

IRRATIONALITY MEASURES FOR $G \pm 5\pi^2/48$ AND $L(2, \chi_{-3})$

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ABSTRACT. Building on OpenAI’s determinant method for the irrationality of Catalan’s constant $G = \sum_{j \geq 0} (-1)^j / (2j + 1)^2$, we prove

$$\mu\left(G - \frac{5\pi^2}{48}\right) \leq 38.35, \quad \mu\left(G + \frac{5\pi^2}{48}\right) \leq 38.35, \\ \mu\left(L(2, \chi_{-3})\right) \leq 23.82,$$

where μ is the irrationality exponent. We know of no earlier irrationality measure for $G \pm 5\pi^2/48$. For $L(2, \chi_{-3})$, which Calegari, Dimitrov and Tang proved irrational, the previous best bound we know of is their effective bound $\mu \leq 24781$.

The approximations come from determinants of *rectangular* Löwner matrices of digamma differences at integer nodes. We bound their denominators with p -adic assignment problems and rank deficits modulo p . We prove nonvanishing modulo a prime by reducing it to bordered Krylov determinants, which we evaluate through Jacobi matrices and Darboux transformations. We bound their size with Andréief’s identity and a certified lower bound for a discrete logarithmic energy.

Every result is formalized in Lean 4, with no unproved steps and only the standard axioms.

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1. INTRODUCTION

OpenAI [32] recently proved that Catalan’s constant

$$G = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = 0.91596\,55941\dots$$

is irrational, and formalized the proof in Lean. (An earlier claimed proof [39] was found to contain a gap [42].) Their argument, and the irrationality-measure results built on it [17, 7, 25], use a determinant of large size whose denominators and decay are balanced very closely.

In the present paper we study a different, much smaller, family of determinants: rectangular Löwner matrices of digamma differences, attached to the numbers

$$\begin{aligned} \lambda &= \frac{G}{2} - \frac{5\pi^2}{96}, & \lambda' &= \frac{G}{2} + \frac{5\pi^2}{96}, & \frac{1}{4}L(2, \chi_{-3}), \\ L(2, \chi_{-3}) &= \sum_{k \geq 0} \left(\frac{1}{(3k+1)^2} - \frac{1}{(3k+2)^2} \right) = 0.78130\,24128\dots \end{aligned}$$

Thus $G - 5\pi^2/48 = 2\lambda$, $G + 5\pi^2/48 = 2\lambda'$, and $\lambda + \lambda' = G$.

Recall that the *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the μ such that $|\xi - a/b| > b^{-\mu}$ for all integers a and all sufficiently large b .

Theorem 1.1. *For every $\varepsilon > 0$ there is q_0 such that for all coprime integers p, q with $q \geq q_0$,*

$$\begin{aligned} \left| G - \frac{5\pi^2}{48} - \frac{p}{q} \right| &> q^{-38.35-\varepsilon}, & \left| G + \frac{5\pi^2}{48} - \frac{p}{q} \right| &> q^{-38.35-\varepsilon}, \\ \left| L(2, \chi_{-3}) - \frac{p}{q} \right| &> q^{-23.82-\varepsilon}. \end{aligned}$$

In particular $G \pm 5\pi^2/48$ are irrational, $\mu(G \pm 5\pi^2/48) \leq 38.35$ and $\mu(L(2, \chi_{-3})) \leq 23.82$.

The computed values of the exponents are 38.342 and 23.8194. Theorem 1.1, together with every result in this paper, has been verified in the Lean 4 proof assistant, with no unproved steps and only the standard axioms; see Section 8. The formalization is available at <https://github.com/slitsey/catalan-irrationality-measure>. The use of AI in this work is disclosed at the end of the paper.

1.1. Previous results. For earlier work on the arithmetic of G and related values see [36, 46, 45, 22, 23] and the survey [48]; for recent rational approximations to G see Eskandari [10]. Finite irrationality exponents for G itself were obtained after OpenAI’s proof by Jha [17], Carreira [7] and the author [25]. Our searches found no previous irrationality result, and no irrationality measure, for $G - 5\pi^2/48$ or $G + 5\pi^2/48$. Eskandari, Murty and Nemoto [11] construct a 2-dimensional mixed motive over $\mathbb{Q}$ with period G and use it to

obtain linear forms in 1 and G . Eskandari [10] improves the approximation exponent of these Rivoal–Zudilin type forms from 0.52 to 0.62 (a preprint; explicit approximations, not a lower bound for all rationals). Sun’s preprint [39] claimed a proof of the irrationality of G ; Wachs [42] reports that its main estimate is not justified.

Calegari, Dimitrov and Tang proved that 1, $\zeta(2)$ and $L(2, \chi_{-3})$ are linearly independent over $\mathbb{Q}$ [5], by a new arithmetic holonomy bound applied to a construction of Zagier. In [6] they made holonomy bounds effective; Theorem 5.1 there gives $\mu(L(2, \chi_{-3})) \leq 24781$. Theorem 1.1 improves this to 23.82.

For comparison, among classical constants, the bound $\mu(\log 2) < 3.5746$ of Marcovecchio [29] and the Rhin–Viola bounds for $\zeta(2)$ and $\zeta(3)$ [34, 35] (after Apéry and Beukers [3]) have been joined by recent preprints. OpenAI proved $\mu(\pi) = 2$, with a Lean formalization [33], improving the bound $\mu(\pi) < 7.1033$ of Zeilberger and Zudilin [43]. Liu, Jiang and Zhang proved $\mu(\log a) = 2$ for every rational $a > 0$, $a \neq 1$, and for nonzero rational combinations of such logarithms [26]. Giraudin [14] lowered the bound for $\mu(\zeta(2))$ to below 5.0193784. Fauzan [12] announced a proof that $\zeta(5)$ is irrational, using rationally normalized Hankel determinants, and Lai [24] combined his method with simultaneous approximations of Zudilin and of Dauguet and Zudilin to show that 1, $\zeta(3)$ and $\zeta(4)$ are linearly independent over $\mathbb{Q}$.

Related work on determinants. Irrationality proofs built on determinants of large size have a precedent in Zudilin’s determinantal criterion with Hankel determinants [47]; the preprint of Liu, Jiang and Zhang on logarithms [26] uses a separated-weight interpolation determinant, and the arguments of Fauzan and Lai [12, 24] use Hankel determinants. Brown [4] proposes a multivariate transfinite-diameter test for Mellin-type integrals, applied to $\zeta(2)$. The framework we use, determinants whose decay is controlled by Cauchy determinants and Andréief’s identity and whose denominators are controlled prime by prime, is OpenAI’s [32]. Our determinants are Löwner (divided-difference) matrices of a digamma difference, a matrix-valued analogue of Padé and Hermite–Padé approximation [31]. Formal proofs of irrationality results in Lean also include the irrationality of $\zeta(3)$ by Liu, Zhang and Zhi [27].

1.2. Outline of the proof. Section 2 explains the construction and the new ideas without technicalities, and states the general deduction. Sections 3–6 give the construction and the three estimates (denominator, nonvanishing, decay); Section 7 deduces Theorem 1.1. Section 8 describes the formalization, and the disclosure of AI use follows it.

1.3. Notation and conventions. $\psi = \Gamma'/\Gamma$ is the digamma function; v_p is the p -adic valuation; for $P \in \mathbb{Q}[x]$, $\text{den } P$ is the least positive integer d with $dP \in \mathbb{Z}[x]$; $\vartheta(x; M, a) = \sum_{p \leq x, p \equiv a \pmod{M}} \log p$; $(x)_n = x(x+1) \cdots (x+n-1)$; $[0, r) = \{0, 1, \dots, r-1\}$. For finite sets I, J of integers and a matrix A indexed by integers, $A[I, J]$ is the submatrix with rows I and columns J .

2. OVERVIEW OF THE METHOD AND OF THE NEW IDEAS

2.1. Löwner matrices of digamma differences. For a function F analytic near distinct points $x_0, \dots, x_{h-1}$ the *Löwner matrix* [28] is the matrix of divided differences

$$\mathbf{L}_F[a, b] = \frac{F(x_b) - F(x_a)}{x_b - x_a} \quad (a \neq b), \quad \mathbf{L}_F[a, a] = F'(x_a).$$

It is a Cauchy-like structured matrix, and Hermite's formula gives $\mathbf{L}_F = \frac{1}{2\pi i} \oint F(z) v(z) v(z)^\top dz$ with $v_a(z) = 1/(z - x_a)$, so that a representation of F as a Stieltjes or contour integral exhibits $\mathbf{L}_F$ as a Gram matrix of Cauchy kernels.

We take three digamma differences, with integer nodes:

$$(2.1) \quad \begin{aligned} F_\lambda(z) &= \psi(z+1) - \psi(z + \tfrac{3}{4}), \quad \omega = \tfrac{1}{16}, \quad \xi = \lambda := \tfrac{G}{2} - \tfrac{5\pi^2}{96}, \\ F_{\lambda'}(z) &= \psi(z+1) - \psi(z + \tfrac{1}{4}), \quad \omega = -\tfrac{1}{16}, \quad \xi = \lambda' := \tfrac{G}{2} + \tfrac{5\pi^2}{96}, \\ F_\chi(z) &= -\tfrac{1}{3}\psi(z + \tfrac{1}{3}) + \tfrac{1}{2}\psi(z + \tfrac{1}{2}) - \tfrac{1}{6}\psi(z+1), \\ &\quad \omega = -\tfrac{1}{6}, \quad \xi = \tfrac{1}{4}L(2, \chi_{-3}). \end{aligned}$$

In each case $\xi = \omega F'(0)$ (this is an evaluation of ψ' at $\frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 1$), while $F(b) - F(a)$ and $F'(a) - F'(0)$ are rational for integers $a, b \geq 0$. Hence, for any finite sets of nonnegative integer nodes R (rows) and Q (columns),

$$(2.2) \quad \omega \mathbf{L}_F[R, Q] = M[R, Q] + \xi E_{R, Q}, \quad E_{R, Q}[a, b] = \mathbf{1}[a = b],$$

with an explicit rational symmetric kernel M on $\mathbb{Z}_{\geq 0}^2$ (§3). Replacing ξ by a variable x gives a matrix polynomial $\Phi(x) = M[R, Q] + x E_{R, Q}$, whose determinant $P(x)$ is a polynomial with rational coefficients of degree at most $|R \cap Q|$, and $P(\xi) = \det(\omega \mathbf{L}_F[R, Q])$. If $|\xi - p/q|$ is very small, $P(p/q)$ is very close to $P(\xi)$, which is tiny; on the other hand $q^{|R \cap Q|} \det(P) P(p/q)$ is an integer. If it is not zero we get a contradiction as soon as

$$\underbrace{\log \det P + |R \cap Q| \log q}_{\text{content}} < \underbrace{-\log |P(p/q)|}_{\text{decay}}.$$

Three estimates are therefore needed: an upper bound for the *content* $\log \det P$, an upper bound for $|P(x)|$ near ξ (the *decay*), and *nonvanishing* of $P(p/q)$ for a size that can be chosen in terms of q . We use rectangular geometries, with $R \neq Q$, and all three estimates are new for these constants.

2.2. New idea 1: rectangular (hybrid) frames. We take a long common block C of r nodes, which carries the variable x , and two short interleaved blocks of $m \approx r/8$ nodes,

$$\begin{aligned} C &= [0, r), \quad X' = \{r, r+2, \dots, r+2m-2\} \text{ (rows only),} \\ Y' &= \{r+1, r+3, \dots, r+2m-1\} \text{ (columns only),} \end{aligned}$$

so that $R = C \sqcup X'$ and $Q = C \sqcup Y'$ (a second “member” adds one more common node $r+2m$). The determinant $P(x) = \det \Phi(x)$ still has degree

$\leq r$ (or $r + 1$), so the integrality loss $|R \cap Q| \log q$ is that of r nodes, but the matrix has size $r + m$ and its decay is that of a configuration of $(1 + \alpha)r$ nodes, $\alpha = m/r$. The content grows too, but less: the extra rows and columns never meet the diagonal. With $\alpha = \frac{1}{8}$ the content and decay constants become (for λ)

$$\Lambda = 3.70343 \quad (\text{square frame: } 313/105 = 2.98095),$$

$$D = 3.9028 \quad (\text{square frame: } 3.1114),$$

and the margin $\eta = D - \Lambda$ rises from 0.1304 to 0.1994. The interleaving of X' and Y' is essential: it keeps the two Cauchy determinants in Andréief's formula comparable (§6) and it gives the reduction modulo ℓ the bordered structure of §5.

