hexagon:2610.00401v3

IRRATIONALITY MEASURES FROM LÖWNER DETERMINANTS

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ABSTRACT. Let μ denote the irrationality exponent. One construction, by determinants of rectangular Löwner matrices of rational combinations of the digamma function and its derivatives at integer nodes, gives irrationality measures on three planes of numbers. For all rationals a and b, not both zero, we prove $\mu(aL(2,\chi_{-3})+b\pi^2) \le 20.6622$, $\mu(a\zeta(3)+b\pi^2) \le 335.20$ and $\mu(a\zeta(3)+b\pi^4) \le 72.726$. The irrationality of these numbers was proved recently by other methods: on the first plane by Calegari, Dimitrov and Tang, on the second by Q. Tang and on the third by Lai. On the first plane the best published bound was 24781, due to Calegari, Dimitrov and Tang. On the second plane, as far as we know, no irrationality measure was known. On the third plane the bound 66.3 of Leibtag et al. is better than ours.

At special points of the planes the bounds are lower. We prove $\mu(L(2,\chi_{-3})) \le 13.6839$, and the bounds 9.4913, 14, 15.359 and 17.492 for $L(2,\chi_{-3}) \pm \pi^2/9$, $\pm 2\pi^2/27$, $\pm\pi^2/27$ and $\pm\pi^2/18$. For completeness we also prove $\mu(\zeta(3)) \le 6.0382$ and $\mu(\pi^2) \le 6.5214$, which are weaker than the known bounds.

The construction also reaches numbers off the planes, many of which are differences of two values of the trigamma function. It gives $\mu(G \pm 5\pi^2/48) \le 31.1973$, where G is Catalan's constant, and bounds 23.7756 and 1806.1719 for the numbers $\psi'(k/5) - \pi^2/6$ and $\psi'(k/7) - \pi^2/6$ with k prime to the denominator. For these numbers we know of no other proof of irrationality. The denominators are bounded prime by prime, and on each plane one table of local bounds and one decay certificate serve every point. Nonvanishing at rational points follows from Hermite interpolation in characteristic zero. The size of the determinants is bounded by Andréief's identity and an energy estimate. The proofs are formalized in Lean 4.

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Date: November 10, 2026.

2020 *Mathematics Subject Classification.* Primary 11J82; Secondary 11J72, 15A15, 31A15, 68V20.

Key words and phrases. Irrationality exponent, $L(2,\chi_{-3})$, $\zeta(3)$, trigamma function, Löwner matrices, p-adic valuations, Hermite interpolation, Andréief's identity, logarithmic energy, formal verification, Lean.

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1. Introduction

1.1. **The results.** The *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the real numbers μ such that $|\xi - p/q| > q^{-\mu}$ for all integers p and all sufficiently large integers q. It does not change when ξ is multiplied by a nonzero rational number or translated by a rational number. A finite bound for $\mu(\xi)$ is an irrationality measure for ξ, and it shows that ξ is irrational.

Let χ_{-3} be the nontrivial Dirichlet character modulo 3. We write

$$L_{-3} = L(2, \chi_{-3}) = \sum_{n \geq 0} \Big(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \Big) = 0.78130\,24128\ldots$$

Calegari, Dimitrov and Tang gave the first proof that L_{-3} is irrational [7], and we call L_{-3} the Calegari–Dimitrov–Tang constant.

Our main results are uniform irrationality measures on three planes of numbers. One Löwner structure, described below, produces all three. The first plane is spanned by L_{-3} and π^2 modulo $\mathbb{Q}$, the second by $\zeta(3)$ and π^2, and the third by $\zeta(3)$ and π^4.

Theorem 1.1 (first plane)**.** *Let a and b be rational numbers, not both zero. Then $aL_{-3} + b\pi^2$ is irrational and*

$$\mu(aL_{-3} + b\pi^2) \leq 20.6622.$$

Theorem 1.2 (second plane)**.** *Let a and b be rational numbers, not both zero. Then $a\zeta(3) + b\pi^2$ is irrational and*

$$\mu(a\zeta(3) + b\pi^2) \leq 335.20.$$

Theorem 1.3 (third plane)**.** *Let a and b be rational numbers, not both zero. Then $a\zeta(3) + b\pi^4$ is irrational and*

$$\mu(a\zeta(3) + b\pi^4) \leq 72.726.$$

The irrationality of the numbers on these planes, that is, the linear independence of 1 and the two spanning numbers over $\mathbb{Q}$, was proved in the last two years by methods unrelated to ours. Calegari, Dimitrov and Tang proved it for the first plane with an arithmetic holonomy bound [7]. Q. Tang proved it for the second plane in Lean [47]. Lai proved it for the third plane with Fauzan's determinant method and the simultaneous approximations of Zudilin and of Dauguet and Zudilin [28].

On two of the planes our measures are better than those known. On the first plane the best published bound was 24781, due to Calegari, Dimitrov and Tang [8]. As far as we know, Theorem 1.2 is the first irrationality measure valid on the whole of it. On the third plane, Leibtag et al. [29] proved $\mu(a\zeta(3)+b\zeta(4)) < 66.3$, which is better than Theorem 1.3. The third plane is included because its function has a pole of order three, and it shows that the structure handles such a pole.

At special points of the planes we prove better bounds. They do not follow from the plane theorems, and each has its own proof. On the first plane we treat one point and four pairs of points. The point L_{-3} uses its own construction (Section 8). The points $L_{-3} \pm \pi^2/9$ use the construction of the off-plane results below (Section 11). The other three pairs use the functions of the plane with further arithmetic that holds only at these points (§7.6). The pair $L_{-3} \pm \pi^2/18$ also has its own geometry and decay certificate.

Theorem 1.4 (special points of the first plane)**.** *The following bounds hold.*

(i) $\mu(L_{-3}) \le 13.6839$.
(ii) $\mu(L_{-3} \pm \frac{\pi^2}{9}) \le 9.4913$.
(iii) $\mu(L_{-3} \pm \frac{2\pi^2}{27}) \le 14$.
(iv) $\mu(L_{-3} \pm \frac{\pi^2}{27}) \le 15.359$.
(v) $\mu(L_{-3} \pm \frac{\pi^2}{18}) \le 17.492$.

At the two ends of the second plane we prove the following bounds (§9.6 and Section 11). We include them for completeness. They are weaker than the bounds $\mu(\zeta(3)) < 5.4197$ and $\mu(\pi^2) < 4.62675$ of Leibtag et al. [29].

Theorem 1.5 (the ends of the second plane)**.** $\mu(\zeta(3)) \le 6.0382$ *and* $\mu(\pi^2) \le 6.5214$.

On the line $b = 0$ Theorem 1.3 follows from Theorem 1.5.

The construction. All these bounds come from one construction. Let ψ be the digamma function, the logarithmic derivative of Euler's gamma function. The approximations to a number ξ are determinants of Löwner matrices [33], the matrices of divided differences, of a rational combination F of the functions $\psi(z+s)$, $\psi'(z+s)$ and $\psi''(z+s)$ with rational shifts s, at integer nodes. The number ξ is $F'(0)$, up to a rational factor. Section 2 explains the construction and why it gives measures.

The trigamma form. The derivative $\psi'(s) = \sum_{n\ge 0}(n+s)^{-2}$ is the trigamma function. When F is a difference of two digamma functions, ξ is a difference

of two trigamma values,

$$\Psi(s,t) = \psi'(s) - \psi'(t) \qquad (0 < s < t \le 1).$$

The trigamma values with denominators 2, 3 and 6 lie in the $\mathbb{Q}$-span of 1, L_{-3} and π^2:

$$\psi'(\tfrac{1}{3}) = \tfrac{2\pi^2}{3} + \tfrac{9}{2}L_{-3}, \qquad \psi'(\tfrac{2}{3}) = \tfrac{2\pi^2}{3} - \tfrac{9}{2}L_{-3}, \qquad \psi'(\tfrac{1}{6}) = 2\pi^2 + \tfrac{45}{2}L_{-3},$$
$$\psi'(\tfrac{5}{6}) = 2\pi^2 - \tfrac{45}{2}L_{-3}, \qquad \psi'(\tfrac{1}{2}) = \tfrac{\pi^2}{2}, \qquad \psi'(1) = \tfrac{\pi^2}{6}.$$

These follow from the reflection formula $\psi'(s) + \psi'(1-s) = \pi^2/\sin^2 \pi s$ and from splitting the series for L_{-3} by residues modulo 6. So each difference $\Psi(s,t)$ of these values is a point of the first plane, and Table 1 gives the form of each special point. A general point of the first plane needs three digamma terms, and $\zeta(3)$ and π^4 come from ψ' and ψ'' themselves, so not every number here is a single difference $\Psi(s,t)$.

The trigamma form also gives numbers off the planes. Let

$$G = L(2,\chi_{-4}) = \sum_{n\ge 0}\Big(\frac{1}{(4n+1)^2} - \frac{1}{(4n+3)^2}\Big) = 0.91596\,55941\ldots$$

be Catalan's constant. Then $\psi'(\frac{1}{4}) = \pi^2 + 8G$ and $\psi'(\frac{3}{4}) = \pi^2 - 8G$. For $q = 5$ and $q = 7$ there is no single classical constant. The number $\psi'(\frac{k}{q})$ is a combination of π^2 and the values $L(2,\chi)$ for the Dirichlet characters χ modulo q, by orthogonality of characters.

Theorem 1.6. *The following bounds hold:*

(i) $\mu(G \pm \frac{5\pi^2}{48}) \le 31.1973$.
(ii) $\mu(\psi'(\frac{k}{5}) - \frac{\pi^2}{6}) \le 23.7756$ *for* $k = 1,2,3,4$.
(iii) $\mu(\psi'(\frac{k}{7}) - \frac{\pi^2}{6}) \le 1806.1719$ *for* $k = 1,\ldots,6$.

Each bound $\mu(\xi) \le \mu_0$ is proved in the stronger form $|\xi - p/q| > q^{-\mu_0}$ for all integers p and all sufficiently large integers q. Table 1 collects the results.

Other named numbers on the first plane are covered by Theorem 1.1, with $\mu \le 20.6622$ in place of 24781. Examples are $\psi'(\frac{1}{3})$, $\psi'(\frac{2}{3})$, $\psi'(\frac{1}{6})$ and $\psi'(\frac{5}{6})$, whose irrationality Calegari, Dimitrov and Tang state explicitly [7, Corollary B]. The Clausen value $\mathrm{Cl}_2(\frac{\pi}{3}) = \frac{3\sqrt{3}}{4}L_{-3}$ and the Mahler measure $m(1+x+y) = \frac{3\sqrt{3}}{4\pi}L_{-3}$ are not on the plane, and our results say nothing about them. Theorem 1.2 says nothing about $\zeta(3)/\pi^2$, which is not on the second plane. The nearest numbers that the method misses are discussed in §12.1.

All the theorems above have been verified in the Lean 4 proof assistant, together with every theorem, proposition and lemma used in their proofs. The formal proofs have no unproved steps and use only the standard axioms. Section 13 describes them. The remarks and Section 12 discuss the method and are not part of the formal proofs. The formalization is available at `https:`

Number	Trigamma form	Bound on μ	Other bound
$aL_{-3} + b\pi^2$		20.6622	24781 [8]
$a\zeta(3) + b\pi^2$		335.20	none
$a\zeta(3) + b\pi^4$		72.726	66.3 [29]
L_{-3}	$\Psi(\frac{1}{3}, \frac{2}{3})$	13.6839	24781
$L_{-3} \pm \frac{\pi^2}{9}$	$\Psi(\frac{1}{3}, 1)$, $\Psi(\frac{2}{3}, 1)$	9.4913	24781
$L_{-3} \pm \frac{2\pi^2}{27}$	$\Psi(\frac{1}{6}, \frac{1}{3})$, $\Psi(\frac{2}{3}, \frac{5}{6})$	14	24781
$L_{-3} \pm \frac{\pi^2}{27}$	$\Psi(\frac{1}{3}, \frac{1}{2})$, $\Psi(\frac{1}{2}, \frac{2}{3})$	15.359	24781
$L_{-3} \pm \frac{\pi^2}{18}$		17.492	24781
$\zeta(3)$		6.0382	5.4197 [29]
π^2	$\Psi(\frac{1}{2}, 1)$	6.5214	4.62675 [29]
$G \pm \frac{5\pi^2}{48}$	$\Psi(\frac{1}{4}, 1)$, $\Psi(\frac{3}{4}, 1)$	31.1973	none
$\psi'(\frac{k}{5}) - \frac{\pi^2}{6}$	$\Psi(\frac{k}{5}, 1)$	23.7756	none
$\psi'(\frac{k}{7}) - \frac{\pi^2}{6}$	$\Psi(\frac{k}{7}, 1)$	1806.1719	none

TABLE 1. The results of this paper. The trigamma form is a rational multiple of the number, where one exists. In the plane rows a and b are rationals, not both zero. The last column gives the best bound from other work that we know. "None" means that we know of no irrationality measure. A bound of $1 + 10^4$ for the first plane follows from an unrefereed supplement to [29] (§1.2).

//github.com/slitsey/catalan-irrationality-measure. The use of AI in this work is disclosed at the end of the paper.

1.2. **Background.** *The constant L_{-3} and the first plane.* Calegari, Dimitrov and Tang [7] proved that 1, π^2 and L_{-3} are linearly independent over $\mathbb{Q}$. Their proof applies a new arithmetic holonomy bound to a construction of Zagier. They list the irrationality of $\psi'(\frac{1}{3})$, $\psi'(\frac{2}{3})$, $\psi'(\frac{1}{6})$ and $\psi'(\frac{5}{6})$ as a corollary, and remark that $\psi'(\frac{n}{6})$ is irrational for every n. In [8] they made holonomy bounds effective. Theorem 5.1 there gives $\mu(L_{-3}) \leq 24781$. The remark that follows it states that the same theorem and proof hold for any $\mathbb{Q}$-linear combination of L_{-3} and π^2. So 24781 is the best published bound at every point of the first plane. Theorem 1.1 improves it to 20.6622 everywhere, and Theorem 1.4 improves it further at these special points. Leibtag et al. [29] find numerically that Apéry-like approximations from conservative matrix fields have positive approximation exponents for $L_{-3} \pm \pi^2/9$. A supplement to their paper uses one such field to give an Apéry-style proof of the theorem of Calegari, Dimitrov and Tang. For every point of the plane it proves an

approximation exponent $\delta > 10^{-4}$, and it certifies the comparison value $0.000103\ldots$. With the bound $\mu \le 1 + 1/\delta$ of their §2 the proved exponent gives $\mu \le 1 + 10^4$, a consequence not stated there. The comparison value would give about $9.7 \cdot 10^3$. The supplement is unrefereed and is not part of the arXiv version.

The values $\zeta(2)$, $\zeta(3)$ and the second plane. Legendre proved that π^2 is irrational. Apéry proved that $\zeta(3)$ is irrational and gave a new proof for $\zeta(2)$ [1], and Beukers gave his integral proofs [4]. Rhin and Viola obtained $\mu(\zeta(2)) < 5.441243$ and $\mu(\zeta(3)) < 5.513891$ with permutation groups of integrals [41, 42]. Zudilin lowered the bound for $\zeta(2)$ to 5.095412 [55]. Giraudin lowered the bounds to 5.0193784 for $\zeta(2)$ [21] and to 5.5138800 for $\zeta(3)$ [22]. Leibtag, Barkan, Kalisch, Weinbaum, Shalyt and Kaminer [29] showed that the classical constructions are one-dimensional trajectories in conservative matrix fields. By searching over other trajectories they obtained $\mu(\pi^2) < 4.62675$ and $\mu(\zeta(3)) < 5.4197$. They also proved the uniform bound $\mu(a\zeta(3) + b\zeta(4)) < 66.3$ on the plane spanned by $\zeta(3)$ and π^4. This is the closest analogue of Theorem 1.2, and it is better than our Theorem 1.3 on the same plane. The linear forms in 1, $\zeta(2)$ and $\zeta(3)$ have a longer history. Dauguet and Zudilin [11] bounded such forms from below when the coefficients satisfy divisibility and size conditions. They noted that this does not give the linear independence of 1, $\zeta(2)$, $\zeta(3)$, which remained open. For a fixed number $a\zeta(3) + b\zeta(2)$ their theorem gives no measure. In 2026 Q. Tang [47] proved the linear independence in Lean. This proof is a formalization with no accompanying written paper. Lai [28] then proved that 1, $\zeta(3)$ and $\zeta(4)$ are linearly independent, and Leibtag et al. gave a second proof with a uniform measure [29]. As far as we know, no irrationality measure valid on the whole plane of $\zeta(3)$ and π^2 was known before Theorem 1.2.

Catalan's constant. In October 2026 OpenAI [38] proved that G is irrational and formalized the proof in Lean. Sun's preprint [46] had claimed a proof of the irrationality of G. Wachs [50] reports that its main estimate is not justified. Finite irrationality exponents for G were obtained after OpenAI's proof by Jha [23], Carreira [10], Bojnal [5], Xu [51], Davis [12], the MultiAgency agent swarm [35] and the author [30]. Carreira [9] also records attempts to carry the determinant method to $\beta(4)$. For the arithmetic of G and related values before these results see [43, 54, 53, 26, 27] and the survey [57]. Eskandari, Murty and Nemoto [15] construct a 2-dimensional mixed motive over $\mathbb{Q}$ with period G. They use it to obtain linear forms in 1 and G. Eskandari [14] improves the exponent of rational approximations to G of Rivoal–Zudilin type from 0.52 to 0.62. These are explicit approximations, not a lower bound for all rationals. Our searches found no proof that $G - 5\pi^2/48$ or $G + 5\pi^2/48$ is irrational. This does not follow from the irrationality of G, since it is not known whether 1, G and π^2 are linearly independent over $\mathbb{Q}$.

The numbers $\psi'(\frac{k}{5}) - \pi^2/6$ and $\psi'(\frac{k}{7}) - \pi^2/6$. Our searches found no irrationality result for them. Prellberg [40] proved that $\mathrm{Li}_2(1/5)$ is irrational, but that is a different number.

Recent results. Since September 2026 a series of new results has appeared in this area. Several of them were obtained with AI systems. OpenAI proved $\mu(\pi) = 2$, with a Lean formalization [39]. This improves the bound $\mu(\pi) < 7.101863$ of Bai [2], which in turn lowered the bound 7.1033 of Zeilberger and Zudilin [52]. Liu, Jiang and Zhang proved $\mu(\log r) = 2$ for every rational $r > 0$ with $r \neq 1$ [31], improving the classical bound $\mu(\log 2) < 3.5746$ of Marcovecchio [34]. Their result also covers nonzero rational combinations of such logarithms. Prellberg [40] proved that $\mathrm{Li}_2(-1/4)$, $\mathrm{Li}_2(1/5)$ and $\mathrm{Li}_2(-1/3)$ are irrational. These are three cases left open in [7]. The proof uses the integrals of Rhin–Viola and Viola–Zudilin. Fauzan proved that $\zeta(5)$ is irrational, using rationally normalized Hankel determinants [17]. Fasten [16], Firsching [19] and Romik [44] verified the irrationality statement in Lean. Romik's formalization takes the prime number theorem as an axiom.

Determinant methods. Irrationality proofs built on determinants of large size have a precedent in Zudilin's determinantal criterion with Hankel determinants [56]. The preprint of Liu, Jiang and Zhang on logarithms [31] uses a separated-weight interpolation determinant. The arguments of Fauzan and Lai [17, 28] use Hankel determinants. Brown [6] proposes a multivariate transfinite-diameter test for Mellin-type integrals and applies it to π^2. The framework we use is OpenAI's [38]. In it the decay of the determinants is controlled by Cauchy determinants and Andréief's identity, and the denominators are controlled prime by prime. Our determinants are Löwner matrices, that is, matrices of divided differences. They are a matrix-valued analogue of Padé and Hermite–Padé approximation [37]. Formal proofs of irrationality results in Lean also include the irrationality of $\zeta(3)$ by Liu, Zhang and Zhi [32] and the linear independence of 1, $\zeta(2)$ and $\zeta(3)$ by Q. Tang [47].

1.3. **The companion paper.** Catalan's constant is also a difference of trigamma values, $G = \frac{1}{16}\Psi(\frac{1}{4}, \frac{3}{4})$, and the construction applies to this pair. For it, however, our bound for the denominators exceeds our bound for the decay, so we obtain no measure for G. The companion paper [30] refines OpenAI's determinant instead and proves $\mu(G) \leq 270$. Our computations for $(\frac{1}{4}, \frac{3}{4})$ are not part of the formal proof.

1.4. **What a measure requires.** Let $P(x)$ be a polynomial with rational coefficients for which $P(\xi)$ is small. It is a member of a family of polynomials indexed by a size r. Suppose that a rational number is very close to ξ. Three estimates turn this into a lower bound for its distance to ξ.

(i) *Denominators.* The coefficients of P have a common denominator of size $e^{(\Lambda+o(1))r^2}$.
(ii) *Smallness on a neighbourhood.* $|P(x)| \leq e^{-(D-o(1))r^2}$ for every x with $|x - \xi| \leq e^{-\rho r}$, not only at ξ.
(iii) *Nonvanishing at the point.* P does not vanish at the rational number for some member of the family whose size can be chosen in terms of its denominator.

In OpenAI's argument for G the size of the determinant is tied to a prime p, and nonvanishing is proved modulo p. Here nonvanishing is proved in characteristic zero, so the size r is free. A plane theorem needs more. A number $aL_{-3} + b\pi^2$ is, up to a rational factor, the value $F'(0)$ for a combination F of three digamma functions whose weights depend on (a, b). The three estimates must then hold with constants that do not depend on the weights. For the second plane the function F has a pole of order two, so the matrices are no longer Cauchy-like at every pole, and both the arithmetic and the analysis need a new argument. For the third plane the pole has order three.

1.5. **New ideas.** The proofs rest on the following ideas. Section 2 explains the first five in more detail.

(1) *Löwner determinants of digamma combinations.* The approximations are determinants of Löwner matrices, the matrices of divided differences, of a combination F of $\psi(z+s)$, $\psi'(z+s)$ and $\psi''(z+s)$ at integer nodes. The rows and columns share a long block of nodes, which carries the variable. A further border of row nodes and column nodes alternates beyond it. The degree in the variable stays small, while the decay is that of a larger node set. This one structure gives all three planes. Only the function F changes.

(2) *Denominators from local bounds.* For each prime we prove a list of upper bounds for the valuation of the common denominator. Each is an explicit integer expression in a few integers attached to the prime. Integrated against $\log p$, their minimum gives the denominator exponent, and checking it reduces to finitely many comparisons of rational numbers.

(3) *Local bounds that do not see the weights.* For integer weights the polar data at a prime p depend only on the shifts. All our local bounds for the first plane read only these data. So one table of bounds serves every point of the plane. A prime dividing a weight can only lower the true valuation, and a weight zero removes a pole without changing the bounds.

(4) *A decay certificate uniform in the weights.* The energy inequalities behind the decay estimate contain no weight. The weights enter only through a constant factor in the external field, which costs $e^{O(r)}$. A weight zero only removes atoms. One trial measure on one contour therefore proves the decay for the whole first plane, and another one for the whole second plane.

(5) *Nonvanishing in characteristic zero.* A kernel vector of the matrix at a rational point gives a solution of a Hermite interpolation problem. A discrete Wronskian bounds the corank, because one pole of the increment of F is isolated in its class modulo $\mathbb{Z}$. The bound depends only on the number of poles, counted with order.

(6) *Level returns and a gauge for the special points.* At a prime p, the partial sums of F step by the residues at its poles. For special weights this level returns to an earlier value, and whole blocks of entries become

p-integral. A row operation inside each residue class modulo p, by a p-integral multiple of a common row, gives some further rows the level of the return. This turns the rank of the reduction modulo p into an assignment bound, and it gives the points of Theorem 1.4(iii)–(v).

(7) *Elimination of complete periods for a double pole.* For the second plane the Schur complement of the Löwner matrix over a union of complete periods modulo p is again of Löwner type, with generators that are constant on half-periods. This follows from the displacement identity of Löwner matrices and from $\sum_{v\in\mathbb{F}_p^\times} v^{-1} = 0$. A uniform pivot on complete double periods extends it to primes smaller than the border. The required nondegeneracy modulo p is the Wronskian argument of item 5 in characteristic p.

(8) *Jets reduced to simple poles.* At a pole of order w the Gram representation of the Löwner matrix carries derivatives up to order $w-1$. The atom term is a nondegenerate symmetric bilinear form in the jets, and it splits into w rank-one terms. These splittings, and Cauchy's estimate on small circles, bound every jet determinant by the Cauchy determinant of a fixed function with simple poles. The decay certificate of that function then applies.

(9) *Assignment bounds for a pole of any order.* For a single shift with a pole of order $h-1$, four explicit systems of potentials in Lemma 4.1 bound the denominators. They use no rank argument. Their minimum gives a denominator bound for the third plane. The elimination of the complete periods below the border, as in item 7, carries over to a pole of any order. It gives a fifth bound. When the border meets no multiple of p, this bound lies below the first one by two units for each border pair. We use all five bounds on the third plane.

The constants of the proofs are listed in Table 2.

1.6. **Organization.** Section 2 explains the construction and the main ideas without technicalities, and states the general deduction. Section 3 gives the construction. Sections 4, 5 and 6 prove the three estimates for differences of two digamma functions. They also prove nonvanishing and decay for general combinations. Section 7 proves Theorem 1.1 and the special points of Theorem 1.4, and Section 8 treats the point L_{-3} itself. Section 9 proves Theorem 1.2, and Section 10 proves Theorem 1.3. Section 11 completes the proofs and gives the data. Section 12 discusses the limits of the method and the nearest constants that it does not reach. Section 13 describes the formalization. The disclosure of AI use follows it.

1.7. **Notation and conventions.** We write v_p for the p-adic valuation. For $P \in \mathbb{Q}[x]$, $\operatorname{den} P$ is the least positive integer whose product with P lies in $\mathbb{Z}[x]$. We write $[0,r) = \{0,1,\dots,r-1\}$. For finite sets X_0, Y_0 of integers and a matrix A indexed by integers, $A[X_0,Y_0]$ is the submatrix with rows

X_0 and columns Y_0. We write $\|y\|$ for the distance from a real number y to $\mathbb{Z}$, and $\|m\|_p = p\|m/p\|$.

2. Overview of the method

2.1. **Löwner matrices of digamma differences.** Let F be a function analytic near a finite set of distinct points. The *Löwner matrix* [33] of F is the matrix of divided differences indexed by the points:

$$\mathsf{L}_F[t,t'] = \frac{F(t') - F(t)}{t' - t} \ (t \neq t'), \qquad \mathsf{L}_F[t,t] = F'(t).$$

It is a Cauchy-like structured matrix. Hermite's formula gives $\mathsf{L}_F = \frac{1}{2\pi i} \oint F(z)\, v(z) v(z)^\top dz$ with $v_t(z) = 1/(z-t)$. A representation of F as a Stieltjes or contour integral therefore exhibits L_F as a Gram matrix of Cauchy kernels.

For a rational shift $0 < s < 1$ with denominator q we take the digamma difference F_s. For the pair $(\frac{1}{3}, \frac{2}{3})$ we take a combination F_χ of three digamma functions. The nodes are integers. We put

$$\begin{gathered} F_s(z) = \psi(z+1) - \psi(z+s), \qquad \omega = \tfrac{1}{q^2}, \quad \xi_s = -\tfrac{1}{q^2}\Psi(s,1), \\ F_\chi(z) = -\tfrac{1}{3}\psi(z+\tfrac{1}{3}) + \tfrac{1}{2}\psi(z+\tfrac{1}{2}) - \tfrac{1}{6}\psi(z+1), \\ \omega = -\tfrac{1}{6}, \quad \xi_\chi = \tfrac{1}{36}\Psi(\tfrac{1}{3}, \tfrac{2}{3}). \end{gathered} \tag{2.1}$$

In each case $\xi = \omega\, F'(0)$. The differences $F(t') - F(t)$ and $F'(t) - F'(0)$ are rational for integers $t, t' \geq 0$. Let R and Q be finite sets of nonnegative integer nodes, which index the rows and the columns. Then

$$\omega\, \mathsf{L}_F[R,Q] = M[R,Q] + \xi\, \mathrm{Id}_{R,Q}, \qquad \mathrm{Id}_{R,Q}[t,t'] = \mathbf{1}[t = t'], \tag{2.2}$$

with an explicit symmetric function M on $\mathbb{Z}^2_{\geq 0}$ with rational values, given in §3. Replacing ξ by a variable x gives the matrix polynomial $\Phi(x) = M[R,Q] + x\mathrm{Id}_{R,Q}$. Its determinant $P(x)$ has rational coefficients and degree at most $|R \cap Q|$, and $P(\xi) = \det(\omega\, \mathsf{L}_F[R,Q])$.

Now let p/q be a rational number, with $q \geq 1$, such that $|\xi - p/q|$ is very small. Then $P(p/q)$ is very close to $P(\xi)$, which is tiny. On the other hand $q^{|R\cap Q|}\, \mathrm{den}(P)\, P(p/q)$ is an integer. If it is not zero, we get a contradiction as soon as

$$\underbrace{\log \mathrm{den}\, P}_{\text{denominators}} + |R \cap Q| \log q \ < \ \underbrace{-\log |P(p/q)|}_{\text{decay}}.$$

Three estimates are therefore needed. The *arithmetic estimate* is an upper bound for $\log \mathrm{den}\, P$. The *analytic estimate*, or decay, is an upper bound for $|P(x)|$ near ξ. The *nonvanishing* is that of $P(p/q)$ for some member of a family of determinants whose size can be chosen in terms of q. Each estimate is proved once, for every shift, and the families differ only in a few parameters.

2.2. **Rectangular Löwner matrices with an alternating border.** We take a long common block $C = [0, r)$, which carries the variable x, and add m row nodes and m column nodes that alternate beyond it:

$$X' = \{r, r+2, \dots, r+2m-2\}, \qquad Y' = \{r+1, r+3, \dots, r+2m-1\}.$$

The rows are $R = C \sqcup X'$ and the columns are $Q = C \sqcup Y'$ (Figure 1). The determinant $P(x) = \det \Phi(x)$ has degree at most r, since the variable occurs only on the diagonal of the common block. So the factor $q^{\deg P}$ in the contradiction is that of r nodes, while the matrix has size $r+m$ and its decay is that of about $(1+c/2)r$ nodes, where $c = 2m/r$ is the length of the border relative to r. The denominators grow too, but less, because border rows and border columns never share a node. To guarantee nonvanishing we also use the principal minors that delete a bounded number of common nodes. A larger c raises D and Λ together, and we chose c numerically to minimise $\rho/(D-\Lambda)$. Its value for each constant is given in Table 2.

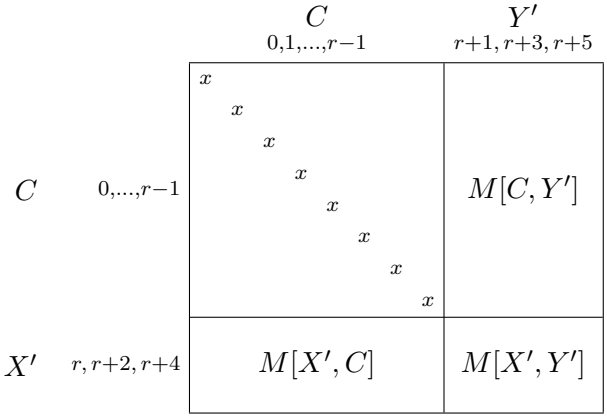


FIGURE 1. The geometry for $r = 8$ and $m = 3$. Rows are indexed by $C \sqcup X'$ and columns by $C \sqcup Y'$. The variable x occurs only on the diagonal of the common block. The border rows $X' = \{r, r+2, r+4\}$ and border columns $Y' = \{r+1, r+3, r+5\}$ alternate and never meet the diagonal.

