A],$$

where $[\text{proj } A]$ is the ideal of maps factoring through projective modules. We use contravariant additive functors on $\text{add } G$. Evaluation at G , with precomposition as the right action, identifies the finite-dimensional functors vanishing on projectives with right Λ -modules. Put $P_M = \text{Hom}_A(-, M)|_{\text{add } G}$.

For a short exact sequence

$$\sigma : \quad 0 \longrightarrow Y \xrightarrow{i} L \xrightarrow{p} X \longrightarrow 0$$

define $D_\sigma = \text{coker}(P_L \rightarrow P_X)$. In the category of all finite-dimensional contravariant additive functors on $\text{add } G$, equivalently right $\text{End}_A(G)$ -modules, it has the projective resolution

$$0 \longrightarrow P_Y \longrightarrow P_L \longrightarrow P_X \longrightarrow D_\sigma \longrightarrow 0. \quad (2.2)$$

Every finite-dimensional right Λ -module arises in this way. To see this, take a projective presentation in this full functor category, $P_L \rightarrow P_X \rightarrow D \rightarrow 0$. Yoneda identifies the first map with a morphism $p : L \rightarrow X$. Evaluating at A and using $D(A) = 0$ shows that p is surjective. Its kernel supplies (2.2).

Every finite projective resolution is the direct sum of a minimal resolution and contractible complexes. Here this follows by eliminating an invertible matrix component of a differential until all differentials lie in the radical of $\text{add } G$. Uniqueness follows successively from uniqueness of projective covers. Consequently, there is a unique minimal sequence

$$\sigma_D : \quad 0 \rightarrow Y_D \rightarrow L_D \rightarrow X_D \rightarrow 0$$

with defect D , and every sequence with that defect is uniquely, up to isomorphism, of the form

$$\sigma_D \oplus (0 \rightarrow B \xrightarrow{1} B \rightarrow 0 \rightarrow 0) \oplus (0 \rightarrow 0 \rightarrow C \xrightarrow{1} C \rightarrow 0). \quad (2.3)$$

Thus its terms are $Y_D \oplus B$, $L_D \oplus B \oplus C$, and $X_D \oplus C$.

For a nonprojective I_i let $S_i = P_{I_i}/\text{rad}(-, I_i)$, where $\text{rad}(-, I_i)$ is the subfunctor of radical morphisms into I_i . For a finite-length functor, the top is its largest semisimple quotient and the socle is its largest semisimple subfunctor. For a defect D , write

$$\delta_i(D) = \dim_k D(I_i), \quad t_i(D) = [\text{top } D : S_i], \quad s_i(D) = [\text{soc } D : S_i].$$

The brackets denote multiplicity in a semisimple functor. At projective labels all three numbers are zero. After extension to E , the dimension is taken over E and the labels are $(I_i)_E$ and $(S_i)_E$. In particular, δ_i is divisible by d_i . Define

$$X(t) = \bigoplus_{I_i \text{ nonprojective}} I_i^{t_i}, \quad Y(s) = \bigoplus_{I_i \text{ nonprojective}} (\tau I_i)^{s_i}.$$

Here τ denotes the Auslander–Reiten translation from the projective stable category to the injective stable category, and its values are taken without injective summands.

Lemma 2.2. *For every defect D , one has $X_D \simeq X(t(D))$ and $Y_D \simeq Y(s(D))$.*

Proof. The epimorphism $P_{X_D} \rightarrow D$ in the minimal resolution is a projective cover, proving the first assertion. Define the covariant defect

$$D'_\sigma = \text{coker}(\text{Hom}_A(L, -) \rightarrow \text{Hom}_A(Y, -)).$$

Auslander's defect formula gives natural isomorphisms $D'_\sigma(\tau I) \simeq \text{Hom}_k(D_\sigma(I), k)$; see [7, Proposition 2.5]. One can obtain them by writing $D_\sigma(I)$ as the cokernel of the corresponding stable Hom map, dualizing, and applying Auslander–Reiten duality to the resulting kernel.

The covariant representable resolution associated with a minimal sequence is minimal: both maps remain in the radical. Thus $\text{Hom}_A(Y_D, -) \rightarrow D'_\sigma$ is its projective cover. The translation equivalence preserves the radical and the residue division fields, while k -duality interchanges socle and top. The multiplicity of τI_i in Y_D is therefore $s_i(D)$. Injective summands of Y_D would split off contractible summands, and hence do not occur. $\square$

Let $H = (h(I_i, I_j))_{i,j=1}^m$. We identify the split Grothendieck group with $\mathbb{Z}^m$ by the classes of the I_i . The matrix H is invertible over $\mathbb{Q}$. For a nonprojective I_j , let E_j be the middle term of its almost split sequence. Applying $\text{Hom}_A(I_i, -)$ shows that the column $[\tau I_j] + [I_j] - [E_j]$ is sent by H to d_j times the j th unit vector. For projective I_j , use $[I_j] - [\text{rad } I_j]$ instead: every nonisomorphism from an indecomposable to I_j has image in its unique maximal submodule. Let C be the matrix with these columns. Then $HC = \text{diag}(d_1, \dots, d_m)$.

For a sequence with terms X, Y, L it follows that

$$\delta_i = h(I_i, Y) + h(I_i, X) - h(I_i, L), \quad (2.4)$$

and, for a minimal sequence of defect data δ, t, s ,

$$[L_D] = [X(t)] + [Y(s)] - H^{-1}\delta. \quad (2.5)$$

In particular the minimal middle term is determined by those data.

2.2 Stabilizers and the mass formula

Lemma 2.3. *If the contractible terms of σ are B, C , then over a field of order Q ,*

$$|\text{Aut}(\sigma)| = Q^{h(L,Y)+h(X,L)-h(X,Y)-h(B,B)-h(C,C)} |\text{Aut}(D_\sigma)| a_B(Q) a_C(Q). \quad (2.6)$$

Proof. There is an exact sequence of vector spaces

$$\begin{aligned} 0 \rightarrow \text{Hom}(X, Y) &\longrightarrow \text{Hom}(L, Y) \oplus \text{Hom}(X, L) \\ &\longrightarrow \text{End}(\sigma) \longrightarrow \text{End}(D_\sigma) \rightarrow 0, \end{aligned}$$

where the first two maps are $u \mapsto (up, -iu)$ and $(s, t) \mapsto (si, is + tp, pt)$. The last map is onto by lifting along the projective resolution (2.2). If a chain map induces zero on the defect, its component on X lifts to pt for a map $t : X \rightarrow L$. Its middle component minus tp factors through i , giving is , and the chain condition gives the component si on Y . Finally, $si = pt = is + tp = 0$ implies $s = up$ and $t = -iu$. This proves exactness and

$$\dim \text{End}(\sigma) = \dim \text{End}(D_\sigma) + h(L, Y) + h(X, L) - h(X, Y).$$

For a minimal resolution, every null-homotopic chain endomorphism has radical components. If f is such an endomorphism and g is arbitrary, $1 - gf$ is invertible in every degree and its inverses form a chain map. Hence the kernel of $\text{End}(\sigma_D) \rightarrow \text{End}(D)$ lies in the Jacobson radical and the semisimple quotients agree.

The category of these finite complexes is Krull–Schmidt. Its minimal indecomposable summands correspond to the indecomposable summands of D . Neither type of nonzero contractible summand is isomorphic to a minimal summand or to the other contractible type. Therefore the semisimple quotient of $\text{End}(\sigma)$ is the product of those of $\text{End}(D)$, $\text{End}(B)$, and $\text{End}(C)$. Applying

$$|R^\times| = Q^{\dim R - \dim(R/\text{rad } R)} |(R/\text{rad } R)^\times|$$

to these four endomorphism algebras proves (2.6). $\square$

Define the mass, over the current extension field, by

$$\mathfrak{M}_{\delta,t,s}(E) = \sum_{\substack{[D] \in \text{mod } \Lambda_E \\ \delta(D)=\delta, t(D)=t, s(D)=s}} \frac{1}{|\text{Aut}(D)|}.$$

For fixed dimension this is a finite sum. The mass of the zero module is one. Incompatible data have mass zero.

Proposition 2.4. *Let E/k be conservative for A . Fix X, Y, L and let δ be given by (2.4). If $B_s = Y \ominus Y(s)$ and $C_t = X \ominus X(t)$ denote existing direct-sum complements, then*

$$F_{X,Y}^L(E) = a_L(Q) Q^{h(X,Y)-h(L,Y)-h(X,L)} \sum_{t,s} \frac{Q^{h(B_s,B_s)+h(C_t,C_t)}}{a_{B_s}(Q)a_{C_t}(Q)} \mathfrak{M}_{\delta,t,s}(E). \quad (2.7)$$

The sum ranges over precisely the pairs for which both complements exist.

Proof. The group $\text{Aut}(L_E)$ acts on the submodules counted by the left-hand side. The stabilizer of a submodule is the automorphism group of its short exact sequence, since the maps induced on the submodule and quotient are uniquely determined. For a fixed defect D , decomposition (2.3) shows that every nonempty layer is one orbit. Its complements are determined by the endpoints. Equation (2.5), together with (2.4), then gives the required middle term. Apply (2.6) and sum the orbit sizes. $\square$

Corollary 2.5. *If every mass $\mathfrak{M}_{\delta,t,s}$ is given by an element of $\mathcal{R}$ on all conservative extensions, then A has Hall polynomials.*

Proof. Every sum in (2.7) is finite, and all its other factors belong to $\mathcal{R}$. The result takes integer values on the unbounded set of conservative field sizes. Apply Lemma 2.1. $\square$

The mass transformation is reversible. For fixed δ, t, s , form the vector in (2.5). If it is not a nonnegative integral vector, the mass is zero. Otherwise let L_0 be the corresponding module and apply (2.7) to $X(t), Y(s), L_0$. After multiplication by $Q^{h(L_0,Y(s))+h(X(t),L_0)-h(X(t),Y(s))}/a_{L_0}(Q)$, the coefficient of $\mathfrak{M}_{\delta,t,s}$ is one, and all other terms have smaller $\sum_i (t_i + s_i)$. This also proves the converse of Corollary 2.5.

3 Nilpotent cycles and square boundaries

3.1 Semilinear cyclic Hall polynomials

Let K be a finite field of order z , let $p \geq 1$, and index vertices by $\mathbb{Z}/p\mathbb{Z}$. Fix automorphisms θ_i of K . A semilinear cyclic representation consists of finite-dimensional K -spaces V_i and θ_i -semilinear maps $T_i : V_i \rightarrow V_{i+1}$. A morphism is a collection of K -linear maps commuting with the T_i . We restrict to nilpotent representations: every sufficiently long consecutive composite of arrows is zero.

For $i \in \mathbb{Z}/p\mathbb{Z}$ and $l \geq 1$, let $I(i, l)$ be the chain with basis positions of colours $i, i+1, \dots, i+l-1$, with the prescribed semilinear transport between consecutive positions and zero after the last position. A label is a finitely supported family $(m_{i,l})$ of nonnegative integers, denoting $\bigoplus_{i,l} I(i, l)^{m_{i,l}}$.

Lemma 3.1. *For fixed chain labels, the automorphism orders and Hall numbers are polynomials in z with integer coefficients, independent of the automorphisms θ_i . The automorphism polynomials are monic and their inverses belong to $\mathbb{Z}[z, z^{-1}, (z^j - 1)^{-1} : j \geq 1]$.*

Proof. Put $V = \bigoplus_i V_i$ and write T for the additive map on V given by the arrows and set $W_h = \ker T \cap \operatorname{im} T^h$, degree by degree. Choose bases of the descending flags W_h and lift a basis of W_h/W_{h+1} through T^h . The resulting strings of length $h+1$ are independent: applying the highest remaining power of T to a linear relation gives a relation among the chosen socle basis vectors. Semilinearity applies an invertible field automorphism to each coefficient, so the coefficients vanish. The number of chain positions is

$$\sum_{h \geq 0} (h+1) \dim_K(W_h/W_{h+1}) = \sum_{h \geq 0} \dim_K W_h = \dim_K V.$$

The last identity follows from $\dim W_h = \dim \operatorname{im} T^h - \dim \operatorname{im} T^{h+1}$. This proves existence of the chain decomposition. Its multiplicities are determined by the colours and dimensions of W_h/W_{h+1} , proving uniqueness.

Put $r_{i,h}(V) = \dim_K(\operatorname{im} T^h)_i$ and $z_{i,h}(V) = r_{i,h}(V) - r_{i,h+1}(V)$. The chain multiplicities can equivalently be recovered by

$$m_{i,l} = z_{i+l-1,l-1}(V) - z_{i+l,l}(V), \quad (3.1)$$

with colour indices read modulo p . Thus fixed total dimension gives a finite label set.

For integers $0 \leq e \leq b$, the Gaussian polynomial $\begin{bmatrix} b \\ e \end{bmatrix}_z = \prod_{j=0}^{e-1} (z^{b-j} - 1)/(z^{e-j} - 1)$ counts e -dimensional subspaces of K^b and is zero outside this range. First count a semisimple submodule S of L with quotient X . Write $s_i = \dim_K S_i$ and

$$W_{i,h} = (\operatorname{soc} L)_i \cap (\operatorname{im} T^h)_i, \quad c_{i,h} = r_{i,h}(L) - r_{i,h}(X).$$

A submodule $U \simeq S$ is a tuple of subspaces $U_i \subseteq (\operatorname{soc} L)_i$. Since $\operatorname{im}(T^h : L/U \rightarrow L/U) = (\operatorname{im} T^h + U)/U$, its quotient is X precisely when $\dim(U_i \cap W_{i,h}) = c_{i,h}$ for all i, h , by (3.1). Define

$$b_{i,h} = \dim W_{i,h} - \dim W_{i,h+1}, \quad e_{i,h} = c_{i,h} - c_{i,h+1}.$$

The count is zero unless $c_{i,0} = s_i$, the tails vanish, and $0 \leq e_{i,h} \leq b_{i,h}$. Otherwise it is

$$F_{X,S}^L(z) = \prod_i \left(z^{\sum_{h < a} e_{i,h}(b_{i,a} - e_{i,a})} \prod_{h \geq 0} \begin{bmatrix} b_{i,h} \\ e_{i,h} \end{bmatrix}_z \right). \quad (3.2)$$

To obtain (3.2), choose the image in each successive flag quotient and then its graph into the deeper unchosen quotient spaces. These choices contribute, respectively, the Gaussian factors and the displayed power of z .