2.3. New idea 2: content by p -adic assignment problems and rank deficits. For a prime p let v be the Gauss valuation on $\mathbb{Q}[x]$ (the minimum of the p -adic valuations of the coefficients); it is multiplicative. Expanding $\det \Phi(x)$ by Leibniz, $-v(\det \Phi)$ is at most the value τ_p of the *assignment problem* with weights $W_{ab} = \max(0, -v_p(M[a, b]))$, i.e. the maximum of $\sum_a W_{a\sigma(a)}$ over bijections $\sigma : R \rightarrow Q$. By LP duality, any *potentials* u_a, v_b with $W_{ab} \leq u_a + v_b$ certify $\tau_p \leq \sum u_a + \sum v_b$. We give explicit potentials for every prime power p^e (they depend only on p^e/r and on $p \bmod 4$ or $p \bmod 6$), and sum over primes with the prime number theorem in arithmetic progressions. This gives a “tropical” content $\Lambda_{\text{trop}} = 3.74993$ for λ .

The tropical bound is not sharp. If (u, v) is an optimal dual solution, the scaled matrix $B = p^u \Phi p^v$ has entries in the valuation ring $\mathcal{O}$ of v , and $v(\det \Phi) = -\tau_p + v(\det B) \geq -\tau_p + \text{corank}(B \bmod p)$. So every unit of corank of the reduction of B modulo p over $\mathbb{F}_p(x)$ saves one factor of p . We compute this corank in several ranges of primes, using the block structure of the reduction (residue classes modulo p), Schur complements, the identity $H_{p-1-n} \equiv H_n \pmod{p}$ for harmonic numbers, an antisymmetry of Cauchy-type sums, and the Frobenius–König theorem. The savings lower the content from 3.74993 to 3.70343; with the decay constant below this lowers the exponent from about 50.0 to 38.34.

2.4. New idea 3: nonvanishing through a commuting Jacobi matrix. Fix a prime ℓ in a suitable residue class with $\ell \nmid q$. After scaling by ℓ the rows and columns of nodes beyond a threshold $K \approx \ell/4$, the matrix $\Phi(x)$ reduces modulo ℓ to a *bordered Cauchy–Gram* matrix:

$$\ell^{2h} \det \Phi(x) \equiv \text{unit} \times \det \begin{pmatrix} yI_K + N_S & W_Y \\ W_X^\top & 0 \end{pmatrix} \pmod{\ell},$$

$y = \text{affine function of } x,$

where $N_S = \sum_{j=1}^S c_j c_j^\top$ is a sum of rank-one Gram terms of Cauchy vectors $c_j = (1/(a+j))_{0 \leq a < K}$ and W_X, W_Y consist of further Cauchy vectors. The two members of the pair differ by one Gram term. Their reductions cannot

vanish simultaneously unless a *bordered Krylov determinant*

$$k = \det[W_Y \mid v, Av, \dots, A^{K-m-1}v], \quad A = \Pi N_S, \quad v = \Pi c_S,$$

vanishes modulo ℓ , where Π is the projection onto $\ker W_X^\top$ along the span of W_Y . We prove that, with the nodes shifted by a parameter θ , $k(\theta)$ is a constant prime to ℓ times a product of powers of $\theta + c$ with $1 \leq c < \ell$. The proof is algebraic and uses a structure familiar from orthogonal polynomials: there is an explicit *Jacobi (tridiagonal) matrix* $\tilde{T}$ that commutes with Π and with A and acts on the Cauchy vectors by a three-term recurrence (a bispectral situation in the sense of Grünbaum and Haine [15]). Its LU factorisation, reversed to UL, is (up to a shift and a diagonal conjugation) the Jacobi matrix of the next smaller instance: a discrete Darboux, or Christoffel, transformation [38, 16]. Krylov determinants of commuting pairs satisfy a cross-ratio identity, chains of Cauchy vectors give products of Cauchy determinants, and the remaining Gram determinants are evaluated with Cauchy–Binet, Hankel determinants of moments, Newton’s divided differences and the Chu–Vandermonde identity (which also proves the Pfaff–Saalschütz-type identity needed). One auxiliary determinant has no product formula and is shown to be nonzero by positivity. An induction on the size closes. We never need the exact exponents: it suffices to know that the determinant is an ℓ -adic unit times powers of $\theta + c$ with $1 \leq c < \ell$.

Because the pair of members is available for every S in a range, the prime ℓ can be chosen *after* the size r , anywhere in an interval $[\frac{3}{2}r, 4r]$. This removes the cost of locating ℓ from the exponent (for λ and λ'), and the final exponent is simply c/η .

2.5. New idea 4: decay through an energy certificate. Moving Hermite’s contour to two rays in the left half-plane writes $\omega \mathsf{L}_F[R, Q]$ as a Gram matrix $\int v_a v_b dm$ of a complex measure m (finitely many signed atoms at poles of F plus an exponentially small density on the rays). Andréief’s identity with two families of functions (rows and columns) gives

$$|\det \mathsf{L}_F[R, Q]| \leq \frac{1}{n!} \int |\mathsf{C}_R(z)| |\mathsf{C}_Q(z)| \prod_{i=1}^n d|m|(z_i),$$

where $\mathsf{C}_R, \mathsf{C}_Q$ are Cauchy determinants. Taking logarithms turns the integrand into a discrete logarithmic energy of the free points z_i against the nodes in an external field. We bound this energy from below by comparing with a *certificate measure* σ (a few hundred point masses with dyadic coordinates) through a regularised kernel $-\frac{1}{2} \log(|z - w|^2 + \rho^2)$, which is conditionally positive definite on finite sums by an elementary argument (Frullani’s integral and Gaussian sums). Everything left is a finite list of inequalities between explicit functions on boxes, which we verify in interval arithmetic inside Lean. The same argument bounds every principal minor along the common block, so the determinant stays small at every point within e^{-cr} of ξ .

2.6. The deduction. Write $\eta = D - \Lambda$. The three estimates are combined in §7; the result is

$$\mu(\xi) \leq c \left(\frac{1}{\eta} + g s \right),$$

where c is the window constant of the decay estimate, $g = \lim r/\ell$ for the sizes at which nonvanishing is proved, and s is the relative length of the interval in which ℓ must be found. For λ and λ' the size can be chosen independently of ℓ , so the term gs disappears and $\mu \leq 7.6444/0.19937 < 38.35$. For $L(2, \chi_{-3})$ the size is tied to ℓ ($g = \frac{2}{5}$), and $\ell \nmid q$ is needed ($s = 2$), so $\mu \leq 6.6510(1/0.35954 + 0.8) < 23.82$.

constant	content Λ	decay D	window c	$\eta = D - \Lambda$	gs	exponent
$G - 5\pi^2/48$	3.70343	3.9028	7.6444	0.19937	0	38.342
$G + 5\pi^2/48$	3.70343	3.9028	7.6444	0.19937	0	38.342
$L(2, \chi_{-3})$	10339/3072	3.7251	6.6510	0.35954	0.8	23.8194

2.7. Formal verification. Every step, including the arithmetic of the dual certificates, the closed-form evaluations behind the nonvanishing, the interval-arithmetic decay certificates and the prime number theorem in progressions modulo 4 and 6, is formalised in Lean 4 with Mathlib [9, 40]. The formal proofs are built on a shared core (rectangular determinants, Gauss valuations and assignment bounds, bordered Krylov determinants, energy certificates, the measure deduction) with thin instances for the three functions in (2.1). The written proofs below are at the level of a research article; for long computations we point to the formal proof, which is the basis for the correctness of Theorem 1.1.

3. THE CONSTRUCTION

3.1. The kernels. For the three functions of (2.1) the kernel M of (2.2) is as follows; here $0 \leq a < b$ are integers and $M[b, a] = M[a, b]$.

$$\begin{aligned} \lambda : \quad M[a, b] &= -\frac{1}{b-a} \sum_{k=a+1}^b \frac{1}{16k(4k-1)}, \\ M[a, a] &= -\sum_{k=1}^a \left(\frac{1}{16k^2} - \frac{1}{(4k-1)^2} \right); \\ \lambda' : \quad M[a, b] &= \frac{1}{b-a} \sum_{k=a+1}^b \frac{3}{16k(4k-3)}, \\ M[a, a] &= -\sum_{k=1}^a \left(\frac{1}{(4k-3)^2} - \frac{1}{16k^2} \right); \\ \chi : \quad M[a, b] &= \frac{1}{6(b-a)} \sum_{k=a+1}^b d(k), \\ M[a, a] &= -\sum_{k=1}^a \delta(k), \end{aligned}$$

with $d(k) = \frac{1}{3k-2} - \frac{1}{2k-1} + \frac{1}{6k}$ and $\delta(k) = \frac{1}{2(3k-2)^2} - \frac{1}{3(2k-1)^2} + \frac{1}{36k^2} = -\frac{1}{6}d'(k)$.

Lemma 3.1. *For each of the three functions, $\omega \mathbb{L}_F[R, Q] = M[R, Q] + \xi E_{R, Q}$ for all finite sets $R, Q \subset \mathbb{Z}_{\geq 0}$, with ω and ξ as in (2.1).*

Proof. From $\psi(w+1) - \psi(w) = 1/w$ one gets $F(k) - F(k-1)$ as an explicit rational function of k ; for instance $F_\lambda(k) - F_\lambda(k-1) = \frac{1}{k} - \frac{1}{k-1/4} = -\frac{1}{k(4k-1)}$, and telescoping gives the off-diagonal entries. Similarly $\psi'(w+1) - \psi'(w) = -1/w^2$ gives $F'(a) - F'(0)$. Finally $\omega F'(0) = \xi$: for λ , $\frac{1}{16}(\psi'(1) - \psi'(\frac{3}{4})) = \frac{1}{16}(\frac{\pi^2}{6} - \pi^2 + 8G) = \lambda$; for λ' use $\psi'(\frac{1}{4}) = \pi^2 + 8G$; for χ use $\psi'(\frac{1}{2}) = \frac{\pi^2}{2}$ and $\psi'(\frac{1}{3}) = \frac{2\pi^2}{3} + \frac{9}{2}L(2, \chi_{-3})$, which give $-\frac{1}{6}F'_\chi(0) = \sum_{k \geq 1} \delta(k) = \frac{1}{4}L(2, \chi_{-3})$. $\square$

Note that $2\lambda = G - 5\pi^2/48$, $2\lambda' = G + 5\pi^2/48$ and $4\xi_\chi = L(2, \chi_{-3})$, and $\mu(a\xi + b) = \mu(\xi)$ for rational $a \neq 0, b$. So Theorem 1.1 is a statement about the three numbers ξ .

3.2. Geometries.

Definition 3.2. Let $r, m \geq 0$ and $e \in \{0, 1\}$. The *geometry* $\mathcal{G}(r, m, e)$ consists of the common nodes $C = [0, r) \cup \{r + 2m : e = 1\}$, the row nodes $X' = \{r + 2i : 0 \leq i < m\}$ and the column nodes $Y' = \{r + 2i + 1 : 0 \leq i < m\}$. Rows are indexed by $R = C \sqcup X'$, columns by $Q = C \sqcup Y'$. The *member* $e = 0$ is $\mathcal{G}_1(r, m)$, the member $e = 1$ is $\mathcal{G}_2(r, m)$. The matrix of the geometry is

$$\Phi(x) = \Phi_{\mathcal{G}}(x) = (M[a, b] + x \mathbf{1}[a = b \in C])_{a \in R, b \in Q},$$

a square matrix of size $|C| + m$, and $P_G(x) = \det \Phi_G(x) \in \mathbb{Q}[x]$.

By Lemma 3.1, $\Phi_G(\xi + t) = \omega \mathbf{L}_F[R, Q] + t E_{R, Q}$ for real t . The geometry with $m = 0$, $e = 0$ is the square Löwner matrix of size r . We use *admissible* geometries, with $m \approx r/8$: for λ, λ' the condition is $r \leq 8m \leq r + 8$, and for $L(2, \chi_{-3})$ it is $r \leq 8m \leq r + 9$ (the extra unit is needed to find a pair of members for every prime $\ell \equiv 1 \pmod{6}$; see §7). All $o(1)$ terms below are uniform over admissible geometries as $r \rightarrow \infty$.

Lemma 3.3 (integrality). *P_G has degree at most $|C|$. If $d_G = \text{den } P_G$ and p/q is a fraction in lowest terms with $P_G(p/q) \neq 0$, then $|d_G P_G(p/q)| \geq q^{-|C|}$.*

Proof. The variable occurs only in the $|C|$ diagonal entries indexed by C , so each Leibniz term has degree at most $|C|$. Then $q^{|C|} d_G P_G(p/q)$ is a nonzero integer. $\square$

Remark 3.4. When the $m \times m$ block $M[X', Y']$ is invertible, the Schur complement gives $P_G(x) = \det M[X', Y'] \cdot \det(xI_C + M_S)$ with $M_S = M[C, C] - M[C, Y']M[X', Y']^{-1}M[X', C]$. Thus the hybrid determinant is, up to a constant, the characteristic polynomial of a rational matrix of size $|C|$, as in the square case. We do not need this factorisation: Lemma 3.3 only uses the degree bound.

