

APPROXIMATIONS IN l_p AND MORREY SEQUENCE SPACES

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ABSTRACT. In this paper we introduce positive order differences for sequences and study approximations in l_p and Morrey sequence spaces.

1. INTRODUCTION

Decomposition methods play an important role in the study of various function spaces. These methods allow us to obtain descriptions of such spaces that lead to the corresponding discrete spaces (see e.g. [9], [11], [13]). On the other hand, the study of analogues of classical operators in discrete harmonic analysis (see e.g. [4], [8]), is also an incentive to study discrete spaces. So the consideration of discrete spaces is quite natural and is of independent interest. In this paper we consider approximations in l_p and in more general Morrey sequence spaces.

2. APPROXIMATIONS IN l_p SPACES

By l_p , $0 < p \leq \infty$, we denote the vector space of complex-valued sequences $\mathbf{a} = (a_1, a_2, \dots) = (a_k)$ with finite (quasi)norm

$$\|\mathbf{a}\|_p = \begin{cases} (\sum_{n=1}^{\infty} |a_n|^p)^{1/p}, & \text{if } 0 < p < \infty; \\ \sup_{n \geq 1} |a_n|, & \text{if } p = \infty. \end{cases}$$

The subspace of l_{∞} of all sequences which tend to zero with the same norm is denoted by c_0 .

Let $\mathbf{e}_n = (\delta_{kn})$ where δ_{kn} is the Kronecker delta (these vectors form a basis both in l_p , $0 < p < \infty$, and c_0) and $X_m = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_m)$, $m = 1, 2, \dots$. For $\mathbf{a} \in l_p$ or $\mathbf{a} \in c_0$ the values

$$E_m(\mathbf{a})_p = \inf_{\mathbf{b} \in X_m} \|\mathbf{a} - \mathbf{b}\|_p$$

are called the best approximations of the vector $\mathbf{a}$ by elements from the linear spaces X_m .

We will denote by $\#A$ (or $|A|$) the number of elements of a finite set A .

Next, we need a notion similar to the notion of the difference of a function with a given step h . The duality between the function space $L_2(-\pi, \pi)$ and the sequence space l_2 under the Fourier transform, and the relation between the modulus of smoothness of a given function and the norm of the trigonometric series with coefficients $\min(1, (kh)^r)c_k$ (where (c_k) are the Fourier coefficients of the original function) as obtained in [2], lead us to the following natural definition.

Definition 2.1. The difference $\Delta_h^r \mathbf{d}$ of the positive order $r > 0$ with the step $h \geq 0$ for a sequence $\mathbf{d}$ is called $\Delta_h^r \mathbf{d} = (\min(1, (kh)^r)d_k)$.

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Recall the definitions of the derivative of a sequence and Sobolev sequence spaces. The r -th derivative ($r > 0$) of the sequence $\mathbf{d} = (d_k)$ is defined as $\mathbf{d}^{(r)} = (k^r d_k)$. The Sobolev sequence space of the order r is denoted by $W^r(l_p)$ and consists of all sequences $\mathbf{a}$ such that $\mathbf{a}^{(r)} \in l_p$. The expression $\|\mathbf{a}\|_{W^r(l_p)} = \|(k^r |a_k|)\|_p$ is a (quasi)norm on the $W^r(l_p)$.

Proposition 2.1. *For any sequence $\mathbf{a} \in l_p$ its differences have the following properties*

- 1) $\|\Delta_h^r \mathbf{a}\|_p \leq \|\mathbf{a}\|_p$, if $0 < h < 1$, and $\|\Delta_h^r \mathbf{a}\|_p = \|\mathbf{a}\|_p$, if $h \geq 1$;
- 2) $\lim_{h \rightarrow 0} \|\Delta_h^r \mathbf{a}\|_p = 0$ if $p < \infty$; $\lim_{h \rightarrow 0} \|\Delta_h^r \mathbf{a}\|_\infty = 0$ if $\mathbf{a} \in c_0$;
- 3) if $0 < h_1 < h_2$ and $0 < p \leq \infty$, then $\|\Delta_{h_1}^r \mathbf{a}\|_p \leq \|\Delta_{h_2}^r \mathbf{a}\|_p$;
- 4) if $\lambda > 1$ and $0 < p \leq \infty$, then $\|\Delta_{\lambda h}^r \mathbf{a}\|_p \leq ([\lambda] + 1)^r \|\Delta_h^r \mathbf{a}\|_p$, where $[\lambda]$ denotes the integer part of the number λ ;
- 5) if $\mathbf{a} \in W^s(l_p)$, $0 < p \leq \infty$ and $r > s$, then $\|\Delta_h^r \mathbf{a}\|_p \leq h^s \|\Delta_h^{r-s} \mathbf{a}^{(s)}\|_p$ and $\|\Delta_h^r \mathbf{a}\|_p \leq h^r \|\mathbf{a}^{(r)}\|_p$;
- 6) if $r > s$ and $0 < p \leq \infty$, then $\|\Delta_h^r \mathbf{a}\|_p \leq \|\Delta_h^s \mathbf{a}\|_p$.

Proof. 1) Immediately follows from the definition.