Use the unmodified Hall multiplication $u_X u_Y = \sum_L F_{X,Y}^L u_L$. For a module M , choose $l \geq 1$ with $\operatorname{rad}^l M = 0$, and let $S_h = \operatorname{rad}^h M / \operatorname{rad}^{h+1} M$, where the radical is the arrow image. Expanding $u_{S_0} \cdots u_{S_{l-1}}$ by left-associated multiplication uses only (3.2). A term u_N in this product corresponds to a descending flag with these semisimple quotients. It satisfies $\operatorname{rad}^h N \subseteq N_h$ and hence $r_{i,h}(N) \leq r_{i,h}(M)$. Equality for all i, h implies $N \simeq M$ by (3.1); then the only possible flag is the radical filtration. Its coefficient is one. Therefore

$$u_{S_0} \cdots u_{S_{l-1}} = u_M + \sum_{\sum_{i,h \geq 1} r_{i,h}(N) < \sum_{i,h \geq 1} r_{i,h}(M)} a_{M,N}(z) u_N$$

with integer polynomial coefficients. Induction expresses every u_M as an integer polynomial combination of words in semisimple modules. Expand the second factor of $u_X u_Y$ in such words and multiply successively on the right by semisimple factors. This proves the Hall assertion, with the same polynomials for every holonomy.

A map between two chains is determined by the image of its first generator: this image has the required colour and is killed by the prescribed power of T . Its free coordinates are counted by a power of z . The endomorphism ring of one chain has residue field K ; the leading generator coefficient is arbitrary in K , and zero leading coefficient gives the nilpotent radical. Thus, if $M = \bigoplus_I I^{m_I}$ and $|\text{End}(M)| = z^{e_M}$,

$$|\text{Aut}(M)| = z^{e_M - \sum_I m_I^2} \prod_I |\text{GL}_{m_I}(K)|.$$

The exponent and factors depend only on the labels, proving the final assertion. $\square$

A Hall flag polynomial with successive quotients $T_1, \dots, T_n$ means the coefficient of u_L in $u_{T_n} \cdots u_{T_1}$, or equivalently the number of flags $0 = L_0 \subseteq \cdots \subseteq L_n = L$ with $L_j/L_{j-1} \simeq T_j$. Lemma 3.1 applies as well to a disjoint union of semilinear cycles, by taking products.

3.2 Total mass at a square boundary

For integers $n, s \geq 1$, consider vertices (i, t) with $0 \leq i < n$ and $t \in \mathbb{Z}/s\mathbb{Z}$. Put a copy of K at each vertex. The forward arrows $a_{i,t} : (i, t) \rightarrow (i+1, t)$ are linear, the central arrows $\varepsilon_t : (0, t) \rightarrow (0, t+1)$ are θ_t -semilinear, and the reverse arrows $b_{i,t} : (i+1, t) \rightarrow (i, t+2)$ have transport $\theta_{t+1}\theta_t$. Composition is written with the rightmost map acting first. For $n \geq 2$ impose

$$\begin{aligned} \varepsilon_{t+1}\varepsilon_t &= b_{0,t}a_{0,t}, \\ a_{i-1,t+2}b_{i-1,t} &= b_{i,t}a_{i,t} \quad (1 \leq i < n-1), \\ a_{n-2,t+2}b_{n-2,t} &= 0. \end{aligned} \tag{3.3}$$

For $n = 1$ impose just $\varepsilon_{t+1}\varepsilon_t = 0$.

Proposition 3.2. *For every fixed vertex-dimension vector, the total mass $\sum_{[V]} 1/|\text{Aut}(V)|$ of these representations is given by an element of $\mathbb{Z}[z, z^{-1}, (z^j - 1)^{-1} : j \geq 1]$, independent of the θ_t .*

Proof. Fix the forward-chain representations M_t of $0 \rightarrow 1 \rightarrow \cdots \rightarrow n-1$. They are sums of interval representations, so only finitely many types occur in the fixed dimension. At the initial vertex define

$$F_{0,t} = 0, \quad F_{j,t} = \ker(M_{0,t} \rightarrow M_{j,t}) \quad (1 \leq j < n), \quad F_{n,t} = M_{0,t}.$$

Write $f_{j,t} = \dim_K F_{j,t}$. Fix t , put $\sigma = \theta_{t+1}\theta_t$, and choose bases in the forward chains. Let A_i and B_i be the matrices of $a_{i,t}$ and $a_{i,t+2}$, respectively. The twisted chain σM_t has the same vertex dimensions as M_t and arrow matrices $\sigma(A_i)$, obtained by applying σ entrywise. If the reverse arrow is $v \mapsto R_i\sigma(v)$, its relation map is the K -linear map

$$\rho : \bigoplus_{i=0}^{n-2} \text{Hom}_K((\sigma M_t)_{i+1}, M_{i,t+2}) \longrightarrow \bigoplus_{i=0}^{n-1} \text{Hom}_K((\sigma M_t)_i, M_{i,t+2})$$

with components

$$\begin{aligned} (\rho R)_0 &= R_0\sigma(A_0), \\ (\rho R)_i &= R_i\sigma(A_i) - B_{i-1}R_{i-1} \quad (1 \leq i < n-1), \\ (\rho R)_{n-1} &= -B_{n-2}R_{n-2}. \end{aligned}$$

If $\varepsilon_{t+1}\varepsilon_t(v) = C_t\sigma(v)$, the required equation is $\rho R = (C_t, 0, \dots, 0)$.

Pair matrices in opposite directions by $\langle f, g \rangle = \text{tr}(fg)$, where tr is the sum of the diagonal entries, and sum over vertices or arrows. For maps $f_i : M_{i,t+2} \rightarrow (\sigma M_t)_i$, let

$$(df)_i = f_{i+1}B_i - \sigma(A_i)f_i.$$

Direct expansion gives

$$\sum_i \text{tr}(f_i(\rho R)_i) = - \sum_{i=0}^{n-2} \text{tr}((df)_i R_i).$$

Thus ρ is the negative transpose of d . The chain Hom complex has kernel $\text{Hom}(M_{t+2}, \sigma M_t)$ and cokernel $\text{Ext}^1(M_{t+2}, \sigma M_t)$: an extension splits at each vertex, its off-diagonal arrow blocks are arbitrary, and changing these splittings changes the blocks by d . Consequently

$$\dim_K \ker \rho = \dim_K \text{Ext}^1(M_{t+2}, \sigma M_t),$$

and the equation for R is solvable exactly when $\text{tr}(f_0 C_t) = 0$ for every $f \in \text{Hom}(M_{t+2}, \sigma M_t)$.

An interval starting at zero and ending at a contributes an endpoint flag direction born in layer $a+1$. Between two such intervals a map with nonzero component at zero exists exactly when the target birth layer is no larger than the source birth layer. Intervals starting later have zero endpoint space. Hence evaluation at zero on $\text{Hom}(M_{t+2}, \sigma M_t)$ has all and only the blocks from source layer a to target layer $b \leq a$. Its trace annihilator consists of maps from layer b to layers strictly below b . The solvability condition is therefore

$$\varepsilon_{t+1} \varepsilon_t F_{j,t} \subseteq F_{j-1,t+2} \quad \text{for all } t, j. \quad (3.4)$$

Interval matrices have entries zero and one, so twisting preserves their labels and all the displayed Ext dimensions. When (3.4) holds, the reverse-arrow solution sets together have size $z^{\sum_t \dim_K \text{Ext}^1(M_{t+2}, M_t)}$. For $n = 1$, the reverse-arrow space is zero, and solvability says $C_t = 0$, again precisely (3.4).

Condition (3.4) forces every composite of $2n$ central arrows to vanish. Let N range over the finite set of nilpotent cyclic chain labels with vertex dimensions $f_{n,t}$. Define ΦN by replacing each central arrow by the two-step map from t to $t+2$. It is a representation of $\gcd(s, 2)$ disjoint cycles. On every chain, taking alternate positions gives the chain decomposition of ΦN ; this operation on labels is independent of the field and of its automorphisms.

Let $H_{\Phi N, f}(z)$ count flags of subrepresentations of ΦN whose j th layer has vertex dimensions $f_{j,t} - f_{j-1,t}$ and is semisimple. Equivalently these flags satisfy (3.4). It is a Hall flag polynomial by Lemma 3.1. The order of the stabilizer of the given flags, in $\prod_t \text{GL}(M_{0,t})$, is

$$p_f(z) = \prod_t \left(z^{\sum_{a < b} (f_{a,t} - f_{a-1,t})(f_{b,t} - f_{b-1,t})} \prod_{j=1}^n |\text{GL}_{f_{j,t} - f_{j-1,t}}(K)| \right).$$

Double-count pairs consisting of a central representation of type N and an admissible flag. Its representation orbit has size $\prod_t |\text{GL}(M_{0,t})|/a_N(z)$; the space of flags has size $\prod_t |\text{GL}(M_{0,t})|/p_f(z)$. Hence the number of central-arrow choices for the fixed flags is

$$p_f(z) \sum_N \frac{H_{\Phi N, f}(z)}{a_N(z)}.$$

Finally quotient by the automorphisms of the fixed forward chains. The total mass is

$$\sum_{[M_t]} \frac{z^{\sum_t \dim_K \text{Ext}^1(M_{t+2}, M_t)}}{\prod_t a_{M_t}(z)} p_{f(M)}(z) \sum_N \frac{H_{\Phi N, f(M)}(z)}{a_N(z)}. \quad (3.5)$$

All sums are finite. Interval automorphism orders and the a_N have only powers of z and $z^j - 1$ as factors. This proves the result, including $n = 1$, for which the single flag step requires the two-step maps to vanish. $\square$

4 Masses on modulated meshes

4.1 Linear relations and Gaussian moments

In this section, $h(U, V)$ denotes the dimension over the base field of the Hom space in the category containing U, V ; the category will always be specified.

A finite species consists of vertex fields K_i/k and finite-dimensional (K_j, K_i) -bimodules B_a for arrows $a : i \rightarrow j$. Its tensor algebra is $H = T_{\prod_i K_i}(\bigoplus_a B_a)$. We use covariant representations: an arrow acts by a K_j -linear map $B_a \otimes_{K_i} V_i \rightarrow V_j$. This convention will be applied to the opposite of the stable mesh category when counting right defect modules.

Let F be a species automorphism, including a vertex permutation ρ , field isomorphisms along that permutation, and invertible compatible transport of the arrow bimodules. It induces an exact equivalence, also denoted F , on representations. In the split case our convention is $(FV)_i = V_{\rho i}$, with transported arrow maps. For a dimension vector v , write Fv for its transported dimension vector. For dimension vectors measured over the vertex fields define

$$\chi_H(v, w) = \sum_i [K_i : k] v_i w_i - \sum_{a:i \rightarrow j} (\dim_k B_a) v_i w_j. \quad (4.1)$$

For representations U, V , the two-term complex

$$\bigoplus_i \operatorname{Hom}_{K_i}(U_i, V_i) \xrightarrow{d} \bigoplus_{a:i \rightarrow j} \operatorname{Hom}_{K_j}(B_a \otimes_{K_i} U_i, V_j), \quad (4.2)$$

$$d(f)_a = f_j U_a - V_a(1 \otimes f_i)$$

has kernel $\operatorname{Hom}_H(U, V)$ and cokernel $\operatorname{Ext}_H^1(U, V)$. For the latter assertion, write an extension on the direct-sum vertex spaces: its off-diagonal arrow blocks are arbitrary, and changing a vertex splitting adds $d(f)$. The difference of the dimensions of the two spaces is (4.1).

The *transpose mesh* adds trace-dual reverse arrows and imposes the transpose of (4.2). The dual of a bimodule is $D_k B_a = \operatorname{Hom}_k(B_a, k)$, with reversed actions and the target field transported by F . The field trace identifies this with either relative dual; the trace pairing is nondegenerate because finite fields are separable. These choices specify the relations independently of bases. In split coordinates an old arrow $a : i \rightarrow j$ has a reverse arrow $y_a : \rho j \rightarrow i$, and the relation at i is

$$\mu_i(x, y) = \sum_{t(a)=i} x_a y_a - \sum_{s(a)=i} y_a x_{\rho a} = 0.$$

Here $s(a)$ and $t(a)$ denote source and target.

We also allow disjoint special cycles of vertices v_t , indexed by $t \in \mathbb{Z}/s\mathbb{Z}$ for a cycle of length $s \geq 1$, with $\rho v_t = v_{t-2}$ and arrows $\varepsilon_t : v_t \rightarrow v_{t+1}$. At v_{t+2} the relation becomes $\mu_{v_{t+2}} + \varepsilon_{t+1} \varepsilon_t = 0$. The central-arrow bimodules are invertible and their twofold transport equals that of the relation. Scalar signs may be absorbed in the reverse arrows.

Fix a dimension vector v and nonnegative vectors r, u . For a representation D , the space $(\operatorname{soc}_{\operatorname{ar}} D)_i$ is the common kernel of arrows leaving i , and $(\operatorname{rad}_{\operatorname{ar}} D)_i$ is the sum of images of arrows entering i . Set $\operatorname{top}_{\operatorname{ar}} D = D / \operatorname{rad}_{\operatorname{ar}} D$. These spaces are taken vertexwise; all dimensions in the following formula are over the vertex fields. Put

$$Z_{v;r,u}(E) = \sum_{[D], \dim D=v} \frac{1}{|\operatorname{Aut} D|} \prod_i \begin{bmatrix} \dim(\operatorname{soc}_{\operatorname{ar}} D)_i \\ r_i \end{bmatrix}_{Q^{d_i}} \begin{bmatrix} \dim(\operatorname{top}_{\operatorname{ar}} D)_i \\ u_i \end{bmatrix}_{Q^{d_i}}, \quad Q = |E|. \quad (4.3)$$

Here $d_i = [K_i : k]$, E preserves all vertex fields, and vertex dimensions are over $K_i \otimes_k E$. This is the mass of triples (D, S, K) with $S \subseteq \operatorname{soc}_{\operatorname{ar}} D$, $K \supseteq \operatorname{rad}_{\operatorname{ar}} D$, $\dim S = r$, and $\dim(D/K) = u$. The two subspaces are chosen independently.

Fix the old-arrow representation M and its subspaces S, K , and put $X = M/S$. Write $\alpha : K \hookrightarrow M$, $\beta : M \twoheadrightarrow X$, and $h = \beta\alpha$. A reverse arrow killing S and landing in K is uniquely a compressed arrow from the corresponding transported X -space to a K -space. Its parameter space is the dual of the degree-one space of (4.2) for (K, FX) ; the relation space is the dual of the degree-zero space. Denote these dual spaces by C^1, C^0 . The relation map $\mu : C^1 \rightarrow C^0$ has transpose d .