4. ARITHMETIC: THE CONTENT

In this section we prove the following.

Theorem 4.1 (content). *For admissible geometries of scale r ,*

$$\log \text{den } P_G \leq (\Lambda + o(1)) r^2,$$

where $\Lambda = 3.70343$ for λ and λ' , and $\Lambda = 10339/3072 = 3.36556\dots$ for $L(2, \chi_{-3})$.

We give the details for λ and indicate the changes for λ' and for $L(2, \chi_{-3})$ at the end.

4.1. Gauss valuations and assignment problems. Fix a prime p . Let $\mathcal{O} \subset \mathbb{Q}(x)$ be the localisation of $\mathbb{Z}_{(p)}[x]$ at the prime ideal (p) ; it is a discrete valuation ring with uniformiser p and residue field $\mathbb{F}_p(x)$, and its valuation v restricted to $\mathbb{Q}[x]$ is the *Gauss valuation* $v(f) = \min_k v_p(f_k)$, $f = \sum f_k x^k$. By Gauss's lemma $v(fg) = v(f) + v(g)$, and $v_p(\text{den } f) = \max(0, -v(f))$.

Lemma 4.2 (assignment bound and LP duality). *Let B be a square matrix over $\mathbb{Q}[x]$ with rows I and columns J , $W_{ab} = \max(0, -v(B_{ab}))$, and let $\tau(W) = \max_{\sigma} \sum_{a \in I} W_{a\sigma(a)}$ over bijections $\sigma : I \rightarrow J$. Then $v(\det B) \geq -\tau(W)$. If real numbers u_a, v_b satisfy $W_{ab} \leq u_a + v_b$ for all a, b , then $\tau(W) \leq \sum_a u_a + \sum_b v_b$.*

Proof. Each Leibniz term has valuation $\sum_a v(B_{a\sigma(a)}) \geq -\sum_a W_{a\sigma(a)}$, and v of a sum is at least the minimum. The second statement is weak LP duality for the assignment problem: $\sum_a W_{a\sigma(a)} \leq \sum_a (u_a + v_{\sigma(a)})$. $\square$

For $\Phi(x)$ the weight of every entry is $W_{ab}(p) = \max(0, -v_p(M[a, b]))$, since $v(x) = 0$. We call τ_p the value of this assignment problem; thus $v_p(\text{den } P_{\mathcal{G}}) \leq \tau_p$.

4.2. The level majorant. Let $p \geq 5$ and, for a prime power $q = p^e$, let $S_q = \{k \geq 1 : q \mid k(4k - 1)\}$; it consists of the classes $k \equiv 0$ and $k \equiv k_q \pmod{q}$, where $k_q = (j_q q + 1)/4$ with $j_q \in \{1, 3\}$, $j_q q \equiv 3 \pmod{4}$. Let $\beta_q(a) = \#(S_q \cap [1, a])$. The ultrametric inequality applied to the formulas for M gives

$$\begin{aligned} W_{ab}(p) &\leq \sum_{e \geq 1} w^{(p^e)}(a, b), \\ w^{(q)}(a, b) &= \mathbf{1}[\beta_q(a) \neq \beta_q(b)] + \mathbf{1}[q \mid a - b] \quad (a \neq b), \\ w^{(q)}(c, c) &= 2 \cdot \mathbf{1}[c \geq k_q], \end{aligned}$$

and $w^{(q)} \in \{0, 1, 2\}$. Since the maximum of a sum is at most the sum of the maxima, $\tau_p \leq \sum_e \tau(w^{(p^e)})$. The primes 2 and 3 and all levels $e \geq 2$ contribute $o(r^2)$ in total (each $w^{(q)}$ has assignment value $O(r)$, there are $O(\log r)$ levels for each of 2, 3, and $\sum_{p^e \leq Cr, e \geq 2} \log p = O(\sqrt{r})$).

4.3. Explicit dual certificates. Fix an odd prime power q , write $k = k_q$, $N = r + 2m$, and let

$$\delta_A(q) = \min(2r, 2(r - n) + 2 \min(n, (r - q)_+)), \quad n = \min(r, k),$$

the value of the square block $C \times C$. Let $U = [r - q, k]$ (empty if $r - q \geq k$) be the set of “low unpaired” indices; a zone node $z \in X' \cup Y'$ is *low* if $z - q \in U$, and L is the number of low zone nodes.

Proposition 4.3 (dual certificates). *For $\mathcal{G}(r, m, 0)$ with $8m \leq r$, $\tau(w^{(q)}) \leq \delta_A(q) + E(q)$ with*

range	$E(q)$	potentials
$q < 2m$	$2m - q \parallel m/q \parallel$	by residue class mod q
$2m \leq q \leq r$	$\min(2m + L, m + L + U)$	two certificates
$r < q < N$	$2m + (N - q)$	$\mathbf{1}[c \geq k]$ on C , $1 + \mathbf{1}[z \geq q]$ on the zone
$q \geq N, k \leq r$	$2m$	$\mathbf{1}[c \geq k]$ on C , 1 on the zone
$q \geq N, k > r$	$(N - k)_+$	0 on C , $\mathbf{1}[z \geq k]$ on the zone

($\parallel \cdot \parallel$ is the distance to the nearest integer.)

Proof sketch. In each range one writes down potentials and checks $w \leq u + v$ entrywise, using two facts: two distinct indices not congruent modulo q have $w \leq 1$, and two zone nodes are never congruent once $q \geq 2m$. For example, for $q \geq N$ and $k \leq r$, give the common nodes the potential $\mathbf{1}[c \geq k]$ and the zone nodes the potential 1: there are no congruences, so $w \leq 1$ off the

diagonal, and the diagonal cells (only at $c \in C$) are covered. For $q < 2m$ the potentials are $\frac{1}{2}, \frac{3}{2}, 1$ on rows and $\frac{3}{2}, \frac{1}{2}, 1$ on columns of a residue class ϱ , according as X' has more, fewer or as many nodes in ϱ as Y' ; since the zone alternates between X' and Y' , the imbalance is at most one per class and the number of unbalanced classes is $2q\|m/q\|$. The certificates for $2m \leq q \leq r$ start from the square-frame potential on C and raise the potential of low zone nodes (first certificate), or move half a unit between rows and columns in the residue classes of high zone nodes (second certificate). All cases are checked in the formal proof. $\square$

Remark 4.4. The certificates are sharp: the exact assignment value computed by the Hungarian algorithm agrees with $\delta_A + E$ for all odd q and all tested (r, m) . Sharpness is not used.

4.4. The tropical content. Put $\theta = q/r$, $\alpha = m/r$. Proposition 4.3 gives $\tau(w^{(q)}) \leq r f_j(\theta; \alpha) + O(1)$ for an explicit continuous piecewise linear function f_j supported in $[0, 4(1 + 2\alpha)/j]$. Summing over primes in the two classes modulo 4 with the prime number theorem in arithmetic progressions, $\frac{1}{r} \sum_{p \equiv a(4)} f(p/r) \log p \rightarrow \frac{1}{2} \int_0^\infty f$ for bounded Riemann-integrable f of compact support (see [8, Ch. 20] for the prime number theorem in progressions), one finds

$$\sum_p \tau_p \log p \leq (\Lambda_{\text{trop}}(\alpha) + o(1))r^2,$$

$$\Lambda_{\text{trop}}(\alpha) = \frac{313}{105} + \frac{554}{105}\alpha + \left(\frac{118}{21} + 2\log 2\right)\alpha^2$$

for $\alpha \leq \frac{1}{8}$ (the term $2\alpha^2 \log 2$ comes from the sawtooth $q\|m/q\|$ in the range $q < 2m$). At $\alpha = \frac{1}{8}$, $\Lambda_{\text{trop}} = 3.7499347\dots$; for $\alpha = 0$ it is the square-frame value $313/105$.

4.5. Rank deficits. The assignment bound is sharp for the matrix of weights, but not for $\det \Phi$: cancellation in the Leibniz expansion makes $\text{den } P$ smaller. We capture it by the following principle.

Lemma 4.5 (rank deficit). *Let (u, v) be integer-valued potentials feasible for the weights of Φ at p , of value τ' . Then $B = \text{diag}(p^u) \Phi \text{diag}(p^v)$ has entries in $\mathcal{O}$, and*

$$v_p(\text{den } P_{\mathcal{G}}) \leq \tau' - \text{corank}_{\mathbb{F}_p(x)}(B \bmod p).$$

Proof. $v(\det \Phi) = -\tau' + v(\det B)$, and over the discrete valuation ring $\mathcal{O}$ the Smith normal form shows $v(\det B) \geq \text{corank}(B \bmod p)$. (The formal proof avoids the Smith form: if $B \bmod p$ has rank ρ , an explicit rank witness and a change of basis over $\mathcal{O}$ exhibit $|I| - \rho$ rows divisible by p .) $\square$

Write δ_p for the corank obtained in this way with an optimal dual (so $v_p(\text{den } P) \leq \tau_p - \delta_p$). We use five families of primes. In all of them $p^2 > 4N$, so only the level $e = 1$ matters; the reduction $B \bmod p$ has a block structure governed by the singular indices $k \in S_p$ below N , which cut $[0, N]$ into blocks

on which the leading coefficients (lead₋₁ of the partial sums of $1/(4k(4k-1))$ and lead₋₂ of the diagonal sums) are constant.

Theorem 4.6 (rank deficits for λ). *Let $\mathcal{G} = \mathcal{G}(r, m, e)$ be admissible and r large. Then $v_p(\text{den } P_{\mathcal{G}}) \leq \tau_p - \delta_p + O(1)$, with δ_p as follows, for all primes p except $p \in \{139, 1009\}$:*

<i>primes</i>	δ_p	<i>mechanism</i>
$p \equiv 3 \pmod{4}, \frac{5}{8}r < p < \frac{4}{5}r; \quad p \equiv 1 \pmod{4}, \frac{5}{7}r < p < r$	m	<i>integral Schur complement</i>
$r < p < r + m$	$r + m - p$	<i>harmonic reflection, rank count</i>
$p \equiv 3 \pmod{4}, \frac{4}{5}(r + m) \leq p < r$	$\frac{5p+1}{4} - r - m$	<i>Frobenius-König</i>
$p \equiv 1 \pmod{4}, \frac{4}{7}(r + m) \leq p < \frac{5}{7}r$	$\frac{7p+1}{4} - r - m$	<i>Frobenius-König</i>
$p \equiv 3 \pmod{4}, \frac{5}{12}r < p < \frac{4}{9}r; \quad p \equiv 1 \pmod{4}, \frac{5}{11}r \leq p < \frac{1}{2}r$	m	<i>block-diagonal leading form</i>

We sketch the mechanisms; the complete proofs are formalised.

The integral Schur complement. Write $\Phi = \begin{pmatrix} A & \Phi[C, Y'] \\ \Phi[X', C] & \Phi[X', Y'] \end{pmatrix}$ with $A = \Phi[C, C]$, so $\det \Phi = \det A \cdot \det T$ with the Schur complement $T = \Phi[X', Y'] - \Phi[X', C]A^{-1}\Phi[C, Y']$. At these primes the square block has $-v(\det A) = 2r$, and the claim is that T has entries in $\mathcal{O}$, whereas the assignment bound allows $v(T_{zy}) = -1$. For a zone row z with partner $c_z = z - p \in C$ decompose $\Phi[z, C] = A[c_z, C] + \ell_z$, and similarly for columns; since $A[c_z, \cdot]A^{-1} = e_{c_z}^\top$,

$$T_{zy} = \Delta^2 M(z, y) - \ell_z A^{-1} \ell_y^\top,$$

$$\Delta^2 M(z, y) = M(z, y) - M(z, c_y) - M(c_z, y) + M(c_z, c_y),$$

and x cancels. The leading form of $p^2 A$ is block diagonal by residue class (blocks of size 1 and 2), with explicit nonzero determinants (here $p \neq 139$ is used: one block has determinant $3^2 \cdot 5 \cdot 139/6400$). Expanding $\ell_z A^{-1} \ell_y^\top$ modulo p^2 , the only surviving terms are $p\beta \left[\frac{\sigma}{4(c_z - c_y)} + \frac{\sigma}{4(c_y - c_z)} \right] = 0$, an antisymmetry of Cauchy-type sums. For $p \equiv 1 \pmod{4}$ the analogous computation uses a scalar Schur complement: modulo p it equals $s_0 I$ with $s_0 = x - \frac{1}{16} H_{K_p-1}^{(2)}$, $K_p = (3p+1)/4$, which follows from partial fractions and the reflections $H_{p-1-n} \equiv H_n$, $H_{p-1-n}^{(2)} \equiv -H_n^{(2)} \pmod{p}$ for harmonic numbers. Here it is essential that x is an indeterminate: s_0 is a unit of $\mathcal{O}$, although it may vanish at a specialisation $x \in \mathbb{Q}$.