The interleaving of X' and Y' is essential. It keeps the two Cauchy determinants in Andréief's formula comparable (§6). It also gives the reduction of the matrix modulo a prime an alternating structure. Several of the arithmetic bounds of §4 rest on this structure.

2.3. **Denominators from local bounds.** As in the work of Rhin and Viola [41, 42], we bound the p-adic valuation of the common denominator prime by prime and sum the bounds with the prime number theorem in arithmetic progressions. For $p \asymp r$ the valuation depends on r, m, p and the first pole k_p, the least index $k_p \geq 1$ at which the second digamma term has a pole modulo p. One has $k_p = \kappa p + O(1)$ with $\kappa = j/q$, where the residue $j \equiv qs\, p^{-1}$

(mod q) is the *root class* of p. The function F_χ has two first poles, and on each class of primes modulo 6 both are affine functions of one index of the same kind (Section 8).

The local bounds come from three sources: the valuations of the entries, the corank of the reduction modulo p, and cancellations at second and third order in the Smith normal form. Each is an explicit piecewise linear expression in (r, m, p, k_p), valid in a range of p/r described by linear inequalities. Replacing (r, m, p, k_p) by $(1, \alpha, \theta, \kappa\theta)$, with $\alpha = c/2$, turns it into a piecewise affine function of $\theta = p/r$, with an explicitly bounded error. The *denominator exponent* is

$$\Lambda = \frac{1}{\varphi(q)} \sum_j \int_0^\infty \mathcal{P}_j(\theta)\, d\theta,$$

where $\mathcal{P}_j$ is the minimum of these functions for the root class j. A table of affine pieces above $\mathcal{P}_j$ certifies Λ by finitely many comparisons of rational numbers (Theorem 4.14). The bounds depend on the shift only through the root class, so one table serves every shift with denominator q, and the numbers $\Psi(s, 1)$ for these shifts receive the same exponent.

2.4. **Primes smaller than the border.** For $p \le 2m$ some residue classes modulo p contain two border nodes, a row t and a column $t + p$. The bounds above are not sharp for these primes. Each such pair gives a pivot that is a p-adic unit, because the interval $(t, t + p]$ contains one pole of each digamma term and their contributions cannot cancel. Eliminating these pivots leaves a matrix of the same shape with a border of $d = \|m\|_p$ rows and columns. To this smaller matrix we apply the separation argument used for large primes, at third order. Two identities for bordered inverses show that the elimination does not disturb the second-order terms (Lemma 4.9). A rational interpolation problem of type (n, n) shows that the remaining border nodes stay separated (Lemma 4.11).

2.5. **Nonvanishing.** Let x_0 be a rational point. The kernel of $\Phi(x_0)$ is identified with a space of pairs of polynomials that satisfy Hermite interpolation conditions at the nodes. Two such pairs have vanishing cross product, by a count of degrees and zeros. A pair of low degree would give a rational antidifference of the increment of F. No such antidifference exists, because a pole of the increment is isolated in its class modulo $\mathbb{Z}$. This bounds the corank of $\Phi(x_0)$ by half the number of poles of F in a period, counted with order. A principal restriction that deletes at most that many common nodes is then nonsingular. This is Theorem 5.6. No prime enters, so the size r can be chosen freely in terms of q.

2.6. **Decay.** Moving Hermite's contour to a polygonal path Γ in the left half-plane writes $\omega \llcorner_F [R, Q]$ as a Gram matrix $\int v_t v_{t'}\, d\Omega$. Here Ω is a complex measure. It has signed atoms at poles of F and an exponentially small

density on Γ. Andréief's identity with two families of functions, one for the rows and one for the columns, gives

$$|\det \mathsf{L}_F[R,Q]| \le \frac{1}{n_*!}\int |\mathsf{C}_R(z)|\,|\mathsf{C}_Q(z)| \prod_{i=1}^{n_*} d|\Omega|(z_i),$$

where n_* is the size of the matrix and C_R, C_Q are Cauchy determinants. Taking logarithms turns the integrand into a discrete logarithmic energy. The free points z_i interact with the nodes in an external field.

We bound this energy from below by comparing it with a *trial measure* σ, a density that is polynomial on each of finitely many segments. The comparison uses the conditional positive definiteness of the logarithmic kernel. What is left is a lower bound for one explicit function Υ on the contour, on the segment carrying the atoms and on the interval of the nodes. We prove it on a finite cover of these sets by cells, each checked in exact rational arithmetic. One theorem, Theorem 6.2, gives the decay for every geometry parameter $c \le 1$ and every function F whose poles are separated from the contour. Each family then supplies only σ, the polygon and the cover.

2.7. **The deduction.** Write $\eta = D - \Lambda$, where D is the *decay rate*, and let ρ be the *neighbourhood exponent* of the decay estimate: the decay holds at every point within $e^{-\rho r}$ of ξ. Section 11 combines the three estimates into

$$\mu(\xi) \le \frac{\rho}{\eta} = \frac{\rho}{D-\Lambda}.$$

The companion paper [30] has the same bound multiplied by the degree of the determinant per unit of scale, which is one here.

2.8. **Formal verification.** Every step is formalized in Lean 4 with Mathlib [13, 48]. This includes the local arithmetic bounds and the conditions on the tables for Λ, and the interpolation argument for nonvanishing. It also includes the contour representation, the cell checks of the decay estimate, and the prime number theorem in arithmetic progressions. The formal proofs share one core, which contains rectangular determinants, Gauss valuations, the local bounds, the tables for Λ, the selection in characteristic zero, the decay theorem and the deduction. A family supplies its function F, its geometry parameter and its numerical tables. The written proofs below are at the level of a research article. For long computations we point to the formal proof, which is the basis for the correctness of our theorems.

3. The construction

3.1. **The entries of M.** Let $0 < s < 1$ be rational with denominator q. Put $q_s = q(1-s)$, so that $k - 1 + s = (qk - q_s)/q$. For the function F_s of (2.1), the rational matrix M of (2.2) is as follows. Here $0 \le t < t'$ are integers,

and $M[t',t] = M[t,t']$.

$$M[t,t'] = \frac{1}{q^2(t'-t)} \sum_{k=t+1}^{t'} \Big(\frac{1}{k} - \frac{q}{qk-q_s} \Big),$$

$$M[t,t] = -\frac{1}{q^2} \sum_{k=1}^{t} \Big(\frac{1}{k^2} - \frac{q^2}{(qk-q_s)^2} \Big).$$

For $s = \frac{3}{4}$ this is $M[t,t'] = -\frac{1}{t'-t} \sum_{k=t+1}^{t'} \frac{1}{16k(4k-1)}$. For the function F_χ,

$$M[t,t'] = \frac{1}{6(t'-t)} \sum_{k=t+1}^{t'} g(k), \qquad M[t,t] = \frac{1}{6} \sum_{k=1}^{t} g'(k),$$

with $g(k) = \frac{1}{3k-2} - \frac{1}{2k-1} + \frac{1}{6k}$. Then $-\frac{1}{6} g'(k) = \frac{1}{2(3k-2)^2} - \frac{1}{3(2k-1)^2} + \frac{1}{36k^2}$.

We write each function as $F = \sum_{\varsigma \in \mathcal{T}} w_\varsigma \psi(z+\varsigma)$, where $\mathcal{T}$ is the set of shifts of F and $\sum_\varsigma w_\varsigma = 0$. For F_s, $\mathcal{T} = \{1, s\}$ with $w_1 = 1$ and $w_s = -1$. For F_χ, $\mathcal{T} = \{\frac{1}{3}, \frac{1}{2}, 1\}$. The number $|\mathcal{T}|$ of digamma terms is 2 for F_s and 3 for F_χ.

Lemma 3.1. *For each function of (2.1), $\omega \mathsf{L}_F[R,Q] = M[R,Q] + \xi \mathrm{Id}_{R,Q}$ for all finite sets $R, Q \subset \mathbb{Z}_{\geq 0}$, with ω and ξ as in (2.1).*

Proof. From $\psi(w+1) - \psi(w) = 1/w$ one gets $F_s(k) - F_s(k-1) = \frac{1}{k} - \frac{1}{k-1+s} = \frac{1}{k} - \frac{q}{qk-q_s}$. Telescoping gives the off-diagonal entries. In the same way $\psi'(w+1) - \psi'(w) = -1/w^2$ gives $F_s'(t) - F_s'(0)$. Finally $\omega F_s'(0) = q^{-2}(\psi'(1) - \psi'(s)) = -q^{-2}\Psi(s,1) = \xi_s$. For F_χ the entries are computed in the same way. For the value at 0, the multiplication formula $\sum_{i<n} \psi'(z+\frac{i}{n}) = n^2 \psi'(nz)$ gives $\psi'(\frac{1}{2}) = 3\psi'(1)$ and $\psi'(\frac{1}{3}) + \psi'(\frac{2}{3}) = 8\psi'(1)$. Hence $F_\chi'(0) = -\frac{1}{3}\psi'(\frac{1}{3}) + \frac{1}{2}\psi'(\frac{1}{2}) - \frac{1}{6}\psi'(1) = -\frac{1}{6}\Psi(\frac{1}{3}, \frac{2}{3})$, and $-\frac{1}{6}F_\chi'(0) = \frac{1}{36}\Psi(\frac{1}{3}, \frac{2}{3}) = \xi_\chi$. □

3.2. **Combinations with several terms.** The planes need combinations with more than two terms. The second plane needs the trigamma function itself, and the third plane needs its derivative ψ''. We call a *frame* a function

$$F(z) = \sum_{s \in S} \big(w_s \psi(z+s) + w_s' \psi'(z+s) + w_s'' \psi''(z+s) \big),$$

where $S \subset \mathbb{Q} \cap (0,1]$ is a finite set of shifts, the weights w_s, w_s', w_s'' are rational, not all zero, and $\sum_s w_s = 0$. The *order* of a shift is 3 if $w_s'' \neq 0$, 2 if $w_s'' = 0 \neq w_s'$, 1 if $w_s'' = w_s' = 0 \neq w_s$, and 0 if all three weights vanish. A shift of order 0 carries no pole of F. We keep such shifts in S, because the planes are families of frames on a fixed set of shifts, and some members of a family have a weight zero. We write d for the sum of the orders, the number of poles of F in a period counted with order. The balance $\sum_s w_s = 0$ gives $F(z) = O(1/|z|)$ in the right half-plane. Put $\xi = F'(0)$ and

$$\delta(z) = F(z+1) - F(z) = \sum_{s \in S} \Big(\frac{w_s}{z+s} - \frac{w_s'}{(z+s)^2} + \frac{2w_s''}{(z+s)^3} \Big),$$

and put $f_t = \sum_{k<t} \delta(k)$ and $e_t = -\sum_{k<t} \delta'(k)$. Then $f_t = F(t) - F(0)$ and $F'(t) = \xi - e_t$ for integers $t \geq 0$. So the Löwner matrix of F at nonnegative integer nodes is $\mathsf{L}_F = M + \xi\mathrm{Id}$, with $M[t,t'] = (f_{t'} - f_t)/(t'-t)$ for $t \neq t'$ and $M[t,t] = -e_t$. The function F_s of (2.1) is the frame with $S = \{s, 1\}$, $w_1 = 1$, $w_s = -1$, multiplied by ω. The geometries, the members $P_{r,I}$ and Lemma 3.5 below apply to every frame without change.

The frames of the first plane. Let $S^+ = \{\frac{1}{3}, \frac{1}{2}, 1\}$ and $S^- = \{\frac{2}{3}, \frac{1}{2}, 1\}$, and write $s_\pm$ for the first shift. For an integer vector $\vec{w} = (w_\pm, w_{1/2}, w_1)$, not zero, with $w_\pm + w_{1/2} + w_1 = 0$ put

$$F^\pm_{\vec{w}}(z) = w_\pm \psi(z + s_\pm) + w_{1/2}\psi(z + \tfrac{1}{2}) + w_1 \psi(z+1).$$

Here $w_\pm$ is the weight of the shift $s_\pm$. These are the *frames on* $S^\pm$. We also write them by their weight vectors, as in "the frame $(2,-1,-1)$". All shifts have order at most 1, so $d \leq 3$, with $d = 3$ when $w_\pm w_{1/2} w_1 \neq 0$. The function F_χ of (2.1) is $-\frac{1}{6} F^+_{(2,-3,1)}$.

Lemma 3.2. *For $F^\pm_{\vec{w}}$ one has $\xi = \pm\frac{9w_\pm}{2} L_{-3} + (\frac{w_\pm}{2} + \frac{w_{1/2}}{3})\pi^2$. If $w_\pm \neq 0$ and $\tau = (3w_\pm + 2w_{1/2})/(27w_\pm)$, this is $\xi = \frac{9w_+}{2}(L_{-3} + \tau\pi^2)$ on S^+ and $\xi = -\frac{9w_-}{2}(L_{-3} - \tau\pi^2)$ on S^-. For all rationals a and b, not both zero, there is a frame on S^+ or on S^- whose number ξ is a nonzero rational multiple of $aL_{-3} + b\pi^2$.*

Proof. On S^+, $\xi = w_+\psi'(\frac{1}{3}) + w_{1/2}\psi'(\frac{1}{2}) + w_1\psi'(1)$. Insert the values of §1 and $w_1 = -w_+ - w_{1/2}$. The coefficient of L_{-3} is $\frac{9w_+}{2}$ and that of π^2 is $\frac{2w_+}{3} + \frac{w_{1/2}}{2} - \frac{w_+ + w_{1/2}}{6} = \frac{w_+}{2} + \frac{w_{1/2}}{3}$. On S^- the sign of L_{-3} changes. If $a = 0$, the frame $F^+_{(0,3,-3)}$ has $\xi = \pi^2$. Its shift $\frac{1}{3}$ has weight zero. If $a \neq 0$, write $b/a = k/n$ with integers k and $n \geq 1$, and let $w_+ = 2n$, $w_{1/2} = 27k - 3n$ and $w_1 = -w_+ - w_{1/2}$. Then $\tau = b/a$. Here $w_+ \neq 0$, $w_{1/2} = 0$ only for $b/a = \frac{1}{9}$, and $w_1 = 0$ only for $b/a = \frac{1}{27}$. For these two values we use S^- with the frames $(1,-3,2)$ and $(1,-2,1)$, for which $\tau = -\frac{1}{9}$ and $\tau = -\frac{1}{27}$. □

The frames of the second plane. For integers γ_2 and γ_1, not both zero, put

$$F^{(2)}_{\gamma_2,\gamma_1}(z) = \gamma_2\, \psi'(z+1) + \gamma_1(\psi(z+1) - \psi(z+\tfrac{1}{2})).$$

The shift 1 has order 2 if $\gamma_2 \neq 0$ and order 1 if $\gamma_2 = 0$. The shift $\frac{1}{2}$ has order 1 if $\gamma_1 \neq 0$ and order 0 if $\gamma_1 = 0$. So $d = 3$ for $\gamma_1\gamma_2 \neq 0$ and $d = 2$ otherwise. Since $\psi''(1) = -2\zeta(3)$ and $\psi'(1) - \psi'(\frac{1}{2}) = -\frac{\pi^2}{3}$,

$$\xi = -2\gamma_2\, \zeta(3) - \frac{\gamma_1}{3}\pi^2.$$

Given rationals a and b, not both zero, choose a rational $\lambda \neq 0$ for which $\gamma_2 = \lambda a$ and $\gamma_1 = 6\lambda b$ are integers. Then $\xi = -2\lambda(a\zeta(3) + b\pi^2)$, and $\gamma_1 \neq 0$ exactly when $b \neq 0$. The frame $F^{(2)}_{1,0} = \psi'(z+1)$ gives $\zeta(3)$. Its increment is $\delta(k) = -(k+1)^{-2}$, so $M[t,t'] = -(t'-t)^{-1}\sum_{k=t+1}^{t'} k^{-2}$ and $M[t,t] = \sum_{k\leq t} 2k^{-3}$.

The frames of the third plane. For rationals γ_2 and γ_3 with $\gamma_3 \neq 0$ put

$$F^{(3)}_{\gamma_2,\gamma_3}(z) = \gamma_2\,\psi'(z+1) + \gamma_3\,\psi''(z+1).$$

It has the single shift 1, of order 3, so $d = 3$. Its increment is $\delta(k-1) = -\gamma_2 k^{-2} + 2\gamma_3 k^{-3}$ for $k \geq 1$. Since $\psi''(1) = -2\zeta(3)$ and $\psi'''(1) = 6\zeta(4) = \frac{\pi^4}{15}$,

$$\xi = -2\gamma_2\,\zeta(3) + \frac{\gamma_3}{15}\pi^4.$$

Given rationals a and b with $b \neq 0$, the choice $\gamma_2 = -\frac{a}{2}$ and $\gamma_3 = 15b$ gives $\xi = a\zeta(3) + b\pi^4$. The line $b = 0$ is the multiples of $\zeta(3)$, which the second plane already contains.

Since μ is invariant under rational affine maps, the theorems of the introduction are statements about the numbers ξ of these frames and of (2.1).

3.3. **Geometries.**

Definition 3.3. Let $r, m \geq 0$. The *geometry* $\mathcal{G}(r,m)$ consists of the common nodes $C = [0,r)$, the row nodes $X' = \{r + 2i : 0 \leq i < m\}$ and the column nodes $Y' = \{r + 2i + 1 : 0 \leq i < m\}$. Rows are indexed by $R = C \sqcup X'$ and columns by $Q = C \sqcup Y'$. Let $I \subseteq C$ be a set of deleted common nodes. The matrix

$$\Phi_{\mathcal{G},I}(x) = \big(M[t,t'] + x\,\mathbf{1}[t = t' \in C \setminus I]\big)_{t \in R\setminus I,\ t' \in Q\setminus I}$$

is square of size $|C \setminus I| + m$. Put $P_{\mathcal{G},I}(x) = \det \Phi_{\mathcal{G},I}(x) \in \mathbb{Q}[x]$. We write $\Phi_{\mathcal{G}}$ and $P_{\mathcal{G}}$ when $I = \emptyset$, and we write $N = r + 2m$ for the number of nodes.

By Lemma 3.1, $\Phi_{\mathcal{G}}(\xi + h) = \omega \llcorner_F [R,Q] + h\mathrm{Id}_{R,Q}$ for real h. The geometry with $m = 0$ is the square Löwner matrix of size r.

Definition 3.4. Let $0 < c \leq 1$ and $\alpha = c/2$. The *alternating geometry* of parameter c at an even scale r is $\mathcal{G}(r, m(r))$ with $m(r) = \lceil \alpha r - \frac{1}{2} \rceil$. Then $|m(r) - \alpha r| \leq \frac{1}{2}$, and the border $[r, r+2m)$ has length $cr + O(1)$. For a depth $K \geq 0$, its *members* are the matrices $\Phi_{r,I} = \Phi_{\mathcal{G},I}$ with $|I| \leq K$. We put $P_{r,I} = \det \Phi_{r,I}$.

For the functions F_s we use $c = \frac{3}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{9}, \frac{1}{6}$ when s has denominator $2, 4, 3, 5, 7$. For the frames of the first plane we use $c = \frac{1}{6}$, except $c = \frac{1}{4}$ for the points $L_{-3} \pm \frac{\pi^2}{18}$. For the frames of the second plane we use $c = \frac{6}{41}$, for $\zeta(3)$ $c = \frac{3}{10}$, and for the frames of the third plane $c = \frac{13}{100}$. For F_χ we use $c = \frac{1}{2}$, as explained in Section 8. All $o(1)$ terms below are uniform over the members as $r \to \infty$.

Lemma 3.5 (integrality)**.** *$P_{\mathcal{G},I}$ has degree at most $|C \setminus I|$. If p and $q \geq 1$ are integers with $P_{\mathcal{G},I}(p/q) \neq 0$, then $|\operatorname{den}(P_{\mathcal{G},I})\,P_{\mathcal{G},I}(p/q)| \geq q^{-|C\setminus I|}$.*

Proof. The variable occurs only in the $|C \setminus I|$ diagonal entries indexed by $C \setminus I$. So each Leibniz term has degree at most $|C \setminus I|$. Then $q^{|C\setminus I|} \operatorname{den}(P_{\mathcal{G},I})\,P_{\mathcal{G},I}(p/q)$ is a nonzero integer. □

Example. Take $s = \frac{3}{4}$, $r = 2$ and $m = 1$, so that $C = \{0, 1\}$, $X' = \{2\}$ and $Y' = \{3\}$. Then

$$\Phi_{\mathcal{G}}(x) = \begin{pmatrix} x & -\frac{1}{48} & -\frac{67}{7392} \\ -\frac{1}{48} & x + \frac{7}{144} & -\frac{47}{14784} \\ -\frac{17}{1344} & -\frac{1}{224} & -\frac{1}{528} \end{pmatrix}$$

and

$$P_{\mathcal{G}}(x) = -\frac{x^2}{528} - \frac{823\,x}{3725568} - \frac{239}{37158912}.$$

The variable occurs only at the common nodes 0 and 1, so the degree is $|C| = 2$ although the matrix has size 3. When $M[X', Y']$ is invertible, a Schur complement shows that $P_{\mathcal{G}}$ is, up to a constant factor, the characteristic polynomial of a rational matrix of size $|C|$, as in the square case. We do not use this factorisation.

4. Arithmetic: the denominators

In this section we bound the common denominator of the coefficients of the members $P_{r,I}$ for the functions F_s. The method is that of Rhin and Viola [41, 42]: the p-adic valuation of the denominator is bounded prime by prime, and the bounds are summed against $\log p$ with the prime number theorem in arithmetic progressions. What is new is the form of the local bounds. Each is an explicit integer expression in four integers attached to the prime, and their minimum, integrated, gives the denominator exponent.

The local bounds come from four mechanisms, of increasing depth.

(a) *Valuations of the entries.* The assignment bounds (T1)–(T5) of §4.4.1 use only the valuations of single entries and LP duality.
(b) *Corank modulo p.* The bounds (LP1), (LP2) and (R) of §§4.4.2 and 4.4.4 compute the corank of the reduction of a rescaled matrix.
(c) *Second order.* The bounds (W) and (S) of §§4.4.3 and 4.4.5 use the pairing of kernel vectors modulo p^2 (§4.3).
(d) *Third order.* The bounds (U_n) of §4.4.6 treat the primes smaller than the border.

Comparing the local bounds with a table of affine pieces is a finite computation with rational numbers, and Theorem 4.14 turns an admissible table into the arithmetic estimate. Section 8 treats the function F_χ.

4.1. **The root class of a prime.** Fix the function F_s, with s of denominator q, and the alternating geometry of parameter $c = 2\alpha$. Let $p > q$ be a prime. The indices $k \geq 1$ at which the entries of M acquire a denominator divisible by p are the multiples of p and the solutions of $qk \equiv q_s \pmod p$. Let k_p be the least of the latter. Then

$$qk_p = jp + q_s, \qquad 0 < j < q, \qquad j \equiv q s\, p^{-1} \pmod q.$$

We call j the *root class* of p and put $\kappa = \kappa_j = j/q$. Then $k_p - \kappa p = q_s/q \in [\frac{1}{q}, 1 - \frac{1}{q}]$. So the first pole is at distance less than one from κp, uniformly in s.

Throughout we assume $q(r+2m) < p^2$. Then every pole below $\bar{N}$ is simple, and only the first power of p matters. The primes with $p^2 \le q(r+2m)$ and the prime powers contribute $O(r^{3/2}\log r)$ in total.

4.2. **Gauss valuations, the assignment bound and the rank of the reduction.** Fix a prime p. Let $\mathcal{O} \subset \mathbb{Q}(x)$ be the localisation of $\mathbb{Z}_{(p)}[x]$ at the prime ideal (p). It is a discrete valuation ring with uniformiser p and residue field $\mathbb{F}_p(x)$. Its valuation v, restricted to $\mathbb{Q}[x]$, is the *Gauss valuation* $v(f) = \min_k v_p(f_k)$ for $f = \sum f_k x^k$. By Gauss's lemma $v(fg) = v(f) + v(g)$, and $v_p(\operatorname{den} f) = \max(0, -v(f))$. For a member we write $e_p(r, I) = v_p(\operatorname{den} P_{r,I})$.

Lemma 4.1 (assignment bound and LP duality)**.** *Let A be a square matrix over $\mathbb{Q}[x]$ with rows X_0 and columns Y_0. Let $W_{tt'} = \max(0, -v(A_{tt'}))$. The weight of a perfect matching $\sigma\colon X_0 \to Y_0$ is $\sum_t W_{t\sigma(t)}$. Then $v(\det A)$ is at least minus the largest weight of a perfect matching. If real numbers u_t, $u'_{t'}$ satisfy $W_{tt'} \le u_t + u'_{t'}$ for all $t \in X_0$ and $t' \in Y_0$, then every perfect matching has weight at most $\sum_t u_t + \sum_{t'} u'_{t'}$.*

Proof. Each Leibniz term is a product of entries along a perfect matching. So its valuation is at least minus the weight of the matching. The valuation of a sum is at least the minimum of the valuations of its terms. The second statement is weak LP duality for the assignment problem. The weight of a perfect matching is at most $\sum_t u_t + \sum_{t'} u'_{t'}$. □

Lemma 4.2 (rank of the reduction)**.** *Let (u, u') be integer potentials with $W_{tt'} \le u_t + u'_{t'}$ for the matrix $\Phi_{r,I}$ at p. Then $\widetilde{\Phi} = \operatorname{diag}(p^u)\,\Phi_{r,I}\,\operatorname{diag}(p^{u'})$ has entries in $\mathcal{O}$, and*

$$e_p(r, I) \le \Big(\sum_t u_t + \sum_{t'} u'_{t'} - \operatorname{corank}_{\mathbb{F}_p(x)}(\widetilde{\Phi} \bmod p)\Big)_+ .$$

Proof. We have $v(\det \Phi_{r,I}) = -\sum_t u_t - \sum_{t'} u'_{t'} + v(\det \widetilde{\Phi})$. Over the discrete valuation ring $\mathcal{O}$ the Smith normal form shows $v(\det \widetilde{\Phi}) \ge \operatorname{corank}(\widetilde{\Phi} \bmod p)$. Taking $\max(0, -v(\det \Phi_{r,I}))$ gives the claim. □

4.3. **Second-order information.** The rank bounds below use a refinement of Lemma 4.2 at second order. Suppose that, after a unimodular change of rows and columns over $\mathcal{O}$, the first n_1 rows and the first n_2 columns of $\widetilde{\Phi}$ are divisible by p. Suppose also that the $n_1 \times n_2$ block where they meet is divisible by p^2. Dividing the first n_1 rows by p leaves the first n_2 columns divisible by p. So

$$v(\det \widetilde{\Phi}) \ge n_1 + n_2. \tag{4.1}$$

The vectors that produce such rows and columns are kernel vectors of the reduction of $\widetilde{\Phi}$ on single residue classes. The condition on the corner is a statement about their pairing at first order, which we now describe. Let $\widetilde{\Phi} = p^2 \Phi_{r,I}$, so that all potentials equal 1. Since $q(r+2m) < p^2$, two distinct nodes $t \ne t'$ have $v_p(t'-t) \ge 1$ only if $t \equiv t' \pmod p$, and then $v_p(t'-t) = 1$.

Hence $\widetilde{\Phi}$ has entries in $\mathcal{O}$. Its reduction is block diagonal over the residue classes modulo p, which we call the *fibres*. For nodes t, t' in distinct fibres,

$$\widetilde{\Phi}_{tt'} \equiv \frac{p\,(V(t') - V(t))}{t' - t} \pmod{p^2}. \tag{4.2}$$

Here $V(t) \in \mathbb{F}_p$ is the residue of p times the partial sum $\sum_{k \le t}(\frac{1}{k} - \frac{q}{qk - q_s})$, up to the normalising factor q^{-2}. The function V is a step function of t which jumps only at the poles $t \equiv 0$ and $t \equiv k_p$. Since $t' - t$ is prime to p, only its residue matters in (4.2).

For a vector $\mathbf{y}$ indexed by nodes, its *moment pair* is

$$\hat{\mathbf{y}} = \Big(\sum_t y_t,\ \sum_t y_t V(t)\Big).$$

Lemma 4.3 (pairing)**.** *Let $\mathbf{y}$ be supported on the rows of one fibre and $\mathbf{y}'$ on the columns of a different fibre. Let t and t' be nodes of the first and of the second fibre. Then*

$$\mathbf{y}^\top \widetilde{\Phi}\, \mathbf{y}' \equiv \frac{p}{t' - t} \det \begin{pmatrix} \hat{\mathbf{y}} \\ \hat{\mathbf{y}}' \end{pmatrix} \pmod{p^2}.$$

In particular the pairing vanishes modulo p^2 whenever the moment pairs of $\mathbf{y}$ and $\mathbf{y}'$ are proportional.

Proof. Expand $\mathbf{y}^\top \widetilde{\Phi}\, \mathbf{y}'$ with (4.2). □

The moment pairs are never both zero. This follows from the displacement structure of Löwner matrices.

Lemma 4.4 (moment lemma)**.** *Let A be a matrix with rows indexed by a set X_0 of distinct integers and columns by a set Y_0, with $|X_0| = |Y_0| + 1$. Suppose that its off-diagonal entries are $A(t, t') = (\mathcal{L}(t') - \mathcal{L}(t))/(t' - t)$ for some function $\mathcal{L}$, and that its diagonal entries, on $X_0 \cap Y_0$, are arbitrary. Then A has a nonzero left kernel vector $\mathbf{y}$ with $(\sum_t y_t, \sum_t y_t \mathcal{L}(t)) \neq (0, 0)$.*

Proof. Write $\operatorname{diag}(X_0)$ and $\operatorname{diag}(Y_0)$ for the diagonal matrices of the indices. Then $\operatorname{diag}(X_0)A - A\operatorname{diag}(Y_0) = \mathcal{L}_{X_0}\mathbf{1}^\top - \mathbf{1}\mathcal{L}_{Y_0}^\top$, because the diagonal entries cancel on common indices. The left kernel $\mathcal{V}$ of A is nonzero. Suppose that every $\mathbf{y} \in \mathcal{V}$ had zero moment pair. Multiplying on the left by $\mathbf{y}^\top$ would give $\mathbf{y}^\top \operatorname{diag}(X_0)A = 0$. So $\operatorname{diag}(X_0)$ would preserve $\mathcal{V}$. As this matrix is diagonal with distinct entries, $\mathcal{V}$ would contain a standard basis vector. That vector is supported on one index t, and its moment pair is $(1, \mathcal{L}(t)) \neq (0, 0)$. This is a contradiction. □

4.4. **The local bounds.** We now list the local bounds. Each is a statement of the following form. Let $P_{r,I}$ be a member with $|I| \le K$, and let $p > p_0$ be a prime with $q(r + 2m) < p^2$ that satisfies the stated conditions. Then

$$e_p(r, I) \le \tau(r, m, p, k_p) + \epsilon_\tau\, |I|. \tag{4.3}$$

Here τ is the expression in the table and $\epsilon_\tau \in \{0,1\}$ is given in its last column. The constant p_0 depends only on F. We write $(\cdot)_+ = \max(\cdot, 0)$, $\|m\|_p$ for the distance of m to $p\mathbb{Z}$, and $\bar{u} = (k_p + p - r)_+$.