On a special cycle write

$$\varepsilon_t = \alpha_{t+1} z_t \beta_t, \quad z_t : X_t \rightarrow K_{t+1}, \quad c_t(z) = z_{t+1} h_{t+1} z_t,$$

with the prescribed transports. At ordinary vertices put $c(z) = 0$. The compressed equations are $\mu(y) + c(z) = 0$. For a nontrivial additive character $\psi : E \rightarrow \mathbb{C}^\times$, orthogonality gives

$$\#\{(y, z) : \mu(y) + c(z) = 0\} = Q^{\dim_E C^1 - \dim_E C^0} \sum_{f \in \ker d} \sum_z \psi\langle f, c(z) \rangle. \quad (4.4)$$

Indeed, insert $Q^{-\dim C^0} \sum_{f \in (C^0)^*} \psi\langle f, \mu(y) + c(z) \rangle$ and sum over y . The sum vanishes unless $d(f) = 0$, when it contributes $Q^{\dim C^1}$. The pairing includes field traces, and $\dim C^1 - \dim C^0 = -\chi_H(v - u, F(v - r))$.

4.2 String and band representations

On each orbit o of ρ , the vertexwise data (K, M, X, f) form a diagram

$$K_t \xrightarrow{\alpha_t} M_t \xrightarrow{\beta_t} X_t, \quad K_t \xrightarrow{f_t} X_{t-1}, \quad (4.5)$$

after successively identifying the vertex fields. The maps α_t are injective and β_t are surjective. An orbit of length p gives a semilinear cycle with $3p$ vertices, with orientation $K \rightarrow M \rightarrow X \leftarrow K$ and residual field transport on one edge.

The string and band classification applies to this local category. Its underlying quiver is a cycle, has no special loops or relations, has at most two incoming and outgoing arrows at each vertex, and each arrow has at most one composable successor and predecessor. Thus the Main Theorem of [1] applies; its hypotheses on special-loop polynomials are vacuous. Strings are finite intervals in the universal covering chain, pushed down with the prescribed transports. A closed reduced walk traverses the entire cycle, so every arrow of a band is invertible. Bands correspond to finite-dimensional modules over a skew Laurent ring. Every local object therefore decomposes uniquely as

$$E_o = W_o \oplus B_o,$$

where W_o is a sum of finite strings and B_o has all arrows invertible. Write b_o for its constant vertex rank, and regard $b = (b_o)_o$ as a vertex-dimension vector by assigning b_o to every vertex in the orbit o . Injectivity of α and surjectivity of β are tested on each summand; every band satisfies them. Fixed dimensions allow only finitely many string labels and multiplicities.

For a string, a diagonal endomorphism coefficient is arbitrary in its vertex field and is transported along the finite interval. Its residue field is that vertex field. Hom spaces between strings are found by propagating coefficients along overlap intervals and imposing zero at obstructed ends. Hence their dimensions and automorphism polynomials $a_{W_o}(Q)$ are unchanged by vertex-conservative extension, and $a_{W_o}^{-1} \in \mathcal{R}$.

We describe all old-arrow choices for fixed local diagrams. In the split case, an old-arrow orbit of length l has endpoint periods dividing l . Pull both endpoint diagrams to the common $3l$ -vertex cycle. The arrow orbit is a morphism between these pullbacks whose M -component

kills the source $\ker \beta$ and lands in the target $\operatorname{im} \alpha$. Denote this vector space by $H_a^0(E_o, E_{o'})$ and its dimension over the current base field by h_a^0 . Its induced K - and X -components are unique, and the f -square is exactly $d(f) = 0$. With modulation, tensor the source with the arrow bimodule and use its invertible transport in the same equations; this defines H_a^0 without choosing an invariant arrow basis.

For local objects E, E' , let $\operatorname{Hom}^0(E, E')$ be the space of morphisms whose M -components kill $\ker \beta$ and land in $\operatorname{im} \alpha'$ of the target. Let $S(E)$ have value $\ker \beta$ at its M -vertices and zero elsewhere, and let E^+ be the kernel of the quotient supported on the M -vertices with value $\operatorname{coker} \alpha$. Then

$$\operatorname{Hom}^0(E, E') = \operatorname{Hom}(E/S(E), (E')^+).$$

These two operations replace strings by sums of intervals and leave bands unchanged.

Lemma 4.1. *The string-string, string-band and band-string dimensions in every H_a^0 , and the corresponding local Hom dimensions, depend only on string labels and band ranks.*

Proof. A pullback of a finite interval through a cyclic cover is a union of finite intervals: a starting lift determines the lift of its entire coefficient interval. For a map from an interval to a band, attach a target vector to each coefficient position. Interior equations transport it invertibly; an endpoint either leaves it free or sets it to zero. Each surviving interval contributes one full fibre. Maps in the opposite direction use row vectors. For two intervals, use the connected overlap intervals.

With an arrow bimodule, each coefficient pair carries the corresponding vector fibre. Its holonomy gives invertible maps between successive fibres. Since their coefficient interval has distinct endpoints, these maps only transport a free fibre. The dimension is the sum of the dimensions of surviving fibres. The operations $E/S(E)$ and $(E')^+$ delete coefficient positions, so the same calculation applies to the shortened intervals. This proves the assertion for different endpoint periods and for semilinear transport. $\square$

The mass for fixed local diagrams and all old-arrow choices is $Q^{\sum_a h_a^0(E_o, E_{o'})} / \prod_o |\operatorname{Aut} E_o|$. To see this, fix the vertex spaces and divide by their change-of-basis groups. Adding frames for K, X introduces exactly the groups removed by the injective and surjective maps in (4.5). Each old-arrow orbit is then the independent vector space just described.

4.3 The quadratic character sum

On a special cycle the phase in (4.4) is

$$\sum_t \operatorname{Tr}(f_t z_{t+1} h_{t+1} z_t), \quad f_t : K_{t+2} \rightarrow X_t, \quad (4.6)$$

where the matrix trace is followed by the field trace to E . For a fixed local diagram E_{loc} , let $\mathcal{W}(E_{\text{loc}})$ be the sum over all compressed central maps z of ψ applied to this phase. If there are no special arrows, set $\mathcal{W}(E_{\text{loc}}) = 1$.

Lemma 4.2. *For fixed local strings W and band ranks b , summation of ψ applied to (4.6) factors as*

$$\mathcal{W}(W \oplus B) = Q^{\gamma(W, b)} \mathcal{W}(B),$$

where $\gamma(W, b)$ is a nonnegative integer independent of band parameters and of ψ .

Proof. Use coefficient bases on strings. Their h and f maps are partial matchings with the prescribed field transport. Index a coefficient of z_t by its target position k in K_{t+1} and source position x in X_t . Its successor in (4.6), when defined, is

$$(k, x) \mapsto (f_t^{-1} x, h_{t+1} k).$$

Here the inverse is that of a partial matching. Each coefficient has at most one predecessor and successor. Two successor steps move both coefficient positions forward on their local h, f covering chains. A position on a finite string reaches an endpoint, so every component containing a string is a finite path.

For a path with m scalar variables and nonzero edge coefficients,

$$\sum_{x_1, \dots, x_m \in E} \psi \left(\sum_{j=1}^{m-1} c_j x_j x_{j+1} \right) = Q^{\lceil m/2 \rceil}.$$

Summing the first variable sets the second to zero and contributes Q ; repeat at the new endpoint. An isolated variable also contributes Q . When one position belongs to a band, retain its entire vector space. The same path carries constant-rank vector fibres with nondegenerate pairings on adjacent fibres, and the same orthogonality calculation gives the appropriate power of the field size. Semilinear transport is bijective and field trace is nondegenerate, so this remains valid over the original field. The sum of the fibre dimensions times $\lceil m/2 \rceil$ is $\gamma(W, b)$. The unintegrated variables form exactly the band-band block. $\square$

4.4 Band masses and Gaussian inversion

Let J be the vertex support allowed for the representations and let P be the union of the ρ -cycles entirely contained in J . Assume that special cycles are contained in P . Every nonzero band has equal positive rank along its orbit, so bands are supported on P .

String and band summands have disjoint indecomposable types. Taking semisimple quotients of their endomorphism rings gives

$$|\mathrm{Aut}(W_o \oplus B_o)| = a_{W_o}(Q) |\mathrm{Aut} B_o| Q^{h(W_o, B_o) + h(B_o, W_o)}.$$

For fixed W, b define

$$\begin{aligned} \eta(W, b) = & \sum_{a: o \rightarrow o'} (h_a^0(W_o, W_{o'}) + h_a^0(W_o, B_{o'}) + h_a^0(B_o, W_{o'})) \\ & - \sum_o (h(W_o, B_o) + h(B_o, W_o)). \end{aligned}$$

Here all dimensions are over the current base field. Lemma 4.1 makes this independent of the chosen band types. The remaining band sum is

$$P_b(Q; \psi) = \sum_{[B], \mathrm{rank} B=b} \frac{Q^{\sum_{a \text{ in } P} h_a^0(B_o, B_{o'})}}{\prod_o |\mathrm{Aut} B_o|} \mathcal{W}(B).$$

Without special arrows the character factor is one. Combining the preceding computations yields

$$Z_{v; r, u}(Q) = Q^{-\chi_H(v-u, F(v-r))} \sum_b \sum_W \frac{Q^{\eta(W, b) + \gamma(W, b)}}{\prod_o a_{W_o}(Q)} P_b(Q; \psi). \quad (4.7)$$

The K, M, X dimensions of W must be $v-u-b, v-b, v-r-b$, respectively; these conditions specify the finite sums. At ordinary orbits $\gamma = 0$.

Proposition 4.3. *Suppose that the total representation mass on the full subquiver P , for each ρ -invariant dimension vector, is given by an element of $\mathcal{R}$ on all vertex-conservative extensions. Then every fixed-dimension, fixed-arrow-top and fixed-arrow-sockle mass on J has this property.*

Proof. Apply (4.7) on P with $r = u = 0$ and ρ -invariant v . Denote its finite sum of string coefficients at rank b by $A_{v,b}(Q) \in \mathcal{R}$. At $b = v$ the only string object is zero, with coefficient one. Hence

$$P_v(Q; \psi) = Q^{\chi_H(v, Fv)} Z_{v;0,0}^P(Q) - \sum_{b < v} A_{v,b}(Q) P_b(Q; \psi). \quad (4.8)$$

Here $b < v$ means componentwise inequality with at least one strict component. Starting with $P_0 = 1$, induction on total dimension gives $P_v \in \mathcal{R}$; the recursion also gives independence of ψ . Multiple periodic components can be treated as a direct product.

Substitute into (4.7) to obtain every Gaussian moment in $\mathcal{R}$. For fixed v , top and socle dimensions are bounded. Each matrix $(\binom{s}{r}_{Q^{d_i}})_{0 \leq r, s \leq v_i}$ is triangular with diagonal entries one, and its inverse has entries in $\mathbb{Z}[Q]$ by finite back-substitution. Invert in both sets of variables in (4.3). $\square$

4.5 Representation masses for Dynkin species

For an ordinary transpose mesh with representation-finite species H , fix an old-arrow representation M . The reverse-arrow space is $\text{Ext}_H^1(M, FM)^*$ by (4.2). Its total mass is therefore

$$\sum_{[M], \dim M=v} \frac{Q^{\dim_E \text{Ext}_H^1(M, FM)}}{a_M(Q)}. \quad (4.9)$$

On conservative extensions of H , its labels, Hom dimensions and the permutation of labels induced by F are fixed. Thus (4.9) is a finite sum in $\mathcal{R}$.

Lemma 4.4. *If H is acyclic and its symmetrized Euler form is positive definite, every indecomposable H -module has endomorphism algebra isomorphic as a k -algebra to a vertex field. Its conservative period is the least common multiple of the vertex-field degrees.*

Proof. At a sink i , a representation without the simple summand at i has a surjective map $\bigoplus_{a:j \rightarrow i} B_a \otimes_{K_j} M_j \rightarrow M_i$. Replace M_i by its kernel and reverse the incident bimodules using relative duals. The inverse at the new source takes the cokernel. The two exact sequences give mutually inverse functors on the subcategories without the corresponding simple summands. They preserve indecomposability and endomorphism rings.

Let E_H be the Euler matrix in a topological order and $G_H = E_H + E_H^t$. A sink reflection changes coordinate i to $-m_i + \sum_{a:j \rightarrow i} (\dim_k B_a) m_j / [K_i : k]$. Solving these coordinate updates successively along a complete sink sequence gives $c = -E_H^{-1} E_H^t$. The transformations preserve the integral lattice and G_H . Positive definiteness implies that the integral isometry group is finite: each column has fixed bounded G_H -length and hence finitely many possible integral values. Thus $c^h = 1$ for some h . Also $c - 1 = -E_H^{-1} G_H$ is invertible, so $1 + c + \dots + c^{h-1} = 0$.

Repeat the sink sequence on a nonzero indecomposable until it becomes the simple at the current sink. If this never happens, $m, cm, \dots, c^{h-1}m$ are all nonnegative, the first is nonzero, and their sum is zero, a contradiction. Endomorphism rings are preserved along these reflections and the final simple has a vertex field as its endomorphism ring. The same bound gives finitely many possible indecomposables. Every vertex field already occurs at its simple, giving the asserted least common multiple. $\square$

5 Standard forms and finite completion

5.1 Descent of the mesh presentation

We first explain why geometric standardness gives a presentation over the original field. For a finite-dimensional algebra C , its radical grading is

$$\mathrm{gr} C = \bigoplus_{j \geq 0} \mathrm{rad}^j C / \mathrm{rad}^{j+1} C.$$

A radical-graded algebra is an algebra isomorphic to this graded algebra, with its grading forgotten.

Lemma 5.1. *Let k be a finite field. If $C_{\bar{k}}$ is radical-graded, then C is radical-graded.*

Proof. Put $B = \mathrm{gr} C$. Scalar extension commutes with the radical because k is perfect. Choose a geometric isomorphism $C_{\bar{k}} \rightarrow B_{\bar{k}}$. Its associated graded map is a graded automorphism of $B_{\bar{k}}$; composing with its inverse makes the associated graded map the identity.

Let U be the reduced algebraic group of automorphisms of B inducing the identity on its associated graded algebra. This group is smooth since k is perfect. It is connected: write an automorphism in homogeneous blocks. Its diagonal blocks are identities and its other blocks go from degree a to a strictly larger degree b . Conjugation by the grading action multiplies these blocks by t^{b-a} . It extends to $t = 0$, with value the identity. Thus every geometric point of U belongs to the identity component. The isomorphism scheme of maps $C \rightarrow B$ inducing the graded identity is geometrically nonempty. Choosing one geometric isomorphism identifies it, by composition, with the automorphism scheme inducing the graded identity. Reduction commutes with extension of the perfect field k , so its reduced scheme becomes $U_{\bar{k}}$ with its translation action. It is therefore a smooth U -torsor over k . Lang's theorem [8] gives a rational point of this torsor. $\square$

Take one representative of each indecomposable A -module and let G be their direct sum. Over $\bar{k}$, $G_{\bar{k}}$ contains every geometric indecomposable. Indeed, any geometric indecomposable is defined over a finite separable extension E/k , and is a summand of the scalar extension of its restriction to k . Also

$$\mathrm{End}_A(G) / \mathrm{rad} \mathrm{End}_A(G) = \prod_i \mathbb{F}_{q^{d_i}},$$

so after splitting the fields its primitive summands have multiplicity one. Thus $\mathrm{End}_A(G)_{\bar{k}}$ is the basic endomorphism algebra of all geometric indecomposables.