Primes just above r . For $r < p < r + m$ the only singular indices below N are k_p and p . After eliminating an invertible block, the Schur complement on $[0, k_p)$ is $s_0 I - \frac{1}{16} E E^\top$ with $E = (1/(u+i))_{u < k_p, 1 \leq i \leq p-r}$, by the same harmonic reflection. Each zone node $z < p$ contributes one rank-one term

indexed by $i = p - z$, so the correction has rank at most $p - r$ and the corank is at least $r + m - p$.

Primes with one singular zone index. For $p \equiv 3 \pmod{4}$ with $\frac{4}{5}(r + m) \leq p < r$ (and $p \equiv 1 \pmod{4}$ in the corresponding range) there is one singular index inside the zone. After elimination the zone $\times$ zone block of $B \bmod p$ has a zero sub-block of size $L_X \times L_Y$ with $L_X + L_Y = L \geq m$ (the low zone nodes), so by the Frobenius–König theorem its rank is at most $2m - L$, and the corank is at least $L - m$.

The low band. For primes p with $2m < p < r$ that are *clean* (no singular index in the zone) we prove a general statement: if every residue-class block of the leading form of $p^2 A$ is invertible modulo p , then T is integral. The proof splits $p^2 A = B_0 + pE$ with B_0 block diagonal by residue class, inverts by a Neumann series, and shows that the order- p term vanishes: the translation $c \mapsto c + (y - z)$ identifies the classes of z and y (cleanness is used here), and what remains is $\sum_{i,j} g_i g_j (L_i - L_j) = 0$, an antisymmetric sum. In the two families listed the classes have size 2 or 3 and the block determinants are explicit rationals, for example $-350972849/368640000$ (with prime numerator), which are nonzero modulo p except at $p = 139$ and $p = 1009$.

4.6. Summation. Each family is a union of intervals $p \in (ar, br)$ in a fixed class, and δ_p is a linear function of p there. With the prime number theorem in progressions, the total saving is

$$\left(\frac{129}{4480} + \frac{59}{4480} + \frac{29}{6336} \right) r^2 = (0.0287946 + 0.0131696 + 0.0045770) r^2$$

(the three terms come from the first row, the next three rows, and the last row of the table). Hence

$$\Lambda_\lambda \leq \Lambda_{\text{trop}}\left(\frac{1}{8}\right) - 0.0465412 = 3.7033935 \dots \leq 3.70343.$$

(The formal proof uses the slightly coarser rational value 3.7034204, from a truncation of the sawtooth term.)

4.7. The second member. The geometry $\mathcal{G}_2$ has one extra common node $b = r + 2m$. Bordering $\Phi_{\mathcal{G}_1}$ by one row and one column changes each per-prime bound by $O(1)$, so the content constant is unchanged; but the rank deficits have to be re-proved for the bordered matrix, since a deficit is not automatically inherited under bordering. For the deficits proved by rank witnesses there is a general bordering lemma: a witness for $\Phi_{\mathcal{G}_1}$ extends, with the border row and column scaled by p^w , to a witness for $\Phi_{\mathcal{G}_2}$ at a cost of $2w = O(1)$. For the low-band deficit, which is proved through integrality of the Schur complement, the border row and column are rescaled by p^w so that the same core argument applies, with one unit of slack.

4.8. The mirror constant and $L(2, \chi_{-3})$. For λ' the singular indices are those k with $p \mid k(4k - 3)$; the two residue classes modulo 4 exchange their roles. The tropical law is the same function Λ_{trop} , the deficits are the same

with the classes swapped, and $\Lambda_{\lambda'} = \Lambda_{\lambda}$. The formal proof uses one generic content theorem with the offset (1 or 3) as a parameter.

For $L(2, \chi_{-3})$ the singular indices are those k with $p \mid k(3k-2)(2k-1)$, and the relevant classes of primes are modulo 6. Here no rank deficits are used. Two explicit duals suffice: below a threshold θ^* ($\frac{27}{32}$ for $p \equiv 1, \frac{3}{4}$ for $p \equiv 5 \pmod{6}$) the residue-class potentials of the first row of Proposition 4.3, and above θ^* a certificate in which only row potentials are raised, on nodes that have a congruent partner of level 0. The levels (leading coefficients of the partial sums) enter only through which of them vanish, and this reduces to eight explicit identities. The resulting bound is piecewise linear in p/r , with 3+6 and 3+5 pieces in the two classes, and integrating gives $\Lambda_{\chi} = \frac{10339}{3072}$. The per-prime bound agrees with the exact assignment value at all primes tested ($r = 256$ and $r = 1024$).

5. NONVANISHING MODULO A PRIME

This section follows the formal proof. We treat λ in detail; λ' and $L(2, \chi_{-3})$ are in §5.7. Throughout, $\ell \equiv 3 \pmod{4}$ is a prime, $K = (\ell + 1)/4$, and $\mathcal{G}_e = \mathcal{G}(r, m, e)$, $e \in \{0, 1\}$, is the pair of members of a scale r , with

$$(5.1) \quad 1 \leq m < K \leq r, \quad S := \ell - r - 2m \geq n := K - m.$$

For fixed r and $r \leq 8m \leq r + 8$, (5.1) holds for every prime ℓ with $\frac{4}{3}(r + m) + \frac{1}{3} \leq \ell \leq 4r - 1$, an interval $[\frac{3}{2}r + O(1), 4r]$.

Theorem 5.1 (nonvanishing). *Assume (5.1) and let $x_0 \in \mathbb{Q}$ with $\ell \nmid \text{den } x_0$. Then $P_{\mathcal{G}_0}(x_0) \neq 0$ or $P_{\mathcal{G}_1}(x_0) \neq 0$.*

The proof has four parts: a reduction modulo ℓ to a pair of bordered Cauchy–Gram determinants (§5.1); a lemma showing that if both vanish, a bordered Krylov determinant vanishes (§5.2); a deformation in a parameter θ (§5.3); and an algebraic evaluation of the deformed Krylov determinant, up to factors that are units at ℓ , by a commuting Jacobi matrix, diagonal steps and Darboux steps (§§5.4–5.6).

5.1. Reduction to bordered Cauchy–Gram determinants. Call a node b *low* if $b < K$ and *high* if $K \leq b \leq \ell - 1$; all nodes of $\mathcal{G}_e$ are at most $r + 2m \leq \ell - 1$, and all low nodes are common nodes since $K \leq r$. For a high node put $j_b = \ell - b \in [1, \ell - K]$, and for $j \geq 1$ let $w_j = (1/(a + j))_{0 \leq a < K} \in \mathbb{F}_{\ell}^K$; all denominators are nonzero modulo ℓ . For a set J let W_J be the $K \times |J|$ matrix with columns w_j , $j \in J$. In these coordinates the row nodes X' and the column nodes Y' have $J_{X'} = S + \{2, 4, \dots, 2m\}$ and $J_{Y'} = S + \{1, 3, \dots, 2m - 1\}$. The high nodes *absent* from the geometry are $\{1, \dots, S\}$ for $\mathcal{G}_0$ and $\{1, \dots, S - 1\}$ for $\mathcal{G}_1$, whose extra node $r + 2m$ has $j = S$. Put

$$N = \sum_{j=1}^{S-1} w_j w_j^{\top}, \quad w = w_S.$$

Lemma 5.2 (ℓ -adic structure). *The only ℓ -singular factor occurring in the entries $M[a, b]$, $a, b \leq \ell - 1$, is $4K - 1 = \ell$. Consequently $M[a, b]$ is ℓ -integral if a, b are both low, or both high and distinct; for a low and b high, $\ell M[a, b] \equiv \frac{1}{4}w_{j_b}(a)$; for b high, $\ell^2 M[b, b] \equiv 1 \pmod{\ell}$; and on the low block, $16 M[\text{low}, \text{low}] \equiv W_{[1, \ell-K]} W_{[1, \ell-K]}^\top + H_{K-1}^{(2)} I_K \pmod{\ell}$, where $H_n^{(2)} = \sum_{i \leq n} i^{-2}$.*

Proof. $4k - 1 \in \{\ell, 2\ell, 3\ell\}$ with $k \leq \ell - 1$ forces $k = K$. For a low and b high, the sum defining $M[a, b]$ contains the term $k = K$ once, and $16K = 4\ell + 4$, $b - a \equiv -(a + j_b)$. For the low block, $\frac{4}{4k-1} \equiv \frac{1}{k-K}$, and partial fractions give $\sum_{j=1}^{\ell-K} \frac{1}{(a+j)(b+j)}$ off the diagonal; on the diagonal one uses $H_{\ell-K}^{(2)} \equiv -H_{K-1}^{(2)}$. $\square$

Proposition 5.3 (reduction). *Let D be diagonal with entry 1 on low and ℓ on high rows and columns, and let h_e be the number of high rows of $\Phi_{\mathcal{G}_e}$. Then $D\Phi_{\mathcal{G}_e}(x)D$ is ℓ -integral, and with $y = 16x + H_{K-1}^{(2)}$,*

$$\ell^{2h_1} P_{\mathcal{G}_1}(x) \equiv \pm 16^{-K} \det \mathcal{B}(y, N),$$

$$\ell^{2h_0} P_{\mathcal{G}_0}(x) \equiv \pm 16^{-K} \det \mathcal{B}(y, N + ww^\top) \pmod{\ell},$$

$$\text{where } \mathcal{B}(y, N') = \begin{pmatrix} yI_K + N' & W_{Y'} \\ W_{X'}^\top & 0 \end{pmatrix}.$$

Proof. By Lemma 5.2, modulo ℓ the matrix $D\Phi D$ has the block form $\begin{pmatrix} xI + M_{\text{low}} & \frac{1}{4}W_H & \frac{1}{4}W_{Y'} \\ \frac{1}{4}W_H^\top & I & 0 \\ \frac{1}{4}W_{X'}^\top & 0 & 0 \end{pmatrix}$, where rows are low, high common, X' , columns are low, high common, Y' , and H is the set of j -coordinates of the high common nodes. Take the Schur complement of the identity block. By the low-block formula and the partition of $[1, \ell - K]$ into H , the absent set, $J_{X'}$ and $J_{Y'}$, the upper-left block becomes $\frac{1}{16}(yI + N_e + W_{X'} W_{X'}^\top + W_{Y'} W_{Y'}^\top)$, with N_e the Gram matrix of the absent columns. Row and column operations using the zero lower-right block remove the last two terms, and scaling gives the claim. $\square$

Reduction modulo ℓ commutes with evaluation at $x_0 \in \mathbb{Z}_{(\ell)}$, and $x \mapsto y$ is the same for both members. So if $P_{\mathcal{G}_0}(x_0) = P_{\mathcal{G}_1}(x_0) = 0$, then for $y_0 = 16x_0 + H_{K-1}^{(2)} \in \mathbb{F}_\ell$ both $\det \mathcal{B}(y_0, N)$ and $\det \mathcal{B}(y_0, N + ww^\top)$ vanish.

5.2. A pair of singular bordered matrices. Let $\Delta = \det(W_{X'}^\top, W_{Y'})$. If $\Delta \neq 0$, let $\Pi = I - W_{Y'}(W_{X'}^\top W_{Y'})^{-1} W_{X'}^\top$ be the projection onto $V = \ker W_{X'}^\top$, along the column span of $W_{Y'}$, and let U be a $K \times n$ matrix whose columns span V . Define

$$k_R = \det[w^\top (\Pi N)^k U]_{0 \leq k < n},$$

$$k_L = \text{the same with } X' \text{ and } Y' \text{ exchanged.}$$

(Changing U multiplies k_R by a nonzero factor; in k_L the columns of U span $\ker W_{Y'}^\top$, instead.)

Lemma 5.4 (singular pair). *Let N' be symmetric, $\tau \neq 0$, and suppose $\Delta \neq 0$. If $\det \mathcal{B}(y_0, N') = \det \mathcal{B}(y_0, N' + \tau w w^\top) = 0$ for some y_0 , then $k_R = 0$ for every admissible U , or $k_L = 0$ for every admissible U (computed with N' in place of N).*

Proof. Put $\mathcal{B}_2 = \mathcal{B}(y_0, N')$ and $\hat{w} = (w, 0)$, and consider the bordered matrix $\mathcal{P} = \begin{pmatrix} \mathcal{B}_2 & \hat{w} \\ -\tau \hat{w}^\top & 1 \end{pmatrix}$. Its Schur complement on the corner 1 is $\mathcal{B}_2 + \tau \hat{w} \hat{w}^\top = \mathcal{B}(y_0, N' + \tau w w^\top)$, so $\det \mathcal{P} = 0$, and its principal minor $\mathcal{B}_2$ is singular. Then $\mathcal{B}_2$ has a right kernel vector $\hat{x}$ with $\hat{w}^\top \hat{x} = 0$, or a left kernel vector $\hat{x}$ with $\hat{x}^\top \hat{w} = 0$. (If $\text{rank } \mathcal{B}_2$ is at most its size minus two, its kernel meets $\hat{w}^\perp$. Otherwise $\text{adj } \mathcal{B}_2 = c \hat{x} \hat{y}^\top$ with $c \neq 0$ and $\hat{x}, \hat{y}$ spanning the right and left kernels, and $0 = \det \mathcal{P} = \det \mathcal{B}_2 + \tau \hat{w}^\top \text{adj}(\mathcal{B}_2) \hat{w} = \tau c (\hat{w}^\top \hat{x}) (\hat{y}^\top \hat{w})$.)