Theorem 4.5 (local bounds)**.** *The following bounds of the form* (4.3) *hold.*

	bound τ	*conditions*	ϵ_τ
(T1)	$2(r+m)$	*none*	0
(T1′)	$2r + 2m - \|m\|_p$	*none*	0
(T2)	$2(r-\bar{u})_+ + 2m + \min(\bar{u}, 2m)$	$2m < p \le r$	0
(T3)	$2r + m$	$2m < p \le r$	0
(T4)	$2(r-k_p)_+ + 2m + (2m-(p-r))_+$	$r < p$	0
(T5)	$(2m - (k_p - r)_+)_+$	$\bar{N} \le p$, $r \le k_p$	0
(LP1)	$\tau_{\mathrm{T2}} - (\min(\bar{u}, 2m) - m)_+$	$p \le r < p + k_p$, $\bar{N} \le 2p$	1
(LP2)	$2(r-k_p) + 3m$	$r < p < r + m$, $k_p \le r$, $\bar{N} \le 2p$	1
(W)	$2r + 2m - (t_2 - t_1)$ *for each window* $[t_1, t_2)$ *of Theorem 4.7*	*as there*	0
(R)	Π_R *of* (4.4)	$2m < p$	0
(S)	Π_S *of* (4.5)	$2m \le p$	1
(U_n)	$\Pi_{U,n}$ *of* (4.6), $n \ge 0$	$np \le m < (n+1)p$, *and* $p \le 2m$ *if* $n = 0$	0

The rest of this subsection proves Theorem 4.5, family by family.

4.4.1. *The assignment bounds (T1)–(T5).* These follow from Lemma 4.1 with explicit potentials. Consider an off-diagonal entry $M[t, t']$. By the ultrametric inequality, $-v_p(M[t,t'])$ is at most the number of the following two events that occur: the interval $(t, t']$ contains a pole, and p divides $t' - t$. A diagonal entry $M[t,t]$ has $-v_p \le 2$ if $t \ge k_p$ and $-v_p \le 0$ if $t < k_p$. The bound (T1) is the trivial bound, with potentials 1 on every row and column.

For (T1′) the potentials depend on the residue class modulo p. On the rows of a class they are $\frac{1}{2}$, $\frac{3}{2}$ or 1, according as X' has more, fewer or as many nodes in the class as Y'. On the columns they are $\frac{3}{2}$, $\frac{1}{2}$ or 1 in the same three cases. The border alternates between X' and Y'. So the imbalance is at most one per class, and the number of unbalanced classes is $2\|m\|_p$.

For (T2)–(T5) one starts from the potential $\mathbf{1}[t \ge k_p]$ on the common nodes t, which accounts for the diagonal. One adds 1, or $1 + \mathbf{1}[t \ge p]$, on the border nodes t. Then one raises the potentials of the common nodes that have a congruent partner in the border. In each range one checks $W \le u + u'$ entrywise. Two facts are used. Two distinct nodes not congruent modulo p have weight at most 1. Two border nodes are never congruent once $2m < p$. For (T5) we have $\bar{N} \le p$ and $r \le k_p$. Then no common node accounts for the

diagonal, and only the border nodes beyond k_p carry potential 1. So (T5) vanishes once $k_p \geq \bar{N}$, which happens for $p/r > \max(1+2\alpha, 1/\kappa) + o(1)$. All cases are checked in the formal proof.

4.4.2. *Primes near r: the bounds (LP1) and (LP2).* For p close to r the reduction of the rescaled matrix has a large block of rank one per residue class. Its corank can be computed. We give the proof of (LP2). The bound (LP1) applies to the primes with $p \leq r < p + k_p$ and $\bar{N} \leq 2p$. It is proved by the same method. With the potentials of (T2), eliminating an invertible block of common nodes brings the reduction to the form $\left(\begin{smallmatrix} \tilde{x}\mathrm{Id} & \mathcal{J}_Y \\ \mathcal{J}_X & 0 \end{smallmatrix}\right)$, where $\tilde{x}$ is the variable x plus a constant. In this form the border nodes below $p + k_p$ contribute rows and columns proportional to unit vectors. The Frobenius–König theorem bounds its rank. The saving over (T2) is $(\min(2m, p + k_p - r) - m)_+$, where $\min(2m, p + k_p - r)$ is the number of border nodes below $p + k_p$. We refer to the formal proof for the details.

Theorem 4.6. *Assume $r < p < r + m$, $k = k_p \leq r$ and $\bar{N} = r + 2m \leq 2p$, and that every pole below $\bar{N}$ is simple. Then $v(\det \Phi_{r,\emptyset}) \geq -(2(r-k) + 3m)$. The same holds for every member, with $\epsilon_\tau = 1$.*

Proof. Use the potentials $u(t) = \mathbf{1}[k \leq t]$ for common nodes $t < r$ and $u(t) = 1 + \mathbf{1}[p \leq t]$ for border nodes t. Their sum is $2(r-k) + 2m + (\bar{N} - p)$. Since $\bar{N} \leq 2p$, the poles below $\bar{N}$ are k, p, and $p + k$ if $p + k < \bar{N}$. Each is simple. The residues of p times the increment $\frac{1}{t} - \frac{q}{qt - q_s}$ at these poles are $-q/j$, 1 and $-q/(q + j)$.

Rows and columns beyond $p + k$ vanish. Let $t \in [p + k, \bar{N})$ be a border node, so that $u(t) = 2$. Let t' be a node of the other kind. If $t' < k$, then $u(t') = 0$. The partial sum over $(t', t]$ then has valuation at least -1, and $t - t' \in (p, 2p)$ is prime to p. Otherwise $u(t') \geq 1$ and $v_p(M[t', t]) \geq -2$. In both cases the rescaled entry is divisible by p.

The remaining entries. For nodes below $p + k$ the only poles are k and p. Put $C_0 = [0, k)$ and $C_1 = [k, r)$. Recall that $\omega = q^{-2}$ is the normalisation of (2.1). The reduction has the following blocks. On $C_0 \times C_0$ it is the regular part plus x on the diagonal. On $C_0 \times C_1$ the entries are $-\omega q/(j(t - t'))$ with $t' \in C_0$ and $t \in C_1$. On $C_1 \times C_1$ it is the scalar matrix $\omega(q/j)^2\mathrm{Id}$. A border node $t \in [r, p)$ against $t' \in C_0$ gives $\omega q/(j(t - t'))$. A border node $t \in [p, p + k)$ against $t' \in C_0$ gives $\omega(1 - q/j)$ if $t' = t - p$, and 0 otherwise. All other border entries vanish.

Rank. Delete the zero rows and columns found above. This does not change the rank. Eliminate the scalar block on C_1. The Schur complement on C_0 is $\tilde{x}\mathrm{Id}$ plus a constant multiple of the matrix with entries $\sum_{i=1}^{p-r} \frac{1}{(t+i)(t'+i)}$ for $t, t' \in C_0$. Here $\tilde{x}$ is x plus a constant, and the computation uses the reflection $H_{p-1-n} \equiv H_n \pmod{p}$ of harmonic numbers. The border nodes $t \in [r, p)$ are indexed by $i = p - t$. They are all present because $\bar{N} > p$. Even offsets are rows and odd offsets are columns, and together

they cover $1 \le i \le p - r$. So row and column operations remove that multiple and leave $\left(\begin{smallmatrix} \tilde{x}\mathrm{Id} & \mathcal{J}_Y \\ \mathcal{J}_X & 0 \end{smallmatrix}\right)$. Here it is essential that x is an indeterminate, so that $\tilde{x}$ is a unit. A Schur complement on $\tilde{x}\mathrm{Id}$ then shows that the rank is $k + \operatorname{rank}(\mathcal{J}_X\mathcal{J}_Y)$. In $\mathcal{J}_X\mathcal{J}_Y$ the block between the high rows and the high columns is zero, since their offsets $t - p$ have opposite parities. So $\operatorname{rank}(\mathcal{J}_X\mathcal{J}_Y) \le p - r$. In total the rank of the reduction is at most $(r - k) + k + (p - r) = p$, and the corank is at least $r + m - p$. Lemma 4.2 gives $v(\det) \ge (r + m - p) - (2(r - k) + 2m + \bar{N} - p) = -(2(r - k) + 3m)$. Deleting a common node changes the potentials by at most one. □

Under the hypothesis $\bar{N} \le p + k$ the third pole does not occur, and the proof is shorter. The border nodes beyond $p + k$ only contribute rows and columns that vanish modulo p. This observation allows the whole range $\bar{N} \le 2p$, which is automatic on the interval $r < p < r + m$.

4.4.3. *Pole-free windows: the bound (W).* Like the rank bounds, the bound (W) does not come from the valuations of the entries alone. Between two consecutive poles the step function V of (4.2) is constant. The reductions of the fibres of nearby residue classes are then translates of one another. Kernel vectors can be transported from one fibre to the next. Their pairings vanish modulo p^2 by Lemma 4.3. The corner bound (4.1) turns this into a saving at second order.

A *cut* is an integer t with $t \equiv 0$, $t \equiv k_p$ or $t \equiv r \pmod p$. A window $[t_1, t_2)$ of integers is *cut-free* if no cut lies in the open interval (t_1, t_2).

Theorem 4.7. *Assume that every pole below $\bar{N}$ is simple and $\bar{N} < p^2$. Let $n \ge 0$ with $np \le m$, and put $\bar{N}_n = r + 2(m - np)$. Let $[t_1, t_2)$ be a cut-free window with $\max(r, \bar{N}_n - p) \le t_1 \le t_2 \le \min(\bar{N}_n, r + p)$. Then every member satisfies*

$$e_p(r, I) \le 2r + 2m - (t_2 - t_1).$$

For $n = 0$ the windows lie in the border above $\max(r, \bar{N} - p)$, and the bound applies to primes $p > m$. For $n = \lfloor m/p \rfloor \ge 1$ it applies to the small primes $p \le m$. For these primes only (T1) and (T1′) are available among the assignment bounds. Read modulo p, the window range is then the arc of residues that receive an odd number of border nodes. The saving over (T1′) is the amount, if any, by which the length of the longest cut-free run of this arc exceeds $\|m\|_p$. For other n the window range is empty. The proof uses F only through its poles. For a function with several digamma terms the same statement holds when the cuts are the residues of all the poles and of r.

Proof. Let $\widetilde{\Phi} = p^2\Phi$, with independent diagonal variables on the common nodes. Every member is a coefficient of this determinant, so a bound for it bounds every member. Let $i < i'$ be two positions of the window. They lie in different residue classes. The translation $t \mapsto t + (i' - i)$ maps the nodes of the fibre of i bijectively onto those of the fibre of i'. It maps common nodes to common nodes, since no $t \equiv r$ lies in $(i, i']$. It maps border nodes to

border nodes, since both classes lie in the window range. It maps X' to Y' when $i'-i$ is odd. There is no pole in $(i,i']$. So the step function V and the diagonal residues are unchanged, and so are the differences of nodes. Hence the reduced block of the fibre of i' equals that of i when $i'-i$ is even, and its transpose when $i'-i$ is odd.

In the window the fibres alternate between one more row than column and one more column than row. Take a nonzero left kernel vector $\mathbf{y}$ of one block and transport it to every fibre of the window. It is a left kernel vector on the fibres with a surplus row and a right kernel vector on the others. Its moment pair is the same everywhere. By Lemma 4.3 all pairings between these vectors vanish modulo p^2. There are n_1 such vectors on rows and n_2 on columns, with $n_1+n_2=t_2-t_1$. Extending them to unimodular matrices, (4.1) gives $v(\det\widetilde{\Phi})\geq t_2-t_1$. Since $\det\widetilde{\Phi}=p^{2(r+m)}\det\Phi$, the theorem follows. □

4.4.4. *The rank bound (R).* Let $2m<p$. Write $r=k_0p+r_p$ with $0\leq r_p<p$. Note that $k_p=\lceil jp/q\rceil$, since $0<q_s<q$. Put $\mathrm{arc}_0=[0,k_p-1]$ and $\mathrm{arc}_1=[k_p,p-1]$. For $k\geq 0$ and $\iota\in\{0,1\}$ let

$$n_{k,\iota}=\big|[r,r+2m)\cap[kp+\min\mathrm{arc}_\iota,\ kp+\max\mathrm{arc}_\iota]\big|,$$
$$n_{\mathrm{sq}}=\big|[\max(0,r+2m-p),\min(r,k_p))\big|,$$

and

$$\Pi_R=2(r+m)-\max\big(m+n_{\mathrm{sq}},\ n_{0,0}+n_{1,0}+2n_{\mathrm{sq}},\ n_{k_0,0},\ n_{k_0,1},\ n_{k_0+1,0},\ n_{k_0+1,1}\big). \tag{4.4}$$

Proof of the bound (R). Every node is $t=i+pk$ with residue $i\in[0,p)$ and digit k. Write $k(i)$ for the number of common nodes in the fibre of i. The border has length $2m<p$. So this fibre contains the common digits $0,\dots,k(i)-1$ and at most one border digit, namely $k(i)\in\{k_0,k_0+1\}$. This border node is a row or a column according to the parity of $i+pk(i)-r$. The step function V is constant on the fibre up to the bit $\iota(i)=\mathbf{1}[i\geq k_p]$. Thus the reduced block of a fibre depends only on $(k(i),\iota(i))$ and on the role of its border digit. Its entries have the Löwner form of Lemma 4.4.

A fibre with a border row has a left kernel vector with nonzero moment pair. A fibre with a border column has such a right kernel vector. For the digit 0 below the pole the step function vanishes. So these fibres have moment pairs proportional to $(1,0)$. A fibre of type $(1,0)$ without border digit has the zero block and contributes one row and one column. Choosing one such vector per fibre annihilates $m+n_{\mathrm{sq}}$ rows and as many columns. This gives the saving $m+n_{\mathrm{sq}}$.

Within a class of fibres of the same type (k,ι) all moment pairs are equal. So by Lemma 4.3 the pairings between the annihilated rows and columns of one class vanish modulo p^2. Then (4.1) gives a saving equal to the number of annihilated coordinates of the class. This number is $n_{k,\iota}$ for $k\geq 1$ and $(k,\iota)\neq(1,0)$. It is $n_{0,0}+n_{1,0}+2n_{\mathrm{sq}}$ for the class with moment pair $(1,0)$.

Taking the best class gives (4.4). A deleted common node changes at most one fibre. Give up that fibre, and one further fibre to restore the balance of rows and columns. The trivial certificate then costs at most two annihilated coordinates of each kind. The smaller matrix compensates for them, so $\epsilon_\tau = 0$. □

4.4.5. *Separated border nodes: the bound (S).* The bound (S) uses a different source of cancellation. Choose a window $[t_0, t_v]$ of border nodes whose residues lie in one arc between consecutive poles. Consider a border node outside it. Suppose the interval between the node and t_0 contains a singular index. Then the residues modulo p of p times the partial sums at the two nodes differ, because two singular contributions in an interval of length less than p cannot cancel. The leading coefficient in x of the bordered determinant of a fibre is this residue. So the cross product of the two bordered fibre determinants has a nonzero leading coefficient. The border node is then eliminated against t_0 with no further information. Count the residues of the window together with the residues of the eliminated border nodes. This gives an explicit bound, a minimum over the four window types $(k, \iota) \in \{k_0, k_0 + 1\} \times \{0, 1\}$, which we now state.

Let $2m \le p$, and let k_0, r_p, arc_0, arc_1 and $n_{k,\iota}$ be as in §4.4.4. For a type (k, ι) let $\mathcal{K}_{k,\iota}$ be the interval of the integers $kp + i$ with $i \in \mathrm{arc}_\iota$, and with $i < r_p$ if $k = k_0 + 1$. The border nodes in $\mathcal{K}_{k,\iota}$ form the window, and there are $n_{k,\iota}$ of them. Let n_K be the number of border nodes in $\mathcal{K}_{k,\iota}$ and in its two translates $\mathcal{K}_{k,\iota} \pm p$. Put

$$n_\emptyset = p - |\mathcal{K}_{k,\iota}| - (2m - n_K), \qquad n_+ = (n_{k,\iota} - m)_+.$$

Since $2m \le p$, each residue class receives at most one border node. So n_K is the number of border nodes whose residues are those of $\mathcal{K}_{k,\iota}$. And $n_\emptyset$ is the number of the other residue classes that receive no border node. Then

$$\Pi_S = \min_{(k,\iota)} (2r + m - n_+ - (n_+ - n_\emptyset)_+), \tag{4.5}$$

where a type whose window is empty contributes $2r + m$. The bound (S) is $e_p(r, I) \le \Pi_S + |I|$. Its proof carries out the elimination above for every border node outside the window. We refer to the formal proof for it.

4.4.6. *Primes smaller than the border: the bound (U_n).* When $p \le 2m$ the border meets some residue classes more than once, and the bound (S) is not available. The bound (U_n) extends it to these primes. It works at third order in the Smith normal form. We show two things. After the elimination of a block of unit pivots, the rescaled matrix has the Smith invariants of a smaller geometry, to which the separation argument of (S) applies. The third-order information of that argument survives the elimination.

Write $m = np + \bar{m}$ with $0 \le \bar{m} < p$, so that $n = \lfloor m/p \rfloor$. Put

$$d = \|m\|_p = \min(\bar{m}, p - \bar{m}), \qquad \bar{r} = r + m - d.$$

For $i \in \{0,1,2\}$ and $\iota \in \{0,1\}$ let $Z_i = p\lfloor r/p \rfloor + (n+i)p$, and put

$$w_0 = Z_i + \min \operatorname{arc}_\iota, \quad t_0 = \max(\bar r, w_0), \quad t_v = \min(Z_i + \max \operatorname{arc}_\iota,\ \bar r + p - 1).$$

We call the geometry with common block $[0,\bar r)$ and border $[\bar r, \bar r + 2d)$ the *auxiliary geometry* $(\bar r, d)$. In it, let $n_+ = \big(\min(t_v + 1, \bar r + 2d) - t_0 - d\big)_+$. Let n_K be the number of border nodes in $[w_0, t_v]$ and in its two translates by $\pm p$, and let $n_\emptyset = p - (t_v + 1 - w_0) - (2d - n_K)$. Then

$$\Pi_{U,n} = \min_{i,\iota} \Big(2(r+m) - d - n_+ - (n_+ - n_\emptyset)_+ \Big), \tag{4.6}$$

all subtractions being truncated. This is the expression (4.5) for the auxiliary geometry $(\bar r, d)$, with the windows $[w_0, t_v]$ in place of $\mathcal{K}_{k,\iota}$. Note that $2(r+m) - d = 2\bar r + d$.

Theorem 4.8. *Let $p > p_0$ be a prime with $q(r+2m) < p^2$, and let $n \geq 0$ with $np \leq m < (n+1)p$. Assume $p \leq 2m$ if $n = 0$. Then every member satisfies $e_p(r, I) \leq \Pi_{U,n}$.*

We first describe the proof for $n = 0$, that is $m < p \leq 2m$. We then give the changes for $n \geq 1$. As in the proof of Theorem 4.7 we work with $\widetilde\Phi = p^2\Phi$. We put an independent indeterminate x_t on each common diagonal entry. We work over the discrete valuation ring $\mathcal{O}$ of Gauss valuations in these indeterminates. Every member is a coefficient of $\det \widetilde\Phi$, up to the deletion of common rows and columns, which we treat at the end.

Pivots from pairs of border nodes. For $n = 0$ there are $2m - p$ residue classes that carry two border nodes $t < t + p$, one a row and the other a column. There are $2d$ classes that carry one border node, and they lie in an interval of length $2d < p$. Eliminate the common block. Its fibres are invertible because the x_t are indeterminates. Consider a class with two border nodes. Let A be its common fibre, let $\mathbf{a}$ and $\mathbf{a}'$ be the couplings of its border column and border row to A, and let $\varkappa$ be its border entry. After the elimination the class has the pivot $\varpi = \varkappa - \mathbf{a}'^{\top} A^{-1} \mathbf{a}$. As the x_t tend to infinity, ϖ tends to a nonzero multiple of the residue of p times the increment of F over $(t, t+p]$. This interval contains exactly one pole of each digamma term. Let $k'p$ be the multiple of p in it, and t' the pole of the second term. If the two contributions cancelled, then $q(t' - k'p) \equiv u$ with $|q(t' - k'p) - u| < q$. Hence $q(t' - k'p) = u$, which is impossible for $0 < u < q$. So these $2m - p$ pivots are units.

After them, the matrix of size $r + m = \bar r + d$ is equivalent over $\mathcal{O}$ to $\operatorname{diag}(\mathrm{Id}_{\bar r}, p\,\Xi)$, with Ξ of size d. Its Schur complement on the border has the expansion

$$S = p\,S_1 + p^2 (S_2 - \mathcal{C}_{UB} \mathcal{B}^{-1} \mathcal{C}_{BU}) + O(p^3). \tag{4.7}$$

Here $\mathcal{B}$ is the diagonal of these pivots. The matrices $\mathcal{C}_{UB}$ and $\mathcal{C}_{BU}$ are the first-order couplings between these pivots and the single border nodes. The correction term in (4.7) is not negligible. The following identities account for it.

Lemma 4.9 (bordered inverses)**.** *Let A be an invertible square matrix over a commutative ring, and let $\mathbf{u}$ be a vector.*

(a) Let A' be obtained from A by replacing one column by a vector $\mathbf{w}$, and suppose that A' is invertible. Let $\mathbf{u}'$ be obtained from $\mathbf{u}$ by replacing the corresponding entry by $\mathbf{u}^\top A^{-1}\mathbf{w}$. Then $\mathbf{u}'^\top A'^{-1}\mathbf{u} = \mathbf{u}^\top A^{-1}\mathbf{u}$.

(b) Suppose that the matrix

$$A' = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{a}'^\top & \varkappa \end{pmatrix}$$

is invertible. Then $(\mathbf{u}, y)^\top A'^{-1}(\mathbf{u}, \mathbf{a}'^\top A^{-1}\mathbf{u}) = \mathbf{u}^\top A^{-1}\mathbf{u}$ for every scalar y.

No symmetry of A or A' is assumed.

Proof. (a) Comparing columns gives $\mathbf{u}'^\top = \mathbf{u}^\top A^{-1}A'$. (b) We have $A'\,(A^{-1}\mathbf{u}, 0)^\top = (\mathbf{u}, \mathbf{a}'^\top A^{-1}\mathbf{u})^\top$. □

Moments. In the separation argument of (S), a border node is counted when its moment relative to a reference node vanishes. That moment is the bilinear form $\mathbf{u}^\top A^{-1}\mathbf{u}$ in the inverse of the common fibre A of its residue class. After the elimination, the common fibre of a class with two border nodes is replaced by a mixed block A' as in Lemma 4.9. Its lower node is a row and its upper node a column, or conversely. The product added to A'^{-1} is exactly the product subtracted in (4.7). So, by Lemma 4.9, the moments of the counted classes are unchanged. The single border nodes outside the window are separated from it as in (S), since two singular contributions in an interval of length $2d < p$ cannot cancel. Applied to the auxiliary geometry $(\bar{r}, d)$, the count of (S) then gives (4.6). It uses the third-order analogue of (4.1). Call the *coupling* of a residue class the residue modulo p of its border entry minus its moment against the reference node. Suppose that n_+ kernel vectors on each side survive, and that at most n_0 residue classes have a nonzero coupling. Then $\sum_i \min(v_i, 3) \geq d + n_+ + (n_+ - n_0)_+$, where $v_1, v_2, \ldots$ are the valuations of the Smith invariants. The first term d is the factor p of $p\,\Xi$.

Members. Deleting ℓ common nodes lowers the trivial bound $2(r+m)$ by 2ℓ. Consider a class with one deleted common node whose bordered system keeps its value at infinity Θ (Lemma 4.10 below). Then the class keeps its moment. Indeed, let A be the common fibre and delete its last coordinate. The residual of the smaller solution is supported on that coordinate. Since the two values at infinity are equal, its pairing with the data vanishes. A class whose value at infinity changes gives a new separation instead. Classes with two or more deleted nodes, and classes outside the window, cost at most one each. Let k_1, k_2 and k' be the numbers of classes of these three kinds. Then $k_1 + 2k_2 + k' \leq \ell$. An elementary inequality for the count $n_+ + (n_+ - n_0)_+$ shows that the loss is at most 2ℓ. Hence the bound holds for every member with $\epsilon_\tau = 0$.

Larger n. Let $np \le m < (n+1)p$. Each class of the remaining border absorbs n pairs of border nodes. Its block after the elimination is a mixed block. Its rows and columns are the nodes of $2n+1$ consecutive quotients $\beta, \beta+1, \dots, \beta+2n$. Here β is the number of common nodes in the class, and the quotient of a node t in the class of a residue i is $(t-i)/p$. The window counts are those of the first case at the scale $r+np$, translated by np. The new difficulty is to show that a remaining border node separated from the window by a pole has a nonzero coupling. The values at infinity are now those of $(n+1)\times(n+1)$ bordered systems, which we describe first.

Lemma 4.10 (bordered systems and rational interpolation)**.** *Let $\mathbb{K}$ be a field and $\mathcal{Z} = \{0, 1, \dots, 2n\}$, with $(2n)! \neq 0$ in $\mathbb{K}$. Let $\mathcal{L} : \mathcal{Z} \to \mathbb{K}$, and let $\mathcal{Z} = X \sqcup Y$ with $|X| = n+1$ and $|Y| = n$. Consider the bordered system*

$$\sum_{y \in Y} \frac{\mathcal{L}(y) - \mathcal{L}(x)}{y - x}\, z_y + \Theta = \mathcal{L}(x) \qquad (x \in X)$$

in the unknowns z_y and Θ, with data $\mathcal{L}$. Its solutions correspond bijectively to the pairs $(\mathcal{N}, \mathcal{D})$ of polynomials with $\mathcal{D}$ monic of degree n, $\deg \mathcal{N} \le n$ and $\mathcal{N}(t) = \mathcal{L}(t)\mathcal{D}(t)$ for $t \in \mathcal{Z}$. Under this correspondence Θ is the coefficient of t^n in $\mathcal{N}$. In particular the system is nonsingular exactly when the interpolation problem of type (n, n) has a unique solution. Then $\Theta = \lim_{t\to\infty} \mathcal{N}(t)/\mathcal{D}(t)$ depends only on $\mathcal{L}$ and not on the split. We call Θ the value at infinity *of the system.*

Proof. Let $\mathcal{W}(t) = \prod_{y \in Y}(t - y)$, and let $\mathcal{W}_y = \mathcal{W}/((t-y)\mathcal{W}'(y))$ be the Lagrange polynomials of Y. Given a solution, put $\mathcal{D} = \mathcal{W} - \sum_y z_y \mathcal{W}'(y)\, \mathcal{W}_y$ and $\mathcal{N} = \Theta\mathcal{W} - \sum_y z_y \mathcal{L}(y)\mathcal{W}'(y)\, \mathcal{W}_y$. Then $\mathcal{N}(y) = \mathcal{L}(y)\mathcal{D}(y)$ for $y \in Y$. Dividing $\mathcal{N}(x) - \mathcal{L}(x)\mathcal{D}(x)$ by $\mathcal{W}(x) \neq 0$ gives the equation of the row x. Conversely, Lagrange interpolation on Y recovers $z_y = -\mathcal{D}(y)/\mathcal{W}'(y)$ and Θ from $(\mathcal{N}, \mathcal{D})$. □

Lemma 4.11 (harmonic data)**.** *Let $\mathcal{A} \in \mathbb{K}$ with $\mathcal{A} + i \neq 0$ for $0 \le i < 2n$, and let $\mathcal{L}_{\mathcal{A}}(t) = \sum_{0 \le i < t} 1/(\mathcal{A}+i)$. Let the characteristic of $\mathbb{K}$ be 0 or larger than $2n$. For every split, the bordered system of Lemma 4.10 with $\mathcal{L} = \mathcal{L}_{\mathcal{A}}$ is nonsingular. Its value at infinity is $\Theta = \mathcal{L}_{\mathcal{A}}(n) + \mathsf{c}_n$, with $\mathsf{c}_n \in \mathbb{K}$ depending only on n. Consequently, for the harmonic data $\mathcal{L}(t) = H_t$ on the quotients $\beta, \dots, \beta+2n$ the value at infinity is $H_{\beta+n} + \mathsf{c}_n$. The values at infinity of two classes with β and $\beta+1$ common nodes differ by $1/(\beta+n+1)$.*

Proof. Write Δ for the forward difference. For $l \ge 1$ one has $\Delta^l \mathcal{L}_{\mathcal{A}}(k) = (-1)^{l-1}(l-1)!/(\mathcal{A}+k)_l$, where $(y)_l$ is the rising factorial. One also has $\Delta^l[\binom{t}{k} f(t)](0) = \binom{l}{k} \Delta^{l-k} f(k)$. By Lemma 4.10 a solution is a pair $(\mathcal{N}, \mathcal{D})$ with $\mathcal{N} = \mathcal{L}_{\mathcal{A}}\mathcal{D}$ on $\mathcal{Z}$. Since $\mathcal{N}$ has degree at most n, this means $\Delta^l(\mathcal{L}_{\mathcal{A}}\mathcal{D})(0) = 0$ for $n < l \le 2n$. Write $\mathcal{D} = \sum_k \mathsf{d}_k \binom{t}{k}$ and insert the two formulas. After removing nonzero factors of the rows and columns, these n equations become a system for the coefficients d_k with $k < n$. Its matrix is the Cauchy matrix

$\big(1/(l-k)\big)_{n<l\le 2n,\,0\le k<n}$, and its right-hand side comes from $\mathsf{d}_n = n!$. The Cauchy matrix is nonsingular. So $\mathcal{D}$ is unique, which is the nonsingularity.

The same computation shows that $\mathsf{d}_k/(\mathcal{A}+k)_{n-k}$ does not depend on $\mathcal{A}$. For $k<n$, $\Delta^{n-k}\mathcal{L}_{\mathcal{A}}(k)$ is a constant multiple of $1/(\mathcal{A}+k)_{n-k}$. Hence $\Theta = \Delta^n(\mathcal{L}_{\mathcal{A}}\mathcal{D})(0)/n! = \mathcal{L}_{\mathcal{A}}(n) + \mathsf{c}_n$ with c_n independent of $\mathcal{A}$. For the harmonic data on the quotients $\beta+t$ we have $H_{\beta+t} = H_\beta + \mathcal{L}_{\beta+1}(t)$. A constant added to $\mathcal{L}$ is added to Θ. So the value at infinity is $H_\beta + \mathcal{L}_{\beta+1}(n) + \mathsf{c}_n = H_{\beta+n} + \mathsf{c}_n$. □

Remark 4.12. We note without proof that the interpolation problem of Lemma 4.11 can be solved in closed form. Its denominator is the polynomial $\mathcal{D}_n(t) = \sum_{k=0}^{n} \binom{n}{k}\binom{2n-k}{n-k}(\mathcal{A}+k)_{n-k}\, t(t-1)\cdots(t-k+1)$, of Hahn type [25]. The constant is $\mathsf{c}_n = 2H_n$. The determinant of the bordered system is $\mathrm{sf}(n)^2\,\mathrm{sf}(n-1)^2/\big(\prod_{x\in X, y\in Y}|x-y| \cdot \prod_{i<n}(\mathcal{A}+i)_{n+1}\big)$, with $\mathrm{sf}(n) = \prod_{i\le n} i!$. All its prime factors are at most $2n$ or divide some $\mathcal{A}+i$. These closed forms are not used in the proof and are not part of the formalization.

Separation for $n \ge 1$. Between a remaining border node and the reference node of the window lies a cut. Suppose first that the cut is a pole of the second digamma term. Deform the shift to $s+pY$, with an indeterminate Y. Evaluate at the value of Y at which that pole becomes central. After multiplication by the vanishing factor, the bordered systems of the two classes become block systems built from Cauchy matrices on the quotients. The square blocks are nonsingular by Cauchy's determinant formula. A rectangular Cauchy matrix with $k+1$ rows and k columns has a left kernel vector with nonzero coordinate sum. Otherwise the rational function $\sum_i y_i/(T_i - T)$, formed from a kernel vector (y_i) and the row nodes T_i, would vanish at the k column nodes while its numerator has degree less than k. This gives the two values at infinity the values 0 and the weight of the term, which differ.