In a mesh category, arrows are irreducible morphisms and the defining relations are homogeneous of degree two. In finite representation type its arrow ideal is nilpotent, so the grading is the radical grading. Lemma 5.1 descends this grading to $\mathrm{End}_A(G)$. One can align the vertex idempotents with the original summand projections: if two orthogonal systems (e_i) and (f_i) coincide modulo the radical, then $u = \sum_i f_i e_i$ lies in $1 + \mathrm{rad} \mathrm{End}_A(G)$ and satisfies $f_i u = u e_i$.

Here is the resulting modulated presentation. A partial translation quiver has a partially defined injective translation τ on vertices. For every arrow $a : y \rightarrow x$ with τx defined there is a companion

$$\sigma a : \tau x \longrightarrow y;$$

There is also an inverse companion $\sigma^{-1} a : x \rightarrow \tau^{-1} y$ whenever $\tau^{-1} y$ is defined. These partial operations are mutually inverse, and σ^2 is translation on arrows wherever defined. The companion pairings are nondegenerate. These assertions hold in the Auslander–Reiten quiver and remain true after deleting projectives whenever all the indicated endpoints remain. Over a splitting field, the relation at x is the sum of the length-two paths $a \sigma a$, with the

coefficients from its almost-split sequence. Over k , the vertices have their residue fields, the arrow spaces are the corresponding irreducible-morphism bimodules, and the relations are the coevaluation tensors for the almost-split pairings. The geometric quadratic presentation implies the presentation over k : scalar extension is faithfully flat, so the descended degree-two relations generate the entire kernel.

The stable Auslander algebra

$$\Lambda = \text{End}_A(G)/[\text{proj } A]$$

is obtained by deleting the projective vertices and every path through them. A map between nonprojective indecomposables factoring through a projective lies in the square of the radical, since neither factor can split. Thus this quotient does not change the remaining arrow spaces. Its relation at an old vertex is exactly the sum of the remaining length-two mesh paths. The sum can be empty. Translation is retained only when both translated endpoints remain.

5.2 Periodic components

Call a vertex periodic if every positive and negative iterate of its translation exists. The full subquiver on periodic vertices of a representation-finite Auslander–Reiten quiver is a disjoint union of quotients of repetitive Dynkin quivers; this is the stable-subquiver statement of Riedtmann’s structure theorem [9, Hauptsatz]. It applies to the stable full subquiver of an arbitrary representation-finite algebra. Its vertices need not constitute all nonprojective vertices.

We record the cuts and their finite-field forms needed for counting. For Dynkin trees other than A_{2n} , orient the tree away from its central fixed vertex, or choose an orientation invariant under its diagram automorphisms. The automorphisms of the repetitive quiver are generated by translation and diagram automorphisms preserving such a cut. For A_{2n} there is instead a generator ρ with $\rho^2 = \tau$. These assertions, including the exceptional generator, are given explicitly in [4, Proposition 10].

In the first case, the forward arrows form a finite union of Dynkin species H ; the companions are the transpose arrows. Deck transformations and finite-field descent preserve this division. The automorphism transporting one mesh to the next becomes the species automorphism F of Section 4. Folding a positive definite Dynkin form remains positive definite: the folded form is its restriction to the invariant subspace, with the vertex-field degrees as symmetrizing weights. Lemma 4.4 and (4.9) therefore give the total mass on every vertex-conservative extension.

The exceptional cut has the following coordinates. Label the two arms of A_{2n} by (i, p) , where $0 \leq i < n$ is distance from the centre and $p = 0, 1$ distinguishes the arms. If $l \in \mathbb{Z}$ is the repetition coordinate, set $t = 2l + p$. Then τ changes t by two and ρ changes it by one, after choosing the direction of indexing. The outward arm arrows are $a_{i,t}$; the central arrows are ε_t . Their companions are the reverse arm arrows $b_{i,t}$. Replacing the reverse arrows at distance i by $(-1)^{i+1}$ times themselves gives precisely the square-boundary equations of Section 3.2. A quotient by a power of ρ makes t cyclic. The equality of the residue-field degrees follows from the action on these coordinates. Suppose Frobenius permutes r geometric components transitively, and write one as $\mathbb{Z}A_{2n}/\langle \rho^s \rangle$. Its r th power lifts to a power ρ^a , by the automorphism description above. At every distance i from the centre it acts on $t \in \mathbb{Z}/s\mathbb{Z}$ by $t \mapsto t + a$. Every geometric vertex therefore has Frobenius orbit length $rs/\text{gcd}(s, a)$. This length is the degree of its descended residue field, independently of i .

The same description holds without splitting the field. Along an arm, choose successive identifications of the vertex fields and trivializations of the invertible arrow bimodules. There is no obstruction, since an arm is a tree. At the centre, retain the invertible bimodule of each ε_t and its field transport. The two summands in the central relation are the tensor product

of two successive central bimodules and the first arm-return bimodule. Each projection of the relation bimodule is an isomorphism: it is one after extension to $\bar{k}$, hence one before extension. The internal relations identify consecutive arm-return bimodules in the same way. Choose the reverse-arrow identifications successively along the arm. This gives the displayed square-boundary equations over the original fields, with all remaining transport in the ε_t . Proposition 3.2 applies to the resulting semilinear central cycle.

5.3 Completion of the finite translation intervals

We complete the finite partial quiver underlying Λ by adding vertices.

Lemma 5.2. *A finite modulated partial mesh quiver of the preceding form is a full vertex quotient of a finite transpose mesh with square-boundary components. Its original periodic subquiver is unchanged. Every added translation cycle has a new vertex.*

Proof. Every component of the partial permutation τ is a cycle or an interval. Give an interval coordinates $0, \dots, l_i$, with τ decreasing the coordinate by one. Let $L = \max_i l_i$, taking $L = 0$ if there are no intervals. For each old translation cycle c , let m_c be its length and let α_c be the automorphism of its starting vertex field obtained by composing the prescribed field transports once around the cycle. Write h_c for the finite order of α_c . Choose an even integer $M > 2L + 2$ divisible by every $m_c h_c$. Then transport through M steps is the identity on each old cycle. Complete each interval to a cycle of length M .

Follow companion arrows. A chain that is already closed has both endpoints on old periodic cycles and is left unchanged. Otherwise, if one arrow is $a_0 : (i, p) \rightarrow (j, q)$, its successive companions have coordinates

$$a_{2r} : (i, p - r) \rightarrow (j, q - r), \quad a_{2r+1} : (j, q - r - 1) \rightarrow (i, p - r).$$

Complete each maximal open companion chain separately modulo $2M$, identifying the coordinates of an old periodic endpoint modulo its old period. Retain the chain label, so arrows from different chains are not identified merely because they have the same endpoints. If both endpoints belong to finite intervals, any new arrow with both endpoints old would require, in the even case, $u - v = p - q + kM$. The difference between the two sides with $k = 0$ has absolute value at most $2L$; hence $k = 0$. For odd arrows the analogous bound is $2L + 1$. Before reduction modulo M , the even terms with old endpoints have

$$\max(p - l_i, q - l_j) \leq r \leq \min(p, q),$$

and the odd terms have

$$\max(p - l_i, q - l_j - 1) \leq r \leq \min(p, q - 1).$$

Each endpoint of the odd range equals the corresponding endpoint of the even range or that endpoint minus one. Thus the allowed even and odd indices interlace to form one consecutive interval of companion indices. Forward and inverse companion completeness puts every term of this interval in the original chain. Thus no new old-to-old arrow occurs. With one periodic endpoint, the finite endpoint determines the unique old occurrence, while divisibility of M by the old period preserves the other phase. With two periodic endpoints the chain was already closed.

Transport the vertex fields along the completed cycles. For a completed interval, choose the transport at one new closing edge to be the inverse of the product of all other transports. Its full field transport is then the identity; for old cycles this holds after M steps by the choice of M . Extend each arrow bimodule by alternating duality relative to its two endpoint

fields, using the original nondegenerate pairings wherever prescribed. After $2M$ companions the endpoint field actions agree with the original ones. Choose the last new pairing to close the chain and retain the resulting invertible linear return map as part of the species transport F . No identity condition on this linear return map is needed.

On each open companion chain, alternate the forward and reverse designation. Any descent automorphism preserves its two ends and commutes with taking companions, so it preserves this designation; a shift along a finite chain cannot preserve its ends. On the old closed chains use the Dynkin cuts, including the central exceptional cut, just constructed. The double-dual identifications give a species transport F . Choose the newly added mesh pairings to be the transpose pairings. On an old open chain, the prescribed pairings determine these choices successively. In split one-dimensional arrow coordinates, if the terms at the consecutive meshes have nonzero coefficients c_{2r} and c_{2r+1} , take the forward arrow to be a_{2r} , replace the reverse arrow a_{2r+1} by $c_{2r}a_{2r+1}$, and set $F(a_{2r}) = -c_{2r+1}c_{2r}^{-1}a_{2r+2}$. The transpose terms then have the original coefficients at both meshes. For general bimodules the same construction uses the two duality isomorphisms in place of the scalar ratio. An open chain imposes no return condition on these choices. On the closed components the preceding Dynkin-coordinate calculation supplies the compatible choice. This gives the required transpose relations and the exceptional square relations.

Finally quotient by all new vertex idempotents. Each relation at a new vertex vanishes, since its endpoint is new. At an old vertex whose predecessor was missing, the relation has a new translated endpoint and vanishes as well. Every other old relation loses exactly the terms with new intermediate vertices. It is therefore precisely its original partial mesh relation. The image of an ideal under a quotient map is generated by the images of its generators. Thus all relations in the vertex quotient are generated by these images of mesh relations, and the original and quotient algebras have identical presentations. An added cycle contains a new vertex because $M > L + 1$. $\square$

The construction also applies to the opposite category. Reverse arrows and translation, and replace each bimodule by its opposite with the interchanged endpoint actions. Pairings and the square-boundary equations are carried to the corresponding opposite equations.

5.4 Hall polynomials in the standard case

Proof of Theorem 1.2. Apply the preceding presentation and completion to the opposite of Λ , so that its covariant representations are the right defect modules used in Section 2. A representation of the quotient is exactly a representation of the completion with zero spaces at all new vertices. Each added translation cycle contains such a vertex, and hence cannot support a nonzero local band. Thus the full cycles supporting bands are exactly the original periodic components.

Fix a conservative extension E/k of A . Its degree is coprime to every residue-field degree d_i from Section 2. All vertex fields of the presentation and completion are copies of these fields. In the ordinary periodic components, Lemma 4.4 shows that no extra extension condition is required to preserve the indecomposables of the Dynkin species. Equation (4.9) gives their total masses in $\mathcal{R}$. On an exceptional component, Proposition 3.2 gives the same conclusion; replacing its field-size variable by Q^d preserves $\mathcal{R}$.

Proposition 4.3 now gives all dimension, arrow-top and arrow-socle masses supported on the original vertices. On Λ the arrow ideal is the radical, since the mesh grading has nilpotent positive part and semisimple degree zero. These are therefore the actual module top and socle. At vertex i , a defect dimension δ_i measured over k is dimension δ_i/d_i over its residue field; a nonintegral quotient means that the corresponding mass is zero. We have obtained

exactly the masses required by the defect criterion, on every conservative extension of A . Corollary 2.5 proves the theorem. $\square$

5.5 Ray-standard forms

Over an algebraically closed field, a basic algebra is *distributive* when its lattice of two-sided ideals is distributive. Its ray category has primitive idempotents as objects and, in each corner, the orbits under the two corner unit groups as rays. For composable morphisms $a : x \rightarrow y$ and $b : y \rightarrow z$, the ray product is zero when $ba \in b \operatorname{rad} A(y, y)a$, and otherwise is the ray of ba . Here ba denotes categorical composition. The linearization has the nonzero rays as a basis and extends this product bilinearly. An algebra is *ray-standard* when it is isomorphic to this linearization. A category is *mild* when every proper ideal quotient is locally representation-finite.

A contour is a pair of paths with the same nonzero ray. It is essential if the equality cannot be obtained by replacing equal-ray proper subpaths, and non-deep if the common ray is not annihilated by both endpoint corner radicals. The full subcategory on its vertices is its support category. The structure and disjointness theorems in [3, Theorem 6] identify the essential non-deep support categories as pennies, dumb-bells and diamonds, with distinct supports disjoint. The penny path coordinates needed here are specified in Section 6. In the stem category, morphisms are the subbimodules generated by rays and composition is their product. The mixed forms of [3, Theorem 10] use stem composition on selected penny supports and ray composition elsewhere.

Lemma 5.3. *A ray-standard representation-finite algebra over an algebraically closed field is mesh-standard.*

Proof. Let Γ be its Auslander–Reiten quiver and $\mathcal{C} = k(\Gamma)$ its mesh category. By [3, Theorem 14], the universal mesh category covers the category of indecomposables. This covering preserves radicals. Indeed, if a noninvertible map upstairs had invertible image, lift an inverse using the direct-sum Hom isomorphism in the definition of a covering. The component returning to the original object gives a left inverse. Local endomorphism rings then make the resulting nonzero idempotent on the other object the identity, so the original map would be invertible. The contradiction proves the assertion.

The finite category of indecomposables has nilpotent radical, say of exponent N . Every path of length N upstairs maps to zero, hence is zero, since the covering is injective on each individual Hom space. Every path in $\mathcal{C}$ lifts, so its length- N paths vanish too. As Γ is finite, $\mathcal{C}$ is finite-dimensional; the universal mesh map to $\mathcal{C}$ is a covering between locally bounded categories.

Apply [3, Theorem 15] first to the covering from the universal mesh category to the indecomposable category of the given algebra. The universal mesh category is therefore an Auslander category. Apply the same theorem to its covering of $\mathcal{C}$; hence $\mathcal{C}$ is also an Auslander category and is the indecomposable category of its algebra of projective points. These points are detected by the projective dimensions of simple contravariant functors: the radical inclusion at a projective point gives a resolution of length at most one, whereas at a nonprojective point the minimal mesh resolution has a nonzero term in degree two. The projective points are therefore exactly those of Γ . By [3, Theorem 18(c),(d)], their algebra is the ray-standard form, which is the given algebra. $\square$

6 Boundary maps and Hall numbers

All modules in the remaining sections are right modules. Products of paths are written in walking order: if $a : x \rightarrow y$ and $b : y \rightarrow z$, then $ab : x \rightarrow z$. Composition of linear maps

retains the convention $gf = g \circ f$. At a quiver vertex v , write e_v for the primitive idempotent and S_v for the corresponding simple module. For a module V , put $\text{rad } V = V \text{ rad } A$; its socle $\text{soc } V$ is its largest semisimple submodule. We write $D_k V = \text{Hom}_k(V, k)$.