Take the right case, $\hat{x} = (x, c)$. Then $(y_0 + N')x + W_{Y'}c = 0$, $W_X^\top x = 0$ and $w^\top x = 0$; and $x \neq 0$, since otherwise $W_{Y'}c = 0$ and $c = 0$ because $\Delta \neq 0$. Applying Π (which kills $W_{Y'}$ and fixes $x \in V$) gives $\Pi N'x = -y_0x$, hence $w^\top (\Pi N')^k x = (-y_0)^k w^\top x = 0$ for all k . Writing $x = Uz$ with $z \neq 0$, the matrix $[w^\top (\Pi N')^k U]_k$ annihilates z , so $k_R = 0$. The left case is the same argument for the transpose, $\mathcal{B}(y_0, N')^\top = \mathcal{B}(y_0, N')$ with X' and Y' exchanged, and gives $k_L = 0$. $\square$

No condition on y_0 is needed; in particular the value $y_0 = 0$ is not special. By Proposition 5.3 and Lemma 5.4 (with $N' = N$, $\tau = 1$), Theorem 5.1 follows from the three statements

$$(5.2) \quad \Delta \neq 0, \quad k_R \neq 0, \quad k_L \neq 0 \quad \text{in } \mathbb{F}_\ell.$$

5.3. Deformation and bordered Krylov determinants. We lift to characteristic zero and deform the nodes. Let θ be a parameter, $c_y = (1/(\theta + a + y))_{0 \leq a < K}$, W_X, W_Y the matrices with columns c_y for $y \in X = S + \{2, \dots, 2m\}$ and $y \in Y = S + \{1, \dots, 2m - 1\}$, $N_s = \sum_{y=1}^s c_y c_y^\top$, Π the projection onto $V = \ker W_X^\top$ along $\text{span } W_Y$, and $A_s = \Pi N_s$. For a $K \times K$ matrix B preserving V and $u \in V$, the *bordered Krylov determinant* is

$$\kappa(B, u) = \det[W_Y \mid u, Bu, \dots, B^{n-1}u];$$

it is nonzero exactly when u is a cyclic vector of $B|_V$. Put

$$k(\theta) = k_{K,m,S}(\theta) = \kappa(A_{S-1}, \Pi c_S) = \kappa(A_S, \Pi c_S)$$

(the two agree by (K2) below). At $\theta = 0$ all denominators are integers $a + y \leq S + K + 2m - 1 = \ell - r + K - 1 < \ell$.

Lemma 5.5 (bridge). *Let $K = m + n$ with $1 \leq n \leq S$, and let θ be generic, in the sense that $\theta + z \neq 0$ for every integer z , with $\Delta(\theta) \neq 0$. Then the row-Krylov determinant $\det[c_S^\top (\Pi N_{S-1})^k u_i]_{k,i < n}$, in which $u_1, \dots, u_n$ is the explicit basis U of V given by Lagrange interpolation at the nodes, equals $\nu(\theta) k(\theta)$, where $\nu(\theta) = C_\nu \prod_{a < K} \prod_{t \in Y} (\theta + a + t)$ and $C_\nu \in \mathbb{Q}^\times$ does not*

depend on θ . The constant C_ν is an ℓ -adic unit for every prime $\ell > S + K + m$ (all its prime factors are at most $S + K + m$).

Proof sketch. Let U and V' be the Lagrange bases of $\ker W_X^\top$ and $\ker W_Y^\top$, $H = V'^\top U$ and $H_c = V'^\top N_{S-1}U$; both are symmetric. Then $\Pi = UH^{-1}V'^\top$, hence $\Pi N_{S-1}U = UB$ with $B = H^{-1}H_c$, and B is self-adjoint for H , which turns the row-Krylov matrix into the column-Krylov matrix up to the factors $(c_X/c_Y)^n$ and $\det H$, where $c_X = \Pi_X(S)/\prod_a(\theta + a + S)$ and c_Y is the same with Π_Y . Two block-determinant identities, $\det[W_X | V'] \det[W_Y | U] = \Delta \det H$ and $\det[W_Y | C] \det[W_X | V'] = \Delta \det(C^\top V')$ with $C = [c_1, \dots, c_n]$, eliminate $\det H$ and Δ . By Lagrange interpolation $C^\top V'$ is a diagonal matrix times a Vandermonde matrix, and $\det[W_Y | C]$ is a Cauchy determinant; this gives the stated ν , whose only θ -dependence is $\prod_{a < K} \prod_{t \in Y} (\theta + a + t)$. The operator in the Krylov determinant of the formalisation is $\Pi N_S \Pi$, which agrees with A_S on V and is related to A_{S-1} by (K2). $\square$

Definition 5.6. Fix $B \geq 1$. A function f of θ is *good* (with bound B) if $f(\theta) = \varkappa \prod_{1 \leq c \leq B} (\theta + c)^{e_c}$ for rational $\theta > 0$, with integers e_c and a nonzero rational $\varkappa$ whose prime factors are at most B .

Good functions with bound B form a group under multiplication. The point of the definition is that a good function with $B < \ell$ is continuous at $\theta = 0$ and its value there is an ℓ -adic unit.

Theorem 5.7 (good form). *Assume $1 \leq m < K$ and $S \geq n = K - m$. Then $k_{K,m,S}$ is good with bound $S + K + 2m - 1$.*

Theorem 5.7 is all that is needed. The exact exponents e_c are known (they are piecewise quadratic in c , and $\varkappa$ is a ratio of Hankel determinants; this was checked in exact arithmetic in 156 cases with $K \leq 9$), but the proof below never computes them.

Proof of Theorem 5.1 from Theorem 5.7 and Proposition 5.8. By Proposition 5.8 below, Δ is an ℓ -unit at $\theta = 0$, so $\Delta \neq 0$ in $\mathbb{F}_\ell$. The row-Krylov determinant of Lemma 5.5 is a continuous function of θ near 0 (its entries are rational in θ with no pole at 0, and Π is continuous because $\Delta(0) \neq 0$). By Lemma 5.5 and Theorem 5.7 it equals, for rational $\theta \in (0, 1)$, a good function with bound $S + K + 2m - 1 < \ell$, which is continuous at 0; hence the values at $\theta = 0$ agree, and the value is an ℓ -adic unit. All entries are ℓ -integral at $\theta = 0$, and reduction modulo ℓ of the matrices of §5.3 gives the matrices of §5.2 (after reversing the order of the border columns, which does not change Π). So $k_R \neq 0$ in $\mathbb{F}_\ell$, and the same argument with X and Y exchanged gives $k_L \neq 0$. This is (5.2). $\square$

Proposition 5.8 (the Cauchy-type Gram determinant). *For $m \leq K$ and $\theta \notin \{-1, -2, \dots\}$,*

$$\det(W_X^\top W_Y) = V(X)V(Y) \frac{H_{K,m}}{\prod_{i < m} (\theta + S + 1 + i)_m (\theta + S + K + 1 + i)_m},$$

$$H_{K,m} = \det \left[\sum_{a=0}^{K-1} a^{p+q} \right]_{p,q < m},$$

where $V(X) = \prod_{s < s' \in X} (s' - s)$ and $V(Y)$ are Vandermonde products. The Hankel determinant $H_{K,m}$ of the uniform weight on $\{0, \dots, K-1\}$ is a product of factorials and binomial coefficients with all prime factors less than $2K$. In particular Δ is an ℓ -adic unit at $\theta = 0$ when $S + K + 2m \leq \ell$.

Proof. Step 1 (Cauchy–Binet). By Cauchy–Binet and Cauchy’s determinant, $\det(W_X^\top W_Y)$ is $\pm V(X)V(Y)$ times $\sum_P V(P)^2 \prod_{a \in P} \rho(a)$ over m -subsets $P \subseteq \{0, \dots, K-1\}$, where $\rho(a) = \prod_{s \in X} (\theta + a + s)^{-1} \prod_{t \in Y} (\theta + a + t)^{-1}$. This sum is the Hankel determinant $\det[\sum_a \rho(a) a^{p+q}]_{p,q < m}$ of the moments of ρ .

Step 2 (basis change and telescoping). Replacing the monomials a^p and a^q by monic rising factorials $\varphi_p(a) = (a + \theta + 1)_p$ and suitable ψ_q is a unitriangular change of basis, which does not change the determinant. After it, $\rho(a) \varphi_p(a) \psi_q(a) = 1/(a + \theta + p + 1)_{2m-p-q}$, and these sums over $a < K$ telescope: $L \sum_{a < K} 1/(c + a)_{L+1} = 1/(c)_L - 1/(c + K)_L$. Hence each entry is a ratio of Gamma values times a divided difference $f[u_p, v_q]$ of $f(z) = (z)_K$ at two points $u_p = \theta + p + 1$, $v_q = \theta + 2m - q$.

Step 3 (Newton expansion). We use divided differences only through three facts: the recursion; Newton’s expansion, which peels off the nodes $u_0, u_1, \dots$ one at a time and gives $f[u_p, v_q] = \sum_{i,j} N_i(u_p) N_j(v_q) f[\text{window}]$ with Newton basis polynomials N_i ; and the values on windows of consecutive points, $f[s, s+1, \dots, s+n] = \binom{K}{n} (s+n)_{K-n}$ for $f = (z)_K$. Thus $[f[u_p, v_q]] = \mathcal{A} \mathcal{D} \mathcal{B}^\top$ with $\mathcal{A}, \mathcal{B}$ triangular, and the determinant reduces to a Pochhammer product times $\det[\binom{K}{p+q+1}]_{p,q < m}$.

Step 4 (Chu–Vandermonde). The identity $\sum_{a=0}^{K-1} \binom{a}{p} \binom{K-1-a}{q} = \binom{K}{p+q+1}$ (Chu–Vandermonde), together with another unitriangular basis change, gives $H_{K,m} = \prod_p (-1)^p p! \prod_q q! \det[\binom{K}{p+q+1}]$. No limit in θ is used. Finally $H_{K,m}$ is the Hankel determinant of the uniform weight, i.e. of the Hahn polynomials with $\alpha = \beta = 0$ [19, §9.5]; its classical product formula has all factors at most $K + m \leq 2K - 1$. $\square$

5.4. A commuting Jacobi matrix. Put $g = S - n + 1$ and, for $0 \leq a < K$,

$$\sigma(a) = (a+1)(a+1+\theta)(K-1-a) \frac{\prod_{i=1}^m (\theta + S + 2i + a)}{\prod_{i=1}^{m-1} (\theta + S + 2i + 1 + a)},$$

$$\tau(a) = (a+1)(a+1+\theta)(K-1-a) \frac{\prod_{i=0}^m (\theta + S + 2i + 1 + a)}{\prod_{i=1}^m (\theta + S + 2i + a)},$$

$$\mu(a) = -(K-1-a)^2 (\theta + S + 1 + a) (\theta + S + 2m + 1 + a).$$

Let $\tilde{T}$ be the $K \times K$ tridiagonal matrix with $\tilde{T}_{a,a+1} = -\sigma(a)$, $\tilde{T}_{a+1,a} = -\tau(a)$ and

$$\begin{aligned} \tilde{T}_{aa} &= \mu(a) + \frac{\sigma(a-1)\tau(a-1)}{\mu(a-1)} + g(g+1)a^2 + g c_0 a, \\ c_0 &= -2(K-1)(\theta+g) - 2K^2 + 2K - m^2. \end{aligned}$$

For $m = 0$ this is, up to conjugation, the Jacobi matrix of a family of Hahn polynomials; for $m \geq 1$ the products over i record the border.