Suppose next that the cut is a multiple of p. Take instead the leading coefficient in Y. Every fractional pole term tends to 0, and the data become the harmonic numbers. The two classes have β and $\beta+1$ common nodes. Lemma 4.11 gives values at infinity that differ by $1/(\beta+n+1)$. This is a unit because $\beta + 2n + 1 < p$, by $q(r+2m) < p^2$.

In both cases the comparison is made on the cleared numerators of the original determinants, not on inverses. The result returns to $Y = 0$ by Gauss's lemma. The cleared denominators are products of the elementary denominators of the entries, which are units at $Y = 0$. So a lower bound for the Gauss valuation of the deformed determinant gives the same lower bound for every coefficient of the original one. With these separations, the count and the treatment of the members are those of the case $n = 0$.

4.5. **Profiles.** Let $\mathcal{E}$ be the set of integer expressions in r, p, m, k built from integer constants with the following operations: addition, multiplication by integer constants, truncated subtraction $(X-Y)_+$, min, max, and the distance $\|X\|_p$ of an expression X to $p\mathbb{Z}$. The quantities $p\lfloor r/p\rfloor$ and $r \bmod p$ are also allowed. Every bound of Theorem 4.5 lies in $\mathcal{E}$. For (R) and (S) this

is (4.4) and (4.5). The ends of the intervals $\mathcal{K}_{k,\iota}$ are $kp = p\lfloor r/p\rfloor + (k-k_0)p$ plus 0, k_p, $k_p - 1$, $p-1$ or $r_p - 1$. For (U_n) this is (4.6) with $d = \|m\|_p$ and $Z_i = p\lfloor r/p\rfloor + (n+i)p$. For (W) the window ends are the cuts nearest to the ends of the window range, which are expressions of the same kind.

The *homogenisation* $\hat\tau(\theta)$ of $\tau \in \mathcal{E}$ replaces r, p, m, k by $1, \theta, \alpha, \kappa\theta$. It replaces every constant by 0, $\|X\|_p$ by $\theta\|\hat X/\theta\|$ and $p\lfloor r/p\rfloor$ by $\theta\lfloor 1/\theta\rfloor$. It is piecewise affine with rational breakpoints, finitely many in every interval bounded away from 0.

Lemma 4.13 (homogenisation)**.** *For each $\tau \in \mathcal{E}$ there is an explicit constant* $\mathrm{err}(\tau)$ *such that* $|\tau(r,p,m(r),k_p) - r\hat\tau(p/r)| \le \mathrm{err}(\tau)$ *for all* $r, p \ge 1$*. It is computed recursively from* $\mathrm{err}(m) = \frac12$, $\mathrm{err}(k) = 1 - \frac1q$ *and the rules* $\mathrm{err}(\tau+\tau') = \mathrm{err}(\tau) + \mathrm{err}(\tau')$ *and* $\mathrm{err}(\min(\tau,\tau')) \le \max(\mathrm{err}(\tau), \mathrm{err}(\tau'))$.

Proof. The real version of τ, with the constants dropped, is positively homogeneous of degree one. The recursive Lipschitz bounds for the operations give the stated error rules. The inputs differ from their homogeneous values by $|m - \alpha r| \le \frac12$ and $|k_p - \kappa p| \le 1 - \frac1q$. The quotient and remainder terms are unchanged, since r and p are exact inputs. Induct on τ. □

The conditions of Theorem 4.5 are linear inequalities in r, p, m, k, strict or not. They homogenise to intervals of θ. At an endpoint the error of such an inequality is a combination of $m - \alpha r$ and $k_p - \kappa p$. With the rounding $m = \lceil \alpha r - \frac12 \rceil$ two endpoints are *exact*. These are $\theta = 2\alpha$, from $2m < p$, and $\theta = (1+2\alpha)/(1+\kappa)$, from $\bar N \le p + k$. At an exact endpoint the condition holds for every prime on the correct side. The other endpoints are exact, or hold outside a strip of primes of bounded width around the endpoint. On that strip another bound must be used.

4.6. **Tables for Λ and the arithmetic estimate.** For the root class j let $\mathcal{P}_j(\theta)$ be the minimum of the homogenised bounds $\hat\tau(\theta)$ over the bounds τ of Theorem 4.5 valid at θ. A *table for* Λ, for the function F_s and the geometry of parameter c, consists of the parameters (q, α, K) and the following data for every root class j:

- breakpoints $0 < \theta_0 < \theta_1 < \cdots < \theta_{\mathrm{end}}$;
- on each interval $(\theta_{i-1}, \theta_i]$, an affine function ℓ_i, a constant $\mathrm{err}_i \ge 0$ and a local bound τ_i of Theorem 4.5, with a second bound at each inexact endpoint of the range of τ_i;
- a constant ℓ_0 for the initial interval $(0, \theta_0]$;

together with a rational number Λ_{fin}. We call the table *admissible* if it satisfies the following conditions.

(C1) For each shift of the table, every unit class modulo the modulus q^* of the prime number theorem used is assigned a root class with a table.

(C2) On each piece ℓ_i is nonnegative, and the piece lies in the validity interval of its bound. Moreover $\hat\tau$ lies below ℓ_i at both ends and at both one-sided limits of each breakpoint inside the piece.

(C3) At each inexact endpoint the second bound is valid on both sides and lies below ℓ_i there. The constant err_i exceeds the error constants of Lemma 4.13, the numbers $\epsilon_\tau K$, the strip widths times the slopes, and the trivial bound at the scales below an explicit threshold.

(C4) $\ell_0 \geq 2 + 2\alpha$ and $0 < \theta_0 \leq 1$. Also $\theta_{\mathrm{end}} \geq \max(q(1+2\alpha), p_0/2)$ and $\kappa_j \theta_{\mathrm{end}} \geq 1 + 2\alpha$.

(C5) $\frac{1}{\varphi(q)} \sum_j (\ell_0 \theta_0 + \sum_i \int_{\theta_{i-1}}^{\theta_i} \ell_i(\theta)\, d\theta) = \Lambda_{\mathrm{fin}}$.

Each condition is a finite set of comparisons of rational numbers.

Theorem 4.14 (denominators)**.** *Let $0 < c \leq 1$, and let F_s be the function with shift s of denominator q. Suppose that an admissible table for Λ of depth K has s among its shifts. Then for every even r and every member with $|I| \leq K$,*

$$\log \operatorname{den} P_{r,I} \leq (\Lambda_{\mathrm{fin}} + o(1))\, r^2.$$

Proof. By definition $\operatorname{den} P_{r,I} = \prod_p p^{e_p(r,I)}$. Fix a piece $(\theta_{i-1}, \theta_i]$ of the root class j. Let p be a prime in a unit class assigned to j, with $\theta_{i-1} r < p \leq \theta_i r$. Let r be beyond the threshold of (C3). Then the conditions of the bound τ_i hold, either exactly or outside a strip where the second bound applies. Lemma 4.13 with (C2) and (C3) gives $e_p(r, I) \leq r\, \ell_i(p/r) + \mathrm{err}_i$. Below the threshold the trivial bound is at most err_i.

For $p \leq \theta_0 r$ the bound (T1) gives $e_p \leq \ell_0 r + O(1)$. The primes with $p^2 \leq q(r + 2m)$, the prime powers and the finitely many primes below p_0 contribute $O(r^{3/2} \log r)$. For $p > \theta_{\mathrm{end}} r$, (C4) gives $\bar{N} \leq p$ and $k_p \geq \bar{N}$, and (T5) gives $e_p = 0$.

Fix a unit class modulo q^*. By the prime number theorem in arithmetic progressions,

$$\sum_{\substack{\theta_{i-1} r < p \leq \theta_i r \\ p \text{ in the class}}} r\, \ell_i(p/r) \log p = \frac{r^2}{\varphi(q^*)} \int_{\theta_{i-1}}^{\theta_i} \ell_i(\theta)\, d\theta + o(r^2).$$

The class j receives $\varphi(q^*)/\varphi(q)$ unit classes modulo q^*, because the map that sends a unit to qs times its inverse permutes the units modulo q. Summing and using (C5) gives the theorem. The constants err_i contribute $O(r)$. □

The tables of this paper are generated from the functions $\mathcal{P}_j$ themselves. The pieces are the maximal intervals on which $\mathcal{P}_j$ is affine and attained by one bound. So $\Lambda_{\mathrm{fin}} = \frac{1}{\varphi(q)} \sum_j \int \mathcal{P}_j$ up to the initial interval. Every bound of Theorem 4.5 is available to the tables: (T1)–(T5), (LP1), (LP2), (R), (S), the windows (W) and the bounds (U_n). The windows with $n = 0$ are used for $m < p$, between consecutive cuts.

On $(0, \alpha]$, that is for $p \leq m$, the only other bounds are (T1), (T1′) and (U_n). With (T1′) alone the profile there is the sawtooth $2 + 2\alpha - \theta \|\alpha/\theta\|$. Its integral is

$$\int_0^\alpha \Big(2 + 2\alpha - \theta \Big\| \frac{\alpha}{\theta} \Big\| \Big) d\theta = 2\alpha + \frac{\alpha^2}{2} + 2\alpha^2 \log 2,$$

since $\int_1^\infty \|y\| y^{-3}\,dy = \frac{3}{2} - 2\log 2$. The windows with $n = \lfloor m/p \rfloor \geq 1$ lower this profile on the strips $\alpha/(n+1) < \theta \leq \alpha/n$. The tables use them for $n \leq 8$, again with each window running between two consecutive cuts. For $q = 3, 4, 5, 7$ the tables also use (U_0) on $(\alpha, 2\alpha]$ and (U_n) on the strips with $1 \leq n \leq 24$. The table for $q = 2$ uses only the other bounds. Beyond the strips used, a table resolves the sawtooth on each interval $(\alpha/(n+1), \alpha/n]$ with $n < n_0$. It ends with a constant piece on $(0, \alpha/n_0]$, which costs a rational multiple of α^2, at most $\alpha^2/(4n_0^2)$. The number Λ_{fin} is then rational, and (C5) is an identity between rational numbers. The bounds depend on the shift s only through the root class, and their profiles only through (q, j, α). So one table serves all shifts with denominator q.

On small θ the pieces of a table repeat with a period in $1/\theta$. Write $\alpha = a/b$ in lowest terms, and call $T_\nu(r, p, m, k) = (r + \nu bp,\ p,\ m + \nu ap,\ k)$ the *translation of order* $\nu \geq 0$. It keeps p, k and $m - \alpha r$, and hence the rounding of m. A list of local bounds is *closed under translation*, with constant K_0, if for every bound τ of the list and every ν there is a bound τ' of the list with $\tau'(T_\nu y) = \tau(y) + \nu K_0 bp$ for all $y = (r, p, m, k)$, whose conditions hold at $T_\nu y$ whenever those of τ hold at y.

Theorem 4.15 (periodic tables)**.** *Let a list of local bounds be closed under translation with constant K_0. Suppose that for every $r \geq R$ and every prime p with $\theta_- r < p \leq \theta_+ r$ some bound τ of the list has its conditions satisfied at $y = (r, p, m(r), k_p)$ and $\tau(y) \leq Ar + Bp + C$. Let $1/\theta_{-,\nu} = 1/\theta_- + \nu b$ and $1/\theta_{+,\nu} = 1/\theta_+ + \nu b$. Then for every member at a scale $r \geq R\theta_+/\theta_{+,\nu}$ and every prime p with $\theta_{-,\nu} r < p \leq \theta_{+,\nu} r$,*

$$e_p(r, I) \leq Ar + \bigl(B + (K_0 - A)\nu b\bigr)p + C.$$

Proof. Put $r' = r - \nu bp$ and $m' = m(r) - \nu ap$. Then $r'/p = r/p - \nu b$, so $\theta_- < p/r' \leq \theta_+$, and $r' \geq r\,\theta_{+,\nu}/\theta_+ \geq R$. Also $m' = m(r')$, since $m' - \alpha r' = m(r) - \alpha r$. So the hypothesis gives a bound τ whose conditions hold at $y = (r', p, m', k_p)$, with $\tau(y) \leq Ar' + Bp + C$. The conditions of its translate τ' hold at $T_\nu y = (r, p, m(r), k_p)$. Since τ' is a local bound, $e_p(r, I) \leq \tau(y) + \nu K_0 bp \leq A(r - \nu bp) + Bp + C + \nu K_0 bp$. □

So a table for Λ on a range where its bounds are closed under translation needs to be checked on one period of $1/\theta$ only. The pieces of the other periods are its translates, and their integrals are again rational numbers. The finitely many scales below the thresholds take the trivial bound, as in (C3).

5. Nonvanishing

The deduction needs, for every rational x_0 and every large scale, one member of the family that does not vanish at x_0. We prove this in characteristic zero for every frame of §3.2. The number of deleted common nodes is at most half the number of poles of F in a period, counted with order. The proof has three steps. The kernel of the matrix is a space of solutions of a Hermite interpolation problem. A discrete Wronskian bounds its dimension.

The values of the solutions at the common nodes then tell us which nodes to delete.

Throughout this section F is a frame with d poles per period, counted with order, and $\mathcal{G}(r,m)$ is a geometry with $r \geq 1$. Shifts of order 0 play no role here, since removing them from S does not change F. By the balance $d \geq 2$. Fix $x_0 \in \mathbb{Q}$ and put $\vartheta_t = x_0 - e_t$. Then $\Phi_{\mathcal{G}}(x_0)$ has the entries $(f_{t'} - f_t)/(t' - t)$ for $t \neq t'$ and ϑ_t on the diagonal of the common block. Put $n = r + m$, and let $Z = [0, \bar{N})$ be the set of all nodes and $C = [0, r)$ the set of common nodes. So $|R| = |Q| = n$ and $|Z| + |C| = 2n$.

5.1. **The kernel as an interpolation space.** Let $\mathcal{V}$ be the space of pairs (u, w) of polynomials over $\mathbb{Q}$ of degree less than n that satisfy the *value conditions* and the *jet conditions*

$$\mathsf{w}(t) = f_t\, \mathsf{u}(t) \quad (t \in Z), \qquad \mathsf{w}'(t) = f_t\, \mathsf{u}'(t) + \vartheta_t\, \mathsf{u}(t) \quad (t \in C).$$

For a finite set T of integers write $\omega_T(z) = \prod_{t \in T}(z - t)$.

Lemma 5.1 (kernel)**.** *The map that sends $(\mathsf{u}, \mathsf{w}) \in \mathcal{V}$ to the vector $\mathbf{y}$ with $y_t = \mathsf{u}(t)/\omega_Q'(t)$ for $t \in Q$ is an isomorphism of $\mathcal{V}$ onto the right kernel of $\Phi_{\mathcal{G}}(x_0)$. The same holds for the left kernel, with R in place of Q. In any pair of $\mathcal{V}$, $\mathsf{u} = 0$ implies $\mathsf{w} = 0$.*

Proof. Let $\mathbf{y}$ be a vector indexed by Q, and put

$$\mathsf{u}(z) = \sum_{t' \in Q} y_{t'} \frac{\omega_Q(z)}{z - t'}, \qquad \mathsf{w}(z) = \sum_{t' \in Q} y_{t'} f_{t'} \frac{\omega_Q(z)}{z - t'}.$$

By Lagrange interpolation at the n nodes of Q, this is a bijection between vectors $\mathbf{y}$ and pairs of degree less than n that satisfy the value conditions on Q, with inverse $y_t = \mathsf{u}(t)/\omega_Q'(t)$. Let $t \in X'$ be a border row. Since $\mathsf{u}(t)/\omega_Q(t) = \sum_{t'} y_{t'}/(t - t')$, and likewise for w,

$$\big(\Phi_{\mathcal{G}}(x_0)\mathbf{y}\big)_t = \sum_{t' \in Q} y_{t'} \frac{f_{t'} - f_t}{t' - t} = -\frac{\mathsf{w}(t) - f_t \mathsf{u}(t)}{\omega_Q(t)}.$$

Let $t \in C$ be a common row, and put $\mathsf{v} = \mathsf{w} - f_t \mathsf{u}$. The term $t' = t$ of v vanishes. Each other term vanishes at t with derivative $y_{t'}(f_{t'} - f_t)\, \omega_Q'(t)/(t - t')$. Since $\mathsf{u}(t) = y_t\, \omega_Q'(t)$,

$$\mathsf{w}'(t) - f_t \mathsf{u}'(t) - \vartheta_t \mathsf{u}(t) = \mathsf{v}'(t) - \vartheta_t \mathsf{u}(t) = -\omega_Q'(t)\big(\Phi_{\mathcal{G}}(x_0)\mathbf{y}\big)_t.$$

So $\mathbf{y}$ lies in the right kernel exactly when the value conditions on X' and the jet conditions on C hold. As $Z = Q \sqcup X'$, these are the conditions defining $\mathcal{V}$. The matrix M is symmetric. So the transpose of $\Phi_{\mathcal{G}}(x_0)$ is the matrix of the same nodes with the roles of R and Q exchanged. This exchange leaves Z and C unchanged, and with them $\mathcal{V}$. This gives the left kernel. Finally, if $\mathsf{u} = 0$ then w vanishes on Z. As $|Z| \geq n > \deg \mathsf{w}$, this gives $\mathsf{w} = 0$. $\square$

5.2. **The corank.** Put

$$\mathsf{D}(z) = \prod_{s \in S} (z+s)^{\nu_s}, \qquad \mathsf{A} = \mathsf{D}\,\delta,$$

where ν_s is the order of s. Then D has degree d, and A is a polynomial of degree at most $d-2$, since $\delta(z) = O(|z|^{-2})$ by the balance $\sum_s w_s = 0$. Differentiating $\mathsf{D}\delta = \mathsf{A}$ gives $\mathsf{D}'\delta + \mathsf{D}\delta' = \mathsf{A}'$. For integers $k \geq 0$ we have $\mathsf{D}(k) \neq 0$, $f_{k+1} - f_k = \delta(k)$ and $\vartheta_{k+1} - \vartheta_k = \delta'(k)$.

Lemma 5.2 (cross product)**.** *For two pairs* $(\mathsf{u}_1, \mathsf{w}_1)$ *and* $(\mathsf{u}_2, \mathsf{w}_2)$ *of* $\mathcal{V}$ *one has* $\mathsf{w}_1\mathsf{u}_2 = \mathsf{w}_2\mathsf{u}_1$.

Proof. The polynomial $\Pi = \mathsf{w}_1\mathsf{u}_2 - \mathsf{w}_2\mathsf{u}_1$ has degree at most $2n-2$. The value conditions give $\Pi(t) = 0$ for $t \in Z$. At $t \in C$ the value and jet conditions give

$$\Pi'(t) = (f_t\mathsf{u}_1' + \vartheta_t\mathsf{u}_1)\mathsf{u}_2 + f_t\mathsf{u}_1\mathsf{u}_2' - (f_t\mathsf{u}_2' + \vartheta_t\mathsf{u}_2)\mathsf{u}_1 - f_t\mathsf{u}_2\mathsf{u}_1' = 0,$$

with all polynomials evaluated at t. So Π has at least $|Z| + |C| = 2n$ zeros, counted with multiplicity, and $\Pi = 0$. □

The next lemma is the heart of the argument. It is used twice, with all the conditions and with the value conditions only.

Lemma 5.3 (no short pairs)**.** *Let* $N_0 \geq 1$ *and* $0 \leq c_0 \leq N_0$ *be integers, and let* u, w *be polynomials of degree at most* ℓ *that satisfy the value conditions on* $[0, N_0)$ *and the jet conditions on* $[0, c_0)$*. If*

$$2\ell + d - 2 < (N_0 - 1) + \max(c_0 - 1, 0),$$

then $\mathsf{u} = 0$.

Proof. Suppose that $\mathsf{u} \neq 0$. Put $\mathsf{W}(z) = \mathsf{w}(z+1)\mathsf{u}(z) - \mathsf{w}(z)\mathsf{u}(z+1)$ and

$$\mathsf{E}(z) = \mathsf{D}(z)\,\mathsf{W}(z) - \mathsf{A}(z)\,\mathsf{u}(z)\,\mathsf{u}(z+1).$$

By Taylor's formula $\mathsf{W} = \sum_{j \geq 1} (\mathsf{w}^{(j)}\mathsf{u} - \mathsf{w}\mathsf{u}^{(j)})/j!$. The term $j = 1$ has degree at most $2\ell - 2$, because the coefficients of $z^{2\ell-1}$ in $\mathsf{w}'\mathsf{u}$ and in $\mathsf{w}\mathsf{u}'$ are both ℓ times the product of the leading coefficients. The terms with $j \geq 2$ have degree at most $2\ell - 2$ too. Hence $\deg \mathsf{E} \leq 2\ell + d - 2$.

Now we count zeros. Let $k \geq 0$ with $k + 1 < N_0$, and put $U(z) = \mathsf{u}(z)\mathsf{u}(z+1)$. The value conditions at k and $k+1$ give $\mathsf{W}(k) = \delta(k)U(k)$, so $\mathsf{E}(k) = (\mathsf{D}\delta - \mathsf{A})(k)\,U(k) = 0$. Let moreover $k + 1 < c_0$. The jet conditions at k and $k+1$ give $\mathsf{W}'(k) = \delta(k)U'(k) + \delta'(k)U(k)$. Hence

$$\mathsf{E}'(k) = (\mathsf{D}'\delta + \mathsf{D}\delta' - \mathsf{A}')(k)\,U(k) + (\mathsf{D}\delta - \mathsf{A})(k)\,U'(k) = 0.$$

So E has at least $(N_0 - 1) + \max(c_0 - 1, 0)$ zeros, counted with multiplicity. This exceeds its degree, and $\mathsf{E} = 0$.

Let g be the greatest common divisor of u and w, and write $\mathsf{u} = \mathsf{g}\mathsf{u}_1$ and $\mathsf{w} = \mathsf{g}\mathsf{w}_1$. Then $\mathsf{E} = \mathsf{g}(z)\mathsf{g}(z+1)\mathsf{E}_1$, where E_1 is the same expression for $(\mathsf{u}_1, \mathsf{w}_1)$. So $\mathsf{E}_1 = 0$, with $\mathsf{u}_1 \neq 0$ and u_1, w_1 coprime. Take a shift s_0 of positive order and put $a = -s_0$. Then $\mathsf{D}(a) = 0$, and $\mathsf{A}(a)$ is w_{s_0}, $-w_{s_0}'$ or

$2w''_{s_0}$, according to the order, times $\prod_{s\neq s_0}(s-s_0)^{\nu_s}$. So $\mathsf{A}(a)\neq 0$. The points $a+j$ with $j\in\mathbb{Z}\setminus\{0\}$ are not roots of D, since distinct shifts in $(0,1]$ differ by less than 1. Evaluating $\mathsf{E}_1=0$ at a gives $\mathsf{A}(a)\mathsf{u}_1(a)\mathsf{u}_1(a+1)=0$.

Suppose first that $\mathsf{u}_1(a+j)=0$ for some $j\geq 1$. Evaluating $\mathsf{E}_1=0$ at $a+j$ gives $\mathsf{D}(a+j)\mathsf{w}_1(a+j)\mathsf{u}_1(a+j+1)=0$. Here $\mathsf{w}_1(a+j)\neq 0$ by coprimality, so $\mathsf{u}_1(a+j+1)=0$. If $\mathsf{u}_1(a+1)=0$, induction gives infinitely many roots of u_1. Suppose next that $\mathsf{u}_1(a-j)=0$ for some $j\geq 0$. Evaluating $\mathsf{E}_1=0$ at $a-j-1$ gives $\mathsf{D}(a-j-1)\mathsf{w}_1(a-j)\mathsf{u}_1(a-j-1)=0$, so $\mathsf{u}_1(a-j-1)=0$. If $\mathsf{u}_1(a)=0$, induction again gives infinitely many roots. Both cases contradict $\mathsf{u}_1\neq 0$. □

Proposition 5.4 (corank)**.** $\dim\mathcal{V}\leq\lfloor d/2\rfloor$. *Hence* $\operatorname{corank}\Phi_{\mathcal{G}}(x_0)\leq\lfloor d/2\rfloor$.

Proof. Put $\varsigma=\lfloor d/2\rfloor$. If $\varsigma\geq n$ there is nothing to prove, since the right kernel has dimension at most n. Let $\varsigma<n$. By Lemma 5.1 every nonzero pair has $\mathsf{u}\neq 0$. By Lemma 5.2 the difference $\deg\mathsf{w}-\deg\mathsf{u}$, with $\deg 0=-\infty$, is the same for all nonzero pairs. So one of the two coordinates has the larger degree in every pair. Map each pair to the coefficients of $z^{n-\varsigma},\dots,z^{n-1}$ in that coordinate. This is a linear map to $\mathbb{Q}^{\varsigma}$. A pair in its kernel has both coordinates of degree at most $n-\varsigma-1$. Apply Lemma 5.3 with $N_0=\bar{N}$, $c_0=r$ and $\ell=n-\varsigma-1$. Its condition reads $2n-2\varsigma+d-4<2n-2$, that is $d-2<2\varsigma$, which holds since $2\lfloor d/2\rfloor\geq d-1$. So $\mathsf{u}=0$, and the pair is zero. The map is injective, and $\dim\mathcal{V}\leq\varsigma$. The second statement follows from Lemma 5.1. □

5.3. **The choice of the deleted nodes.** It remains to see the kernel on the common nodes. At a common node t the coordinates of a right and of a left kernel vector are $\mathsf{u}(t)/\omega'_Q(t)$ and $\mathsf{u}(t)/\omega'_R(t)$, by Lemma 5.1.

Lemma 5.5 (visibility)**.** *Let $r\geq d-2$. If a pair of $\mathcal{V}$ has $\mathsf{u}(t)=0$ for every $t\in C$, then it is zero.*

Proof. The value conditions give $\mathsf{w}(t)=0$ for $t\in C$. So ω_C divides u and w. Write $\mathsf{u}=\omega_C\mathsf{u}_0$ and $\mathsf{w}=\omega_C\mathsf{w}_0$, with degrees less than $n-r=m$. If $m=0$ then $\mathsf{u}_0=0$. Let $m\geq 1$. For $t\in Z\setminus C$ we have $\omega_C(t)\neq 0$, so $\mathsf{w}_0(t)=f_t\mathsf{u}_0(t)$. For $t\in C$ we have $\mathsf{u}(t)=0$, $\mathsf{u}'(t)=\omega'_C(t)\mathsf{u}_0(t)$ and $\mathsf{w}'(t)=\omega'_C(t)\mathsf{w}_0(t)$, with $\omega'_C(t)\neq 0$. So the jet condition at t reads $\mathsf{w}_0(t)=f_t\mathsf{u}_0(t)$. The pair $(\mathsf{u}_0,\mathsf{w}_0)$ therefore satisfies the value conditions on all of Z. Apply Lemma 5.3 with $N_0=\bar{N}=r+2m$, $c_0=0$ and $\ell=m-1$. Its condition reads $2m+d-4<r+2m-1$, that is $r\geq d-2$. So $\mathsf{u}_0=0$. Then $\mathsf{u}=0$, and the pair is zero by Lemma 5.1. □

Theorem 5.6 (nonvanishing for frames)**.** *Let F be a frame, and let $r\geq\max(1,d-2)$ and $m\geq 0$. For every $x_0\in\mathbb{Q}$ some member $P_{r,I}$ with $|I|\leq\lfloor d/2\rfloor$ satisfies $P_{r,I}(x_0)\neq 0$.*

Proof. Let k be the corank of $\Phi_{\mathcal{G}}(x_0)$. Then $k=\dim\mathcal{V}\leq\lfloor d/2\rfloor$ by Proposition 5.4. The map $(\mathsf{u},\mathsf{w})\mapsto(\mathsf{u}(t))_{t\in C}$ is injective on $\mathcal{V}$, by Lemma 5.5. Its

image is a subspace of $\mathbb{Q}^C$ of dimension k. So there is a set $I \subseteq C$ with $|I| = k$ such that the coordinates in I already determine the pair. By the remark before Lemma 5.5, no nonzero right kernel vector and no nonzero left kernel vector vanishes on I.

Write $\Phi = \Phi_{\mathcal{G}}(x_0)$, of size n and rank $n-k$. A linear relation among the columns $Q \setminus I$ is a right kernel vector that vanishes on I. So these $n-k$ columns are independent, and they span the column space. In the same way the rows $R \setminus I$ span the row space. Let $\mathbf{z}$ be a vector on $Q \setminus I$ with $\Phi[R \setminus I, Q \setminus I]\,\mathbf{z} = 0$. Every row of Φ is a combination of the rows $R \setminus I$, so $\Phi[R, Q \setminus I]\,\mathbf{z} = 0$, and $\mathbf{z} = 0$ by the independence of the columns. Hence $P_{r,I}(x_0) = \det \Phi[R \setminus I, Q \setminus I] \neq 0$. □

For the frames of the first plane $d \leq 3$, and one deleted node suffices. For the frames $F^{(2)}_{\gamma_2,\gamma_1}$ with $\gamma_1\gamma_2 \neq 0$ we have $\mathsf{D} = (z+1)^2(z+\frac{1}{2})$ and $\mathsf{A} = -(\gamma_2 + \frac{\gamma_1}{2})z - \frac{\gamma_2+\gamma_1}{2}$, with $\mathsf{A}(-1) = \frac{\gamma_2}{2} \neq 0$. For $\gamma_2 = 0$ we have $\mathsf{D} = (z+1)(z+\frac{1}{2})$ and $\mathsf{A} = -\frac{\gamma_1}{2}$. For $\gamma_1 = 0$ we have $\mathsf{D} = (z+1)^2$ and $\mathsf{A} = -\gamma_2$. So again one deleted node suffices. For the frames $F^{(3)}_{\gamma_2,\gamma_3}$ of the third plane we have $\mathsf{D} = (z+1)^3$ and $\mathsf{A} = -\gamma_2(z+1) + 2\gamma_3$, with $\mathsf{A}(-1) = 2\gamma_3 \neq 0$ and $d = 3$. One deleted node suffices here too. The formal proof uses the same interpolation argument for every frame, with poles of any finite order.

5.4. **The functions of (2.1).**

Theorem 5.7 (selection)**.** *Let F be a function of (2.1), and consider the alternating geometry of parameter c. There is K, depending only on $|\mathcal{T}|$, with the following property. For every large even r and every $x_0 \in \mathbb{Q}$ some member $P_{r,I}$ with $|I| \leq K$ satisfies $P_{r,I}(x_0) \neq 0$.*

Proof. The function F_s is ω times the frame on $\{s, 1\}$ with $w_1 = 1$ and $w_s = -1$, so $d = 2$. The function F_χ is $-\frac{1}{6}F^{+}_{(2,-3,1)}$, with $d = 3$. Multiplying a frame by $\omega \neq 0$ multiplies M by ω. So the matrix $\Phi_{\mathcal{G},I}(x_0)$ of ωF is ω times that of F at x_0/ω. Theorem 5.6 gives the claim with $K = \lfloor |\mathcal{T}|/2 \rfloor = 1$ for $r \geq 2$. □

The formal proof of Theorem 5.7 uses a coarser count of the same kind, with $K \leq 64$. The deduction needs only that the depth is bounded.

6. Analytic decay

In this section we bound $|P_{r,I}(x)|$ for x near ξ. The argument is that of OpenAI [38] for square determinants, with three changes. The determinant has two families of nodes, one for the rows and one for the columns. The comparison measure is a density on finitely many segments rather than an explicit equilibrium measure. The inequalities that remain are verified on a finite cover by cells. The result is one theorem for every geometry parameter $c \leq 1$ and every function of (2.1).