We first give a counting argument for two algebras with a common quotient. Suppose A_0, A_1 have ideals J_0, J_1 and specified identifications $A_i/J_i = C$. Suppose corresponding modules E_i have identified C -modules

$$S = E_i J_i, \quad R = \{v \in E_i : v J_i = 0\}, \quad T = E_i/S,$$

where $S \subseteq R$. Let $h_i : R \rightarrow T$ be the induced maps, with kernel S . Assume their common image is a submodule $H \subseteq T$ and

$$h_1 = \phi h_0, \tag{6.1}$$

where $\phi \in \text{Aut}_C(H)$ preserves every submodule of H . For C -modules V, W , let $\mathcal{P}_C(V, W)$ be the subspace of $\text{Hom}_C(V, W)$ consisting of maps factoring through finite-dimensional projective C -modules.

Lemma 6.1. *Suppose that, for every $\Omega \subseteq H$, every C -module N' and every $g : \Omega \rightarrow N'$, there is $\psi \in \text{Aut}_C(N')$ such that*

$$\psi g - g\phi|_{\Omega} \in \mathcal{P}_C(\Omega, N'). \tag{6.2}$$

Then for all C -modules X, Y, N ,

$$F_{X,Y}^{E_0 \oplus N}(A_0) = F_{X,Y}^{E_1 \oplus N}(A_1).$$

Here the algebra in parentheses specifies where the Hall number is computed.

Proof. For a submodule counted on side i , the conditions $Y J_i = X J_i = 0$ give

$$S \subseteq U \subseteq R \oplus N.$$

Fix its actual projection $V = \text{pr}_{E_i} U$, so $S \subseteq V \subseteq R$, and put $\Omega = h_0(V)$. Since ϕ preserves Ω , also $h_1(V) = \Omega$. Write $q_i = h_i|_V : V \rightarrow \Omega$. For $K_0 = U \cap N$, put $N'_0 = N/K_0$. The submodule U is determined by a unique graph map $g : \Omega \rightarrow N'_0$. Its quotient and its own isomorphism type are given by

$$(T \oplus N'_0)/\{(\omega, g\omega) : \omega \in \Omega\} \simeq X, \quad \text{mid}((gq_i)^* \beta_{K_0}) \simeq Y, \tag{6.3}$$

where β_{K_0} is the class of $0 \rightarrow K_0 \rightarrow N \rightarrow N'_0 \rightarrow 0$. A minus sign may be placed in the graph and in g simultaneously. For a map $f : W \rightarrow N'_0$, the pullback $f^* \beta_{K_0}$ has middle module $\{(w, n) : f(w) = n + K_0\}$; mid denotes that middle module.

For fixed abstract modules K, N' , write $\text{Ext}_C^1(N', K)_N$ for extension classes with middle module isomorphic to N . Put $a_W = |\text{Aut}_C(W)|$ in this proof. The contribution of all K_0 with these endpoint types is

$$\frac{a_N}{a_K a_{N'} |\text{Hom}_C(N', K)|} \# \{(\beta, g) : \beta \in \text{Ext}_C^1(N', K)_N, g : \Omega \rightarrow N', \text{ (6.3) holds}\}. \tag{6.4}$$

Indeed, every K_0 has $a_K a_{N'}$ endpoint frames. A fixed extension class has $a_N / |\text{Hom}_C(N', K)|$ exact-sequence representatives on the fixed middle module N : the stabilizer is the group of maps $1 + iup$, with $u : N' \rightarrow K$, where i, p are its two structure maps. Quotient frames transport g at the same time. This proves (6.4).

Fix g and choose ψ in (6.2). Pullback by ψ is a bijection of $\text{Ext}_C^1(N', K)_N$. Its inverse is pullback by ψ^{-1} , and its middle module remains isomorphic to N . Moreover,

$$(gq_0)^*(\psi^* \beta) = (\psi gq_0)^* \beta = (g\phi q_0)^* \beta = (gq_1)^* \beta.$$

The second equality holds because extensions pull back to zero through projective modules. The quotient condition in (6.3) depends only on g . Thus (6.4) agrees on both sides. Summing over K, N' and over the same submodules $V \subseteq R$ proves the assertion. $\square$

6.1 Root-isolated pennies

Let k have characteristic two. The local configurations used below have a root x , a loop r , and an oriented arm

$$x \xrightarrow{a_1} 1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} n-1 \xrightarrow{a_n} x, \quad n \geq 2.$$

Write $p_j = a_1 \cdots a_j$ and $q_i = a_{i+1} \cdots a_n$ for $1 \leq i, j < n$. The two configurations have parameter $\epsilon \in \{0, 1\}$ and relations

$$a_1 \cdots a_n = r^2 + \epsilon r^3, \quad a_n a_1 = 0, \quad r^4 = 0. \quad (6.5)$$

The root corner has basis $1, r, r^2, r^3$; the root-to- j and i -to-root spaces have bases $p_j, r p_j$ and $q_i, q_i r$. In particular,

$$r^2 p_j = q_i r^2 = q_i p_j = 0.$$

The displayed vertices form a full subcategory with exactly the displayed arrows. Nonroot paths through x are generated by $q_i r p_j$ and their nonroot multiples. We call the root *isolated* when both corner spaces between x and any outside vertex are zero. Arrows incident with nonroot arm vertices are retained. In particular, $p_j b = 0$ for every arrow b from j to an outside vertex. The ray configuration is described in [2, §1 and Proposition 2]; the nonstandard multiplication is given by [3, Theorem 10]. To obtain (6.5) from the stem coordinates, write its arm arrows as $b_1, \dots, b_n$. In walking order their relations are

$$b_1 \cdots b_n = r^2, \quad b_n b_1 = b_n r b_1, \quad r^4 = 0.$$

Multiplication of the second relation by the rest of the arm gives $r^2 b_1 = r^3 b_1$ and, dually, $b_n r^2 = b_n r^3$. Since $1 + r$ is invertible, $r^2 b_1 = b_n r^2 = 0$. Set $a_1 = (1 + r)b_1$ and $a_i = b_i$ for $2 \leq i \leq n$. Then

$$a_1 \cdots a_n = r^2 + r^3, \quad a_n a_1 = 0, \quad a_n r a_1 = b_n r b_1.$$

Moreover $q_i r p_j$ is unchanged, since the additional term contains $r^2 b_1 = 0$. Thus the non-root relations are preserved. The occurrence of these configurations in representation-finite algebras is used in Section 9.

At an isolated split root, $P = e_x A$ is projective-injective with simple socle $S = k r^3$ and top the root simple. Indeed, pair $e_x A$ with $A e_x$ by the coefficient of r^3 in their product. On the root powers the pairing has ones on its antidiagonal and zeros elsewhere. On the bases $p_j, r p_j$ and $q_j, q_j r$ its matrix is

$$\begin{pmatrix} \epsilon & 1 \\ 1 & 0 \end{pmatrix}.$$

Root isolation leaves no other blocks. Each block is nondegenerate, and associativity of multiplication makes the pairing a right-module isomorphism $e_x A \simeq D_k(A e_x)$. Its socle is therefore the root simple, represented by r^3 . For a Frobenius orbit of such roots, the same assertion follows over the finite ground field: the corresponding projective and injective become isomorphic after a finite splitting extension; restriction of scalars and Krull–Schmidt cancellation give their isomorphism over the ground field.

For several selected roots, their socles form a two-sided ideal J . Assume their penny supports are disjoint. The common quotient of the two choices is $C = A/J$. At one split root put $T = e_x C$, $H = \text{rad } T$, and $R = \text{rad } P$. Their spaces are

$$T_x = k1 \oplus kr \oplus kr^2, \quad H_x = kr \oplus kr^2, \quad R_x = kr \oplus kr^2 \oplus kr^3, \quad T_j = H_j = R_j = kp_j \oplus krp_j.$$

All outside spaces vanish. Arm arrows propagate the two columns. In H , the first arrow sends r to rp_1 , and the last sends p_{n-1} to r^2 and rp_{n-1} to zero. The loop sends r to r^2 .

Identify the two radicals by the identity on arms and by

$$r \mapsto r + r^2, \quad r^2 \mapsto r^2 + r^3, \quad r^3 \mapsto r^3 \quad (6.6)$$

at the root, from the nonstandard to the standard side. The last-arrow value $r^2 + r^3$ is thereby sent to r^2 ; the loop and first arrow also commute. Identify P/S by the path bases. The two boundary maps $R \rightarrow T$ differ by ϕ , which acts on H_x as $v \mapsto v + vr$ and is the identity elsewhere. This is a C -automorphism preserving every submodule of every H^m , for $m \geq 0$.

Lemma 6.2. *For $\Omega \subseteq H^m$ and $g : \Omega \rightarrow N'$, condition (6.2) holds with $\psi = 1 + n$, where $n^2 = 0$, n kills $\text{rad } N'$, and its image lies in the root part of $\text{soc } N' \cap \text{rad } N'$.*

Proof. Write a root vector as $r\xi + r^2\eta$, with $\xi, \eta \in k^m$. Let $W \subseteq k^m$ be the space of first coefficients of vectors in Ω_x . The first-coefficient map has kernel $L = \ker(r : \Omega_x \rightarrow \Omega_x)$. Choose a linear section $\sigma : W \rightarrow \Omega_x$. Closure under paths gives $r^2W \subseteq \Omega_x$ and $rp_jW \subseteq \Omega_j$.

On the arm chain $1 \rightarrow \cdots \rightarrow n-1$, the second columns rp_jW form a constant subrepresentation. It is injective in the category of chain representations: for one copy, a map from a chain V is determined by an arbitrary functional on V_{n-1} , and the earlier components are its composites with the intervening arrows. This Hom functor is exact. Hence there are chain-compatible retractions $\rho_j : \Omega_j \rightarrow W$ with $\rho_j(rp_j\xi) = \xi$.

We first construct a projective factorization. Suppose $u : W \rightarrow (\text{rad } N')_x$ satisfies $u(\xi)r^2 = u(\xi)rp_j = 0$. The map with root values $\sigma(\xi) \mapsto u(\xi)r$, zero on L and zero on the arms, factors through projectives. Indeed, root isolation gives

$$(\text{rad } N')_x = N'_x r + \sum_{i=1}^{n-1} N'_i q_i.$$

Choose linear $w : W \rightarrow N'_x$, $z_i : W \rightarrow N'_i$ with $u(\xi) = w(\xi)r + \sum_i z_i(\xi)q_i$. The $w(\xi)r^2$ term factors through $T \otimes_k W$: send $\sigma(\xi)$ to $r^2 \otimes \xi$, all other components to zero, and then $1 \otimes \xi$ to $w(\xi)$. This commutes with the first arrow because $r^2p_1 = 0$ and with the last because its source image lies in L .

For the i th remaining term use $e_i C \otimes_k W$. Send $\sigma(\xi)$ to $q_i r \otimes \xi$, send L to zero, and send $v \in \Omega_j$ to

$$q_i rp_j \otimes \rho_j(v).$$

The loop commutes by $q_i r^2 = 0$; the first arrow commutes by the retraction identity; internal arrows commute by chain compatibility. The last arrow gives $q_i rp_{n-1} a_n = q_i r^3 = 0$, and its source image lies in L . Outside arrows commute because $p_j b = 0$ and the source has zero outside spaces. Send the projective generator $e_i \otimes \xi$ to $z_i(\xi)$. All these factors use the same retraction, so their total arm value is zero:

$$\sum_i z_i(\eta) q_i rp_j = u(\eta) rp_j - w(\eta) r^2 p_j = 0 \quad (\eta \in W).$$

Their root sum is $u(\xi)r$, as required.

Now let $\pi : N'_x \rightarrow N'_x / (\text{rad } N')_x$ be the root top projection. Put $B = \pi(gL)$ and define $t : W \rightarrow N'_x / (\text{rad } N')_x / B$ by $t(\xi) = \pi(g\sigma(\xi)) + B$. Choose $W = W_0 \oplus W_v$ with $W_0 = \ker t$, and denote the projections by p_0, p_v . The spaces B and $\pi(g\sigma(W_v))$ are disjoint. Define a map from the root top to the root socle by zero on B , by

$$\pi(g\sigma(\xi)) \mapsto g\sigma(\xi)r \quad (\xi \in W_v),$$

and by zero on a complementary space. The values lie in $\text{rad } N' \cap \text{soc } N'$ because $\sigma(\xi)r^2 = \sigma(\xi)rp_1 = 0$ and the root has no other outgoing arrow. Composing with the top projection and socle inclusion gives a C -endomorphism n with $n^2 = 0$. It satisfies $ngL = 0$ and $ng\sigma(\xi) = g\sigma(p_v\xi)r$.

For $\xi \in W_0$, choose linearly $\ell(\xi) \in L$ with $\pi(g\sigma(\xi)) = \pi(g\ell(\xi))$. Then $u(\xi) = g(\sigma(\xi) - \ell(\xi))$ lies in $(\text{rad } N')_x$, satisfies $u(\xi)r = g\sigma(\xi)r$, and is killed by r^2 and rp_j . Extend it by up_0 . The factorization just constructed gives

$$g(\phi - 1) - ng \in \mathcal{P}_C(\Omega, N').$$

Thus $\psi = 1 + n$ has the asserted properties. $\square$

Lemma 6.3. *Suppose the common boundary data and their shear are defined over a finite field k , and become the preceding disjoint root data after scalar extension. Then (6.2) holds over k and over every finite extension of k .*

Proof. Let $\text{top}_{\text{sel}} N'$ and $\text{soc}_{\text{sel}} N'$ be the sums of the selected simple types in the top and socle. Elements of

$$\text{Hom}_C(\text{top}_{\text{sel}} N', \text{rad } N' \cap \text{soc}_{\text{sel}} N')$$

give square-zero endomorphisms n by projection and inclusion. The condition $ng - g(\phi - 1) \in \mathcal{P}_C(\Omega, N')$ is linear in n . Moreover,

$$\mathcal{P}_C(\Omega, N') = \text{im}(\text{Hom}_C(\Omega, C) \otimes_k \text{Hom}_C(C, N') \longrightarrow \text{Hom}_C(\Omega, N')),$$

where the map is composition: a projective is a summand of a finite free module. This image commutes with field extension, as do Hom spaces, radicals and the indicated semisimple parts over a perfect field. Over a splitting field, Ω decomposes by the disjoint penny supports, since arrows between these supports act by zero in its ambient module. Apply Lemma 6.2 to each summand and add the corrections. The resulting linear system has a solution after extension, hence over k . Then $1 + n$ is the required automorphism. The argument applies over any finite extension as the new base field. $\square$

Proposition 6.4. *Suppose A, A^0 have the actual common quotient and root-isolated boundary data above. Retain every C -module and replace each selected projective-injective P_s by its corresponding P_s^0 , with the same multiplicity. This correspondence preserves all Hall numbers over every finite extension. If A^0 is geometrically standard and representation-finite, then A has Hall polynomials.*

Proof. For an indecomposable A -module M with $MJ \neq 0$, a map $P_s \rightarrow M$ is nonzero on the simple essential socle of some P_s . Its kernel is zero. Injectivity of P_s splits the embedding, so $M \simeq P_s$. All other indecomposables are exactly the C -modules. The same holds on the other side. Lemmas 6.1 and 6.3 give the Hall comparison when both endpoints are C -modules and the middle module is arbitrary.