- Proposition 5.9** (bispectral identities). *(i) Column chain. For $1 \leq y \leq S$, $\tilde{T}c_y = l(y)c_{y-1} + d(y)c_y + r(y)c_{y+1} + \sum_{t \in Y} \kappa_t(y)c_t$ with $l(y) = -\tau(-\theta - y)$, $r(y) = -\sigma(-\theta - y - 1)$, $d(y) = \tilde{T}_{aa}|_{a=-\theta-y}$, and explicit corrections $\kappa_t(y)$ given by residues of τ and σ at the border points. The transpose $\tilde{T}^\top$ satisfies the analogous row chain with corrections in $\text{span } W_X$, with the same coefficients.*
- (ii) Invariance. $\tilde{T}$ maps $V = \ker W_X^\top$ and $\text{span } W_Y$ into themselves; hence $\tilde{T}\Pi = \Pi\tilde{T}$.*
- (iii) Commutation. $[N_S, \tilde{T}]$ maps V into $\text{span } W_Y$; hence $\tilde{T}$ commutes with A_S on V .*

Proof sketch. (i) says that the three-term recurrence of $\tilde{T}$ in the node variable a is mirrored, under $a \mapsto -\theta - y$, by a three-term recurrence in the column variable y : a bispectral duality in the sense of Grünbaum and Haine [15]. It is an identity between rational functions of a . Conjugating by $\hat{Q}(z) = \prod_{t \in Y} (z + t)$ turns $\tilde{T}$ into a difference operator with polynomial coefficients, and (i) becomes the statement that a certain polynomial has degree at most $m + 2$. Its four top coefficients are computed from the low-order coefficients of $\prod_i (1 + c_i W)$ modulo W^4 and vanish identically; the remainder, of degree less than m , is identified by Lagrange interpolation at the border points. (ii) follows from (i) at the border points, where l and r vanish. (iii) follows from (i) and the row chain by telescoping $\sum_{y \leq S} ((\tilde{T}c_y)c_y^\top - c_y(\tilde{T}^\top c_y)^\top)$. $\square$

5.5. Krylov calculus. We use five elementary facts about bordered Krylov determinants of matrices preserving V .

- (K1) *Cross ratio.* If A and T commute on V , then $\kappa(A, u)\kappa(T, u') = \kappa(A, u')\kappa(T, u)$ for all $u, u' \in V$. (If $T|_V$ is cyclic, $A = p(T)$ on V , and $\kappa(A, u)/\kappa(T, u)$ is the determinant of the matrix of the powers of $p(T)$ in the Krylov basis of T , which does not depend on u .)
- (K2) *Rank-one update along the start vector.* $\kappa(B + uz^\top, u) = \kappa(B, u)$.
- (K3) *Chains.* If $Tc_y \equiv l(y)c_{y-1} + d(y)c_y + r(y)c_{y+1}$ modulo $\text{span } W_Y$ for the relevant y , then $\kappa(T, \Pi c_{y_0}) = \prod_{k < n} \prod_{j < k} l(y_0 - j) \cdot \det[W_Y | c_{y_0}, c_{y_0-1}, \dots, c_{y_0-n+1}]$, and similarly upward with r ; the Krylov vectors are triangular in the chain.
- (K4) *Cauchy determinants.* $\det[W_Y | c_{y_1}, \dots, c_{y_n}]$ is a Cauchy determinant: a product of differences of the shifts divided by a product of factors $\theta + a + y$ [20, 21].

(K5) *Transfer*. If $B_2\Phi = \Phi B_1$ for a linear map Φ carrying V_1 into V_2 , then $\kappa(B_2, \Phi z) = \det[W_2 | \Phi] \cdot \kappa(B_1, z)$, where $\det[W_2 | \Phi]$ is computed in adapted bases.

5.6. Proof of Theorem 5.7. Write $u(m, n, S; \theta) = k_{m+n, m, S}(\theta)$ and $B = S + n + 3m - 1 = S + K + 2m - 1$. We prove by induction on $n \geq 1$ that $u(m, n, S)$ is good with bound B for all $S \geq n$.

Base case $n = 1$. Then $\kappa(A, \Pi c_S) = \det[W_Y | \Pi c_S] = \det[W_Y | c_S]$, a Cauchy determinant, which is good.

Diagonal steps (from S to $S + 1$). Since $c_y(\theta) = c_{y+1}(\theta - 1)$, the instances (θ, S) and $(\theta - 1, S + 1)$ have the same border. Apply (K1) to the commuting pairs $(A, \tilde{T})$ of both instances, (K2) to pass between A_{S-1} and A_S , and (K3), (K4) to evaluate the four values $\kappa(\tilde{T}, \Pi c_y)$ that occur as explicit products (each is nonzero, so $\tilde{T}$ is cyclic). The result is

$$u(m, n, S + 1; \theta - 1) = u(m, n, S; \theta) \cdot \varkappa_{\text{DS}} \prod_{j < n-1} \left(\frac{\theta + K - 1 - j}{\theta + j} \right)^{2(n-1-j)}$$

with a constant $\varkappa_{\text{DS}}$ whose prime factors are at most B . Substituting $\theta + 1$ for θ expresses $u(m, n, S + 1; \theta)$ through $u(m, n, S; \theta + 1)$ times factors $\theta + c$ with $1 \leq c \leq K$; so goodness passes from S to $S + 1$. (It is this orientation, downward in S and upward in θ , that keeps all bases $c \geq 1$.)

Darboux steps (from $n - 1$ to n at $S = n$). Consider the instance (K, m, S) with $S = n - 1$, i.e. $g = 0$. After normalising $\tilde{T}$ so that it annihilates the kernel vector of A on V , it has an LU factorisation

$$\tilde{T} = LU, \quad U = \text{diag}(\mu) - \text{superdiag}(\sigma), \quad L = I - \sum_a \frac{\tau(a-1)}{\mu(a-1)} E_{a, a-1},$$

with explicit pivots $\mu(a) = -(K - 1 - a)^2(\theta + S + 1 + a)(\theta + S + 2m + 1 + a)$; note $\mu(K - 1) = 0$. Reversing the factors gives the Jacobi matrix of the instance with one node fewer and one more column:

$$(UL)_{<K-1} = D^{-1}(\tilde{T}' + \beta I)D, \quad \tilde{T}' = \tilde{T}(K - 1, m, S + 1),$$

for an explicit diagonal D and scalar β . This is the discrete Darboux (Christoffel) transformation of Jacobi matrices [38, 44, 16]. The bidiagonal map $X = DU$ with its last row removed is an intertwiner:

$$X\tilde{T} = (\tilde{T}' + \beta)X, \quad X\Pi = \Pi'X, \quad XA = A'X,$$

where primes refer to $(K - 1, m, S + 1)$. The first identity is the reversal of the factors; the other two follow from the column and row chains of Proposition 5.9 at the border points ($Xc_t \in \text{span } W_{Y'}$ for $t \in Y$, and dually) and, for A , from telescoping $XN_S - N'_{S+1}X$ in y . By (K5), (K2) and (K1),

$$\kappa(A, \Pi c_0) \kappa(\tilde{T}', w') = \det[W_Y | \Pi c_0 | E] \cdot \kappa(A', w') \cdot \prod_{k < n-1} \prod_{j < k} l(-j) \cdot \mathcal{G},$$

where $w' = \Pi' c'_n$ is a projected Cauchy vector of the smaller instance, E completes $[W_Y | \Pi c_0]$ by Cauchy columns (so the first factor is a Cauchy

determinant), and $\mathcal{G} = \det[R^\top \Pi c_{-j}]_{j < n-1}$ with $R = [c_1, \dots, c_{n-1}]$. Two further inputs are needed.

- (a) *The Gram determinant $\mathcal{G}$ (Chu–Vandermonde).* A Schur complement gives $\mathcal{G} \cdot \Delta = \det(C_{\mathcal{A}}^\top C_{\mathcal{B}})$ for two Cauchy matrices whose column sets $\mathcal{A}, \mathcal{B}$ have $K-1$ elements. By Cauchy–Binet this is a sum over the one omitted node a , with $V(P_a)^2 = V_K^2/(a!(K-1-a)!)^2$, and the sum is evaluated by

$$\sum_{a=0}^N \binom{N}{a}^2 (x+a)_{2N} = \binom{2N}{N} ((x+N)_N)^2 \quad (N = K-1).$$

This is a terminating balanced ${}_3F_2$ identity of Pfaff–Saalschütz type [1, §2.2]. We prove it with three applications of the Chu–Vandermonde identity: the moment identity $\sum_a \binom{N}{a}^2 a^j = N^j \binom{2N-j}{N}$ (falling factorials), the relation $\binom{2N}{i} \binom{2N-i}{N} = \binom{2N}{N} \binom{N}{i}$, and Vandermonde once more for $(x+2N-1)_N$. With Proposition 5.8 this makes $\mathcal{G}$ explicit.

- (b) *A nondegeneracy (positivity).* The transfer (K5) needs $\det[W' | X\Pi R] \neq 0$. Its numerator, as a function of θ , is not a product, so this is proved by positivity rather than by a formula: for rational $\theta > 0$ (this is the only range used, and it is proved over $\mathbb{Q}$), if $W's + X\Pi R t = 0$, then $s = 0$ (apply $W_{X'}^\top$ and $X\Pi = \Pi'X$), so $y = \Pi R t$ lies in $V \cap \ker X$; the range of $X^\top$ is contained in the span of $[R | W_X]$ (the row chain at the border points) and both have dimension $K-1$, so they are equal and $R^\top y = 0$, i.e. $(R^\top \Pi R)t = 0$. Finally $\det(R^\top \Pi R) \cdot \Delta = \det([R | W_X]^\top [R | W_Y])$ by a Schur complement. For two families of m' distinct shifts x, y with $a + x_i > 0$ and $a + y_i > 0$ for all nodes a , Cauchy–Binet gives

$$\det(C_x^\top C_y) = V(x)V(y) \sum_P V(P)^2 \prod_{a \in P} \prod_i \frac{1}{(a+x_i)(a+y_i)},$$

the sum over m' -subsets P of the nodes, with V the Vandermonde product. The sum is positive and $V(x)V(y) \neq 0$, so the determinant is nonzero. This applies to Δ and to $[R | W_X]^\top [R | W_Y]$, hence $\det(R^\top \Pi R) \neq 0$ and $t = 0$.

Every factor in the displayed identity is now good except the products over $l(-j) = -\tau(-\theta + j)$ and the Pochhammer denominator of $\mathcal{G}$, which contain factors $\theta - j - 1$ with negative shift; these two sets of factors coincide and cancel. The values $\kappa(\tilde{T}', w')$ and $\kappa(\tilde{T}, \cdot)$ are chain products by (K3) and nonzero. Combining with the diagonal step at $(K, m, n-1)$, (K1) and (K2), one obtains

$$u(m, n, n; \theta - 1) = u(m, n-1, n; \theta) \cdot (\text{good factor}).$$

Since $u(m, n-1, n)$ is good by the induction hypothesis (as $n \geq n-1$), so is $u(m, n, n)$, and the diagonal steps give all $S \geq n$. All bases stay in $[1, B]$ and all constants have prime factors at most B . This proves Theorem 5.7. $\square$

5.7. The mirror constant and $L(2, \chi_{-3})$. *The mirror constant.* For λ' one takes primes $\ell \equiv 1 \pmod{4}$ and $K = (\ell + 3)/4$. The only ℓ -singular factor is again $4K - 3 = \ell$, and the reduction gives the same pair of bordered matrices (with the opposite sign of the shift). Everything else is unchanged.