6.1. **A contour representation.** Hermite's formula gives $\mathsf{L}_F[t,t'] = \frac{1}{2\pi i}\oint F(z)\upsilon_t(z)\upsilon_{t'}(z)\,dz$ with $\upsilon_t(z) = 1/(z-t)$. The contour encloses the nodes and no pole of F. Write $F = \sum_{\varsigma\in\mathcal{T}} w_\varsigma \psi(z+\varsigma)$ as in §3. The reflection formula $\psi(w) = \psi(1-w) - \pi\cot\pi w$ splits $F = \tilde{F} + E$. Here $\tilde{F}$ is analytic in the left half-plane and $O(1/|z|)$ there, and

$$E(z) = -\pi \sum_{\varsigma\in\mathcal{T}} w_\varsigma \cot\pi(z+\varsigma).$$

The function E is 1-periodic. Since the weights sum to zero, $E(z) = O(e^{-2\pi|\operatorname{Im} z|})$ for $|\operatorname{Im} z| \ge 1$. For F_s one has $E(z) = \pi[\cot\pi(z+s) - \cot\pi z]$.

Let Γ be a polygonal path, symmetric under complex conjugation. It crosses the real axis at a point $x^* < 0$ that is not a pole of F, runs through the left half-plane and ends in two rays. The poles of F are the points $-(i+\varsigma)$ with $i \ge 0$ and $\varsigma \in \mathcal{T}$. Moving the contour to Γ picks up the residues of F at the poles in $(x^*, 0)$. The part of the integrand with $\tilde{F}$ integrates to 0 over Γ. Hence, for all integers $t, t' \ge 0$,

$$\mathsf{L}_F[t,t'] = \int \upsilon_t \upsilon_{t'}\, d\Omega. \tag{6.1}$$

Here Ω is the complex measure with the mass $-\operatorname{res}_y(F)$ at each pole $y \in (x^*, 0)$ of F and the density $(2\pi i)^{-1}E(z)\,dz$ on Γ. This is a Gram representation with respect to a measure with finitely many atoms and an exponentially small density on Γ. After scaling by r, the atoms lie on a segment $[-r_0, 0]$ with density $|\mathcal{T}|$. The separation condition on x^* is a finite check on the fractional parts of $\varsigma - x^*$ for $\varsigma \in \mathcal{T}$.

6.2. **Two-family Andréief bound.** For a set $C'' \subseteq C$ of common nodes put $\mathcal{M}_{C''} = \omega\,\mathsf{L}_F[C'' \sqcup X', C'' \sqcup Y']$. Consider a member with common nodes $C' = C \setminus I$. Put $R' = C' \sqcup X'$, $Q' = C' \sqcup Y'$ and $n_* = |C'| + m$. By multilinearity in the diagonal,

$$P_{r,I}(\xi + h) = \sum_{C''\subseteq C'} h^{|C'|-|C''|} \det \mathcal{M}_{C''}. \tag{6.2}$$

Andréief's identity [20] for the two families $(\upsilon_t)_{t\in R'}$ and $(\upsilon_{t'})_{t'\in Q'}$ gives

$$\det \mathsf{L}_F[R', Q'] = \frac{1}{n_*!}\int \det[\upsilon_t(z_i)]\,\det[\upsilon_{t'}(z_i)] \prod_{i=1}^{n_*} d\Omega(z_i).$$

Cauchy's determinant formula evaluates both factors. We take absolute values. The logarithmic potential theory behind what follows is in [45]. We obtain

$$|\det \mathsf{L}_F[R', Q']| \le \frac{1}{n_*!}\int \prod_{i<i'} |z_i - z_{i'}|^2 \prod_{\mathcal{F}\in\{R',Q'\}} \frac{\prod_{t<t'\in\mathcal{F}} |t'-t|}{\prod_{i,\,t\in\mathcal{F}} |z_i - t|} \prod_i d|\Omega|(z_i).$$

The integrand is the exponential of minus a discrete logarithmic energy. The free points z_i repel each other with weight 1. Each node family repels itself with weight $\frac{1}{2}$ and attracts the free points with weight 1. The density $|E| \approx e^{-2\pi|\operatorname{Im} z|}$ acts as an external field. The rows and columns differ only in the interleaved border. So the two node families have the same macroscopic density. After scaling by r, both are

$$\nu_c = \mathrm{Leb}|_{[0,1]} + \tfrac{1}{2}\mathrm{Leb}|_{[1,1+c]},$$

of mass $|\nu_c| = 1 + c/2$.

6.3. **The energy certificate.** For a regularisation radius $\epsilon > 0$ let $k_\epsilon(z,w) = -\frac{1}{2}\log(|z-w|^2+\epsilon^2)$. Then $k_\epsilon(z,w) \le -\log|z-w|$.

Lemma 6.1 (conditional positive definiteness)**.** *Let $z_i \in \mathbb{C}$ be finitely many points and y_i real numbers with $\sum_i y_i = 0$. Then $\sum_{i,j} y_i y_j k_\epsilon(z_i,z_j) \ge 0$.*

Proof. We use Frullani's integral $\log T = \int_0^\infty (e^{-u} - e^{-Tu})\,du/u$ for $T > 0$. Together with $\sum_i y_i = 0$ it gives $\sum_{i,j} y_i y_j(-\log(|z_i - z_j|^2+\epsilon^2)) = \int_0^\infty \frac{e^{-u\epsilon^2}}{u}\sum_{i,j} y_i y_j e^{-u|z_i-z_j|^2}du$. Gaussian kernels are positive definite. Indeed, expanding $e^{2u\langle z,z'\rangle}$ in powers and using the binomial theorem writes the inner sum as a convergent sum of squares. □

A *trial measure* is a probability measure σ, symmetric under conjugation, carried by finitely many segments $\mathcal{S}_i \subset \Gamma \cup [-r_0,0] \cup [0,1]$ with rational endpoints. On each segment it has a density $\sigma_i(y)\,dy$ in the parameter $y \in [0,1]$ of the segment. Here σ_i is a polynomial with rational coefficients, and $0 \le \sigma_i \le \bar\sigma_i$ for a rational $\bar\sigma_i$. We write $\Theta_\sigma = \sum_i \bar\sigma_i/|\mathcal{S}_i|$ for the sum of the bounds of its densities with respect to arclength.

We compare the free points and the deleted nodes with $|\nu_c|\sigma$ through Lemma 6.1. This bounds the free–free energy from below by a sum of one-point terms and the constant $|\nu_c|^2\mathcal{I}(\sigma)$. Riemann-sum comparisons replace the node sums by the potential U^{ν_c} and the node self-energy by $\mathcal{I}(\nu_c)$, up to $O(r\log r)$. With the external field one arrives at the function

$$\Upsilon(z) = |\nu_c|\,U^\sigma(z) - U^{\nu_c}(z) + (1-\varepsilon_0)\pi|\operatorname{Im} z|$$

for a small fixed $\varepsilon_0 > 0$. Here $U^\sigma(z) = \int \log\frac{1}{|z-w|}\,d\sigma(w)$ is the logarithmic potential of σ, and U^{ν_c} is that of ν_c. A *certificate* consists of σ and rational numbers Υ_Γ, Υ_0, J, $\mathcal{I}_\sigma$, $\mathcal{I}_\nu$ with the following properties.

(a) $\Upsilon \ge \Upsilon_\Gamma$ on Γ.
(b) $\Upsilon \ge \Upsilon_0$ on $[0,1]$, the scaled position of the deleted common nodes.
(c) $J \le \int_{-r_0}^0 \min(\Upsilon(x) - \Upsilon_\Gamma, 0)\,dx$.
(d) $\mathcal{I}(\sigma) \le \mathcal{I}_\sigma$ and $\mathcal{I}(\nu_c) \ge \mathcal{I}_\nu$.

Theorem 6.2 (decay on a neighbourhood)**.** *Let $0 < c \le 1$. Let F be a function of (2.1) whose poles are separated from x^*. Let $(\sigma, \Upsilon_\Gamma, \Upsilon_0, J, \mathcal{I}_\sigma, \mathcal{I}_\nu)$*

be a certificate for Γ *and* ν_c. *Let* $\epsilon > 0$, $\delta_\epsilon \geq \pi\epsilon\,\Theta_\sigma$, $\delta_{\mathrm{m}} > 0$ *and* $\delta_J > 0$ *be rational, and put*

$$\delta = 3|\nu_c|^2\delta_\epsilon + 4|\nu_c|^2\delta_{\mathrm{m}} + 2|\mathcal{T}|\,\delta_J.$$

Let D *and* ρ *be rational numbers with*

$$D \leq 2(|\nu_c|\Upsilon_\Gamma + |\mathcal{T}|J) - |\nu_c|^2\mathcal{I}_\sigma + \mathcal{I}_\nu - \delta, \qquad \rho \geq 2(\Upsilon_\Gamma - \Upsilon_0).$$

Then for every depth K, *every member of the alternating geometry of parameter* c *and every real* h *with* $|h| \leq e^{-\rho r}$,

$$|P_{r,I}(\xi + h)| \leq \exp\bigl(-(D - o(1))r^2\bigr).$$

The loss δ comes from replacing σ by finitely many atoms in the comparison with the free points. The regularisation radius ϵ of Lemma 6.1 and the mesh of the atoms lower Υ_Γ by $|\nu_c|(\delta_\epsilon + \delta_{\mathrm{m}})$. They raise $\mathcal{I}_\sigma$ by $\delta_\epsilon + 2\delta_{\mathrm{m}}$. Darboux sums for the integral in (c) lower J by δ_J. Inserted in the expression for D, these changes give δ. The tolerances δ_{m} and δ_J may be any positive rational numbers, since the atoms can be taken as fine as needed. In the certificates of this paper $\epsilon = 10^{-12}$ and $\delta < 10^{-9}$.

The node energy $\mathcal{I}(\nu_c)$ is a finite rational combination of the values $\mathcal{H}(y_k)$, where $\mathcal{H}(y) = \frac{1}{2}y^2 \log|y| - \frac{3}{4}y^2$ and the y_k are the rational differences of the endpoints of the two pieces of ν_c. The lower bound $\mathcal{I}_\nu$ is checked with rational enclosures of the logarithms. These come from the series of artanh and a reduction to $[1, 2)$. The energy $\mathcal{I}(\sigma)$ is a finite sum of pair energies of polynomial densities on segments. Each is a closed form of the type polynomial times logarithm plus arctangent, and is bounded in the same way. The conditions (a)–(c) are statements about Υ on sets of positive measure. They are proved on a cover by cells.

6.4. **The cover by cells.** Each of the conditions (a)–(c) is checked on a finite cover by cells with exact rational arithmetic. On a cell, the potential U^σ is split into the contribution of the segments of σ near the cell and the rest. A near segment is bounded by a Taylor polynomial in the cell variable with an explicit remainder. For a cell that meets the line of a segment beyond its end, the same bound is used for the segment with its orientation reversed. At a corner of Γ the two incident segments are treated together. Far segments are bounded by a multipole expansion with an explicit tail. The potential U^{ν_c} is an explicit elementary function, and it is bounded above by monotonicity on the cell.

Some cells end at a breakpoint of σ, that is, at an endpoint of one of its segments or at a corner of Γ. Let s be the distance from the breakpoint along the cell, with $0 < s \leq S$. The potential of the segments that meet at the breakpoint is then a polynomial in s plus a term $\sum_m b_m s^m \log s$, up to an explicit remainder. In most cases the logarithmic part is bounded in absolute value by $\sum_m |b_m| S^m |\log S|$. For the cells on Γ of the shifts $\frac{k}{7}$ we

use a sharper bound. For $0 < t \le 1$ and an order $K \ge 0$ write

$$-\log t = T_K(t) + R_K(t), \qquad T_K(t) = \sum_{k=1}^{K} \frac{(1-t)^k}{k}.$$

Then $0 \le R_K(t) \le (1-t)^{K+1}/((K+1)t)$, and the weighted inequality of arithmetic and geometric means gives

$$0 \le t^m R_K(t) \le \frac{(m-1)^{m-1}(K+1)^K}{(m+K)^{m+K}} \qquad (m \ge 1),$$

with $0^0 = 1$. We put $t = s/S$ and write $\log s = \log S + \log t$. This bounds each term $b_m s^m \log s$ from below by a polynomial of degree $m+K$ in s, minus a constant. Here $\log S$ is replaced by a rational number within 10^{-11} of it, checked with the enclosures of the logarithm described above. The condition on the cell becomes an inequality for one polynomial.

For (a) the cells are intervals of the arclength on each edge of Γ. The bound is proved on a horizontal neighbourhood of the edge of fixed width. Beyond the last vertex the ray is handled by an analytic tail bound. For (b) the cells are intervals of $[0,1]$. For (c) the positive part $(\Upsilon - \Upsilon_\Gamma)_+$ is bounded above on each cell of $[-r_0, 0]$ by a constant or a chord. A separate lower bound for $\int_{-r_0}^{0} \Upsilon(x)\,dx$, together with $\min(t,0) = t - \max(t,0)$, then gives a lower bound J for the integral in (c).

Proposition 6.3 (certified constants)**.** *For each row of Table 2 there is a trial measure σ on a polygon Γ, with a cover by cells satisfying the checks above, for which Theorem 6.2 gives the constants D and ρ of the table. For the second plane the certificate is that of the reference frame of §6.6.*

All inequalities are verified by the Lean kernel, as described in Section 13. The certificate data are the polygon, the density σ and the cells. They were computed with floating-point optimisation and interval arithmetic in Arb [24]. Lean checks them with rational arithmetic only. The data play no role in the proof beyond satisfying these checks.

Remark 6.4 (how good a certificate can be). For an exponent μ_0 the certificate meets μ_0 exactly when $\mu_0(D-\Lambda)-\rho \ge 0$. This quantity is a concave function of σ. Its maximum over all probability measures on $\Gamma \cup [-r_0, 0] \cup [0,1]$ is attained at a measure σ^*. For this measure $|\nu_c|\sigma^*$ is the weighted equilibrium measure of the constrained energy problem on this set, with the external field $(1-\varepsilon_0)\pi|\operatorname{Im} z| - U^{\nu_c}$ and the upper constraint $|\mathcal{T}|\,dx$ on the segment [45]. Weak duality expresses the deficit of any certificate as a sum of nonnegative terms. The first is the energy of $2\mu_0|\nu_c|(\sigma - \sigma^*)$. This is the difference between $2\mu_0|\nu_c|\sigma$ and the optimal dual measure $2\mu_0|\nu_c|\sigma^*$. So a certificate close to the best possible one must be close to σ^* in energy. Then $\Upsilon - \Upsilon_\Gamma$ is a small ripple on the support of σ^*. This is why a pointwise verification on a cover is needed. This remark is not used in any proof and is not part of the formalization.

6.5. **Node removal and the neighbourhood.** The argument applies to every subset $C'' \subseteq C'$ of the common nodes. Removing common nodes turns them into points of the configuration that are not integrated over. Only common nodes can be removed. A node removed from one family only would leave an uncancelled term of weight $\frac{1}{2}$. By condition (b) one obtains

$$|\det \mathcal{M}_{C''}| \le \exp\bigl(-(D-o(1))r^2 + \rho r\,|C' \setminus C''|\bigr),$$

uniformly in C''. Insert this into (6.2) with $|h|e^{\rho r} \le 1$. The sum has at most $2^{|C'|} = e^{O(r)}$ terms, each at most $e^{-(D-o(1))r^2}$. This gives Theorem 6.2. The deletion of at most K common nodes in a member changes the energy by $O(r)$.

6.6. **Frames with several terms and multiple poles.** For a frame of §3.2 with all orders at most 1 the argument above applies verbatim, with $|\mathcal{T}| = |S|$. It reads the frame only through the source bound $|E(z)| \le C_E e^{-2\pi|\operatorname{Im} z|}$ on Γ, through the positions of the poles and through the bound on the atom masses. The weights enter only through C_E and the atom masses. A change of these constants multiplies each term of Andréief's integral by at most $e^{O(r)}$, which is $e^{o(r^2)}$. So a certificate for a set of shifts serves every frame on those shifts.

This includes the frames with a shift of order 0. The heads of such a shift carry atoms of mass zero. The integrand of the bound for $|\det \mathsf{L}_F[R', Q']|$ is nonnegative, and the integral over $|\Omega|$ only loses the terms in which a free point sits at one of these heads. So the bound obtained with all $|S|$ shifts still holds. The construction itself is unchanged. Only the number ξ loses the term of that shift. We use this for the first plane in Section 7, and for the second plane at the multiples of $\zeta(3)$ and of π^2.

At a pole of higher order the Gram representation carries derivatives. Write

$$E(z) = -\pi \sum_s w_s \cot \pi(z+s) + \pi^2 \sum_s \frac{w_s'}{\sin^2 \pi(z+s)} - 2\pi^3 \sum_s \frac{w_s'' \cos \pi(z+s)}{\sin^3 \pi(z+s)},$$

and let Γ cross the negative axis at x^*, with heads $x_{s,n} = -s-n > x^*$.

Lemma 6.5 (Gram formula)**.** *For nonnegative integers t, t', with $g = \upsilon_t \upsilon_{t'}$,*

$$\mathsf{L}_F[t,t'] = \frac{1}{2\pi i}\int_\Gamma g\,E\,dz - \sum_{s,n}\bigl(w_s' g'(x_{s,n}) - w_s g(x_{s,n}) - w_s'' g''(x_{s,n})\bigr).$$

Proof. Put $\tilde F(z) = \sum_s (w_s\psi(1-s-z) - w_s'\psi'(1-s-z) + w_s''\psi''(1-s-z))$. The reflection formula $\psi(1-w) = \psi(w) + \pi\cot\pi w$ and its first two derivatives give $F = \tilde F + E$. The residues of $F\upsilon_t\upsilon_{t'}$ at t and t' give the Löwner entry, also for $t = t'$. At a head, ψ has residue -1, ψ' has principal part $(z-x_{s,n})^{-2}$ and ψ'' has principal part $-2(z-x_{s,n})^{-3}$. This gives the atom terms. The function $\tilde F$ is holomorphic to the left of Γ and $O(1/|z|)$ there by the balance, so its integral against $\upsilon_t\upsilon_{t'}$ vanishes. The function E decays like $e^{-2\pi|\operatorname{Im} z|}$, again by the balance. □

The atom term at a double head is a symmetric bilinear form of rank two in the row and column functions, and it splits into two rank-one terms. For functions g and h put $\hat g = g' - \frac{w_s}{2w_s'}g$ and define $\hat h$ likewise. Then

$$-w_s'(gh)' + w_s gh = -\frac{w_s'}{2}\bigl[(g+\hat g)(h+\hat h) - (g-\hat g)(h-\hat h)\bigr]. \tag{6.3}$$

So the measure Ω of (6.1) is replaced by a real measure on a parameter space with two labels at each double head, carrying the weights $-w_s'/2$ and $w_s'/2$, and one label at each simple head. Andréief's identity holds on any measure space, so it writes every minor of L_F as an integral of products of two determinants. A repeated label gives two equal rows, so its term vanishes.

The same holds at a head of order $w = 3$. Write the principal part of F at the head as $\sum_{j\le w} c_j (z-x)^{-j}$. The atom term is then $-\sum_{i,j\ge 0} B_{ij}\, g^{(i)}(x) h^{(j)}(x)$ with $B_{ij} = c_{i+j+1}/(i!\,j!)$ and $c_j = 0$ for $j > w$. The symmetric matrix B of size w vanishes below its antidiagonal, and its antidiagonal entries are nonzero multiples of c_w. So $\det B \neq 0$, and over $\mathbb{Q}$ the form B is a sum $\sum_{l=1}^{w} \lambda_l\, \ell_l \ell_l^{\top}$ of w rank-one terms, with $\lambda_l \neq 0$. This gives w labels at the head, with the functions $\sum_j \ell_{l,j} g^{(j)}(x)$ and the weights $-\lambda_l$. For the frames of the third plane the head $-n$ has $c_1 = 0$, $c_2 = \gamma_2$ and $c_3 = -2\gamma_3$, so

$$B = \begin{pmatrix} 0 & \gamma_2 & -\gamma_3 \\ \gamma_2 & -2\gamma_3 & 0 \\ -\gamma_3 & 0 & 0 \end{pmatrix}, \qquad \det B = 2\gamma_3^3 \neq 0.$$

For $w = 2$ this is (6.3).

The determinants now contain derivatives of Cauchy kernels at the heads of order at least 2, which we call *multiple heads*. We compare them with a frame that has only simple poles. A *reference frame* for F is a frame $\mathcal{R}$ with simple poles on a set of shifts $S_{\mathcal{R}}$ with the following properties. It contains every shift of F of positive order. For each shift s of order $w \ge 2$ it contains $w-1$ further shifts, the *helpers* of s. Their heads lie on the same side of x^* as those of s, at distances δ from them with $\frac14 \le \delta \le \frac12$, and at mutual distances at least $\frac14$. It may contain further shifts. The poles of $S_{\mathcal{R}}$ are separated from x^*. For every frame $F^{(2)}_{\gamma_2,\gamma_1}$ we take the shifts 1, $\frac34$, $\frac12$, with heads $-n$, $-n+\frac14$, $\frac12 - n$ for $n \ge 1$, and the helper $\frac34$ of the shift 1. This also serves $\gamma_2 = 0$, where all orders are at most 1, and $\gamma_1 = 0$, where the shift $\frac12$ has order 0. For the frames $F^{(3)}_{\gamma_2,\gamma_3}$ of the third plane we take the same shifts 1, $\frac34$, $\frac12$. Now $\frac34$ and $\frac12$ are both helpers of the shift 1, at distances $\frac14$ and $\frac12$. In the separate proof for $\zeta(3)$ in §9.6 we take for $F^{(2)}_{1,0}$ the shifts 1, $\frac12$ of the function $F_{1/2}$, with the helper $\frac12$.

Each label of Andréief's integral now has a *reference point*. For a point of Γ it is the point itself. For a simple head it is the head, which is a head of $\mathcal{R}$. For the w labels at a multiple head $x_{s,n}$ they are $x_{s,n}$ and the heads

$x_{s^*,n}$ of the helpers s^* of s. Distinct labels have distinct reference points. In our frames the only shift of order at least 2 is 1, so the multiple heads are the points $-n$ with $n \geq 1$.

Lemma 6.6 (jets)**.** *Let $k = O(r)$. Let $t_1, \dots, t_k$ be distinct integers in $[0, \bar{N})$, and let $z_1, \dots, z_k$ be an injective list of labels. Each label is a point of Γ or a head of F in $[x^*, 0]$, and each multiple head may occur with any of its labels. Let ϕ_l be the function attached to z_l, that is $\upsilon_t(z_l)$ for a point of Γ or a simple head, and the combination $\sum_j \ell_{l,j} \upsilon_t^{(j)}$ at the head for a label of a multiple head. Let x_l^0 be the reference point of z_l. Then*

$$\left|\det[\phi_l(t_i)]\right| \leq e^{O(r \log r)} \left|\det[(x_l^0 - t_i)^{-1}]\right|,$$

where the implied constant depends only on Γ, on $\mathcal{R}$ and on the weights of F.

Proof. A row at a multiple head is $\sum_j \ell_{l,j} \upsilon_t^{(j)}$ at the head, with fixed rational coefficients $\ell_{l,j}$ and $j \leq 2$. Give each label of a multiple head its own variable y_l, and keep the other labels fixed at their points. By multilinearity the determinant is $\prod_l (\sum_j \ell_{l,j} \partial_{y_l}^j)$, over these labels, applied to the Cauchy determinant $\det[(y_l - t_i)^{-1}]$ and evaluated at the heads. The multiple heads are at distance at least 1 from the nodes. So Cauchy's estimate on circles of radius $\frac{1}{8}$, one variable at a time, bounds the determinant by $\prod_l \sum_j |\ell_{l,j}|\, j!\, 8^j = e^{O(r)}$ times the supremum of the Cauchy determinant over the polydisc. On the polydisc a label of a multiple head is within e_l of its reference point, where $e_l = \frac{1}{8}$ for the label at $x_{s,n}$ and $e_l = \delta + \frac{1}{8}$ for the label at a helper head at distance δ. So $e_l \leq \frac{5}{8}$. All other labels have $e_l = 0$.

Compare each factor of Cauchy's product formula

$$\left|\det[(y_l - t_i)^{-1}]\right| = \frac{\prod_{l<l'} |y_l - y_{l'}| \prod_{i<i'} |t_i - t_{i'}|}{\prod_{l,i} |y_l - t_i|}$$

with its value at the reference points. The labels of one multiple head stay at mutual distance at most $\frac{1}{4}$, and their reference points are at mutual distance at least $\frac{1}{4}$. So their factors do not increase. For any other pair of labels put $d_{ll'} = |x_l^0 - x_{l'}^0| > 0$. Then $|y_l - y_{l'}| \leq d_{ll'}(1 + e_l/d_{ll'})(1 + e_{l'}/d_{ll'})$. For a node t_i we have $|x_l^0 - t_i| \leq |y_l - t_i|(1 + e_l/|y_l - t_i|)$, and $|y_l - t_i| \geq t_i + \frac{7}{8}$. Hence the logarithm of the ratio is at most the sum of $e_l/d_{ll'}$ over pairs, plus the sum of $e_l/(t_i + \frac{7}{8})$ over the moved labels l and the nodes. Fix a moved label l. The reference heads are distinct points of a lattice $\frac{1}{q_{\mathcal{R}}}\mathbb{Z}$ in an interval of length $O(r)$. So the sum of $1/d_{ll'}$ over head labels l' is $O(\log r)$, and so is the sum over the nodes. Now fix a label at a point z of Γ. The heads lie at distance at least a constant $\delta_0 > 0$ from Γ, by the separation of the poles from x^*. So the sum of $e_l/|x_l^0 - z|$ over the moved labels l is again $O(\log r)$. This is the case of mixed configurations, of heads with free points on Γ. A bound by a constant factor for each such pair would give $e^{O(r^2)}$, so the logarithmic sum matters here. In total the exponent is $O(k \log r) = O(r \log r)$. □

Theorem 6.7 (decay for frames)**.** *Let F be a frame and $\mathcal{R}$ a reference frame for it. If all orders of F are at most 1, $\mathcal{R}$ may be F itself. Suppose that $\mathcal{R}$ has a certificate for Γ and ν_c as in Theorem 6.2, with $|\mathcal{T}|$ the number of shifts of $\mathcal{R}$. Then the conclusion of Theorem 6.2 holds for every member of the alternating geometry of parameter c for F, with the same D and ρ.*

Proof. By Lemma 6.5, the rank-one splittings and Andréief's identity, each minor is an integral over the labels of products of two determinants, one for the rows and one for the columns. Lemma 6.6 bounds each of them by the Cauchy determinant at the reference points, at a cost $e^{O(r\log r)} = e^{o(r^2)}$. The label weights are bounded and $|E| \le C_E e^{-2\pi|\operatorname{Im} z|}$. Distinct labels have distinct reference points, all of them heads of $\mathcal{R}$ or points of Γ. So the integral is at most $e^{o(r^2)}$ times the integral of the energy argument above for $\mathcal{R}$, in which every head of $\mathcal{R}$ carries an atom of bounded mass. Those are the only inputs of that argument. □

The certificate of the second plane is that of the reference frame $(1, \frac{3}{4}, \frac{1}{2})$, with $|\mathcal{T}| = 3$. It serves every frame $F^{(2)}_{\gamma_2,\gamma_1}$, including $\gamma_2 = 0$ and $\gamma_1 = 0$. The certificate of the third plane is another certificate for the same reference frame, for its own geometry parameter. The certificate of the separate proof for $\zeta(3)$ is that of the function $F_{1/2}$, with $|\mathcal{T}| = 2$. It is the certificate of the row for π^2.

7. The first plane

In this section we prove Theorem 1.1 and parts (iii)–(v) of Theorem 1.4. We use the frames $F^{\pm}_{\vec{w}}$ of §3.2 with the alternating geometry of parameter $c = \frac{1}{6}$, so $\alpha = \frac{1}{12}$ and $m = \lceil r/12 - \frac{1}{2}\rceil$. The only exception is the pair $L_{-3} \pm \frac{\pi^2}{18}$, for which we take $c = \frac{1}{4}$. The point of the section is that the denominator exponent and the decay certificate do not depend on the weights $\vec{w}$, and that a weight zero is allowed. Nonvanishing is Theorem 5.6 with $d \le 3$, so one deleted node suffices.

7.1. **Polar data.** Fix a frame $F^{\pm}_{\vec{w}}$ with integer weights. Let $\varrho = 2$ on S^+ and $\varrho = 1$ on S^-. For $j \ge 1$ the increment is

$$\delta(j-1) = \frac{3w_{\pm}}{3j-\varrho} + \frac{2w_{1/2}}{2j-1} + \frac{w_1}{j},$$

and $-\delta'(j-1) = \frac{9w_{\pm}}{(3j-\varrho)^2} + \frac{4w_{1/2}}{(2j-1)^2} + \frac{w_1}{j^2}$. Let $p \ge 5$ be a prime with $3\bar{N} < p^2$. Call an integer t *singular* at p if p divides $t(3t-\varrho)(2t-1)$. The singular integers form the three lattices $p\mathbb{Z}$, $k_A + p\mathbb{Z}$ and $k_B + p\mathbb{Z}$, where $k_A \in [1, p)$ solves $3k_A \equiv \varrho \pmod p$ and $k_B = \frac{p+1}{2}$. We write $k_p = \min(k_A, k_B)$ for the first singular index. Then $k_A/p \to \kappa_A$ with

$$\kappa_A = \tfrac{1}{3} \text{ on } S^+,\ p \equiv 1\ (6) \text{ and on } S^-,\ p \equiv 5\ (6), \qquad \kappa_A = \tfrac{2}{3} \text{ otherwise,}$$

and $\kappa_B = \frac{1}{2}$. Passing from S^+ to S^- exchanges the two classes of primes modulo 6.

Lemma 7.1 (polar data)**.** *For every frame on $S^{\pm}$ with integer weights and every $1 \le j \le \bar{N}$, the increment above has p-adic valuation at least -1, and its derivative at least -2. If j is not singular, both are p-integral.*

Proof. Each of j, $3j - \varrho$ and $2j - 1$ is smaller than $3\bar{N} < p^2$. So its valuation is at most 1, and it is 0 if j is not singular. The weights are integers. □

The lemma uses nothing about the weights but their integrality. If p divides a weight, the corresponding pole may disappear, and the lemma still holds since it gives only lower bounds. A weight zero, as for the frames with $w_{\pm} = 0$, is the case where p divides the weight for every p. The singular indices of that shift are still counted, which only weakens the bounds below. Rational weights are cleared first. Multiplying a frame by a nonzero integer multiplies ξ by that integer, which does not change μ.

7.2. **Local bounds for every frame.** For distinct nodes t and t' put

$$W(t,t') = \mathbf{1}[\text{a singular index lies in } (\min(t,t'), \max(t,t')]] + \mathbf{1}[p \mid t - t'],$$

and put $W(t,t) = 2 \cdot \mathbf{1}[t \ge k_p]$. By Lemma 7.1, the entry of $\Phi_{r,I}$ at (t,t') has Gauss valuation at least $-W(t,t')$. Lemma 4.1 then applies with this W. We use four facts about W: $W \le 2$, $W(t,t') \le 1$ if $t \ne t'$ and $p \nmid t - t'$, $W(t,t') = 0$ if $t \ne t'$ are both below k_p, and $W(t,t) = 0$ if $t < k_p$.