Hom dimensions agree under the correspondence. For a C -module V ,

$$\text{Hom}_A(P_s, V) \simeq Ve_s, \quad \text{Hom}_A(V, P_s) \simeq D_k(Ve_s).$$

The second identity uses $P_s \simeq D_k(Ae_s)$, since it is the injective envelope of the root simple. The dual of a finite residue field is isomorphic to itself as a module over that field. Hom spaces between selected projectives are the corner spaces; these have the dimensions of the common quotient corners plus their corresponding socle corners. Residue-field degrees agree for C -labels, and those of P_s, P_s^0 equal that of their common top. Formula (2.1) therefore gives equal automorphism orders.

Write $X = E_X \oplus X_c$, $Y = E_Y \oplus Y_c$ with X_c, Y_c annihilated by J and E_X, E_Y sums of selected projective-injectives. Projectivity and injectivity give

$$\mathrm{Ext}_A^1(X, Y) \simeq \mathrm{Ext}_A^1(X_c, Y_c),$$

with middle module $E_X \oplus E_Y$ plus the middle module on the right. Cancel these summands in a prescribed middle module; if their multiplicities are insufficient, the extension set is empty. For arbitrary endpoints the exact-sequence count is

$$F_{X,Y}^M = \frac{|\mathrm{Ext}_A^1(X, Y)_M| |\mathrm{Aut}_A(M)|}{|\mathrm{Aut}_A(X)| |\mathrm{Aut}_A(Y)| |\mathrm{Hom}_A(X, Y)|}. \quad (6.7)$$

It follows by the same endpoint-frame count as (6.4). Apply it first to the C -endpoints and then to X, Y to obtain the comparison. Over an arbitrary finite extension repeat the argument with its indecomposable decomposition.

The indecomposable correspondence preserves all residue-field degrees, so $D_A = D_{A^0}$. Theorem 1.2 supplies the polynomial on A^0 over exactly the conservative extensions required for A . $\square$

7 Standard forms over finite fields

Let A be a basic representation-finite algebra over a finite field k of characteristic two. In its geometric extension, select all nonstandard pennies with isolated roots. A penny is called *active* when its choice in the mixed ray/stem classification is nonstandard and the critical path $a_n r a_1$ is nonzero. The mixed classification and the disjointness of the supports are [3, Theorems 6 and 10]. In this section all active roots are assumed isolated.

Proposition 7.1. *There is a representation-finite geometrically standard k -algebra A^0 with the common quotient and boundary data of Proposition 6.4.*

Proof. We construct the algebra over k . First consider an isomorphism, possibly semilinear, between two normalized active geometric pennies. Its root and end-arrow coordinates have the form

$$r \mapsto ur + vr^2 + wr^3, \quad a_1 \mapsto \alpha(a_1 + tra_1), \quad a_n \mapsto \gamma(a_n + sa_n r),$$

with $u, \alpha, \gamma \neq 0$. The zero turn and the nonzero critical path imply $s = t$. Higher terms in an internal arm arrow contribute zero to the completed root circle: an internal detour traverses an additional root circuit, whose completed product contains $(r^2 + r^3)r^j(r^2 + r^3) = 0$, and an outside detour is killed by root isolation. Thus the image of the arm circle is $c(r^2 + r^3)$, where c is the product of the leading arrow coefficients. The two end corrections cancel in characteristic two. The root side of the relation maps to $u^2 r^2 + u^3 r^3$, whence

$$c = u^2 = u^3, \quad u = 1. \quad (7.1)$$

The same calculation precludes an isomorphism from an active penny to a standard penny: the image of the latter's arm circle has no cubic term, whereas the image of $r^2 + r^3$ has nonzero cubic coefficient. Essential penny supports are intrinsic in the ray category. Frobenius therefore preserves the active set.

Let J be the sum of the selected projective-injective socles, descended to k , and put $C = A/J$. On a geometric selected root set $R = \mathrm{rad} P$, $T = P/\mathrm{soc} P$, and let $h : R \rightarrow T$ be the natural map. Define $\delta : R \rightarrow T$ by

$$\delta(r) = r^2, \quad \delta(r^2) = \delta(r^3) = 0, \quad \delta|_{R_j} = 0 \quad (1 \leq j < n).$$

The relations $r^2 p_j = q_i r^2 = 0$ make it C -linear. A semisimple complement to $\text{rad } A$ supplies, after extension, Frobenius-permuted primitive idempotents. Equation (7.1) shows that Frobenius sends r to $r' + vr'^2 + wr'^3$ and r^2 to r'^2 . Writing F_R, F_T for the induced Frobenius maps on R, T , respectively, gives $F_T \delta = \delta F_R$, including the zero arm components. For each actual selected projective-injective P_s , the orbit sum descends to

$$\delta_s : R_s \rightarrow T_s, \quad R_s = \text{rad } P_s, \quad T_s = P_s / \text{soc } P_s.$$

Let $h_s : R_s \rightarrow T_s$ be the natural map and put $h_s^0 = h_s + \delta_s$. In geometric coordinates (6.6) identifies this map with the standard boundary.

Every indecomposable other than a selected P_s is a C -module by the essential-socle argument in Proposition 6.4. For any such module X ,

$$\text{Hom}_A(P_s, X) = \text{Hom}_C(T_s, X), \quad \text{Hom}_A(X, P_s) = \text{Hom}_C(X, R_s). \quad (7.2)$$

If $a_s : R_s \hookrightarrow P_s$ and $b_s : P_s \twoheadrightarrow T_s$ are the natural maps, every radical endomorphism of P_s is uniquely $a_s f b_s$ for $f : T_s \rightarrow R_s$, and $b_s a_s = h_s$. The endomorphism ring of P_s is commutative, since its geometric extension is a product of the root corners $\bar{k}[r]/(r^4)$. Its residue field D_s has a unique coefficient-field lift: lift the roots of $X^{|D_s|} - X$, whose derivative is -1 . Write c_R, c_T for the actions of $c \in D_s$ on R_s, T_s . After geometric extension these are scalar actions on the individual root summands. Thus δ_s and h_s^0 commute with them.

Retain the entire C -module category and add objects P_s^0 with

$$\begin{aligned} \text{Hom}(P_s^0, X) &= \text{Hom}_C(T_s, X), & \text{Hom}(X, P_s^0) &= \text{Hom}_C(X, R_s), \\ \text{End}(P_s^0) &= D_s \oplus \text{Hom}_C(T_s, R_s). \end{aligned}$$

On the last space define

$$(c, f)(d, g) = (cd, c_R g + f d_T + f h_s^0 g). \quad (7.3)$$

If $u : X \rightarrow R_s$ and $v : T_s \rightarrow Y$ represent maps through P_s^0 , their composite is $vh_s^0 u$. For the opposite order, $u : X \rightarrow R_s$ and $v : T_s \rightarrow X$ compose to $(0, uv)$. The actions on the adjacent Hom spaces are

$$v(c, f) = v c_T + v h_s^0 f, \quad (c, f)u = c_R u + f h_s^0 u.$$

For distinct selected roots the Hom space between P_s^0, P_t^0 is zero. Indeed, $\text{Hom}_C(T_s, R_t) = 0$ geometrically, because the root of T_s is outside the support of R_t , and hence also over k . These formulas are associative: expansion reduces every product to ordinary composition and the equalities $c_T h_s^0 = h_s^0 c_R$; composites between different roots vanish by the same Hom equality. With h_s in place of h_s^0 , (7.2) recovers the original additive category.

Take the additive hull, whose objects are finite direct sums of the objects just defined. Let W^0 be the sum of the new P_s^0 and all nonselected projectives $e_j C = e_j A$, and define $A^0 = \text{End}(W^0)$ with composition multiplication. The convention is appropriate for right modules, since $\text{End}_A(A_A) = A$. This is a finite-dimensional algebra over k . Geometrically, the formulas are exactly the radical and socle-quotient description of the algebra obtained by changing the selected mixed choices to their standard choices.

For $v \in kr \oplus kr^2 \oplus kr^3$, let $f_v : T \rightarrow R$ denote the C -linear map determined by $f_v(1) = v$, and set $z = (0, f_r)$ in the new root corner. Then

$$z^2 = (0, f_{r^2+r^3}), \quad z^3 = (0, f_{r^3}).$$

The unchanged arm circle represents $(0, f_{r^2+r^3})$, so it equals z^2 . Its common quotient sends z to $r + r^2$. The standard loop $z + z^2 + z^3$ maps to r and has the same square and cube

as z ; its extra terms vanish on adjacent arm arrows. The induced map from the standard radical to the retained radical sends the first three loop powers to $r + r^2, r^2 + r^3, r^3$. Its direction is opposite to (6.6). Composing the two maps sends r to $r + r^3$ and fixes r^2, r^3 and the arm components. Their discrepancy is therefore killed by the socle quotient, so both give the boundary h_s^0 . In the retained radical the new loop acts by the old multiplication by r : if $f_v(1) = v$, its action is $f_v h_s^0 f_{r+r^2}$, and $h_s^0 f_{r+r^2}(1) = r$. Writing $\ell = z + z^2 + z^3$ for the standard loop, the arm products $\ell p_j = r p_j$, $q_i \ell = q_i r$ and $q_i \ell p_j = q_i r p_j$ preserve every nonroot relation. Root isolation preserves the remaining outside zero relations. The mixed classification therefore identifies A_k^0 with the ray-standard form. It is representation-finite by [3, Theorem 4], and mesh-standard by Lemma 5.3.

The common quotient is also defined over k . Send P_s^0 to T_s , fix retained C -objects, and on morphisms send

$$(c, f) \mapsto c_T + h_s^0 f, \quad u : X \rightarrow R_s \mapsto h_s^0 u, \quad v : T_s \rightarrow X \mapsto v.$$

The composition formulas show that this is a functor. On W^0 it gives $A^0 \rightarrow \text{End}_C(C_C) = C$. Geometrically it is surjective and kills exactly the selected socles; faithful scalar extension proves both assertions over k . To identify the actual modules, use the right A^0 -module functor $\mathbb{Y}(X) = \text{Hom}(W^0, X)$, with action by precomposition. For a retained C -module X , its blocks give $\mathbb{Y}(X) = \text{Hom}_C(C, X) \simeq X$, with action through $A^0 \rightarrow C$.

Let $a_s^0 : R_s \rightarrow P_s^0$ and $b_s^0 : P_s^0 \rightarrow T_s$ be represented by 1_{R_s} and 1_{T_s} , respectively. Check their blocks after a finite splitting extension. At the selected root, $\mathbb{Y}(a_s^0)$ sends f to $(0, f)$; at an unselected projective it is the identity on $\text{Hom}_C(e_j C, R_s)$; at every other selected root both spaces are zero. It is therefore injective, with quotient the root simple. Since the root corner is the local algebra generated by ℓ with $\ell^4 = 0$, $\mathbb{Y}(P_s^0)$ is indecomposable projective, and this image is its radical.

At the root, $\mathbb{Y}(b_s^0)$ is $(c, f) \mapsto c_T + h_s^0 f$, surjects onto the span of $1, r, r^2$, and has kernel spanned by $(0, f_{r^3})$. At the other vertices of the arm it is an isomorphism, and the outside spaces are zero. The socle of the retained R_s is the root simple spanned by r^3 . Every simple submodule of a non-simple indecomposable projective lies in its radical: otherwise its projection onto the simple top splits the projective as that simple plus its maximal submodule. Thus the kernel just computed is the entire socle of $\mathbb{Y}(P_s^0)$. Finally, the composition rules give $b_s^0 a_s^0 = h_s^0$. Faithful scalar extension descends these kernel, image and quotient identifications to k . Hence R_s, T_s, h_s^0 are the actual radical, socle quotient and boundary map of the new selected projective.

Finally, geometric representation-finiteness implies representation-finiteness over k . Choose a finite extension L/k defining all geometric indecomposables. Modules over a finite field that become isomorphic after further extension are already isomorphic: restriction gives isomorphic repeated direct sums, and Krull–Schmidt cancels their multiplicities. Thus every M_L is a sum of the finitely many chosen L -modules. Since M is a summand of the restriction of M_L , each indecomposable M is a summand of one of their restrictions. There are only finitely many possibilities. This completes the construction. $\square$

8 Pennies with an external sink

Over a field of characteristic two, consider vertices x, y , a loop r at x , and arrows $p : x \rightarrow y$, $q : y \rightarrow x$, with

$$pq = r^2 + \lambda r^3, \quad qp = 0, \quad r^4 = 0, \quad \lambda \in \{0, 1\}. \quad (8.1)$$

The three attachments below have the additional vertices, arrows and relations displayed in the table. Their outside vertices are sinks. The defining ideal is generated by the displayed

relations and (8.1).

type	additional arrows	additional relations
a	$\beta : x \rightarrow b$	$r\beta = 0$
b	$\gamma : y \rightarrow c$	$rp\gamma = 0$
c	$\beta : x \rightarrow b, \gamma : y \rightarrow c$	$r\beta = q\beta = p\gamma = 0$

We also use type a with $q\beta = 0$ and type b with $p\gamma = 0$. Denote each resulting algebra by A_λ , retaining its type. The standard choice is $\lambda = 0$. In every type, $r^2p = qr^2 = 0$ follows from (8.1): multiply pq by p , or by q on the left, and use that $1 + \lambda r$ is invertible.