$L(2, \chi_{-3})$: *three levels of nodes.* Let $\ell = 6t + 1$ be prime, $K = (\ell + 2)/3 = 2t + 1$, $T = (\ell + 1)/2 = 3t + 1$, and $m_0 = 3t$. Now there are two ℓ -singular indices among $1, \dots, \ell - 1$, namely K (where $3K - 2 = \ell$) and T (where $2T - 1 = \ell$); the factor k in $d(k)$ is a unit. Accordingly the nodes are *low* $[0, K)$, *middle* $[K, m_0]$ and *top* $[T, \ell - 1]$. The geometries are tied to ℓ : we take $r + 2m = m_0$, so all nodes of $\mathcal{G}_e$ are low or middle, the extra node of the second member is m_0 , and $K \leq r$ is the condition $2m + 1 \leq t$. With the scaling 1 on low and ℓ on middle and top nodes, a window identity for sliding sums of $1/u$ modulo ℓ ,

$$\sum_{a < k \leq b} \frac{1}{k - t_1 - 1} - \sum_{a < k \leq b} \frac{1}{k - t_0} = (b - a) \sum_{t=t_0}^{t_1} \frac{1}{(a - t)(b - t)} \quad (a, b \notin [t_0, t_1]),$$

gives the entries modulo ℓ : low–low $\bar{x} \delta_{pq} + A_{pq}$; low–middle $-\frac{1}{6} \cdot \frac{1}{p-q}$; middle–middle $-\frac{1}{2} \delta_{pq}$; low–top and middle–top 0. Here, in terms of $u_y = (1/(a + y))_{a < K}$,

$$A = \kappa \left(hI + \sum_{n \in [K, T)} u_{\ell-n} u_{\ell-n}^\top \right) + \tau_0 \sum_{n \in [T, \ell)} u_{\ell-n} u_{\ell-n}^\top, \quad \kappa = -\frac{1}{18}, \quad \tau_0 = \frac{1}{36},$$

and $h = \frac{1}{2} (\sum_{y=1}^{3t} y^{-2} - 2 \sum_{y=3t+1}^{4t} y^{-2})$ in $\mathbb{F}_\ell$ (note $\ell - K = 4t$). The middle and top nodes enter with weights of opposite sign. The identity for A is the reflection $n = \ell - y$ of the two windows above, and the cross coefficient $\frac{1}{6}$, the middle diagonal $\frac{1}{2}$ and κ satisfy $(\frac{1}{6})^2 = \frac{1}{2} \cdot \frac{1}{18}$, which makes the middle level behave as in the two-level case; the top level does not, and only enters through A . The Schur complement of the middle block then gives, as in Proposition 5.3, bordered matrices $\mathcal{B}(z, N_{\text{mid}} + \frac{\tau_0}{\kappa} N_{\text{top}})$, where N_{top} is the Gram matrix of the top columns, which are the w_j with $j \in [1, T - 1]$, and N_{mid} that of the absent middle columns: w_T for the first member and none for the second. Hence the two members give

$$\mathcal{B}(z, cN) \quad \text{and} \quad \mathcal{B}(z, cN + ww^\top),$$

$$N = \sum_{j=1}^{S-1} w_j w_j^\top, \quad w = w_S, \quad S = T, \quad c = \frac{\tau_0}{\kappa} = -\frac{1}{2}.$$

Dividing the upper-left block by c (and rescaling the border) turns this into the pair $\mathcal{B}(z/c, N)$, $\mathcal{B}(z/c, N + c^{-1}ww^\top)$, which is the situation of Lemma 5.4 with $\tau = c^{-1} = -2 \neq 0$. The hypotheses (5.1) hold with $S = T$: $r + 2m + S = \ell$, and $S = 3t + 1 \geq K - m$. So Theorem 5.7 and Proposition 5.8 apply verbatim, all bases are at most $S + K + 2m - 1 = \ell - r + K - 1 < \ell$, and we obtain the analogue of Theorem 5.1: if $\ell \nmid \text{den } x_0$, one of the two

members does not vanish at x_0 . No information on the block $M[X', Y']$ is needed.

Existence of pairs. For $\ell = 6t + 1$ we need integers $r, m \geq 1$ with $r + 2m = 3t$, $2m + 1 \leq t$ and an admissible relation between r and m . With $r \leq 8m \leq r + W$ this means $3t \leq 10m \leq 3t + W$, which is solvable for all t if and only if $W \geq 9$; for $W = 8$ it fails exactly when $\ell \equiv 43 \pmod{60}$. We therefore use $r \leq 8m \leq r + 9$ for $L(2, \chi_{-3})$; then every prime $\ell \equiv 1 \pmod{6}$ with $\ell \geq 61$ has a pair, with $5r \leq 2\ell \leq 5r + 11$.

6. ANALYTIC DECAY

Theorem 6.1 (decay and window). *For admissible geometries of scale r and real t with $|t| \leq e^{-cr}$,*

$$|P_{\mathcal{G}}(\xi + t)| \leq \exp(-(D - o(1))r^2),$$

$$(D, c) = \begin{cases} (3.9028, 7.6444) & \text{for } \lambda, \lambda', \\ (3.7251, 6.6510) & \text{for } L(2, \chi_{-3}). \end{cases}$$

We describe the proof for λ ; the other two cases differ only in the reflection formula and the certificate data.

6.1. A contour representation. Hermite's formula gives $\mathsf{L}_F[a, b] = \frac{1}{2\pi i} \oint F(z) v_a(z) v_b(z) dz$ with $v_a(z) = 1/(z - a)$, the contour enclosing the nodes and no pole of F . The reflection formulas $\psi(1 + z) = \psi(-z) - \pi \cot \pi z$ and $\psi(\frac{3}{4} + z) = \psi(\frac{1}{4} - z) + \pi \cot(\frac{\pi}{4} - \pi z)$ split $F = \tilde{F} + E$, where $\tilde{F}$ is analytic in the left half-plane and $O(1/|z|)$ there, and

$$E(z) = -\pi [\cot \pi z + \cot(\frac{\pi}{4} - \pi z)]$$

is 1-periodic with $|E(x + iy)| \leq C_E e^{-2\pi|y|}$ for $|y| \geq 1$. Let $N_0 = \lfloor \frac{7}{50}r \rfloor$, $x^* = -N_0 - \frac{3}{8}$, and let Γ be the two rays $x^* + u(-1 \pm 3i)$, $u \geq 0$. Moving the contour to Γ picks up the residues of F at the $2N_0$ poles $-(j + \frac{3}{4})$ (residue +1) and $-(j + 1)$ (residue -1), $j < N_0$; the part of the integrand with $\tilde{F}$ integrates to 0 over Γ . Hence, for all integers $a, b \geq 0$,

$$(6.1) \quad \mathsf{L}_F[a, b] = \int v_a v_b dm, \quad m = -\sum_{\mu} \epsilon_{\mu} \delta_{-\mu} + \frac{1}{2\pi i} E(z) dz|_{\Gamma},$$

a Gram representation with respect to a complex measure m with finitely many atoms and an exponentially small density on Γ .

6.2. Two-family Andréief bound. For a set $S \subseteq C$ of common nodes let $R_S = S \sqcup X'$, $Q_S = S \sqcup Y'$, $n = |S| + m$, and $\mathcal{M}_S = \omega \mathsf{L}_F[R_S, Q_S]$. By multilinearity in the diagonal,

$$(6.2) \quad P_{\mathcal{G}}(\xi + t) = \sum_{S \subseteq C} t^{|C| - |S|} \det \mathcal{M}_S.$$

Andréief's identity [13] for the two families $(v_a)_{a \in R_S}$ and $(v_b)_{b \in Q_S}$ gives

$$\det \mathbf{L}_F[R_S, Q_S] = \frac{1}{n!} \int \det[v_a(z_i)] \det[v_b(z_i)] \prod_{i=1}^n dm(z_i),$$

and Cauchy's determinant evaluates both factors. Taking absolute values (see [37] for the logarithmic potential theory behind what follows),

$$|\det \mathbf{L}_F[R_S, Q_S]| \leq \frac{1}{n!} \int \prod_{i < i'} |z_i - z_{i'}|^2 \prod_{\mathcal{F} \in \{R_S, Q_S\}} \frac{\prod_{a < a' \in \mathcal{F}} |a' - a|}{\prod_{i, a \in \mathcal{F}} |z_i - a|} \prod_i d|m|(z_i).$$

The integrand is the exponential of minus a discrete logarithmic energy: the free points z_i repel each other with weight 1, each node family repels itself with weight $\frac{1}{2}$ and attracts the free points with weight 1, and the density $|E| \approx e^{-2\pi|\operatorname{Im} z|}$ acts as an external field. Because the rows and columns differ only in the interleaved blocks, the two node families have the same macroscopic density: after scaling by r , both are

$$\nu_\alpha = \operatorname{Leb}|_{[0,1]} + \frac{1}{2}\operatorname{Leb}|_{[1,1+2\alpha]}, \quad \alpha = \frac{1}{8},$$

of mass $1 + \alpha$. Its logarithmic energy has the closed form

$$\begin{aligned} I(\nu_{1/8}) &= \iint -\log|u - u'| d\nu_{1/8}(u) d\nu_{1/8}(u') \\ &= \frac{243}{128} - \frac{\log 2}{32} - \frac{25}{32} \log \frac{5}{4} = 1.70244\dots \end{aligned}$$

6.3. The energy certificate. Fix $\rho = 1/1024$ and the regularised kernel $k_\rho(z, w) = -\frac{1}{2} \log(|z - w|^2 + \rho^2) \leq -\log|z - w|$.

Lemma 6.2 (conditional positive definiteness). *For points $p_1, \dots, p_N \in \mathbb{C}$ and real weights q_α with $\sum q_\alpha = 0$, $\sum_{\alpha, \beta} q_\alpha q_\beta k_\rho(p_\alpha, p_\beta) \geq 0$.*

Proof. Frullani's integral $\log a = \int_0^\infty (e^{-t} - e^{-at}) dt/t$ and $\sum q_\alpha = 0$ give $\sum q_\alpha q_\beta (-\log(|p_\alpha - p_\beta|^2 + \rho^2)) = \int_0^\infty \frac{e^{-t\rho^2}}{t} \sum q_\alpha q_\beta e^{-t|p_\alpha - p_\beta|^2} dt$, and Gaussian kernels are positive definite: expanding $e^{2t\langle p, p' \rangle}$ in powers and the binomial theorem write the inner sum as a convergent sum of squares. $\square$

Apply Lemma 6.2 to the free points and removed nodes (weight $1/r$ each) against a *certificate measure* $\sigma = \sum_k m_k \delta_{c_k}$ of total mass $1 + \alpha$. This bounds the free-free energy from below by a sum of one-point terms $U_\rho^\sigma(s) = \sum_k m_k k_\rho(s, c_k)$ and the constant $I_\rho(\sigma) = \sum_{k, l} m_k m_l k_\rho(c_k, c_l)$. Riemann-sum comparisons replace the node sums by the potential U^{ν_α} and the node self-energy by $I(\nu_\alpha)$, up to $O(r \log r)$. With the external field one arrives at the function

$$g(s) = (1 + \alpha)U_\rho^\sigma(s) - U^{\nu_\alpha}(s) + (1 - \kappa_0)\pi|\operatorname{Im} s|, \quad \kappa_0 = 10^{-3},$$

on the scaled support of $|m|$ (the rays Γ/r , the segment $[-\frac{7}{50}, 0]$ containing the atoms, and, for removed nodes, the segment $[0, 1 + 2\alpha]$). The certificate consists of three finite families of inequalities, checked in interval arithmetic:

- (a) $g \geq m'$ on a cover of the rays by boxes, with an analytic tail bound beyond $u = \frac{3}{2}$;
- (b) $g \geq \gamma_\ell$ on a partition of $[-\frac{7}{50}, 0]$ into cells; put $S_{\text{cert}} = 2 \sum_\ell |\text{cell}_\ell| \min(\gamma_\ell - m', 0)$;
- (c) $g \geq g_{\min}$ on a cover of $[0, 1 + 2\alpha]$.

On a box, U_ρ^σ is bounded below by moving each c_k to the farthest corner, and U^{ν_α} is bounded above from its closed form and monotonicity. Then

$$D = 2((1 + \alpha)m' + S_{\text{cert}}) - (1 + \alpha)^2 I_\rho(\sigma) + I(\nu_\alpha), \quad c = 2(m' - g_{\min}).$$

Proposition 6.3 (certified constants). *For λ and λ' there is a measure σ with 480 atoms with dyadic coordinates and masses for which*

$$m' \geq 2.2147, \quad S_{\text{cert}} \geq -0.12218, \quad (1 + \alpha)^2 I_\rho(\sigma) \leq 2.53835, \\ g_{\min} \geq -1.6075, \quad I(\nu_{1/8}) \geq 1.7024457,$$

whence $D \geq 3.9028$ and $c \leq 7.6444$. For $L(2, \chi_{-3})$ the certificate of §6.4 gives $D \geq 3.7251$ and $c \leq 6.6510$.

All the inequalities are verified by the Lean kernel (§8); the data were generated with interval arithmetic in Arb [18] and then checked independently in Lean with rational arithmetic. For λ the measure is assembled from a square-frame certificate and an extra family of atoms that represents the additional density $\frac{1}{2}\text{Leb}|_{[1, 1+2\alpha]}$; the function g splits as the square-frame function plus a correction term, which is certified on a coarse cover.