Proposition 7.2. *Let $p \ge 5$ be a prime with $3\bar{N} < p^2$. For every frame on $S^{\pm}$ and every member, $e_p(r, I)$ is at most each of the following, under its condition.*

	bound	*condition*
(T1)	$2r + 2m$	*none*
(T1′)	$2r + 2m - \|m\|_p$	*none*
(D2)	$2r - 2(k_p + p - r)_+ + 2m + (\min(\bar{N}, p + k_p) - r)_+$	$2m < p \le r$
(D4)	$2(r - \min(k_p, r)) + 2m + (\bar{N} - p)_+$	$p > r$
(D6)	$(\bar{N} - k_p)_+$	$p \ge \bar{N}$, $k_p \ge r$
(W)	$2r + 2m - (t_2 - t_1)$ *for each window of Theorem 4.7*	*as there*

In (W) the cuts are the three lattices of singular indices and the residue of r.

Proof. (T1) and (T1′) are proved as in §4.4.1. For the others we give potentials, with one function u used on rows and on columns. For (D2), $u = 0$ on the common nodes $t < k_p$ with $t + p \ge r$, $u = 1$ on the other common nodes, and $u(t) = 1 + \mathbf{1}[t - p < k_p]$ on the border. For (D4), $u(t) = \mathbf{1}[t \ge k_p]$ on the common nodes and $u(t) = 1 + \mathbf{1}[t \ge p]$ on the border. For (D6), $u = 0$ on the common nodes and $u(t) = \mathbf{1}[t \ge k_p]$ on the border. In each

case the four facts give $W \le u \oplus u$ entrywise, and the sums are the stated values. For (W), the proof of Theorem 4.7 uses the function only through Lemma 7.1 and the cuts. Between two consecutive cuts the step function V and the diagonal residues are constant, which is what the translation of kernel vectors needs. No relation among the weights is used. □

None of these bounds reads the weights. They depend on (r, m, p) and on the class of p modulo 6 only.

7.3. **The denominator exponent.** Homogenise as in §4.5, with $k_A \mapsto \kappa_A\theta$, $k_B \mapsto \kappa_B\theta$ and $\theta = p/r$. Let φ_1 and φ_2 be the minima of the homogenised bounds of Proposition 7.2 for $\kappa_A = \frac{1}{3}$ and $\kappa_A = \frac{2}{3}$, with the windows for $n \le 32$. Put $t_0 = \alpha/33$ and

$$\Lambda^* = (2 + 2\alpha)\, t_0 + \frac{1}{2} \int_{t_0}^{9} (\varphi_1 + \varphi_2)\, d\theta.$$

On the strips $(\frac{\alpha}{n+1}, \frac{\alpha}{n}]$ with $1 \le n \le 32$ only (T1), (T1′) and the windows with $n = \lfloor m/p \rfloor$ act. There the profile has the closed form described after Theorem 7.3. On $(\alpha, 9]$ it is piecewise affine with finitely many pieces. So Λ^* is a rational number, and

$$\Lambda^* = 3.22274274961\ldots$$

Primes of the classes 1 and 5 modulo 6 each have density $\frac{1}{2}$, and S^- exchanges them. So Λ^* serves the frames on both shift sets S^+ and S^-. Above $\theta = \frac{7}{2}$ both profiles vanish.

Theorem 7.3 (denominators on the first plane)**.** *For every frame on $S^{\pm}$, every $\epsilon > 0$ and every even $r \ge R(\epsilon)$, every member satisfies* $\log \operatorname{den} P_{r,I} \le (\Lambda^* + \epsilon) r^2$. *The threshold $R(\epsilon)$ does not depend on the frame.*

Proof. Take the product over primes of the bounds of Proposition 7.2. For the primes with $p^2 \le 3\bar{N}$, and for 2 and 3, every entry has valuation at least $-2\log(3\bar{N})/\log p$. These primes contribute $O(r^{3/2})$. Each bound is built from affine functions of r, m, p, k_A, k_B and $p\lfloor r/p \rfloor$ by max, min and truncation, with bounded coefficients. So it differs from r times its homogenisation by a bounded amount. Each integer condition differs from its homogenised form only for p in an interval of bounded length, where (T1) costs $O(r \log r)$. The prime number theorem in progressions modulo 6 then gives the claim, as in Theorem 4.14. For the primes $p \le t_0 r$ we use (T1), which gives $(2 + 2\alpha) t_0 r^2$. On the strips $\theta \le \alpha$ the content is an instance of Theorem 4.15 for $\alpha = \frac{1}{12}$, with $K_0 = 2 + 2\alpha$. The bounds acting there are closed under translation: (T1) and (T1′) grow by $12\nu K_0 p$ under the translation of order ν, and it takes the window with $n = \lfloor m/p \rfloor$ to the window with $n + \nu$. So the table is checked on one period of $1/\theta$, and its translates give the pieces of the other strips. □

On $\theta \le \alpha$ the profile has a closed form. Only (T1), (T1′) and the windows act there. Put $s = 1/\theta$. Then $\varphi(\theta) = 2 + 2\alpha - \theta\, \Gamma_\kappa(s)$. Here $\Gamma_\kappa(s)$ is the

larger of $\|\alpha s\|$ and the longest gap of the lattice $\{0, \kappa_A, \kappa_B\} + \mathbb{Z}$ inside a window. The window is determined by the fractional parts of s and αs. This function has period 12 in s, as Theorem 4.15 predicts. The table has 82 cells per class on one period.

7.4. **Decay.** Let Γ be the polygon of the formal proof for the alternating geometry of parameter $\frac{1}{6}$. It crosses the real axis at a point x^* with $x^* \equiv \frac{1}{4} \pmod 1$ for the frames on S^+ and $x^* \equiv \frac{3}{4} \pmod 1$ for those on S^-. In both cases x^* is at distance at least $\frac{1}{4}$ from the poles of the shifts. The scaled cover is the same for both.

Proposition 7.4. *There is a trial measure with* 25 *polynomial pieces on* Γ*, with a cover by cells satisfying the checks of §6.4 with* $\varepsilon_0 = 10^{-6}$ *and* $|\mathcal{T}| = 3$*, for which Theorem 6.2 gives*

$$D = \frac{99853629795114224867}{28125000000000000000} = 3.55035\ldots,$$
$$\rho = \frac{33845462710966783}{5 \cdot 10^{15}} = 6.76909\ldots$$

The certificate contains no weight. By §6.6 the decay estimate therefore holds for every frame on S^+ and on S^- with these D and ρ, also when one weight is zero. The weights change only the threshold in r.

7.5. **Proof of Theorem 1.1.** Let a and b be rationals, not both zero. By Lemma 3.2 there is a frame on S^+ or S^- whose number ξ is a nonzero rational multiple of $aL_{-3} + b\pi^2$. For $a = 0$ it is the frame $F^+_{(0,3,-3)}$, whose shift $\frac{1}{3}$ has weight zero. Theorem 7.3 gives the denominators with Λ^*, and Proposition 7.4 gives the decay. Both hold for every frame on $S^{\pm}$, including those with a weight zero. Theorem 5.6 gives nonvanishing with one deleted node, since $d \le 3$. For $a = 0$ the frame has two poles per period, and the argument is that of two digamma terms. The deduction of Theorem 11.1 at the even scales gives

$$\mu(\xi) \le \frac{\rho}{D - \Lambda^*} = 20.66213\,7\ldots < 20.6622.$$

The strict inequality gives the bound $|aL_{-3} + b\pi^2 - p/q| > q^{-20.6622}$ for all large q, and in particular the irrationality. □

The thresholds depend on (a, b) through the height of the weights. No uniformity in (a, b) is claimed.

7.6. **Level returns.** At special weights the partial sums of the increment return to an earlier level, and then more entries are p-integral than Proposition 7.2 uses. Fix a frame on $S^{\pm}$, a class of primes modulo 6, and a prime $p \ge 7$ with $3\bar{N} < p^2$ and $2m < p$. Every singular index $t \le \bar{N}$ has $3t - \varrho$, $2t - 1$ or t equal to p times an integer prime to p and smaller than p. Hence

$$f_t = \frac{V(t)}{p} + (\text{a } p\text{-integer}), \qquad M[t,t] = \frac{U(t)}{p^2} + (\text{a } p\text{-integer}),$$

where the *level* $V(t)$ and $U(t)$ are rational numbers. They are sums of the terms $\frac{3w_\pm}{3l+n}$, $\frac{2w_{1/2}}{2l+1}$, $\frac{w_1}{l+1}$, respectively of their squared analogues with a sign, over the singular indices up to t. Here $n \in \{1,2\}$ is fixed by the class. So V and U are constant between consecutive singular indices, and they depend on p only through the class. Along the first period V steps by the residues at the three poles. On S^+, for $p \equiv 1 \pmod 6$, the poles come in the order k_A, k_B, p with residues $3w_+$, $2w_{1/2}$, w_1, so the level after two poles is $3w_+ + 2w_{1/2}$. It is 0 exactly for the frames whose number is a multiple of L_{-3}. For $p \equiv 5 \pmod 6$ the order is k_B, k_A, p with residues $2w_{1/2}$, $\frac{3}{2}w_+$, w_1. A frame for which a partial sum of these residues vanishes has a *level return*. The frames $(2,-1,-1)$, $(1,-2,1)$ and $(4,-3,-1)$ have level returns, with $w_{1/2} + \frac{w_+}{2} = 0$, $2w_+ + w_{1/2} = 0$ and $2w_{1/2} + \frac{3}{2}w_+ = 0$.

Lemma 7.5 (period sums)**.** *Let $t' = t + sp$ with $1 \le s < p$. Then $f_{t'} - f_t = (V(t') - V(t))/p + \Delta$ with $v_p(\Delta) \ge 1$.*

Proof. The polar terms give $(V(t') - V(t))/p$ exactly. For each shift, every block of p consecutive indices runs once through the residues modulo p. The nonpolar terms of a block therefore contribute a multiple of $\sum_{v \in \mathbb{F}_p^\times} v^{-1} \equiv 0 \pmod p$. □

So two congruent nodes at the same level have a p-integral entry, not one of valuation -1. A second saving comes from row operations inside a residue class. Let t be a border node and $\hat{t} < t$ a common node with $t - \hat{t} = jp$. Suppose that $V(t) \ne V(\hat{t})$, $U(\hat{t}) \ne 0$, and that

$$\lambda_t = \frac{V(t) - V(\hat{t})}{j\,U(\hat{t})}$$

satisfies $\lambda_t \ne 1$ and has no prime factor larger than 5 in its denominator. Then λ_t is p-integral. We call $(t, \hat{t})$ a *pivot pair*, and we put $V^*(t) = (V(t) - \lambda_t V(\hat{t}))/(1 - \lambda_t)$. Pivot pairs whose border node is a column are defined in the same way. Each border node lies in at most one pivot pair, and $V^* = V$ on the nodes in no pair. Define weights W' on rows times columns as follows, for a row node t and a column node t'.

(i) For $t \not\equiv t' \pmod p$, $W'(t,t') = \mathbf{1}[V^*(t) \ne V^*(t')]$.
(ii) For a pivot pair, the entry at the border node and its common node has $W' = 0$.
(iii) For other $t \equiv t'$, $W'(t,t')$ is the largest $W_0(t_1, t_1')$ with t_1 equal to t or its common node and t_1' equal to t' or its common node. Here $W_0(t_1,t_1') = 2 \cdot \mathbf{1}[V(t_1) \ne V(t_1')]$ for $t_1 \ne t_1'$, and $W_0(t_1,t_1) = 2 \cdot \mathbf{1}[U(t_1) \ne 0]$.

Lemma 7.6 (levels)**.** *If nonnegative numbers u on the rows and u' on the columns satisfy $W' \le u + u'$ entrywise, then $e_p(r, I) \le \sum u + \sum u' + 2|I|$ for every member.*

Proof. Subtract λ_t times the row of $\hat{t}$ from the row of t for every pivot pair, and likewise for columns. The pivots are common nodes, which are never

modified themselves, so these operations are unimodular and the determinant is unchanged as a polynomial in the variable x. It suffices to bound the Gauss valuation of each new entry from below by $-W'$. For congruent $t \neq t'$ without pivots, Lemma 7.5 gives $M[t,t'] = (V(t')-V(t))/(sp^2)+\Delta/(sp)$ with $t'-t = sp$, so the valuation is at least $-2 \cdot \mathbf{1}[V(t) \neq V(t')]$. With pivots the entry is a combination of at most four such entries with p-integral coefficients, which gives (iii). For a pivot pair the entry is $M[t,\hat{t}] - \lambda_t(M[\hat{t},\hat{t}]+x)$. The terms of order p^{-2} cancel by the choice of λ_t, the rest is p-integral, and the variable has the p-integral coefficient $-\lambda_t$. This is (ii). For $t' \not\equiv t$ write $\frac{1}{t'-t} = \frac{1}{t'-\hat{t}} + \frac{jp}{(t'-\hat{t})(t'-t)}$. Then

$$M[t,t'] - \lambda_t M[\hat{t},t'] = \frac{(1-\lambda_t)(V(t')-V^*(t))}{p\,(t'-\hat{t})} + (\text{a } p\text{-integer}),$$

since $t'-\hat{t}$ and $t'-t$ are units. If t' is pivoted too, the same identity applied to its column gives the leading term $(1-\lambda_t)(1-\lambda_{t'})(V^*(t')-V^*(t))/(p(t'-\hat{t}))$. This is (i). Finally, let a common node be deleted. Since $2m < p$, each residue class contains at most one border node. So the deleted node is the common node of at most one pivot pair. That pair is undone. This raises the weights of the border line off its fibre by at most 1, and does not raise its weights in the fibre. Raising the potential of that line by 1 restores $W' \leq u+u'$. So each deleted node costs at most 1, within the allowance $2|I|$. □

The bound of Lemma 7.6 with no pivot pairs is already better than Lemma 4.1 at a level return. The pivots add more. They are the assignment form of the rank of the reduction modulo p on the fibres where the level returns. For instance, take the frame $(2,-1,-1)$ on S^+ and $p \equiv 5 \pmod 6$. The poles fall at about $\frac{p}{2}$, $\frac{2p}{3}$, p, $\frac{3p}{2}$, with residues -2, 3, -1, so V returns to 0 at p. A border node in $[\frac{3p}{2}, \frac{5p}{3})$ has level $-\frac{2}{3}$. Its common node in $[\frac{p}{2}, \frac{2p}{3})$ has level -2 and $U = 4$. Then $\lambda = \frac{1}{3}$ and $V^* = 0$, so after the row operation the border node behaves like a node at the returned level.

For the frame $(4,-3,-1)$ we use a longer border, and a simpler consequence of the level return. Let $\mathcal{K}$ be the set of nodes $t < p$ with $V(t) = 0$, and with $U(t) = 0$ if t is a common node. It contains every node below the first singular index. At a level return after the second singular index $k'_p < p$ it also contains the border nodes in $[k'_p, p)$. Two distinct nodes of $\mathcal{K}$ that are not congruent modulo p have a p-integral entry, since $f_t = V(t)/p$ up to a p-integer. The diagonal entry of a common node of $\mathcal{K}$ is p-integral too. Suppose that $\bar{N} \leq 2p$. The potentials of Proposition 8.1(ii), with $\mathcal{K}$ in place of the regular nodes, then give

$$e_p(r,I) \leq 2r + 2m - n_{\mathcal{K}}. \tag{7.1}$$

Here $n_{\mathcal{K}}$ is the number of pairs formed by a node $t \in \mathcal{K}$ and a side on which t lies, row or column, such that $t+p$ is not a node of the other side. The proof is that of Proposition 8.1(ii).

Theorem 7.7 (the special points)**.** *For the frames* $(2,-1,-1)$ *and* $(1,-2,1)$*, on* S^+ *and on* S^-*, with* $c=\frac{1}{6}$*, the minimum of the bounds of Proposition 7.2 and Lemma 7.6 gives the denominator exponents*

$$\Lambda = 3.06683997\ldots \qquad \textit{and} \qquad 3.10960617\ldots$$

For the frame $(4,-3,-1)$*, on* S^+ *and on* S^-*, with* $c=\frac{1}{4}$*, the minimum of the bounds of Proposition 7.2 and* (7.1) *gives the denominator exponent*

$$\Lambda = 3.41401753\ldots$$

Proof. On each cell of the common refinement of the conditions, the nodes split into boundedly many classes by residue interval, parity and period. Pivot pairs, λ and V^* are constant on these classes. Potentials constant on the classes are affine in $\theta = p/r$, with a bounded rounding error per prime, as in Theorem 7.3. The potentials of the formal proof take half-integer values. They are verified on every pair of classes, and the minimum with the plane bounds is integrated as there. A cell is fixed by the integer parts of r/p and $\bar{N}/p$ and by the order of the residues of k_A, k_B, r and $\bar{N}$ in a period. At a common endpoint θ_e of two cells the bounds of the two sides take the same value. A prime with p/r near θ_e may lie in neither cell, through the rounding of m or a coincidence of two residues. So on the window $|p/r-\theta_e| \le 10^{-4}\theta_e$, in each class of primes modulo 6, we use the largest of the bounds of all cells that occur in the window. There are at most three. That one of them applies to each (r,p,m) in the window is a statement of linear integer arithmetic in r, p, m, k_A and k_B, and it is proved for every prime p. This bound is the second bound of (C3) at θ_e. For the first two frames all pivots used have $\lambda \in \mathbb{Z}[\frac{1}{30}]$, so every prime $p \ge 7$ is admissible, and the level branches act only for $\theta \ge \frac{3}{10}$. For the frame $(4,-3,-1)$ the bound (7.1) takes the place of Lemma 7.6, and no pivot is used. On the strips $\theta \le \alpha$ the content is again an instance of Theorem 4.15. The frames on S^- give the same profiles with the classes exchanged. □

Proof of Theorem 1.4(iii)–(v). The frame $(2,-1,-1)$ has $\tau = \frac{2}{27}$. On S^+ its number is $9(L_{-3}+\frac{2\pi^2}{27}) = \frac{1}{2}\Psi(\frac{1}{6},\frac{1}{3})$, and on S^- it is $-9(L_{-3}-\frac{2\pi^2}{27}) = -\frac{1}{2}\Psi(\frac{2}{3},\frac{5}{6})$. The frame $(1,-2,1)$ has $\tau = -\frac{1}{27}$ and gives $\Psi(\frac{1}{2},\frac{2}{3})$ on S^+ and $\Psi(\frac{1}{3},\frac{1}{2})$ on S^-, up to sign. The frame $(4,-3,-1)$ has $\tau = \frac{1}{18}$ and gives $L_{-3} \pm \frac{\pi^2}{18}$. For the first two frames the decay is that of Proposition 7.4. For the frame $(4,-3,-1)$ with $c=\frac{1}{4}$, a trial measure on a polygon Γ, with a cover by cells satisfying the checks of §6.4 with $\varepsilon_0 = 10^{-6}$ and $|\mathcal{T}| = 3$, gives

$$D = \frac{15243910584818244128783}{4\cdot 10^{21}} = 3.81097\ldots,$$
$$\rho = \frac{34717183947859923}{5\cdot 10^{15}} = 6.94343\ldots$$

This certificate contains no weight either, so it serves the frame on S^+ and on S^-. With the denominator exponents of Theorem 7.7 and Theorem 5.6,

the deduction gives

$$\frac{\rho}{D-\Lambda} = 13.99986\,39\ldots, \qquad 15.35829\,31\ldots, \qquad 17.49152\,28\ldots,$$

below 14, 15.359 and 17.492. □

The point $\tau = 0$ is the frame $(2,-3,1)$, which has a level return after two poles in the class $p \equiv 1 \pmod 6$. It is treated with its own geometry in Section 8. The points $\tau = \pm\frac{1}{9}$ are given by the differences $F_{1/3}$ and $F_{2/3}$ of two digamma functions, with their own geometry, in Section 11.

8. The point L_{-3} of the first plane

For $L_{-3} = \frac{1}{9}\Psi(\frac{1}{3},\frac{2}{3})$ we use the function F_χ of (2.1), which combines three digamma terms, with the alternating geometry of parameter $c = \frac{1}{2}$ at all even scales r. Then $\alpha = \frac{1}{4}$ and $m = \lceil r/4 - \frac{1}{2}\rceil$. The decay and nonvanishing estimates of Theorems 6.2 and 5.7 apply unchanged. The denominators need more care. At a prime p the increments $g(k)$ of §3 have poles in three residue classes instead of two, so the local bounds depend on two first poles. This section describes the two poles, the bounds we use, and the resulting constant.

8.1. **The first poles.** Let $p > 3$ be a prime. The increment $g(k) = \frac{1}{3k-2} - \frac{1}{2k-1} + \frac{1}{6k}$ has a pole modulo p exactly when p divides k, $3k-2$ or $2k-1$. Let k_A and k_B be the least $k \geq 1$ with $p \mid 3k-2$ and with $p \mid 2k-1$. Then

$$k_B = \frac{p+1}{2}, \qquad k_A = \begin{cases} \frac{p+2}{3}, & p \equiv 1 \pmod 6, \\ \frac{2p+2}{3}, & p \equiv 5 \pmod 6. \end{cases}$$

The residues of $p\,g(k)$ at k_A and k_B are 1 and -1 if $p \equiv 1 \pmod 6$, and $\frac{1}{2}$ and -1 if $p \equiv 5 \pmod 6$. In the first case $k_A < k_B$ and the two residues cancel. So the partial sums $\sum_{k\leq t} g(k)$ are p-integral for $t < k_A$, and again for $k_B \leq t < p$.

Both first poles are affine in one index. Define k_p by $6k_p = jp + 5$ with $0 < j < 6$. Then $j = 1$ for $p \equiv 1 \pmod 6$ and $j = 5$ for $p \equiv 5 \pmod 6$, $k_p = \frac{j}{6}p + O(1)$, and

$$(k_A, k_B) = \begin{cases} (2k_p - 1,\ 3k_p - 2), & p \equiv 1 \pmod 6, \\ (2k_p - p - 1,\ 3k_p - 2p - 2), & p \equiv 5 \pmod 6. \end{cases}$$

Thus j plays the role of the root class of §4, with q replaced by 6.

8.2. **The local bounds.** Fix a prime $p > 6$ with $6\bar{N} < p^2$. Every pole below $\bar{N}$ is then simple. Call a node t *regular* if $t < \min(k_A, k_B)$, or if $p \equiv 1 \pmod 6$ and t is a border node with $k_B \leq t < p$. At a regular node the partial sum of the increments up to t is p-integral. When t is a common node, its diagonal entry is p-integral too.

Proposition 8.1. *Every member of the alternating geometry of parameter* $\frac{1}{2}$ *for* F_χ *satisfies* $e_p(r, I) \le \tau$ *for each of the following* τ.

(i) The bounds (T1) and (T1′), namely $2(r+m)$ *and* $2r+2m-\|m\|_p$.

(ii) If $\bar{N} \le 2p$, *the bound* $2r+2m-n_{\mathrm{reg}}$. *Here* n_{reg} *is the number of pairs formed by a regular node* t *and a side on which* t *lies, row or column, such that* $t+p$ *is not a node of the other side.*

(iii) If $\bar{N} < \min(k_A, k_B)$, *the bound* 0.

(iv) The bound $2r+2m-(t_2-t_1)$ *of Theorem 4.7, for every* n *and every window* $[t_1, t_2)$ *as there, with the cuts* $t \equiv 0$, $t \equiv k_A$, $t \equiv k_B$ *and* $t \equiv r \pmod p$.

Proof. The bounds (i) are proved as in §4.4.1. For (ii) we apply Lemma 4.1 with the following potentials. At a node that is not regular, the potential is 1 on its row and on its column. At a regular node t, the potential on its row is 1 if $t+p$ is a column node, and 0 otherwise. The potential on its column is 1 if $t+p$ is a row node, and 0 otherwise. Their sum is $2(r+m)-n_{\mathrm{reg}}$, and they dominate the weights W. Indeed, a diagonal weight is 0 at a regular common node and at most 2 in general. Two nodes that are not congruent modulo p have weight at most 1, since all poles are simple. Their weight is 0 if both are regular. Two distinct congruent nodes have weight at most 2. Since $\bar{N} \le 2p$ and regular nodes are smaller than p, at most one of them is regular. Suppose that the row node t of such an entry is regular. Then its column node is $t+p$, so the potential on the row of t is 1. The case of a regular column node is symmetric.

For (iii), the condition $\bar{N} < \min(k_A, k_B)$ implies $\bar{N} < p$. Then no index below $\bar{N}$ is a pole and no two nodes are congruent modulo p, so every entry is p-integral. For (iv), the proof of Theorem 4.7 uses F only through its poles. The step function V and the diagonal residues are constant between consecutive cuts. This is what the translation between the fibres of the window needs. □

The bounds (T2)–(T5), (LP1), (LP2), (R) and (S) of Theorem 4.5 are not used for F_χ.

8.3. **The denominators.** The tables for Λ of §4.6 and Theorem 4.14 apply to F_χ with q replaced by 6. The root class is $j \in \{1, 5\}$ as above, and the bounds are those of Proposition 8.1. These bounds are expressions of the class $\mathcal{E}$ in r, p, m, k_A, k_B. On each class of primes modulo 6 the substitution above makes them expressions in r, p, m, k_p. Their homogenisations are taken with $\kappa = j/6$. The proof of Theorem 4.14 uses F only through these bounds and the root class. It goes through with the bound (iii) in place of (T5) beyond the end of the table.

The table of the formal proof starts at $\theta_0 = \frac{1}{68} = \alpha/17$. Below it the trivial bound $\ell_0 r$ with $\ell_0 = \frac{5}{2} = 2 + 2\alpha$ is used. On $(\alpha/17, \alpha]$ the table uses the bounds (i) and the windows (iv) with $1 \le n \le 16$. Beyond α it uses (i), (ii), the windows (iv) with $n = 0$ between consecutive cuts, and the bound

(iii), which ends the table. The table is admissible for the two classes $j = 1, 5$. The arithmetic estimate therefore holds for every even r and every member with $|I| \le 64$, with the denominator exponent

$$\Lambda_\chi = 4.09592776731\ldots$$

This is an explicit rational number, whose numerator and denominator have 117 and 116 digits.

8.4. **Decay and nonvanishing.** The decay estimate is Theorem 6.2 for the function F_χ, with $|\mathcal{T}| = 3$ and $c = \frac{1}{2}$. The node measure $\nu_{1/2}$ has mass $\frac{5}{4}$ and energy

$$\mathcal{I}(\nu_{1/2}) = \frac{75}{32} - \frac{9}{8}\log\frac{3}{2} - \frac{1}{16}\log 2.$$

A trial measure on a polygon, with a cover by cells satisfying the checks of §6.4 with $\varepsilon_0 = 10^{-6}$, gives

$$D = \frac{580530026928040140331}{125 \cdot 10^{18}} = 4.64424\ldots,$$
$$\rho = \frac{18757555738219881}{25 \cdot 10^{14}} = 7.50302\ldots$$

Nonvanishing is Theorem 5.7 for F_χ, which has $d = 3$. The pole of its increment at -1 is isolated in its class modulo $\mathbb{Z}$, and the members with $|I| \le 64$ are used. The deduction of Section 11, at the even scales, gives

$$\mu\big(\Psi(\tfrac{1}{3}, \tfrac{2}{3})\big) \le \frac{\rho}{D - \Lambda_\chi} = 13.683844\ldots < 13.6839,$$

since $\Psi(\frac{1}{3}, \frac{2}{3}) = 36\,\xi_\chi$.

9. The second plane

In this section we prove Theorem 1.2 and the bound for $\zeta(3)$ in Theorem 1.5. We use the frames $F^{(2)}_{\gamma_2,\gamma_1}$ of §3.2 with integers γ_2 and γ_1, not both zero, and we put $\varepsilon = \mathbf{1}[\gamma_1 \ne 0]$, the order of the shift $\frac{1}{2}$. For Theorem 1.2 we take the alternating geometry of parameter $c = \frac{6}{41}$, so $\alpha = \frac{3}{41}$, for every frame. This includes $\gamma_2 = 0$, the multiples of π^2, and $\gamma_1 = 0$, the multiples of $\zeta(3)$. For the separate bound for $\zeta(3)$ in Theorem 1.5 we take $c = \frac{3}{10}$. In both cases $2m \le r$. The new feature is the double pole at the shift 1. The entries of the Löwner matrix then have valuation down to -3, and most of the work goes into recovering what is lost against a simple pole.

9.1. **Cells and weights.** Let $p \ge 5$ be a prime with $p^2 > 2\bar{N} + 2p$. The increment is

$$\delta(k-1) = \frac{\gamma_1}{k} - \frac{2\gamma_1}{2k-1} - \frac{\gamma_2}{k^2} \qquad (k \ge 1).$$

Its poles below $\bar{N}$ are the double poles $k = jp$ and, if $\gamma_1 \ne 0$, the simple poles $k = jp + \frac{p+1}{2}$. Write an integer $t \ge 0$ as $t = jp + \omega$ with $0 \le \omega < p$. We call j the *period* of t, $\iota = \mathbf{1}[\omega \ge \frac{p+1}{2}]$ its *half*, and $\mathbf{c}(t) = (j, \iota)$ its *cell*. A

pole lies at the start of every cell. Let $S(\mathbf{c})$ be the sum of the polar terms of δ up to the start of the cell $\mathbf{c}$, and $G(\mathbf{c})$ the same for $-\delta'$. Then

$$f_t = S(\mathbf{c}(t)) + \rho_t, \qquad e_t = G(\mathbf{c}(t)) + \rho'_t,$$

with ρ_t and ρ'_t in $\mathbb{Z}_{(p)}$, and $p^2 S$ and $p^3 G$ in $\mathbb{Z}_{(p)}$. Modulo p the numbers $p^2 S$ and $p^3 G$ depend only on the period, because the simple poles have lower order than the double ones.

Lemma 9.1 (weights)**.** *Let* $W(t,t') = \max(0, -v_p(M[t,t']))$. *Then* $W(t,t') \le 3$ *if* $t \equiv t' \pmod p$, *and for* $t \not\equiv t'$

$$W(t,t') \le W_3(\mathbf{c}(t), \mathbf{c}(t')), \qquad W_3(\mathbf{c},\mathbf{c}') = \begin{cases} 0 & \mathbf{c} = \mathbf{c}', \\ 1 & \textit{same period, other half}, \\ 2 & \textit{other period}. \end{cases}$$

On the diagonal $W(t,t)$ *is* 0 *on the cell* $(0,0)$, *at most* 2ε *on* $(0,1)$, *and at most* 3 *elsewhere. This holds also when* p *divides* γ_2 *or* γ_1.

Proof. For $t \neq t'$ the difference $t' - t$ is a unit unless $t \equiv t'$, and then $v_p(t'-t) = 1$ since $p^2 > \bar{N}$. The numerator $f_{t'} - f_t$ has valuation at least -2, at least -1 if t and t' lie in one period, and at least 0 if they lie in one cell. The diagonal is $-e_t$. The weights γ_1 and γ_2 are integers, so a prime dividing them only raises valuations. □

The primes that divide γ_2 need one remark. If $p \mid \gamma_2$, the double poles contribute terms γ_2/k^2 of valuation at least -1 to f_t, and terms $2\gamma_2/k^3$ of valuation at least -2 to e_t. So every entry of M has valuation at least -2, and Lemma 4.1 with all potentials 1 gives $e_p(r,I) \le 2(r+m)$. The bounds of Proposition 9.2 below use Lemma 9.1 only and hold for all p. Those of Theorem 9.6 assume $p \nmid \gamma_2$. Each of them is at least $2(r+m)$, as we check after its proof. So for $p \mid \gamma_2$ the trivial bound $2(r+m)$ takes their place, and every bound used below holds for all frames with $\gamma_2 \neq 0$.