8.1 Exceptional projective and injective modules

For types a, c and the smaller type a , take $J = kr^3$, $P = e_x A_\lambda$, and $I = P/S_b$. For type b and its smaller quotient, put $j = qrp$, $J = kj$, $P = e_y A_\lambda$, and $I = P/S_c$. Each J is a two-sided ideal. In the first case the vertex spaces of P are

$$P_x = k1 \oplus kr \oplus kr^2 \oplus kr^3, \quad P_y = kp \oplus krp, \quad P_b = k\beta;$$

its c -space in type c is zero. In the second case they are

$$P_y = ke_y \oplus kj, \quad P_x = kq \oplus kqr, \quad P_c = k\gamma.$$

In both cases I is obtained by deleting the indicated sink vector.

Lemma 8.1. *For these algebras, I is injective. The indecomposable modules not annihilated by J are exactly P and I .*

Proof. First identify I with the dual of the left projective at the same vertex as P . In types a, c , pair I with $A_\lambda e_x$ by taking the coefficient of r^3 in the product. The root powers pair nondegenerately by complementary degrees summing to three. Between the right basis p, rp and the left basis q, qr , the matrix is

$$\begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}.$$

In type b , use the coefficient of j . The pairs (e_y, j) with (e_y, j) and (q, qr) with (p, rp) both have matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Products at different vertices do not mix these blocks. Each pairing is nondegenerate and satisfies $\langle va, u \rangle = \langle v, au \rangle$. Hence it gives a right-module isomorphism $I \simeq D_k(A_\lambda e_x)$ or $D_k(A_\lambda e_y)$, respectively. Thus I is injective with essential simple socle at its generating vertex. The extra relations of the two smaller types leave these bases and pairings unchanged.

Write s for that generating vertex and t for the deleted sink. There is an exact sequence

$$0 \longrightarrow S_t \longrightarrow P \xrightarrow{\pi} I \longrightarrow 0. \quad (8.2)$$

The copy $PJ \simeq S_s$ in P maps isomorphically to $\text{soc } I$. If an indecomposable M satisfies $MJ \neq 0$, a vector on which the generating path of J acts nontrivially defines a map $f : P \rightarrow M$ nonzero on PJ . Let $K = \ker f$. If $\pi(K) \neq 0$, essentiality implies that it contains $\text{soc } I$. Lift a nonzero socle vector to K ; it differs from the corresponding vector in PJ only by a vector at t . Multiplication by e_s , with $s \neq t$, removes the latter. This contradicts injectivity of f on PJ . Therefore $K \subseteq S_t$.

The map $\text{im } f \rightarrow I$ induced by π extends, by injectivity of I , to a surjection $M \rightarrow I$. Let its kernel be N . Applying $\text{Hom}(-, N)$ to (8.2), using projectivity of P , gives

$$\text{Ext}^1(I, N) = N_t / \text{im}(N_s \rightarrow N_t) = (\text{top } N)_t. \quad (8.3)$$

The last equality uses the unique arrow entering the sink. Since S_t is projective, write $N = N_0 \oplus S_t^d$, with $(\text{top } N_0)_t = 0$. Then $\text{Ext}^1(I, N_0) = 0$ and $\text{Ext}^1(I, S_t^d) = k^d$. The zero column gives middle module $I \oplus N$, and any nonzero column can be taken by $\text{GL}_d(k)$ to the first coordinate, giving $P \oplus N_0 \oplus S_t^{d-1}$. Indecomposability forces $M \simeq I$ or P . $\square$

8.2 Hall numbers over the common quotient

For types a, c , the common quotient is $C = A_\lambda/(r^3)$ in the displayed arrow coordinates. Put $R_I = \{v \in I : vr^3 = 0\}$ and $R_P = \{v \in P : vr^3 = 0\}$. Then $R_P = R_I \oplus S_b$. Identify their two versions by (6.6) on the root and the identity elsewhere. The modules $T_P = P/PJ = e_x C$ and $T_I = I/IJ = T_P/S_b$ are common. Their natural boundaries differ by the root shear $1+r$. For $E = P^u \oplus I^v$, with $u, v \geq 0$,

$$\text{rad}(E/EJ) = H^{u+v} \oplus S_b^u,$$

where H is the two-vertex H of Section 6. All its outside arrows act by zero.

Lemma 8.2. *For each of the five displayed models, identify the common-quotient modules and replace P, I by their standard counterparts. If X, Y, N are common-quotient modules, then*

$$F_{X,Y}^{P^u \oplus I^v \oplus N}(A_1) = F_{X,Y}^{(P^0)^u \oplus (I^0)^v \oplus N}(A_0) \quad (u, v \geq 0).$$

This equality holds after every scalar extension.

Proof. In types a, c , repeat Lemma 6.2 for $\Omega \subseteq H^{u+v} \oplus S_b^u$. The additional sink space is sent to zero in each correction. The root radical of the target is $N'_x r + N'_y q$. In the projective factorization, $r^2 \beta = qr \beta = 0$; in type c the arm value qrp is killed by γ , since $p\gamma = 0$. The top-to-socle correction is a module map because its root image is killed by β as well as r, p . For the smaller type a , $q\beta = 0$ adds no further condition to these maps. Thus (6.2) holds in the actual common quotient, and Lemma 6.1 applies.

For type b , take instead $C = A_\lambda/(j)$. In $A_1/(j)$ the arrow $p' = (1+r)p$ satisfies

$$p'q = r^2, \quad qp' = 0, \quad rp'\gamma = 0.$$

It identifies this quotient with $A_0/(j)$, where the standard p corresponds to p' . In these common coordinates the annihilators of j in P, I are identified by

$$q \mapsto q + qr, \quad qr \mapsto qr, \quad j \mapsto j,$$

and by the identity at the sink. The common $T_P = P/PJ = e_y C$ and $T_I = I/IJ = T_P/S_c$ have root chain $q \mapsto qr \mapsto 0$. Their y -space is generated by e_y , which q sends to q ; the common p acts by zero. Only T_P has the sink vector. The two boundaries differ on the root by $q \mapsto qr$ and agree elsewhere. Thus

$$(\text{rad}(E/EJ))_x \simeq (k[r]/(r^2))^{u+v}, \quad (\text{rad}(E/EJ))_y = 0, \quad (\text{rad}(E/EJ))_c = k^u,$$

and all arrows between distinct vertices act by zero there.

For $\Omega \subseteq \text{rad}(E/EJ)$, use the same top-to-socle construction as in Lemma 6.2. It reduces the remaining difference to a root map sending a chosen length-two-chain generator to ur , killing the bottom and all length-one chains, where

$$u = wr + zq \in (\text{rad } N')_x, \quad ur^2 = up = 0.$$

Here $qr^2 = 0$, so $wr^3 = 0$. In $e_x C$, r^3 is nonzero. The first projective factor must therefore send the source length-two chain to r^2, r^3 , and its length-one summands to zero. This commutes

with r and p by $r^4 = r^2p = r^3p = 0$. Sending the projective generator to w gives the required wr^2 on the upper layer and zero on the lower layer. The second factor sends the upper layer to $qr \in e_y C$ and the other components to zero; it is a module map because $qr^2 = qrp = 0$ in C . Sending e_y to z gives zqr . These constructions are linear in all generator coefficients and apply to arbitrary multiplicities. The γ -square commutes because the source y -space is zero. They also work in the smaller quotient $p\gamma = 0$.

The root socle correction is killed by both r and p , so it is again an endomorphism of the entire module. This proves (6.2) for type b . The shear preserves every source submodule, and Lemma 6.1 gives the equality. All constructions use linear algebra over the base field and repeat after scalar extension. $\square$

8.3 Associativity recursions

Suppose A is representation-finite, $C = A/J$ with $J \subseteq \text{rad } A$, and the only indecomposables outside $\text{mod } C$ are P, I . Assume that P is projective, I is injective, $\text{rad } P \in \text{mod } C$, and (8.2) holds with $\text{top } P = S_s$, where $t \neq s$ is a sink with a unique incoming arrow from s . Under $P \simeq e_s A$, assume that the embedded S_t is spanned by this arrow. Assume $A/\text{rad } A$ is a product of copies of k . In particular, every simple A -module is a C -module. Suppose Hall polynomials in $\mathbb{Z}[T]$ are known whenever both endpoints are C -modules, allowing an arbitrary middle module.

For any modules U, V , use the automorphism polynomials (2.1) and put

$$c(U, V) = \frac{a_{U \oplus V}(T)}{a_U(T)a_V(T)T^{h(U, V)}} \in \mathcal{R}^\times. \quad (8.4)$$

Evaluating at $T = Q$ gives the Hall coefficient of the split middle module over a conservative field of order Q . In the counting identities below, every rational function in T is evaluated at Q ; the same identities will construct the rational functions inductively. To justify the split coefficient, a nonsplit sequence $0 \rightarrow V \rightarrow Z \rightarrow U \rightarrow 0$ satisfies

$$\dim \text{End } Z < \dim \text{End}(U \oplus V). \quad (8.5)$$

Indeed, left exactness gives

$$h(Z, Z) \leq h(U, Z) + h(V, Z), \quad h(U, Z) \leq h(U, V) + h(U, U) - 1, \quad h(V, Z) \leq h(V, V) + h(V, U).$$

The strict term occurs because $\text{Hom}(U, Z) \rightarrow \text{End } U$ cannot contain the identity without splitting the sequence. Thus the only extension class with middle module $U \oplus V$ is zero, and (6.7) gives (8.4).

Proposition 8.3. *Under these hypotheses, A has Hall polynomials.*

Proof. First compute the entire row with quotient I^m . Write $N = N_0 \oplus S_t^d$, where N_0 has no S_t summand. Applying $\text{Hom}_A(-, N)$ to (8.2) gives

$$\text{Ext}_A^1(I, N) = N_t / \text{im}(N_s \rightarrow N_t) = (\text{top } N)_t.$$

The unique incoming arrow at t gives the last equality. A complement to its image splits off a copy of S_t for each basis vector, so $(\text{top } N_0)_t = 0$ and $(\text{top } N)_t = k^d$. Additivity therefore identifies $\text{Ext}^1(I^m, N)$ with $d \times m$ matrices. Rank r has middle module

$$L_r = N_0 \oplus P^r \oplus I^{m-r} \oplus S_t^{d-r}, \quad 0 \leq r \leq \min(m, d).$$

Row and column operations give this decomposition from (8.2), and the P multiplicity distinguishes the different ranks. Factoring a rank- r matrix as an injection times a surjection and dividing by GL_r gives its count

$$\prod_{j=0}^{r-1} \frac{(Q^m - Q^j)(Q^d - Q^j)}{Q^r - Q^j}.$$

For $r = 0$ the product is one. Therefore

$$F_{I^m, N}^{L_r} = \frac{a_{L_r}(T)}{a_{I^m}(T)a_N(T)T^{h(I^m, N)}} \prod_{j=0}^{r-1} \frac{(T^m - T^j)(T^d - T^j)}{T^r - T^j} \quad (8.6)$$

where the right side belongs to $\mathcal{R}$ and is evaluated at $T = Q$. All other middle coefficients in this row are zero.

Induct on the dimension $\dim_k M$ of the middle module. Zero endpoints give the usual Kronecker coefficients. At a conservative extension the indecomposable labels form the same finite, nonrepeating list, as proved in Section 2. Thus every bounded-dimension sum below has a fixed finite index set, and Hom and End dimensions remain unchanged.

If the quotient contains P , projectivity implies that the middle module also contains P . For $M = P \oplus M'$ the exact split-factor identity is

$$F_{P \oplus X', Y}^{P \oplus M'} = \frac{c(P, M')}{c(P, X')} F_{X', Y}^{M'}. \quad (8.7)$$

It follows from (6.7) and $h(P, M') = h(P, X') + h(P, Y)$. If the required summand is absent, the coefficient is zero. Similarly injectivity gives

$$F_{X, Y' \oplus I}^{M' \oplus I} = \frac{c(M', I)}{c(Y', I)} F_{X, Y'}^{M'}. \quad (8.8)$$

Both right sides have smaller middle dimension.

For three nonzero modules U, V, W of total dimension $\dim_k M$, associativity gives

$$\sum_Z F_{U, V}^Z F_{Z, W}^M = \sum_N F_{V, W}^N F_{U, N}^M. \quad (8.9)$$

It follows by counting two-step submodule flags. Every inner coefficient has middle dimension less than $\dim_k M$, so it is already in $\mathcal{R}$.

First let $Y \in \text{mod } C$. The cases $X \in \text{mod } C$, X containing P , and $X = I$ are known. Otherwise $X = I \oplus X'$ with $X' \neq 0$. For fixed $\dim X$, proceed in increasing order of $\dim \text{End } X$. Formula (8.9) gives

$$c(I, X') F_{X, Y}^M = \sum_N F_{X', Y}^N F_{I, N}^M - \sum_{Z \neq X} F_{I, X'}^Z F_{Z, Y}^M. \quad (8.10)$$

The first sum uses the known row (8.6); every nonzero term in the second has $\dim \text{End } Z < \dim \text{End } X$ by (8.5). This constructs all columns with Y a C -module. Formula (8.8) then constructs those with $Y = I^a \oplus Y_0$, $Y_0 \in \text{mod } C$.

Next consider $Y = P$. Its unique maximal submodule is $\text{rad } P$, so $F_{S_s, \text{rad } P}^P = 1$. For $X \neq 0$, associativity gives

$$F_{X, P}^M = \sum_N F_{X, S_s}^N F_{N, \text{rad } P}^M - \sum_{Z \neq P} F_{S_s, \text{rad } P}^Z F_{X, Z}^M. \quad (8.11)$$

The first sum uses C -submodule columns. In the second, $\dim Z = \dim P$, so $Z \not\supseteq P$ cannot contain P and is a sum of I and C -modules. These columns are already known. The inner dimensions are strictly smaller than $\dim_k M$, since $X \neq 0$ and $\text{rad } P \neq 0$.