6.4. The certificate for $L(2, \chi_{-3})$. For F_χ the reflection formula $\psi(z + a) = \psi(1 - a - z) - \pi \cot \pi(z + a)$ gives $F_\chi = \bar{F}_\chi + E_\chi$ with

$$E_\chi(z) = \pi \left[\frac{1}{3} \cot \pi(z + \frac{1}{3}) - \frac{1}{2} \cot \pi(z + \frac{1}{2}) + \frac{1}{6} \cot \pi(z + 1) \right],$$

which is 1-periodic and, because the weights $\frac{1}{3} - \frac{1}{2} + \frac{1}{6}$ sum to zero, decays like $e^{-2\pi|\text{Im } z|}$. The contour argument and the Andréief bound are as above, with poles at $-(j + \frac{1}{3})$, $-(j + \frac{1}{2})$, $-(j + 1)$. The square-frame certificate for F_χ (a measure σ_χ of 699 atoms, $\rho = 1/2048$) gives lower bounds for the square-frame function $g_\chi = U_\rho^{\sigma_\chi} - U^\tau + (1 - \kappa_0)\pi|\text{Im } s|$ on rays, segment and line. The hybrid certificate is

$$\sigma_h = \frac{\sigma_\chi + \lambda}{9/8}, \quad \lambda = \sum_{k=0}^{31} \frac{1}{256} \delta_{t_k}, \quad t_k = 1 + \frac{k}{128} - \frac{1}{2048},$$

so that λ has mass $\frac{1}{8}$ and represents the extra density $\frac{1}{2}\text{Leb}|_{[1, 5/4]}$; the atom t_k sits at the left end $a_k = 1 + k/128$ of the k -th cell, shifted left by ρ . The function of the hybrid frame splits as $g_h = g_\chi + h$ with $h = U_\rho^\lambda - U^{\text{ex}}$, where U^{ex} is the potential of $\frac{1}{2}\text{Leb}|_{[1, 5/4]}$. For u in the k -th cell and $\text{Re } s + \rho \leq a_k$

one has $(\operatorname{Re} s - t_k)^2 + \rho^2 \leq (\operatorname{Re} s - u)^2$, hence $k_\rho(s, t_k) \geq -\log |s - u|$ on the cell, and integrating over the cells gives $h(s) \geq 0$ whenever $\operatorname{Re} s \leq 1 - \rho$. On the short real interval $[1 - \rho, 1]$ only the first cell is affected, and an elementary bound for $\int_0^c -\log v \, dv$ gives $h \geq 0$ there too. Therefore every lower bound of the square-frame certificate holds for g_h , and only the energy $I_\rho(\sigma_h)$ is new: it consists of the energy of σ_χ , the cross terms (699×32 logarithms) and the self-energy of λ (32^2 logarithms), summed exactly in rational interval arithmetic. The result is $D \geq 3.7251$ (the computed value is 3.725194) and $c \leq 6.6510$; with 64 extra atoms instead of 32 one would gain only 0.0008 in D .

Since the admissible relation for $L(2, \chi_{-3})$ is $r \leq 8m \leq r + 9$, the decay estimate is proved uniformly for $\alpha r \leq m \leq \alpha r + \frac{9}{8}$, $\alpha = \frac{1}{8}$: the comparison of node sums with U^{ν_α} and $I(\nu_\alpha)$ is carried out with a window parameter for the excess of m over αr , which only changes the $O(r \log r)$ error terms.

6.5. Node removal and the window. The argument applies to every subset $S \subseteq C$: removing $j = \beta r$ common nodes turns them into points of the configuration that are not integrated over, and only common nodes can be removed, because a node removed from one family only would leave an uncanceled term of weight $\frac{1}{2}$. One obtains

$$|\det \mathcal{M}_S| \leq \exp(-r^2(D - \beta c - o(1))) \quad (\beta = |C \setminus S|/r),$$

uniformly in S . Inserting this into (6.2) with $|t|e^{cr} \leq 1$ gives Theorem 6.1: the sum has at most $2^{|C|} = e^{O(r)}$ terms, each at most $e^{-(D-o(1))r^2}$. The second member is handled by viewing it as the pattern $(r, m+1)$ with one column node moved by one place, at an energy cost $O(r)$.

7. PROOF OF THEOREM 1.1

7.1. $G \pm 5\pi^2/48$. Let $\xi = \lambda$ (the case λ' is identical, with primes $\ell \equiv 1 \pmod{4}$). Let $\eta = D - \Lambda \geq 3.9028 - 3.70343 = 0.19937$, let $\nu > c/\eta$, and suppose $|\xi - p/q| < q^{-\nu}$ with q large; put $L = \log q$. Choose $\delta > 0$ with $\nu > c(1 + \delta)/\eta$, let $r = \lceil (1 + \delta)L/\eta \rceil$ and choose m with $r \leq 8m \leq r + 8$.

Choice of ℓ . By the prime number theorem in progressions, the primes $\ell \equiv 3 \pmod{4}$ in $[\frac{4}{3}(r + m) + 1, 4r - 1]$ have product $\exp((\frac{5}{4} + o(1))r)$ (as $m \leq \frac{r}{8} + 1$), which exceeds $q = e^L$ since $r \gg L$. So one of them does not divide q . By Theorem 5.1 with $x_0 = p/q$, there is $e \in \{0, 1\}$ with $P_{\mathcal{G}_e}(p/q) \neq 0$, where $\mathcal{G}_e = \mathcal{G}(r, m, e)$.

Lower bound. By Lemma 3.3, $|d_{\mathcal{G}_e} P_{\mathcal{G}_e}(p/q)| \geq q^{-(r+1)} = e^{-(r+1)L}$.

Upper bound. Write $p/q = \xi + t$ with $|t| < e^{-\nu L} \leq e^{-cr}$ (as $cr \leq c(1 + \delta)L/\eta + c < \nu L$ for large q). By Theorems 4.1 and 6.1, $|d_{\mathcal{G}_e} P_{\mathcal{G}_e}(p/q)| \leq e^{(\Lambda + o(1))r^2} e^{-(D - o(1))r^2} = e^{-(\eta - o(1))r^2}$.

Contradiction. Comparing, $(\eta - o(1))r^2 \leq (r + 1)L$; but $\eta r \geq (1 + \delta)L$. Hence $\mu(\lambda) \leq c/\eta$, and

$$\frac{c}{\eta} \leq \frac{7.6444}{3.9028 - 3.7034204} = 38.3419 \dots < 38.35.$$

Since $G - 5\pi^2/48 = 2\lambda$ and $G + 5\pi^2/48 = 2\lambda'$, this gives the first two bounds of Theorem 1.1.

7.2. $L(2, \chi_{-3})$. Now the geometry is tied to the prime: for each prime $\ell \equiv 1 \pmod{6}$, $\ell \geq 61$, there is an admissible pair of members of scale r with $5r \leq 2\ell \leq 5r + 11$, so $r/\ell \rightarrow g = \frac{2}{5}$, and Theorem 5.1 (in the form of §5.7) applies when $\ell \nmid q$. Let $\eta = 3.7251 - \frac{10339}{3072} = 0.359540\dots$, $\nu > c(1/\eta + \frac{4}{5})$, and choose δ small. The primes $\ell \equiv 1 \pmod{6}$ in an interval $(Y, Y + (2 + \delta)L]$ with $Y = \frac{5}{2}(1 + \delta)L/\eta$ have product $e^{(1+\delta/2+o(1))L} > q$, so one of them, ℓ , does not divide q . The corresponding scale satisfies

$$(1+\delta)L/\eta - O(1) \leq r \leq \frac{2}{5}(Y + (2+\delta)L) + O(1) = \left(\frac{1+\delta}{\eta} + \frac{2(2+\delta)}{5}\right)L + O(1),$$

so $cr < \nu L$ for δ small, and the comparison above, with $|C| \leq r + 1$, gives a contradiction. Hence

$$\mu(\tfrac{1}{4}L(2, \chi_{-3})) \leq c\left(\frac{1}{\eta} + gs\right), \quad g = \frac{2}{5}, \quad s = 2,$$

and $6.6510(1/0.3595401 + 0.8) = 23.8194\dots < 23.82$. This completes the proof of Theorem 1.1. $\square$

Remark 7.1 (relation to linear independence criteria). The deduction is a one-dimensional version of the familiar scheme behind the criteria of Nesterenko [30] and their use by Ball and Rivoal [2]: the nonzero integer $q^{|C|}d_{\mathcal{G}}P_{\mathcal{G}}(p/q)$ plays the role of a small linear form with integer coefficients. The difference is that the lower bound for the form comes from an arithmetic nonvanishing statement (Theorem 5.1) rather than from an analytic asymptotic, and that the size r of the determinant is chosen in terms of q .

Remark 7.2 (where the constants enter). For λ , $\partial\mu/\partial D = -c/\eta^2 \approx -192$, so the third decimal of D matters. The rank deficits of §4 lower the exponent from 50.0 to 38.34, and the free choice of ℓ (Theorem 5.1 for all S in a range) removes a term cgs which would otherwise be about 5.1. For $L(2, \chi_{-3})$ the term $cgs = 5.32$ is still present; a version of Theorem 5.1 that also covers primes dividing q to the first power would halve it.

8. FORMALIZATION

All results of this paper are formalized in Lean 4 [9] with Mathlib [40]. The development is available at

<https://github.com/slitsey/catalan-irrationality-measure>

and combines analytic and arithmetic proofs with generated rational data and kernel checks for the dual certificates of Section 4 and the decay certificates of Section 6; numerical computations alone are not used as proof. The constants are written out as series, so the main statements use no definitions of the development: G is $\sum_j (-1)^j/(2j+1)^2$ and $L(2, \chi_{-3})$ is $\sum_k ((3k+1)^{-2} - (3k+2)^{-2})$. The main statements are stated in `ChallengeHybrid.lean` and proved in `SolutionHybrid.lean` (namespace `CatalanHybridChallenge`):

- `lam_hybrid_measure`: for every $e > 0$, all sufficiently large $q \in \mathbb{N}$ and all $p \in \mathbb{Z}$ coprime to q , $q^{-(38.35+e)} < |G - 5\pi^2/48 - p/q|$ (Theorem 1.1); it is `lam_hyb_measure_final`;
- `mir_hybrid_measure`: the same for $G + 5\pi^2/48$; it is `mir_hyb_measure_final`;
- `chi3_hybrid_measure`: the same for $L(2, \chi_{-3})$ with exponent 23.82; it is `chi3_hybrid_measure_wide`;
- `nv_hyb_nonvanishing_gen`: Theorem 5.1 and its analogues in §5.7;
- `hyb_primal_false` and `primal_pair`: the end of the nonvanishing chain and Lemma 5.4;
- `lam_hybrid_decay`, `p6_hybrid_decay`: Theorem 6.1 for λ and $L(2, \chi_{-3})$;
- `lam_both_eps4`, `p6_content`: Theorem 4.1 for λ and $L(2, \chi_{-3})$.

The solution has no `sorry` and no `native_decide`, and the command `#print axioms` reports only the three standard axioms `propext`, `Classical.choice` and `Quot.sound`. The file `comparator-hybrid.json` is a configuration for `lakecomparator` stating that `SolutionHybrid` proves the statements of `ChallengeHybrid` with these axioms only. We have not run `Comparator`, which requires Linux; the axiom check above was done with `#print axioms`. The prime number theorem in arithmetic progressions (moduli 4 and 6) enters the content asymptotics and the selection of the prime ℓ . The formal source for it is the PNT+ project [41], through OpenAI’s workspace and compatibility additions, whose Wiener–Ikehara theorem we use.

The following file groups implement the indicated parts of the proof.

CatalanCore/Rect

Geometries, the matrices Φ_G , Lemma 3.3, pair families and admissible relations.

CatalanCore/Arith/Hybrid

Gauss valuations, Lemma 4.2, the level majorant, Proposition 4.3, prime sums over linear pieces, the rank-deficit interface (Lemma 4.5) and bordering.

`CatalanFrames/Lambda/Hyb*`, `CatalanFrames/LambdaMirrorHybrid`,
`CatalanFrames/P6Hybrid`

Theorem 4.1 and Theorem 4.6 for the three constants.

CatalanFrames/LambdaHybrid/NV

Proposition 5.3 and its three-level analogue, Lemma 5.4 (Primal), Theorem 5.1 (FinalNV).

CatalanCore/Star/Leonard

Proposition 5.8, Proposition 5.9, (K1)–(K5), the diagonal and Darboux steps, the positivity argument, Lemma 5.5, Theorem 5.7 and the specialisation $\theta \rightarrow 0$.

`CatalanCore/Decay/Rect`, `CatalanCore/Cert/SigmaHyb`, `CatalanCore/Cert/SigmaP6H`

The contour identity, the two-family Andréief bound, Lemma 6.2, node removal, and the certificates of Proposition 6.3 and §6.4.

`CatalanCore/Measure, CatalanFrames/*/Assembly*`

The deduction of Section 7.

The proofs were checked with Lean v4.35.0-rc2 and the matching Mathlib release. The vendored files from OpenAI’s repository and from the PNT+ project were converted to Lean’s module system and ported to the current toolchain; the repository’s README and provenance documents describe the changes.

DISCLOSURE OF AI USE

The development was produced by the AI systems Claude Opus 5.5 and Claude Sonnet 5.5 (Anthropic): the mathematics, including the rectangular frames, the content bounds and rank deficits, the nonvanishing argument and the decay certificates, and all proofs; the Lean formalization; the numerical computations and the certificate generators; the literature search; and the text of this paper. Claude worked in a series of conversations at the request of the author, who posed the problem, directed the work, and is responsible for this submission. The repository infrastructure, vendored formalizations and certificate tooling of the companion project [25], which this work reuses, were developed with OpenAI Codex as described there. The underlying determinant method and the irrationality proof for G are OpenAI’s [32]. We use Lean 4 [9], Mathlib [40], the PNT+ project [41], and Arb [18] (through `python-flint`) for interval arithmetic in generating certificate data, which Lean re-checks. The formal verification in Lean is intended as an independent check of every step of the argument above.

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