Proposition 9.2 (assignment bounds)**.** *Assume* $2m \le r$. *For every member,*

	bound	*condition*
(P1)	$3(r+m) - \lVert m \rVert_p$	*none*
(P4)	$6(r+m-p) + \varepsilon p$	$p \le r$
(P5)	$\varepsilon r + 3(\bar{N} - p)$	$r \le p \le \bar{N}$
(P6)	$\varepsilon(2r + 2m - p + 1)$	$\bar{N} \le p \le 2r$
(P7)	$\varepsilon(\bar{N} - \frac{p-1}{2})_+$	$p \ge 2r$

Proof. All follow from Lemmas 4.1 and 9.1. For (P1) take the potentials of (T1′), raised by $\frac{1}{2}$ on every row and every column. Congruent pairs get 3 and the others at least 2. For the other bounds take one function φ on rows and columns, with $\varphi = 0$ on the cell $(0,0)$, $\varphi = \varepsilon$ on $(0,1)$ and $\varphi = 3$ on the periods $j \ge 1$. Pairs inside the period 0 have weight at most $\varphi \oplus \varphi$,

and any pair with a node in a later period has weight at most 3. Counting the common nodes twice and the border nodes once gives the four bounds. The hypothesis $2m \le r$ enters twice. For (P5) it gives $p \le \bar{N} \le 2r$, so $(p-1)/2 < r$. For (P7) it gives $\bar{N} \le 2r \le p$, so all nodes lie in the period 0 and the common nodes lie in the cell $(0,0)$. □

Proposition 9.2 bounds the contribution of a prime $p \le 2m$ by $3(r+m) - \|m\|_p$, and that of a prime $2m < p \le r$ by no better. The double pole costs one more unit per node than a simple pole. The next two subsections recover most of this unit on the primes below $r + 2m$.

9.2. **Elimination of complete periods.** Let $\mathcal{O}$ be the discrete valuation ring of §4, in a variable T, and put $x = p^{-3}T$, so that a member is a polynomial in x. Fix integers $\kappa \ge 1$ and $k' \ge 0$ with $(\kappa + 2k')p \le \bar{N} + p$, and put $\bar{N}' = \max(\bar{N}, (\kappa + 2k')p)$. Then $p^2 > 2\bar{N}'$. Let $H = M + p^{-3}T\mathrm{Id}$ on $[0, \bar{N}')$, so that H restricts to the matrices of the members. Extend the border alternately, so $\hat{X} = \{t \ge r : t - r \text{ even}\}$ and $\hat{Y} = \{t \ge r : t - r \text{ odd}\}$. The *pivot* consists of the complete periods below κp and of $2k'$ further complete periods:

$$\mathcal{Q}_\mathrm{r} = [0, \kappa p) \cup (\hat{X} \cap [\kappa p, (\kappa + 2k')p)),$$
$$\mathcal{Q}_\mathrm{c} = [0, \kappa p) \cup (\hat{Y} \cap [\kappa p, (\kappa + 2k')p)).$$

We write $H_\mathcal{Q} = H[\mathcal{Q}_\mathrm{r}, \mathcal{Q}_\mathrm{c}]$.

Lemma 9.3 (displacement and bordering; cf. [18])**.** *For all nodes t, t' one has $(t - t')H(t, t') = f_t - f_{t'}$, whatever the diagonal. Suppose that $H_\mathcal{Q}$ is invertible. Let* Sw *be the Schur complement of $H_\mathcal{Q}$ in the matrix bordered by unit vectors, as in Lemma 4.9. Then*

- *(a) every minor of H that contains $\mathcal{Q}_\mathrm{r} \setminus A_0$ and $\mathcal{Q}_\mathrm{c} \setminus A_0'$ equals $\pm \det H_\mathcal{Q}$ times a minor of* Sw *with rows $B_0 \cup A_0'$ and columns $B_0' \cup A_0$, where B_0 and B_0' are the remaining rows and columns.*
- *(b) there are vectors $g_z \in \mathbb{Q}(T)^2$, one for each label z, such that $(z' - z)\,\mathrm{Sw}(z, z') = g_z \wedge g_{z'}$ for all labels $z \ne z'$.*

Proof. The first identity is the definition of the Löwner matrix. With the diagonal matrix $\mathrm{diag}(t)$ of the nodes it says that $\mathrm{diag}(t)H - H\,\mathrm{diag}(t) = f\mathbf{1}^\top - \mathbf{1}f^\top$ has rank two. The bordered matrix satisfies the same equation with generators $(1, f_t)$ on actual nodes and 0 on the border, since $(t-t)\cdot 1 = 0$. A Schur complement of a block preserves this displacement structure, with the generators given below. Laplace expansion along the unit border gives (a). □

The generators are explicit. On a label z outside the pivot $g_z = (1 - H(z, \mathcal{Q}_\mathrm{c})H_\mathcal{Q}^{-1}\mathbf{1},\ f_z - H(z, \mathcal{Q}_\mathrm{c})H_\mathcal{Q}^{-1}f)$, and on a pivot label t it is $((H_\mathcal{Q}^{-1}\mathbf{1})_t, (H_\mathcal{Q}^{-1}f)_t)$ up to sign. We rescale them as $\tilde{g}_z = (p^{-3}g_{z,1}, p^{-1}g_{z,2})$ on pivot labels and $\tilde{g}_z = (g_{z,1}, p^2 g_{z,2})$ on the others, and the pivot rows

and columns of Sw by p^{-3}. The rescaled matrix $\widetilde{\mathrm{Sw}}$ then has entries $p^{-2}(\tilde g_z \wedge \tilde g_{z'})/(z'-z)$.

The reduction of the pivot is governed by matrices on period indices. For $\iota \in \{0,1\}$ put $s_\iota(j) = p^2 S(j,\iota)$, $s'_\iota(j) = p^3 G(j,\iota)$, $\Gamma_\iota(j) = (1, s_\iota(j))$, and

$$B^\iota(l,j) = \frac{s_\iota(j) - s_\iota(l)}{j-l} \ (l \neq j), \qquad B^\iota(j,j) = T - s'_\iota(j).$$

Modulo p one has $s_\iota(j) \equiv -\gamma_2 h_j$, where $h_j = \sum_{i \le j} i^{-2}$. So the off-diagonal entries of B^ι vanish modulo p when $p \mid \gamma_2$. This is why $p \nmid \gamma_2$ is assumed from now on. Let $J_X = \{\kappa, \kappa+2, \dots, \kappa+2k'-2\}$, $J_Y = J_X + 1$, and $B^\iota_X = B^\iota[[0,\kappa) \cup J_X, [0,\kappa) \cup J_Y]$. Write $\bar B_X$ for its reduction modulo p, which does not depend on ι. If $t \equiv t' \pmod p$ then $p^3 H(t,t') \equiv B^\iota(j_t, j_{t'}) \pmod{p^3}$, where j_t is the period of t. If $t \not\equiv t'$ then $p^3 H(t,t') \in p\mathcal{O}$. Hence $p^3 H_{\mathcal{Q}}$ is congruent modulo p to a direct sum of copies of $\bar B_X$ and its transpose, one for each residue class.

Theorem 9.4 (nondegeneracy)**.** *Let $\kappa \ge 1$, $k' \ge 0$, $p > \kappa + 2k'$ and $p \nmid \gamma_2$. Then $\det \bar B_X \neq 0$ in $\mathbb{F}_p(T)$.*

Proof. For $k' = 0$ the determinant is monic in T. Let $k' \ge 1$. The variable occurs only on the diagonal of the block $[0,\kappa)$, so $\det \bar B_X$ is $\pm \det \bar B[J_X, J_Y]$ times a monic polynomial. Since $\bar B[J_X, J_Y]$ is $-\gamma_2$ times the matrix $[(h_j - h_l)/(j-l)]_{l \in J_X, j \in J_Y}$ and $p \nmid \gamma_2$, it suffices that the determinant of the latter is nonzero in $\mathbb{F}_p$. Suppose that a vector $\mathbf{y} \neq 0$ lies in its kernel. Let $P = \prod_{j \in J_Y}(z-j)$, $\mathsf{u} = \sum_j y_j P/(z-j)$ and $\mathsf{w} = \sum_j y_j h_j P/(z-j)$. These polynomials over $\mathbb{F}_p$ have degree at most $k'-1$, $\mathsf{u} \neq 0$, and $\mathsf{w}(t) = h_t \mathsf{u}(t)$ on the $2k'$ consecutive nodes $\kappa, \dots, \kappa + 2k' - 1$. Since $h_{t+1} - h_t = (t+1)^{-2}$, the polynomial

$$\mathsf{E}(z) = (z+1)^2\big(\mathsf{w}(z+1)\mathsf{u}(z) - \mathsf{w}(z)\mathsf{u}(z+1)\big) - \mathsf{u}(z)\mathsf{u}(z+1)$$

vanishes at the $2k'-1$ nodes other than the last. The top terms cancel, so $\deg \mathsf{E} \le 2k'-2$ and $\mathsf{E} = 0$. Divide u and w by their greatest common divisor. At $z = -1$ the identity gives $\mathsf{u}(-1)\mathsf{u}(0) = 0$. If $\mathsf{u}(x+1) = 0$ with $x \neq -1$ then $\mathsf{u}(x) = 0$, and conversely, by coprimality. So every element of $\mathbb{F}_p$ is a root of u, whose degree is less than p. This is a contradiction. □

The argument is that of Theorem 5.6, in characteristic p. The double pole at -1 is isolated in its orbit, and the orbit is now finite, which replaces the infinitely many roots by p roots.

Proposition 9.5 (generators)**.** *Let $p > \kappa + 2k'$ and $p \nmid \gamma_2$. Then all $\tilde g_z$ lie in $\mathcal{O}^2$. There are vectors $\mathcal{V}(\mathbf{c}) \in \mathcal{O}^2$ on the cells of the periods outside the pivot, of the period $\kappa - 1$ and of the period $\kappa + 2k' - 1$, such that every label z in these periods satisfies $\tilde g_z \equiv \eta_z \mathcal{V}(\mathbf{c}(z)) \pmod{p^2}$ with a unit $\eta_z \equiv 1 \pmod p$,*

and $\mathcal{V}(j,0) \equiv \mathcal{V}(j,1) \pmod p$. *Consequently, for two such labels* $z \neq z'$,

$$v(\widetilde{\mathrm{Sw}}(z,z')) \geq -3 \text{ if } z \equiv z' \pmod p,$$
$$v(\widetilde{\mathrm{Sw}}(z,z')) \geq -W_3(\mathbf{c}(z),\mathbf{c}(z')) \text{ otherwise,}$$

and $v(\widetilde{\mathrm{Sw}}(t,t)) \geq -3$ *on deleted nodes.*

Proof. Put $\widetilde{H} = p^3 H_{\mathcal{Q}}$. By Theorem 9.4 and the direct sum above, $\widetilde{H}$ is invertible over $\mathcal{O}$. We look for the solution Θ of $\widetilde{H}\Theta = (\mathbf{1}, p^2 f)$ modulo p^2. Modulo p^2 the right side is a function of the cell, and modulo p of the period. Put $\Theta(t) = \eta(t)\mathbf{y}^{\iota}(j_t)$ with $\mathbf{y}^{\iota} = (B^{\iota}_X)^{-1}\Gamma_{\iota}$ on a class oriented like $\hat{X}$, and the transposed matrix on the others. The congruent part of $\widetilde{H}(z,\cdot)\Theta$ is then $\eta\, B^{\iota}(j_z,\cdot)\mathbf{y}^{\iota}$, which is $\eta\,\Gamma_{\iota}(j_z)$ on the pivot. The non-congruent part lies in $p\mathcal{O}$, and modulo p^2 it equals

$$p \sum_{\omega' \neq \omega_z} \frac{1}{\omega' - \omega_z}(\Omega_{\omega'}[1,\cdot] - \bar{s}(j_z)\Omega_{\omega'}[0,\cdot]), \qquad \Omega_X = \bar{\Gamma}^{\top}\bar{B}_X^{-1}\bar{\Gamma},$$

with $\Omega_Y = \Omega_X^{\top}$. Here ω_z is the residue of z, $\Omega_{\omega'}$ is Ω_X or Ω_Y according to the orientation of the class of ω', and bars denote reductions modulo p. We use that the pivot consists of complete periods, so every residue carries Ω_X or Ω_Y and nothing else. Since $\sum_{\omega' \neq \omega}(\omega' - \omega)^{-1} \equiv 0 \pmod p$, only the antisymmetric part $\Omega_X - \Omega_X^{\top}$ survives. It acts on 2-vectors as a scalar δ_{rot} times the rotation, so the sum is $p\,\delta_{\mathrm{rot}}\sigma(\omega_z)\bar{\Gamma}(j_z)$, where $\sigma(\omega)$ is the sum of $(\omega' - \omega)^{-1}$ over the residues ω' of one orientation. Taking $\eta = 1 - p\,\delta_{\mathrm{rot}}\sigma$ gives $\widetilde{H}\Theta \equiv (\mathbf{1}, p^2 f) \pmod{p^2}$ on the pivot, and the same computation outside the pivot gives $\tilde{g}_z \equiv \eta(\Gamma_{\iota}(j_z) - B^{\iota}(j_z,\cdot)\mathbf{y}^{\iota}) \pmod{p^2}$. These are the vectors $\mathcal{V}$. They depend on the cell modulo p^2, and on the period modulo p, since B^{ι} and Γ_{ι} do not depend on ι modulo p. For $k' = 0$ we have $\delta_{\mathrm{rot}} = 0$.

By Lemma 9.3, $\widetilde{\mathrm{Sw}}(z,z') = p^{-2}(\tilde{g}_z \wedge \tilde{g}_{z'})/(z' - z)$. The wedge lies in $p^2\mathcal{O}$ for two labels in one cell and in $p\mathcal{O}$ for two labels in one period. The difference $z' - z$ has valuation 1 if $z \equiv z'$ and 0 otherwise. The diagonal entry on a deleted node is $-p^{-3}\widetilde{H}^{-1}(t,t)$. □

9.3. **The local bounds from elimination.** We combine Lemma 9.3(a) with Proposition 9.5. A potential on the labels bounds the Leibniz terms of the minor of $\widetilde{\mathrm{Sw}}$, and the factor $\det H_{\mathcal{Q}}$ has valuation exactly $-3|\mathcal{Q}_{\mathrm{r}}|$.

Theorem 9.6 (local bounds of the second plane)**.** *Let* $p \geq 5$ *with* $p^2 > 2\bar{N} + 2p$ *and* $p \nmid \gamma_2$, *and let* $2m \leq r$. *For every member the following hold.*

- *(i)* (E_1) *If* $p > 2m$, *then* $e_p(r, I) \leq 3r + \sum u + \sum u'$ *for all nonnegative potentials* u *on* X' *and* u' *on* Y' *with* $W_3 \leq u \oplus u'$ *on the border.*
- *(ii)* (E_2) *If* $p \leq r$ *and* $\bar{N} \leq 2p$, *then* (E_1) *holds with* $3r$ *replaced by* $3r - (2p - \bar{N})$.
- *(iii)* (R) *If* $p \leq 2m$, *write* $2m = kp + s$ *with* $0 \leq s < p$, *and let* $E = [r, r+s)$ *for even* k *and* $E = [r+s, r+p)$ *for odd* k. *For* $t \in E$ *let* $c_b(t) \in \{0,1,2,3\}$ *be twice the difference of*

the periods of t and of r, plus the half of t. Then, for each $w \in \{(2,2,0,1),(2,2,1,0),(0,1,2,2),(1,0,2,2),(1,1,1,1)\}$,

$$e_p(r,I) \le 3(r+m) - 3\|m\|_p + \sum_{t\in E} w(c_b(t)).$$

Proof. In each case $\kappa+2k' \le \bar N/p+1 < p$, so Theorem 9.4 and Proposition 9.5 apply.

(i) Take $\kappa = \lfloor r/p\rfloor + 1$ and $k' = 0$. Since $2m < p$, no two border nodes are congruent. The nodes of $[\bar N, \kappa p)$ lie in one period with the border nodes beyond $\kappa p - p$. Give them the potentials $\frac{3}{2}$ on both sides, and the deleted nodes the potentials 3. Proposition 9.5 bounds every Leibniz term of the minor of $\widetilde{\mathrm{Sw}}$, and Lemma 9.3(a) with the count of the pivot gives the bound for the Gauss valuation. The coefficient of x^i carries an extra $3i \ge 0$.

(ii) Take $\kappa = 2$, $k' = 0$ and T replaced by pT. Then $\bar B_X = \gamma_2 \left(\begin{smallmatrix} 0 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, because there is no pole at $k = 0$. It is invertible since $p \nmid \gamma_2$, and its inverse has entry 0 at $(1,1)$. So $\widetilde H^{-1}(t,t) \in p\mathcal O$ for t in the period 1, and the virtual nodes $[\bar N, 2p)$ need the potentials 1 only. This saves $2p - \bar N$.

(iii) Take $\kappa = \lfloor r/p\rfloor + 1$ and $k' = \lfloor k/2\rfloor$, and put $h^* = (\kappa + 2k')p$. Then $h^* \le \bar N + p$. The *virtual nodes* $E_v = [\bar N, h^*)$ are empty if $h^* \le \bar N$. The labels of Lemma 9.3(a) outside I are the nodes of

$$\mathcal L = [r, \kappa p) \cup E_v \cup [h^*, \bar N).$$

On $[r, \kappa p)$ and on $[h^*, \bar N)$ a node of $\hat X$ is a row label and a node of $\hat Y$ a column label. On E_v the two types are exchanged, since there the pivot rows and columns that are not actual rows and columns become labels. Put $\mathrm{sh}(t) = t - 2k'p$ for $t \ge h^*$ and $\mathrm{sh}(t) = t$ otherwise. This map keeps parity, residue and half, and it preserves the weights of Proposition 9.5. The set E has $2\|m\|_p$ nodes and lies in the periods $\lfloor r/p\rfloor$ and $\lfloor r/p\rfloor + 1$, so c_b maps it into four fixed cells. Each w in the list dominates W_3 on these four cells.

Choose a period j_0 as follows. If E meets one period, j_0 is that period. If E is empty, which happens when $\|m\|_p = 0$, put $j_0 = \kappa - 1$. If E meets two periods j and $j+1$, put $j_0 = j$ when $w \ge 1$ on the cells of E in $j+1$, and $j_0 = j+1$ otherwise. This is always possible, since for each w in the list a zero in one period forces $w = 2$ on the other. Define the potential of a label t, the same on both sides, by

$$\phi(t) = \begin{cases} w(c_b(\mathrm{sh}\, t)) & \text{if } \mathrm{sh}\, t \in E, \\ 1 & \text{if } \mathrm{sh}\, t \notin E \text{ lies in the period } j_0, \\ 2 & \text{otherwise.} \end{cases}$$

Then $W \le \phi(t) + \phi(t')$ for every row label t and column label t', where W is the bound of Proposition 9.5. Two labels of E are covered by the dominance of w. A label outside E in the period j_0 against a label of E has $W \le 1$ inside j_0, and $W = 2 \le 1 + w$ outside j_0, by the choice of j_0. A label outside E and outside j_0 has potential 2, and every pair with it

that is not congruent has $W \le 2$. Every congruent pair $(t, t+p)$ off E has exactly one node in j_0. So does every pair of $U_2 = [r+s, \kappa p)$ and E_v when k is even and $h^* > \bar{N}$, since $t \mapsto t + kp$ maps U_2 onto E_v. Such pairs have $W \le 3 = 1+2$. All other pairs off E have $W \le 2$. The sum of the potentials, with the potentials 3 on deleted nodes and the count of the pivot as in (i), is $3(r+m) - 3\|m\|_p + \sum_{t\in E} w(c_b(t))$. □

Each bound of Theorem 9.6 is at least $2(r+m)$, which justifies the remark before Proposition 9.2 on the primes $p \mid \gamma_2$. Indeed (E_1) is at least $3r \ge 2r + 2m$, (E_2) is at least $4r + 2m - 2p \ge 2r + 2m$ since $p \le r$, and (R) is at least $3r + 3m - 3\|m\|_p \ge 3r \ge 2r + 2m$.

For $k = 1$ part (iii) needs only the pivot of (i), with the weight 3 of congruent pairs kept. For $k \ge 2$ it needs the complete double periods. The pivot contracts the window of length $2m$ to one of length $p + s$ or s, and the cost of each eliminated pair of periods is exactly 3 per node.

9.4. **The denominator exponent.** Homogenise with $\theta = p/r$ as before. The profile $\mathcal{P}(\theta)$ is the minimum of the homogenised bounds of Proposition 9.2 and Theorem 9.6. For (E_1) and (E_2) the potentials are constant on the at most three cells cut from the border at the multiples of $\theta/2$. On the band $\frac{2\alpha}{k+1} < \theta \le \frac{2\alpha}{k}$, the bound (R) is

$$3 + \tfrac{3}{2}k\theta + \tau(E) \ (k \text{ even}), \qquad 3 + 6\alpha - \tfrac{3}{2}(k+1)\theta + \tau(E) \ (k \text{ odd}),$$

where $\tau(E)$ is the least over the five vectors w of the sums $\sum w(\mathbf{c})\,|\mathbf{c} \cap E|$, taken over the four cells $\mathbf{c}$, scaled by r, $|\mathbf{c} \cap E|$ is the length of the intersection, and $E = [1, 1 + 2\alpha - k\theta]$, respectively $E = [1 + 2\alpha - k\theta, 1 + \theta]$. Each node of E carries one potential, so there is no factor $\frac{1}{2}$. Between the points $\theta = 2/j$ and $\theta = 2(1+2\alpha)/j$ the cell type of E is fixed. So $\mathcal{P}$ is piecewise affine, with finitely many pieces on every interval bounded away from 0.

On $\theta < 2\alpha$ the bounds (P1) and (R) are closed under translation in the sense of Theorem 4.15, for $\alpha = \frac{3}{41}$, with $K_0 = 3 + 3\alpha$. The translation of order ν takes the band k to the band $k + 6\nu$. So the table for Λ is checked on one period of $1/\theta$, the bands $k = 1, \dots, 6$, and on the head $\theta \ge 2\alpha$. Its translates cover the bands down to $t_1 = 3 \cdot 10^6/(1025 \cdot 10^6 + 3)$, just below $\alpha/25$. On $(0, t_1]$ we use (P1) without its last term.

Theorem 9.7 (denominators on the second plane)**.** *Let $\alpha = \frac{3}{41}$ and $\gamma_2 \ne 0$, with γ_1 any integer. For every $\epsilon > 0$ and every even $r \ge R(\epsilon)$, every member with $|I| \le 1$ satisfies* $\log \operatorname{den} P_{r,I} \le (\Lambda_2 + \epsilon) r^2$, *where*

$$\Lambda_2 = (3 + 3\alpha)t_1 + \int_{t_1}^{\infty} \mathcal{P}(\theta)\, d\theta \le \frac{3476837531}{10^9}.$$

The threshold does not depend on (γ_2, γ_1).

Proof. For each prime with $p^2 > 2\bar{N} + 2p$ the local bounds give $e_p(r, I) \le r\,\mathcal{P}(p/r) + O(1)$. For $p \mid \gamma_2$ the bounds of Theorem 9.6 are replaced by the trivial bound $2(r+m)$, which is at most each of them. For the bounds (E_1),

(E$_2$) and (R) the cells of the integer window and of its homogenisation differ by a bounded number of cells of bounded length, which take the potential 2. The hypotheses $p > \kappa + 2k'$ of Proposition 9.5 hold for r large, since $p > t_1 r$ on the bands used. For the other primes every entry has valuation at least $-3\log(2\bar{N})/\log p$, which costs $O(r^{3/2})$ in total. The prime number theorem then gives the integral, as in Theorem 4.14, and Theorem 4.15 gives the pieces of the translated bands from those of the period. All pieces are affine, so the integral is a rational number. It is at most the stated one. □

For $\gamma_2 = 0$ we take the frame $F^{(2)}_{0,1} = \psi(z+1) - \psi(z+\frac{1}{2})$. Its matrix M is 4 times that of $F_{1/2}$ in (2.1), so the coefficient of x^i in a member with $|C \setminus I|$ common nodes is $4^{|C\setminus I|+m-i}$ times the corresponding coefficient for $F_{1/2}$. Its denominators are therefore at most those of $F_{1/2}$. Theorem 4.14, with an admissible table for $q = 2$ and $c = \frac{6}{41}$, bounds them with the exponent $\Lambda_\pi = 2.60981\,51766\ldots$, which is less than Λ_2.

9.5. **Decay and proof of Theorem 1.2.** By Theorem 5.6 with $d \le 3$, one deleted node suffices for nonvanishing. The decay is Theorem 6.7 with the reference frame of shifts 1, $\frac{3}{4}$, $\frac{1}{2}$.

Proposition 9.8. *Take the reference frame of shifts* 1, $\frac{3}{4}$, $\frac{1}{2}$ *and* $c = \frac{6}{41}$. *There is a trial measure with* 49 *polynomial pieces and a cover by* 508 *cells satisfying the checks of §6.4 with* $\varepsilon_0 = 10^{-6}$ *and* $|\mathcal{T}| = 3$. *For it Theorem 6.2 gives* $D \ge 3.4982052$ *and* $\rho \le 7.1600631$.

Proof of Theorem 1.2. Let a and b be rationals, not both zero. Choose a rational $\lambda \ne 0$ for which $\gamma_2 = \lambda a$ and $\gamma_1 = 6\lambda b$ are integers. By §3.2 the number ξ of $F^{(2)}_{\gamma_2,\gamma_1}$ is $-2\lambda(a\zeta(3) + b\pi^2)$. For $a = 0$ we take λ with $\gamma_1 = 1$, so the frame is $\psi(z+1) - \psi(z+\frac{1}{2})$. Its trigamma term has weight zero, and it has two simple poles per period. For $b = 0$ we take λ with $\gamma_2 = 1$, so the frame is $\psi'(z+1)$. Its shift $\frac{1}{2}$ has weight zero, and it has one double pole per period. Both are handled exactly as the first plane handles a weight zero, through a frame on the same shifts. The geometry is the same for all (a, b). For $\gamma_2 \ne 0$, Theorem 9.7 gives the denominators with Λ_2. For $\gamma_2 = 0$, the exponent Λ_π above gives them, and $\Lambda_\pi < \Lambda_2$. Proposition 9.8 with Theorem 6.7 gives the decay for every frame, also when one weight is zero, and Theorem 5.6 gives nonvanishing with one deleted node, since $d \le 3$. Theorem 11.1 gives

$$\mu(\xi) \le \frac{\rho}{D - \Lambda_2} \le 335.0886 < 335.20. \qquad \square$$

Theorem 1.5 gives better bounds at the two ends, by separate proofs.

9.6. **The value $\zeta(3)$.** For the separate bound of Theorem 1.5 we take $\gamma_1 = 0$, $\gamma_2 = 1$ and $c = \frac{3}{10}$. The frame is $\psi'(z+1)$, with $d = 2$ and $\xi = -2\zeta(3)$. Everything above holds with the half ι set to 0. The cells are then the periods, W_3 takes the values 0 and 2, and $\varepsilon = 0$ in Proposition 9.2. The bounds (P4)

and (P5) become $6(r+m-p)$ and $3(\bar{N}-p)_+$, and (P6), (P7) vanish. In (R) the four cells become two, and the potential vectors are $(0,0,2,2)$ and $(2,2,0,0)$. With $c=\frac{3}{10}$, and with (P1) without its last term on $(0,\frac{3}{500}]$, the profile has integral

$$\Lambda_{\zeta(3)} \le \frac{2912839}{10^6}.$$

The reference frame is the function $F_{1/2}$, and its certificate is that of the row for π^2 in Table 2. Theorem 5.6 with $d=2$ gives nonvanishing with one deleted node.

Proof of Theorem 1.5 for $\zeta(3)$. The three estimates above and Theorem 11.1 give $\mu(\zeta(3)) \le \rho/(D-\Lambda_{\zeta(3)}) \le 6.034884 < 6.0382$. The bound for π^2 is proved in Section 11, together with Theorem 1.6. □

10. The third plane

In this section we prove Theorem 1.3. We use the frames $F^{(3)}_{\gamma_2,\gamma_3}$ of §3.2, with rationals γ_2 and $\gamma_3 \neq 0$, and the alternating geometry of parameter $c=\frac{13}{100}$. Thus $\alpha=\frac{13}{200}$ and $m=\lceil 13r/200-\frac{1}{2}\rceil$. The pole at the shift 1 has order three. We combine four assignment bounds with a fifth bound obtained by eliminating complete periods. None of these bounds requires a nonzero coefficient to be a unit modulo p.

10.1. **Weights.** We state the bounds for a single shift with a pole of any order. Let $h \ge 2$, and let $c_1,\dots,c_{h-1}$ be rationals, any of which may vanish. Let M be the matrix of §3.2 built from the increment

$$\delta(k-1)=\sum_{j=1}^{h-1} c_j k^{-j} \qquad (k\ge 1),$$

the increment of a rational combination of $\psi(z+1),\psi'(z+1),\dots,\psi^{(h-2)}(z+1)$. The frames of the third plane are the case $h=4$, with $c_1=0$, $c_2=-\gamma_2$ and $c_3=2\gamma_3$. As in Section 9, write a node as $t=jp+\omega$ with $0\le\omega<p$, and call j its period.

Lemma 10.1 (weights)**.** *Let p be an odd prime with $p^2>\bar{N}$ that divides no denominator of the c_j. Let $W(t,t')$ be 0 for distinct nodes in the same period, $h-1$ for nodes in different periods that are not congruent modulo p, and h for distinct congruent nodes. Let $W(t,t)=h\cdot\mathbf{1}[t\ge p]$. Then the entry of $\Phi_{r,I}$ at (t,t') has Gauss valuation at least $-W(t,t')$.*

Proof. For $t<t'$ the entry is $(f_{t'}-f_t)/(t'-t)$, and $f_{t'}-f_t$ is a sum of the terms c_jk^{-j} with $t<k\le t'$. Since $p^2>\bar{N}$, every $k\le\bar{N}$ has $v_p(k)\le 1$. So each term has valuation at least $-(h-1)$, and it is a p-integer unless $p\mid k$. If t and t' lie in one period, no multiple of p lies in $(t,t']$. The difference $t'-t$ has valuation 1 if $t\equiv t' \pmod p$ and 0 otherwise. The diagonal entry is $x-e_t$, and e_t is a sum of the terms jc_jk^{-j-1} with $k\le t$. These have valuation at least $-h$, and they are p-integers for $t<p$. □

No coefficient needs to be a p-unit. A prime that divides a numerator can only raise the valuations.

10.2. **Four assignment bounds.**

Proposition 10.2. *Let p be as in Lemma 10.1. For every member $P_{r,I}$, $e_p(r,I)$ is at most each of the following, under its condition.*

	bound	*condition*
(Q1)	$h(r+m) - \|m\|_p$	*none*
(Q2)	$2h(r+m-p)$	$p \le r$
(Q3)	$(2h-1)r + 2(h-1)m - 2(h-1)p$	$\frac{r}{2} \le p \le r$, $2m \le p$, $2p \le \bar{N} \le 3p$
(Q4)	$h(\bar{N}-p)_+$	$p \ge r$

Proof. In each case we give potentials u on the rows and u' on the columns with $W \le u + u'$, and apply Lemma 4.1. The potentials are nonnegative. So deleting common nodes only removes terms, and the bound holds for every member.

(Q1) For a residue a modulo p let ε_a be the number of border rows in the class a minus the number of border columns, and put $\varsigma_a = -\frac{1}{2}\operatorname{sign}(\varepsilon_a)$. Take $u = \frac{h}{2} + \varsigma_a$ on every row of the class a and $u' = \frac{h}{2} - \varsigma_a$ on every column. Two nodes of one class receive the sum h, and two nodes of different classes at least $h-1$. The common nodes contribute h each. The border contributes $hm - \frac{1}{2}\sum_a |\varepsilon_a|$. Every block of $2p$ consecutive border nodes meets each class once as a row and once as a column, since p is odd. The remaining $2(m \bmod p)$ consecutive nodes give $\sum_a |\varepsilon_a| = 2\|m\|_p$.