Finally, submodule endpoints without P or with an I summand have been treated. For the remaining $Y = Y' \oplus P$, $Y' \neq 0$, use increasing End dimension at fixed $\dim Y$ and

$$c(Y', P)F_{X,Y}^M = \sum_N F_{X,Y'}^N F_{N,P}^M - \sum_{Z \not\supseteq Y} F_{Y',P}^Z F_{X,Z}^M. \quad (8.12)$$

The first sum uses (8.11), and the second has strictly smaller End dimension by (8.5). Fixed dimension has finitely many labels, so both inner orderings terminate. All divisions are by units (8.4). These formulas construct a single element of $\mathcal{R}$ for every coefficient, equal to the actual Hall number on every conservative extension. Such field sizes are unbounded, for example with extension degrees congruent to one modulo D_A . Lemma 2.1 makes each function an element of $\mathbb{Z}[T]$. $\square$

Corollary 8.4. *Every representation-finite algebra among the five split models A_1 above has Hall polynomials.*

Proof. For each model, Lemma 8.1 identifies its indecomposables with the common-quotient labels together with P, I . The same argument applies to A_0 , so its finite representation type follows from that of C . The standard choice is ray-standard and hence geometrically standard by Lemma 5.3. Lemma 8.2 and Theorem 1.2 supply the C -endpoint input for Proposition 8.3. The residues of P, I are the base field, already represented by the common simple at their generating vertex. Consequently $D_{A_1} = D_C = D_{A_0}$, and the input holds on every required conservative extension. Here $J \subseteq \text{rad } A_1$, the semisimple quotient is split, and $\text{rad } P$ is annihilated by J . The kernel in (8.2) is spanned by the unique sink arrow β in types a, c and by γ in type b . Thus all the recursion hypotheses hold for each model itself. $\square$

9 Node splitting and Hall polynomials

9.1 Quotients and forms of penny attachments

Lemma 9.1. *Every finite-field form of a representation-finite quotient of one of the three geometric nonstandard attachments has Hall polynomials.*

Proof. First work over an algebraic closure with one maximal attachment B and a two-sided ideal K . Put $j = qrp$. If both r^3 and j survive in B/K , then K has zero intersection with every corner on x, y . In the root corner $k[r]/(r^4)$, every nonzero ideal contains r^3 . For a nonzero $(a + br)p$, multiplication by q gives $ar^2 + (a + b)r^3$; if $a \neq 0$, multiply once more by r , and otherwise the result is already a nonzero multiple of r^3 . The corner spanned by q, qr is treated by left multiplication by p . Finally, the corner spanned by e_y, j is $k[j]/(j^2)$, and every nonzero ideal there contains j .

Every ideal is the direct sum of its intersections with the Peirce corners $e_u B e_v$. The remaining nonidentity corners have bases

$$\begin{array}{c|l} a & \beta, q\beta \\ b & \gamma, p\gamma \\ c & \beta, \gamma. \end{array}$$

Each listed path lies in a different one-dimensional corner. Killing β kills $q\beta$, and killing γ kills $p\gamma$. Thus the connected quotients retaining both critical paths are precisely the three maximal models, type a with $q\beta = 0$, type b with $p\gamma = 0$, and their arrow deletions. Deleting an

outside vertex gives the same connected component, with possible isolated simple components. Deleting β from type c gives the smaller type b ; deleting γ gives the smaller type a . With no outside arrow one obtains an isolated penny.

If $r^3 \in K$, the circle relation is already standard. If $j \in K$, the change $p' = (1 + r)p$ makes it standard and preserves all outside relations. In either case B/K is a quotient of the corresponding standard model B_0 . It is ray-standard: in every Peirce corner of B_0 , subbimodules are exactly spans of terminal segments of the path-ray basis. Explicitly the root corner has ideals (r^i) , the y -corner has ideals $0, kj, k[j]/(j^2)$, the two arrow corners are length-two single Jordan chains under multiplication by r , and outside nonidentity corners are one-dimensional. Hence an ideal is spanned by a subset of ray basis vectors. The subset is an ideal of the ray category, and [3, Proposition 2(c)] identifies the quotient ray category with that of the quotient algebra. Its linearization is the given quotient. Lemma 5.3 and Theorem 1.2 apply to every finite-field form of these representation-finite quotients.

It remains to determine the forms of the connected models retaining r^3, j . Their automorphisms fixing all vertex idempotents are exactly

$$\begin{aligned} r &\longmapsto r + vr^2 + wr^3, \\ p &\longmapsto \alpha(p + trp), & q &\longmapsto \alpha^{-1}(q + tqr), \\ \beta &\longmapsto \eta\beta, & \gamma &\longmapsto \theta\gamma, \end{aligned}$$

where only the existing outside arrows are included, v, w, t are arbitrary, and α, η, θ are nonzero. To prove necessity, first allow root coefficient u and distinct correction coefficients in p, q . The relation $qp = 0$ and $j \neq 0$ force the latter to agree. Comparing the quadratic and cubic terms of $pq = r^2 + r^3$ then gives $u^2 = u^3 = \alpha_1\alpha_2$, hence $u = 1$ and $\alpha_2 = \alpha_1^{-1}$. The corner bases force the other displayed terms. For sufficiency, the two tj terms in qp cancel, as do the two tr^3 terms in pq . The relations $r\beta = 0$ and $rp\gamma = 0$ are preserved by $r^2p = 0$. Where required, $q\beta = 0$ is preserved since $qr\beta = 0$, and $p\gamma = 0$ is preserved since $rp\gamma = 0$. The induced map on arrows modulo their square is invertible, so Nakayama's lemma makes this an automorphism.

For s outside arrows this fixed-idempotent group has underlying variety $(\mathbb{G}_m)^{1+s} \times \mathbb{A}^3$ and is connected. The loop fixes x , the two-cycle fixes y , and the sources of the outside arrows distinguish the sinks; there is no vertex permutation. Every complete lifted idempotent set is conjugate to the given one by $1 + \text{rad } B$. Thus the full reduced automorphism group is connected as the image of the product of this connected group and $1 + \text{rad } B$. Over a perfect field it is smooth. Lang's theorem [8] makes its first Galois cohomology trivial. Since each model has a presentation over $\mathbb{F}_2$, every geometrically connected finite-field form is the displayed split model. Corollary 8.4 and Proposition 7.1 apply. In particular the two smaller models are treated over their own conservative extensions.

A finite-field form can permute several geometric connected components. Its reduction to a geometrically connected algebra over a central field, and the corresponding change of variable in its Hall polynomials, is given in Lemma 9.3 below. $\square$

9.2 Node splitting

A split vertex x is a *node* here if it has no loop and every product of an incoming arrow with an outgoing arrow is zero. Replace it by a receiving vertex x^- and an emitting vertex x^+ , retaining the incoming and outgoing arrows respectively. This defines the node-splitting algebra A^s . For a module M , let $R_x(M)$ be the sum of all incoming-arrow images at x . Outgoing arrows kill this subspace. Define $\mathcal{F}(M)$ on the two new vertices by

$$\mathcal{F}(M)_{x^-} = R_x(M), \quad \mathcal{F}(M)_{x^+} = M_x / R_x(M),$$

and retain all other spaces and induced arrow maps. Write S_x, S_-, S_+ for the simple modules at the old and two new vertices.

Lemma 9.2. *Suppose A is representation-finite. Then A^s is representation-finite with $D_A = D_{A^s}$. If A^s has Hall polynomials, so does A .*

Proof. Fix M and a submodule U . Define a submodule $U' \subseteq \mathcal{F}(M)$ using the same spaces away from the node and

$$U'_- = U_x \cap R_x(M), \quad U'_+ = (U_x + R_x(M))/R_x(M).$$

For a fixed U' , the choices of U_x are graphs of maps $U'_+ \rightarrow R_x(M)/U'_-$, after choosing a vector-space splitting. Every graph satisfies the arrow conditions, because all outgoing arrows kill $R_x(M)$. Its isomorphism type and its quotient's isomorphism type are independent of the graph: a shear from the emitting complement to $R_x(M)$ is an automorphism of the ambient representation. The quotient's receiving space is generated by its incoming images and is exactly $R_x(M/U)$. Thus $\mathcal{F}(M)/U' \simeq \mathcal{F}(M/U)$.

Write $Y = Y_0 \oplus S_x^m$, where Y_0 has no S_x summand, and set $Y'_a = \mathcal{F}(Y_0) \oplus S_+^a \oplus S_-^{m-a}$ for $0 \leq a \leq m$. These are precisely the split-node types that merge to Y . To see this, split off the receiving space not generated by incoming arrows, which is a sum of S_- , and split off the emitting kernel of all outgoing arrows, which is a sum of S_+ . The remaining merged module has no node-simple summand. Consequently the graph count gives

$$F_A^M(X, Y) = \sum_{a=0}^m Q^{\dim(Y'_a)_{x+} - \dim \mathcal{F}(X)_{x-}} F_{A^s}^{\mathcal{F}(M)}(\mathcal{F}(X), Y'_a), \quad Q = |k|. \quad (9.1)$$

Merging the two spaces takes any split-node module with no S_- summand back to an A -module, inverse on isomorphism classes to $\mathcal{F}$. Indeed, absence of an S_- summand says precisely that incoming images generate its receiving space. For each M , the natural map $\text{End}_A(M) \rightarrow \text{End}_{A^s}(\mathcal{F}(M))$ is surjective: lift the two diagonal maps using a splitting of M_x . Its kernel is $\text{Hom}_k(M_x/R_x(M), R_x(M))$ and has square zero. Hence it preserves localness and residue fields. The indecomposables correspond, with one additional simple S_- of the same residue degree as S_x . This proves representation-finiteness and equality of conservative periods. Formula (9.1) is therefore a finite sum of integer polynomials on all conservative extensions. $\square$

The same construction works at an actual primitive idempotent e with $eAe = D$ a finite field. Put $f = 1 - e$ and suppose $(fAe)(eAf) = 0$. The other cross product is also zero: for $a \in eAf$ and $b \in fAe$, one has $(ab)^2 = a(ba)b = 0$, and the field D has no nonzero nilpotent element. Split D into a receiving and an emitting copy, retaining the two off-diagonal bimodules and fAf . All node spaces, splittings and graph maps in the proof are then D -linear. The exponent in (9.1) becomes

$$[D : k] \dim_D(Y'_a e_+) \dim_D(\mathcal{F}(X) e_-). \quad (9.2)$$

The square-zero-kernel argument uses the corresponding D -linear Hom space and still preserves residue fields. On every conservative extension E/k , $D \otimes_k E$ is a field of order $|E|^{[D:k]}$, so the identical formula holds.

9.3 Restriction of scalars and the main theorem

Lemma 9.3. *Let a connected finite-field algebra have a Frobenius orbit of d geometric connected components. It is the restriction of a geometrically connected algebra over a central field k'/k of degree d . A Hall polynomial $f(T)$ for the latter gives $f(T^d)$ over k , on all conservative extensions.*

Proof. The centre is a commutative local Artin algebra whose residue field is $k' = \mathbb{F}_{q^d}$. Lifting the roots of the separable polynomial $X^{q^d} - X$ embeds k' in the centre. With this central scalar structure the algebra is geometrically connected over k' , and its restriction is the original algebra. The field k' embeds in every nonzero indecomposable endomorphism residue field: a nonzero central scalar is invertible and cannot lie in its radical. Thus d divides every such residue degree. If $[E : k] = r$ is conservative, then $\gcd(r, d) = 1$, so $E' = k' \otimes_k E$ is a field of order $|E|^d$. The residue degrees over k' are the degrees over k divided by d ; therefore E'/k' is conservative as well. Modules, morphisms and submodules carry the same central action on both descriptions. The Hall count is $f(|E|^d)$, proving the assertion. $\square$

Over an algebraically closed field, a representation-finite basic algebra is distributive and has a representation-finite, hence mild, ray category [3, Theorems 2 and 4]. In characteristic two, Theorem 10 of that reference describes it by standard or nonstandard choices at pennies, leaving all other products unchanged. Distinct essential non-deep contours have disjoint vertex sets by its Theorem 6(b). The full penny subcategory and its path data are those of Section 6. If the root has nonzero Hom to an outside vertex, Theorem 8 gives a two-vertex penny and exactly one of the three attachments in Section 8. Its listed outside arrows are the only arrows joining the penny to its complement. Each outside receiving point has no other incoming arrow, and every subsequent outgoing arrow composes to zero. The dual statement holds when Hom from an outside vertex to the root is nonzero. Splitting these receiving or emitting nodes isolates the attachment, as in [3, §3.8].

Proof of Theorem 1.1. Morita equivalence preserves exact sequences, subobjects, endomorphism rings and scalar extension, so take A basic. Its geometric extension is also basic, because the semisimple quotient is a product of finite separable fields. It is representation-finite: each geometric indecomposable is defined over a finite extension and is a summand of the scalar extension of its restriction. It is therefore a summand of the geometric extension of one of the finitely many indecomposable A -modules.

In odd characteristic the standard-form theorem [3, Theorem 3] makes $A_{\bar{k}}$ ray-standard. Lemma 5.3 and Theorem 1.2 give the result. Assume henceforth characteristic two.

The active penny set is Frobenius-stable by the normalization calculation in Proposition 7.1, which applies to each full penny subcategory. First select all outside receiving nodes supplied by Theorem 8 for active pennies with nonzero outgoing root Hom. This set is Frobenius-stable. None of its vertices lies in another penny, since the latter's internal incoming arrow would contradict the unique incoming arrow at that receiving node. There is no oriented path between two selected receiving nodes: the last arrow would come from the target's penny, but that penny has no outside incoming arrow. The same argument excludes a return path to a selected node. Choose a semisimple complement to $\text{rad } A$. Each Frobenius orbit of selected vertices gives a primitive idempotent e in this complement, and $e(A/\text{rad } A)e$ is a finite field. After geometric extension, the blocks of $e \text{ rad } A e$ are radical path spaces between vertices of that orbit. The no-path assertion makes every block zero. Faithful scalar extension yields $e \text{ rad } A e = 0$, so $e A e$ is this finite field.

The node products are also zero in the mixed algebra. Each such product passes through an outside receiving vertex, so its domains and codomains do not all lie in one selected penny. The mixed multiplication of [3, Theorem 10] agrees with ray multiplication on these products. The geometric node relations therefore hold in the algebra itself and descend faithfully. The actual-field construction following Lemma 9.2 applies to each orbit.

Split these orbits, then apply the dual construction to the remaining outside emitting nodes. The dual no-path argument again supplies field corners. These operations terminate on the finite set of indicated nodes. The geometric connected components of the result are

root-isolated components, quotients of the three attachments or their opposites, and ray-standard components. This follows from the complete list of incident arrows in Theorem 8: splitting separates each nonisolated penny from the rest, while every remaining active root has zero Hom in both directions to outside vertices.

For an actual connected block, apply Lemma 9.3 to work over its central finite field with geometrically connected extension. In a root-isolated block, Propositions 7.1 and 6.4 give Hall polynomials. The attachment quotients are covered by Lemma 9.1, which treats every retained critical-path quotient over its own conservative extensions. The ray-standard blocks are covered by Lemma 5.3 and Theorem 1.2.

For an opposite algebra, finite-dimensional duality gives

$$F_A^M(X, Y) = F_{A^{\text{op}}}^{D_k M}(D_k Y, D_k X).$$

It preserves residue-field degrees and commutes with finite scalar extension, so the same conclusion holds for the opposite blocks. Central restriction replaces $f(T)$ by $f(T^d)$, and independent blocks multiply their Hall polynomials. Finally apply (9.1) with (9.2) in reverse order to glue the nodes. Each step is a finite sum of integer polynomials multiplied by nonnegative powers of T , with the same conservative period. This gives (1.1) for the original fixed modules over every conservative extension. Returning along the Morita equivalence completes the proof. $\square$

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