(Q2) and (Q4) Take $u = u' = h \cdot \mathbf{1}[t \ge p]$. Two nodes below p lie in the period 0 and have weight 0. Every other pair has a node of potential h. For $p \le r$ there are $r - p + m$ nodes at least p on each side. For $p \ge r$ the nodes at least p are border nodes, $(\bar{N}-p)_+$ in all.

(Q3) Take $u = u'$ with the values 1, 0, $h-1$, h, $h-1$ on the intervals $[0, r-p)$, $[r-p, p)$, $[p, r)$, $[r, 2p)$ and $[2p, \bar{N})$. The conditions make these intervals consecutive, and every node lies in the periods 0, 1 or 2. A diagonal entry at $t \ge p$ receives $2(h-1) \ge h$. Let $t < p \le t'$ be two nodes. If they are not congruent, t' has potential at least $h-1$. If $t' = t + p < r$, then $t < r - p$, and the sum is $1 + (h-1) = h$. If $t' = t + p \ge r$, then t' has potential h. If $t' = t + 2p$, then $t < \bar{N} - 2p \le r - p$ since $2m \le p$, and the sum is again h. Two nodes at least p in different periods receive at least $2(h-1) \ge h$. Counting the common nodes twice and the border nodes once gives $2(r-p) + 2(h-1)(r-p) + h(2p-r) + (h-1)(\bar{N}-2p)$, which is the stated bound. □

10.3. **The refined period bound.** Eliminating the complete periods below the border improves the first assignment bound. The argument applies to every rational choice of the coefficients.

Proposition 10.3. *Let $h \geq 3$, and let $p \geq 5$ be a prime with $p^2 > 2\bar{N}+2p$ that divides no denominator of the c_j. Suppose that $2m < p$. Let $A = (\lfloor r/p \rfloor + 1)p$ be the least multiple of p above r, and put $\beta = \max(0, \min(A - r, \bar{N} - A))$. Then every member satisfies*

$$e_p(r, I) \leq hr + (h-3)m + 2\beta. \tag{Q5}$$

So β is the number of border nodes on the smaller side of A, and $\beta = 0$ when the border lies below A. Since $2m < p$, the bound (Q1) is $hr+(h-1)m$. So (Q5) lies below (Q1) by $2(m - \beta)$.

Proof. We follow §9 with $\kappa = A/p$ and $k' = 0$. Put $\bar{N}' = \max(\bar{N}, A) \leq \bar{N}+p$, so that $p^2 > 2\bar{N}'$, and let $H = M + p^{-h}T\mathrm{Id}$ on $[0, \bar{N}')$. The pivot consists of the complete periods $[0, A)$, on the rows and on the columns. Lemma 9.3 uses only the displacement identity, so it holds for this H.

The pivot. Let $S(j)$ be the sum of the terms $c_i k^{-i}$ of f_t with $p \mid k$, for a node t of the period j, and $G(j)$ the same for e_t. Then $f_t - S(j)$ and $e_t - G(j)$ are p-integers. Since $p^2 > \bar{N}'$, the numbers $s(j) = p^{h-1}S(j)$ and $s'(j) = p^h G(j)$ are p-integers. Define B on the period indices $[0, \kappa)$ by

$$B(l, j) = \frac{s(j) - s(l)}{j - l} \ (l \neq j), \qquad B(j, j) = T - s'(j).$$

If $t \equiv t' \pmod p$, then $p^h H(t, t') \equiv B(j_t, j_{t'}) \pmod{p^{h-1}}$. If $t \not\equiv t'$, then $p^h H(t, t') \in p\mathcal{O}$, because every entry has valuation at least $-(h-1)$ off the diagonal. So $p^h H[[0, A), [0, A)]$ is congruent modulo p to a direct sum of copies of $\bar{B}$, one for each residue class. The variable T occurs only on the diagonal of B, so $\det \bar{B}$ is monic in T, and the pivot is invertible over $\mathcal{O}$. No coefficient c_j needs to be a p-unit.

The weights of the Schur complement. Since $h \geq 3$, the right side $(\mathbf{1}, p^{h-1}f)$ of the pivot system is, modulo p^2, a function of the period. The pivot has the same complete periods on both sides, so B is symmetric and every residue class carries the same matrix. The proof of Proposition 9.5 then applies with the halves omitted, with p^3 and p^2 replaced by p^h and p^{h-1}, and with $\delta_{\mathrm{rot}} = 0$. It gives vectors $\mathcal{V}(j)$ with $\tilde{g}_z \equiv \eta_z \mathcal{V}(j_z) \pmod{p^2}$ and $\eta_z \equiv 1 \pmod p$, for the labels z in the periods $\kappa - 1$ and κ. The rescaled Schur complement has entries $p^{-(h-1)}(\tilde{g}_z \wedge \tilde{g}_{z'})/(z' - z)$. So for labels $z \neq z'$ its entry has valuation at least $-h$ if $z \equiv z' \pmod p$, at least $-(h-3)$ if $z \not\equiv z'$ lie in one period, and at least $-(h-1)$ otherwise. On the diagonal it has valuation at least $-h$.

Potentials. The labels outside I are the border nodes and, when $\bar{N} < A$, the virtual nodes $[\bar{N}, A)$. A virtual node is a label once as a row and once as a column. All labels lie in an interval of length at most p that starts at r. So they lie in the periods $\kappa - 1$ and κ, the latter only from A on, and a row label is congruent to a column label only when both are the same virtual node. If $\bar{N} > A$, let $\chi(t) = 1$ for the border nodes on the side of A that has fewer of them, and $\chi(t) = 0$ for the others. If $\bar{N} \leq A$, let $\chi = 0$. Give

a border row the potential $h-3+2\chi$, a border column 2χ, a virtual node $h-3$ as a row and 3 as a column, and a deleted node h on both sides. A virtual node against itself has weight $h=(h-3)+3$. Two labels in one period that are not congruent have weight $h-3$. Two labels on opposite sides of A have weight $h-1$, and exactly one of them has $\chi=1$. A deleted node carries the largest weight h. So the potentials dominate the weights. As in the proof of Theorem 9.6(i), Lemma 9.3(a) and the count of the pivot give $e_p(r,I)\le hr+\sum u+\sum u'$, with the sums over the border rows and columns. The border carries m rows and β nodes with $\chi=1$, so the sum is $(h-3)m+2\beta$. □

10.4. **The denominator exponent.** Homogenise with $\theta=p/r$ and $m\mapsto\alpha r$ as in §4.5. For $h\ge 3$ and $0<\alpha\le\frac13$ the minimum $\mathcal{P}_h(\theta)$ of the four homogenised bounds is

range of θ	$\mathcal{P}_h(\theta)$
$0<\theta\le\frac{1+\alpha}{2}$	$h(1+\alpha)-\theta\lVert\alpha/\theta\rVert$
$\frac{1+\alpha}{2}\le\theta\le\frac12+\alpha$	$2h-1+2(h-1)\alpha-2(h-1)\theta$
$\frac12+\alpha\le\theta\le1$	$2h(1+\alpha-\theta)$
$1\le\theta\le1+2\alpha$	$h(1+2\alpha-\theta)$
$\theta\ge1+2\alpha$	0

The values agree at the ends of the ranges. The low range is the sawtooth of §4.6. We take the minimum with (Q5). On $\frac1j\le\theta\le\frac{1}{j-1}$, away from the endpoints, its homogenised form is

$$h+(h-3)\alpha+2\max\bigl(0,\min(j\theta-1,1+2\alpha-j\theta)\bigr).$$

At an endpoint we use the next multiple of p in Proposition 10.3. The formal proof integrates an affine majorant of these bounds, rather than the ideal profile.

Theorem 10.4 (denominators on the third plane)**.** *Let $\alpha=\frac{13}{200}$, and let γ_2 and $\gamma_3\ne0$ be rationals. Let $K\ge0$ and $\epsilon>0$. For every sufficiently large even r, every member of $F^{(3)}_{\gamma_2,\gamma_3}$ with $|I|\le K$ satisfies* $\log\operatorname{den}P_{r,I}\le(\Lambda_3+\epsilon)r^2$, *where*

$$\Lambda_3=\frac{2093141963035255548476961845489855904191277848149}{624885257513513299995586450872\cdot10^{18}}$$
$$=3.3496420948\ldots$$

Proof. Apply (Q1)–(Q4) and add (Q5) on the head $\theta\ge2\alpha$. The chambers $\frac1j\le\theta\le\frac{1}{j-1}$ for $j=2,\ldots,8$ cover $[2\alpha,1]$. In each chamber (Q5) has three affine forms: $hr+(h-3)m$ if $\bar N\le jp$, $hr+(h-3)m+2(jp-r)$ if $jp\le r+m$, and $hr+(h-3)m+2(\bar N-jp)$ if $r+m\le jp\le\bar N$. The endpoints are checked with the actual next multiple in Proposition 10.3. Its prime-square condition holds for large r on this head. Oddness of p turns $2m\le p$ into $2m<p$.

The low part uses (Q1) and Theorem 4.15. Translation preserves $\|m\|_p$ and increases $r+m$ by $213\nu p$ when r and m increase by $200\nu p$ and $13\nu p$. The checked table has a constant prefix, a low part and a head. Put $\varepsilon = 10^{-8}$, $\delta = 10^{-9}$ and $H = h(1+\alpha)$. The prefix is the constant $426001/100000$ on $(0, \frac{\alpha}{40}]$. On $(\frac{\alpha}{n+1}, \frac{\alpha}{n}]$ we have $\lfloor \alpha/\theta \rfloor = n$, and (Q1) is the nth tooth $H - \theta\|\alpha/\theta\|$ of the sawtooth. For $n = 1, \dots, 13$ the base cells are the two halves of this tooth, each raised by ε. They cover $1/\theta \in [1/\alpha, 14/\alpha]$, an interval of length 200, which is one period of $1/\theta$. With $h = 4$ the bound $h(r+m)$ grows by $852\nu p$, so (Q1) is closed under translation with $K_0 = H$. Theorem 4.15 with $\nu = 0, 1, 2$ translates the base cells to the teeth $n + 13\nu$. The translated line is the line of the tooth $n + 13\nu$ plus $\varepsilon(1 - 200\nu\theta)$, and the three strips tile $(\frac{\alpha}{40}, \alpha]$. The head is (Q1) on $(\alpha, 2\alpha]$, then the three forms of (Q5) and a bridge of (Q1) of width δ in each chamber, then (Q3), (Q2) and (Q4), each raised by ε. It ends at $1 + 2\alpha + \varepsilon/8$ and has twenty-nine pieces. The table includes the boundary corrections for $m = \lceil \alpha r - \frac{1}{2} \rceil$. Its integral is the rational number Λ_3 above.

The local bounds and the table contain no coefficient parameters. Only primes dividing the denominators of γ_2 or $2\gamma_3$ need to be excluded. Their contribution, together with the small primes, is $o(r^2)$ for each fixed frame. The weighted prime number theorem and the periodic summation give the stated bound, as in Theorem 7.3. Deleting common nodes does not change the upper bound. □

The threshold depends on the fixed frame, K and ϵ. No threshold uniform over the rational coefficients is asserted.

10.5. **Decay and proof of Theorem 1.3.** The decay is Theorem 6.7 with the reference frame of shifts 1, $\frac{3}{4}$, $\frac{1}{2}$ of §6.6, in which $\frac{3}{4}$ and $\frac{1}{2}$ are the helpers of the triple pole.

Proposition 10.5. *Take the reference frame of shifts* 1, $\frac{3}{4}$, $\frac{1}{2}$ *and* $c = \frac{13}{100}$. *There is a trial measure on a polygon* Γ, *with a cover by cells satisfying the checks of §6.4 with* $\varepsilon_0 = 10^{-6}$ *and* $|\mathcal{T}| = 3$, *for which Theorem 6.2 gives*

$$D = \frac{68912340510797043323527}{2 \cdot 10^{22}} = 3.4456170255\dots,$$
$$\rho = \frac{34899270780366191}{5 \cdot 10^{15}} = 6.9798541560\dots$$

Proof of Theorem 1.3. Let a and b be rationals, not both zero. If $b = 0$, then $a \neq 0$, and $\mu(a\zeta(3)) = \mu(\zeta(3)) \leq 6.0382$ by Theorem 1.5. Let $b \neq 0$. The frame $F^{(3)}_{\gamma_2,\gamma_3}$ with $\gamma_2 = -\frac{a}{2}$ and $\gamma_3 = 15b$ has $\xi = a\zeta(3) + b\pi^4$, by §3.2. Theorem 10.4 gives the denominators with Λ_3. Proposition 10.5 with Theorem 6.7 gives the decay. The certificate contains no weight, so it serves every frame of the third plane. Theorem 5.6 with $d = 3$ gives nonvanishing

with one deleted node. Theorem 11.1 at the even scales gives

$$\begin{aligned}\mu(\xi) &\le \frac{\rho}{D-\Lambda_3}\\ &= \frac{1090401990431147876150573765135989459095634276\cdot 10^6}{14993329815540176358024924732241649501611402803569 3}\\ &= 72.7258056646\ldots < 72.726.\end{aligned}$$ □

The thresholds depend on (a,b). No uniformity in (a,b) is claimed.

11. The deduction and the remaining proofs

The three estimates combine as follows.

Theorem 11.1 (deduction)**.** *Let F be a frame with a geometry. Let K be a depth such that, for every large r and every rational x_0, some member $P_{r,I}$ with $|I|\le K$ satisfies $P_{r,I}(x_0)\neq 0$, as in Theorems 5.7 and 5.6. Suppose that the members satisfy $\log\operatorname{den} P_{r,I}\le(\Lambda+o(1))r^2$ and the decay estimate of Theorem 6.2 with constants D and ρ, and that $\Lambda<D$. Then ξ is irrational and $\mu(\xi)\le\rho/(D-\Lambda)$.*

Proof. Let $\eta=D-\Lambda$, and let $\mu_0>\mu_1>\rho/\eta$. Suppose that $|\xi-p/q|<q^{-\mu_0}$ with q large. Let r be the least admissible scale with $r\ge\mu_1\log q/\rho$. With $x_0=p/q$ there is a member with $P_{r,I}(p/q)\neq 0$. By Lemma 3.5, $|\operatorname{den}(P_{r,I})\,P_{r,I}(p/q)|\ge q^{-r}$. Write $p/q=\xi+h$. Then $|h|<q^{-\mu_0}\le e^{-\rho r}$, because $\rho r\le\mu_1\log q+O(1)<\mu_0\log q$ for large q. By the two estimates,

$$|\operatorname{den}(P_{r,I})\,P_{r,I}(p/q)|\le e^{(\Lambda+o(1))r^2}e^{-(D-o(1))r^2}=e^{-(\eta-o(1))r^2}.$$

Comparing the two bounds gives $(\eta-o(1))r^2\le r\log q$. But $\eta r\ge(\mu_1\eta/\rho)\log q$ with $\mu_1\eta/\rho>1$. This is a contradiction for large q. If $\xi=p/q$ were rational, then for every large r some member would satisfy $q^{-r}\le|\operatorname{den}(P_{r,I})\,P_{r,I}(\xi)|\le e^{-(\eta-o(1))r^2}$, which is impossible. So ξ is irrational. □

Proof of Theorem 1.6, Theorem 1.4(ii) and Theorem 1.5 for π^2. For each of these rows of Table 1 the function is F_s of (2.1), and the geometry parameter and the constants are those of Table 2. The denominator exponent is given by Theorem 4.14 with the table for Λ of the formal proof. The decay rates are given by Proposition 6.3, and nonvanishing by Theorem 5.7. Theorem 11.1, applied at the even scales, gives $\mu(\xi)\le\rho/(D-\Lambda)$, and in every row $\rho/(D-\Lambda)$ is below the stated bound. By (2.1), ξ is a nonzero rational multiple of $\Psi(s,1)$. Since the inequality is strict, the definition of μ gives $|\xi-p/q|>q^{-\mu_0}$ for all integers p and all sufficiently large q. □

Theorem 1.4(i) is proved in Section 8, parts (iii)–(v) and Theorem 1.1 in Section 7, Theorem 1.2 and the bound for $\zeta(3)$ in Section 9, and Theorem 1.3 in Section 10. In each case the bound is attained in the same strong form.

Remark 11.2 (relation to linear independence criteria). The deduction is a one-dimensional version of the familiar scheme behind the criteria of

case	c	Λ	D	ρ	ρ/η	$\mu \le$
plane 1	$\frac{1}{6}$	3.2227428	3.5503512	6.7690926	20.66214	20.6622
$(\frac{1}{3}, \frac{2}{3})$	$\frac{1}{2}$	4.0959278	4.6442402	7.5030223	13.68385	13.6839
$(\frac{k}{3}, 1)$	$\frac{3}{10}$	3.3104458	4.0295805	6.8255087	9.491278	9.4913
$L_{-3} \pm \frac{2\pi^2}{27}$	$\frac{1}{6}$	3.0668400	3.5503512	6.7690926	13.99987	14
$L_{-3} \pm \frac{\pi^2}{27}$	$\frac{1}{6}$	3.1096062	3.5503512	6.7690926	15.35830	15.359
$L_{-3} \pm \frac{\pi^2}{18}$	$\frac{1}{4}$	3.4140176	3.8109776	6.9434368	17.49153	17.492
plane 2	$\frac{6}{41}$	3.4768376	3.4982052	7.1600631	335.0886	335.20
$\zeta(3)$	$\frac{3}{10}$	2.912839	3.9790633	6.4345402	6.034884	6.0382
$(\frac{1}{2}, 1)$	$\frac{3}{10}$	2.9923824	3.9790633	6.4345402	6.521399	6.5214
plane 3	$\frac{13}{100}$	3.3496421	3.4456170	6.9798542	72.7259	72.726
$(\frac{k}{4}, 1)$	$\frac{1}{5}$	3.5149036	3.7453322	7.1887383	31.19724	31.1973
$(\frac{k}{5}, 1)$	$\frac{2}{9}$	3.5128488	3.8141613	7.1638755	23.77556	23.7756
$(\frac{k}{7}, 1)$	$\frac{1}{6}$	3.6391158643	3.6432250312	7.4218615374	1806.172	1806.1719

TABLE 2. The data of the proofs. A pair (s, t) denotes the function of (2.1) for $\Psi(s, t)$, with k prime to the denominator. For each row the table gives the geometry parameter c, the denominator exponent Λ, the decay rate D, the neighbourhood exponent ρ and the bound for μ. The constants are rational numbers, rounded outward to the digits shown: Λ and ρ up, D down. The column ρ/η, with $\eta = D - \Lambda$, is the exact quotient, rounded up. The rows $L_{-3} \pm \frac{2\pi^2}{27}$ and $L_{-3} \pm \frac{\pi^2}{27}$ share the decay certificate of plane 1. The row for $\zeta(3)$ shares the certificate of the row $(\frac{1}{2}, 1)$.

Nesterenko [36] and their use by Ball and Rivoal [3]. The nonzero integer $q^{|C \setminus I|} \operatorname{den}(P_{r,I})\, P_{r,I}(p/q)$ plays the role of a small linear form with integer coefficients. There are two differences. The lower bound for the form comes from an arithmetic nonvanishing statement rather than from an analytic asymptotic. The size r of the determinant is chosen in terms of q.

11.1. **Notes on the individual cases.** For the rows $(\frac{k}{q}, 1)$ the function is F_s of (2.1) with $s = k/q$, $\omega = q^{-2}$ and $\xi_s = -q^{-2}\Psi(\frac{k}{q}, 1)$. The table for Λ, the trial measure and the cover are data of the formal proof. All shifts with the same denominator share one table for Λ, indexed by the root classes, and one certificate. Indeed the bounds depend on the shift only through the root class, and the decay depends on it only through the separation of the poles from the contour. For $q = 4$ the root classes $j \equiv kp^{-1} \pmod 4$ of $k = 1$ and $k = 3$ exchange the two classes of primes modulo 4. For $q = 3$ the table uses the primes in the classes modulo 6, and the root classes $j = 1, 2$ correspond

to first poles near $p/3$ and $2p/3$. For $q = 7$ there are six root classes. Here $D - \Lambda$ is only about $4 \cdot 10^{-3}$, which explains the large exponent. A small further gain in either constant would lower it substantially (Section 12). All certificates take $\varepsilon_0 = 10^{-6}$ in the function Υ. The trial measure for $q = 7$ has 22 polynomial pieces, and its cells on Γ at the breakpoints of σ use the truncated logarithm of §6.4. For $q = 2$ there is one root class, $j = 1$, with first pole $k_p = \frac{p+1}{2}$.

On the second plane $D - \Lambda$ is about 0.021, and on the third plane about 0.096. This explains the size of their exponents. On the first plane the points $L_{-3} \pm \frac{2\pi^2}{27}$ and $L_{-3} \pm \frac{\pi^2}{27}$ gain only in Λ, since they use the certificate of the plane. The points $L_{-3} \pm \frac{\pi^2}{18}$ use a longer border, with $c = \frac{1}{4}$, and their own certificate.

12. Limits of the method

The exponents are determined by three constants through $\rho/(D - \Lambda)$. The derivative of $\rho/(D - \Lambda)$ with respect to D is $-\rho/\eta^2$, so the exponent is most sensitive to the decay where η is small. For the shifts $\frac{1}{4}$ and $\frac{3}{4}$, $\rho/\eta^2 \approx 135$. For the shifts $\frac{k}{7}$ it is about $4.4 \cdot 10^5$.

On the arithmetic side, the profiles $\mathcal{P}_j$ are minima of the bounds of Theorem 4.5. Every further local bound lowers Λ by the integral of the amount by which it lies below $\mathcal{P}_j$. The denominator exponent of the method is the integral of the true exponent e_p, which the local bounds approximate from above. At third order this approximation is close to exhaustion for the shifts $\frac{k}{5}$ and $\frac{k}{7}$. We computed the Smith normal forms exactly at scales $r \le 2400$ and primes $p \le 2m$. There the minimum of the local bounds, with (U_n) included, equals the true exponent at all but a few isolated primes. For the shifts $\frac{k}{7}$ the invariants of valuation at least 4 contribute about $5 \cdot 10^{-6}$ to Λ. The bounds (U_n) gain about $7 \cdot 10^{-4}$. So at third order the local bounds are essentially exhausted. These computations are not part of the proof.

On the analytic side, Remark 6.4 explains why a certificate of the present form cannot do better than the constrained equilibrium measure on the chosen contour. A larger decay rate needs a different contour or a different treatment of the energy problem. The geometry parameter c trades the two sides against each other. A longer border increases D, through the larger node measure. It also increases Λ, through the border rows and columns of the assignment bound.

12.1. **Constants not reached.** The construction applies to $\Psi(\frac{k}{q}, 1)$ for every denominator q, but it proves a bound only when $D > \Lambda$. The shifts $\frac{k}{6}$ give points of the first plane, through the identities of Section 1. For the next denominators we estimated the gap $D - \Lambda$ at the geometry parameter $c = \frac{1}{7}$. We took the best local bounds of Section 4 for Λ, and a floating-point trial measure for D. For $q = 11$, 9 and 8 the gap is about -0.41, -0.53 and -0.56. For $q = 10$ and $q = 12$ it is about -1.26 and -1.64. For the

six differences $\Psi(\frac{k}{5}, \frac{l}{5})$ with $1 \le k < l \le 4$ it lies between about -0.85 and -0.66. The closest miss is $\Psi(\frac{k}{5}, \frac{1}{2}) = \psi'(\frac{k}{5}) - \frac{\pi^2}{2}$. For its two-term frames at $c = \frac{1}{8}$ the gap is about -0.0014, and about -0.001 even if the local bounds and the decay estimate were sharp. So none of these numbers is reached, and the shortfall is large against the certified gap of about $4 \cdot 10^{-3}$ for $q = 7$. Over the denominators computed the gap tends to decrease as q grows, and the denominator 7 is close to the edge. Numbers of weight three built from $\zeta(3)$ and π^3, such as $27\zeta(3) \pm \pi^3$, are further away, with a gap of about -2.3. These values are numerical observations. Their error is a few hundredths, except for $\Psi(\frac{k}{5}, \frac{1}{2})$, where it is about $2 \cdot 10^{-4}$. They are not theorems, and they are outside the formal proof.

13. Formalization

The theorems of Section 1 are formalized in Lean 4 [13] with Mathlib [48], together with every theorem, proposition and lemma their proofs use. The remarks and Section 12 are not part of the formalization. The development is available at

https://github.com/slitsey/catalan-irrationality-measure

It combines analytic and arithmetic proofs with generated rational data. Kernel checks verify the tables for Λ and the decay certificates. Numerical computations alone are not used as proof.

The statements are in the file `ChallengeHybrid.lean`. Their proofs are in `SolutionHybrid.lean`, in the namespace `TrigammaDifferences`. Each number ξ is written as a series, or through `Real.pi^2` and `Real.pi^4`. The series is the classical one where there is one. Otherwise it is $\Psi(s, t)$, and $\zeta(3)$ is the series $\sum_{n \ge 0}(n+1)^{-3}$. Each theorem states that $q^{-\mu_0} < |\xi - p/q|$ for all $p \in \mathbb{Z}$ and all sufficiently large q. The plane theorems take rational coefficients a and b, not both zero, and ξ is $aL_{-3} + b\pi^2$, $a\zeta(3) + b\pi^2$ or $a\zeta(3) + b\pi^4$. The statements have no separate irrationality clause, since the bound for all large denominators implies irrationality. The theorems are the following.

- `plane1_measure`: Theorem 1.1, with $\mu_0 = 20.6622$.
- `chi3_measure`: Theorem 1.4(i), with $\mu_0 = 13.6839$.
- `chi3_plus_pi2_measure` and `chi3_minus_pi2_measure`: Theorem 1.4(ii), with $\mu_0 = 9.4913$.
- `dip_2_27_plus_measure`, `dip_2_27_minus_measure`, `dip_1_27_plus_measure`, `dip_1_27_minus_measure`, `dip_1_18_plus_measure` and `dip_1_18_minus_measure`: Theorem 1.4(iii)–(v), with $\mu_0 = 14$, 15.359 and 17.492.
- `plane2_measure`: Theorem 1.2, with $\mu_0 = 335.20$.
- `zeta3_measure` and `pi2_measure`: Theorem 1.5, with $\mu_0 = 6.0382$ and 6.5214.
- `plane3_measure`: Theorem 1.3, with $\mu_0 = 72.726$.

- `lambda_plus_measure` and `lambda_minus_measure`: Theorem 1.6(i), with $\mu_0 = 31.1973$.
- `fifths1_measure`, ..., `fifths4_measure`: Theorem 1.6(ii), with $\mu_0 = 23.7756$.
- `sevenths1_measure`, ..., `sevenths6_measure`: Theorem 1.6(iii), with $\mu_0 = 1806.1719$.

The shared results are in the namespace `CatalanCore.Uniform`.

- `Content.BankShared.Check.hasScaledContent_of_check` is Theorem 4.14. The local bounds of Theorem 4.5 are proved in `Check/Catalogue.lean`, `Check/LP2W.lean`, `Check/Prefix.lean` and `Check/UF.lean`, with the closed rank bound and the branches of Theorem 4.8. Its form for functions with several first poles is `Content.BankShared.KTerm.hasScaledContent_of_kcheck`.
- `Content.BankShared.TwoTerm.LP2HypW.gle` is Theorem 4.6.
- `Content.BankShared.TwoTerm.Pre.prefix_gle` is Theorem 4.7.
- `Content.UFSimple.WrapCheck.catValid_wrap` and `Content.UFSimple.WrapPrefixCheck.catValid_prefix_full` are Theorem 4.8 for $n = 0$ and for $n \geq 1$. Lemma 4.11 is `Content.UFSimple.WrapPrefixClosed.HSplit.det_ne` with `HSplit.dir`.
- `Support.support_nv0_even` is Theorem 5.6. `BlockGeom.hasNV0` is Theorem 5.7.
- `Decay.AltDecay.altGeom_hasScaledDecay` is Theorem 6.2. It uses the node energy check `Decay/Nodes/EnergyCheck.lean`, the contour check `Decay/ContourCheck.lean` and the cell checks `Decay/PolyWire3`. For the shifts $\frac{k}{7}$ it is applied in the form `Decay.AltDecay.altGeom_hasScaledDecay_Tr`, with the same conclusion. There the cells on the contour at breakpoints use the truncated logarithm, proved in `Decay/Breakpoint/TruncLog.lean` and `Decay/Breakpoint/CellT.lean`.
- `measure_of_family_nv0_even` is Theorem 11.1 at the even scales.
- `Content.BankShared.KTerm.hasScaledContent_of_periodic` is Theorem 4.15 in the form used for Theorems 7.3 and 9.7.
- `Bank.Plane1.Dips.dip_point` is Theorem 7.7 together with the deduction of Theorem 1.4(iii)–(v).
- `Bank.Plane2.plane2_point_ab_of_content` is the proof of Theorem 1.2, for every rational $(a, b) \neq (0, 0)$, from the denominator bounds of the three cases $\gamma_2 \neq 0$ with $\gamma_1 \neq 0$, $\gamma_2 \neq 0$ with $\gamma_1 = 0$, and $\gamma_2 = 0$.
- `Bank.Plane3.plane3_content` is Theorem 10.4, `Bank.Plane3.plane3_decay` is the decay of Proposition 10.5, and `Bank.Plane3.plane3_final` is the proof of Theorem 1.3.
- `Support.IntegerPoleRows.assignment_bounds` is Proposition 10.2 for every $h \geq 2$, and `Support.hpole_catalogue` turns it into the table branches. `Support.HPolePeriodRows.row_refined_first` is

Proposition 10.3, and `Support.hpole_refined_catalogue` gives its branches on the intervals of Theorem 10.4. In the formal statements the members are indexed by the remaining common nodes $C \setminus I$, and the bounds are stated for the Gauss valuation over the local ring at p.

- `Support.pole_gram_split_minimal` is the rank-one splitting of §6.6 for a pole of any order. `Support.pole_altGeom_seqFamily_nv0_even` is Theorem 5.6 for such poles.
- `Support.support_measureLB` is the general form of Theorem 11.1. For simple-pole frames, `Bank.Plane1.alt_point` applies it to an alternating geometry. The multiple-pole instances use the support theorem directly.

The solution has no `sorry` and no `native_decide`. The command `#print axioms` reports only the three standard axioms `propext`, `Classical.choice` and `Quot.sound`. The prime number theorem in arithmetic progressions enters the asymptotics of the denominators. Its formal source is the PNT+ project [49], through OpenAI's workspace and compatibility additions. We use its Wiener–Ikehara theorem.

The proofs were checked with Lean `v4.35.0-rc2` and the matching Mathlib release. The vendored files from OpenAI's repository and from the PNT+ project were converted to Lean's module system and ported to the current toolchain. The repository's README and provenance documents describe the changes. The certificate data were generated with the interval arithmetic of Arb [24], through `python-flint`, and Lean re-checks them.

Disclosure of AI use

The mathematics, the Lean formalization, the certificate generators and the text of this paper were produced by AI systems at the request of the author. The author posed the problems, directed the work and is responsible for the paper. The systems were Claude Opus 5.5, Claude Sonnet 5.5 and Claude Haiku 5.5 (Anthropic), running in Claude Code, and GPT-6.1 Sol (OpenAI), running in Codex. The underlying irrationality proof for Catalan's constant and its Lean formalization are due to OpenAI [38].

Acknowledgements. We thank the authors of Mathlib and of the PNT+ project, and OpenAI for making its formalization available.

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