2) Given $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that $\sum_{k=m_\varepsilon+1}^\infty |a_k|^p < \varepsilon^p$ and $1/m_\varepsilon^r < \varepsilon$. Let $h < 1/m_\varepsilon^2$, then for $k \in \{1, \dots, m_\varepsilon\}$ we have $\min(1, (kh)^r) < 1/m_\varepsilon^r$. Hence

$$\begin{aligned} \|\Delta_h^r \mathbf{a}\|_p^p &= \sum_{k=1}^{m_\varepsilon} (\min(1, (kh)^r) |a_k|)^p + \sum_{k=m_\varepsilon+1}^\infty (\min(1, (kh)^r) |a_k|)^p < \\ &\quad \frac{1}{m_\varepsilon^{rp}} \sum_{k=1}^{m_\varepsilon} |a_k|^p + \sum_{k=m_\varepsilon+1}^\infty |a_k|^p < \varepsilon^p \|\mathbf{a}\|_p^p + \varepsilon^p, \end{aligned}$$

which proves 2) for $0 < p < \infty$.

For sequences from c_0 , the proof requires an obvious modification.

3) For $0 < h_1 < h_2$ we have $\min(1, (kh_1)^r) \leq \min(1, (kh_2)^r)$ from which follows the validity of the estimate.

4) Let us first consider the case when $\lambda = l \in \mathbb{N}$ and $l \geq 2$. Suppose that $lh \leq 1$, then

$$\begin{aligned} \|\Delta_{lh}^r \mathbf{a}\|_p^p &= \sum_{k=1}^{[1/(lh)]} (\min(1, (klh)^r) |a_k|)^p + \sum_{k=[1/lh]+1}^{[1/h]} (\min(1, (klh)^r) |a_k|)^p + \\ &\quad \sum_{k=[1/h]+1}^\infty (\min(1, (klh)^r) |a_k|)^p = S_1 + S_2 + S_3. \end{aligned}$$

For $k \in \{1, \dots, [1/(lh)]\}$ we have $\min(1, (klh)^r) = l^r \min(1, (kh)^r)$, and for $k \in \{[1/(lh)] + 1, \dots, [1/h]\}$ we have $1/l < kh \leq 1$. Then

$$\begin{aligned} S_1 &= l^{rp} \sum_{k=1}^{[1/(lh)]} (\min(1, (kh)^r) |a_k|)^p, \\ S_2 &= \sum_{k=[1/(lh)]+1}^{[1/h]} |a_k|^p \leq l^{rp} \sum_{k=[1/lh]+1}^{[1/h]} ((kh)^r |a_k|)^p = l^{rp} \sum_{k=[1/(lh)]+1}^{[1/h]} (\min(1, (kh)^r) |a_k|)^p, \\ S_3 &\leq l^{rp} \sum_{k=[1/h]+1}^\infty |a_k|^p. \end{aligned}$$

The obtained inequalities lead to the inequality $\|\Delta_{lh}^r \mathbf{a}\|_p^p \leq l^{rp} \|\Delta_h^r \mathbf{a}\|_p^p$.

Suppose that $lh \geq 1$. If $h \geq 1$, then the estimate is obvious. For $0 < h < 1$ we have

$$\begin{aligned} \|\Delta_{lh}^r \mathbf{a}\|_p^p &= \|\mathbf{a}\|_p^p = \sum_{k=1}^{[1/h]} |a_k|^p + \sum_{k=[1/h]+1}^{\infty} |a_k|^p \leq \\ &\sum_{k=1}^{[1/h]} (klh)^{rp} |a_k|^p + \sum_{k=[1/h]+1}^{\infty} l^{rp} |a_k|^p = l^{rp} \|\Delta_h^r \mathbf{a}\|_p^p. \end{aligned}$$

Now let $\lambda > 1$ be arbitrary. Since $1 \leq [\lambda] \leq \lambda < [\lambda] + 1$, then using property 3) and the inequality obtained above we have

$$\|\Delta_{\lambda h}^r \mathbf{a}\|_p \leq \|\Delta_{([\lambda]+1)h}^r \mathbf{a}\|_p \leq ([\lambda h] + 1)^r \|\Delta_h^r \mathbf{a}\|_p.$$

5) It is obvious that the estimates are valid for $h \geq 1$. Let $0 < h < 1$ and $k \in \{1, \dots, [1/h]\}$. Then $\min(1, (kh)^r) = (kh)^s \min(1, (kh)^{r-s})$. Therefore

$$\|\Delta_h^r \mathbf{a}\|_p^p \leq h^{sp} \sum_{k=1}^{[1/h]} (\min(1, (kh)^{r-s}) k^s |a_k|)^p + h^{sp} \sum_{k=[1/h]+1}^{\infty} (k^s |a_k|)^p = h^{sp} \|\Delta_h^{r-s} \mathbf{a}^{(s)}\|_p^p.$$

The second inequality follows directly from the definitions.

6) The inequality is a consequence of $\min(1, (kh)^r) \leq \min(1, (kh)^s)$. $\square$

The basic inequalities of approximation theory, such as Jackson's and Bernstein's inequalities (see e.g. [1], [9]), remain valid for approximations by the elements from the spaces X_m . In addition, there are discrete analogues of other important inequalities.

Proposition 2.2. *The following inequalities are valid*

1) for all $n \in \mathbb{N}$, $r > 0$ and $0 < p \leq \infty$

$$E_n(\mathbf{a})_p \leq \|\Delta_{1/n}^r \mathbf{a}\|_p \text{ (Jackson's inequality);}$$

2) for all $r > 0$, $0 < p \leq \infty$ and $\mathbf{a}$ such that $\text{supp } \mathbf{a} = \{k \in \mathbb{N} \mid a_k \neq 0\} \subset \{1, 2, \dots, N\}$

$$\|\mathbf{a}^{(r)}\|_p \leq N^r \|\mathbf{a}\|_p \text{ (Bernstein's inequality);}$$

3) for all $r > 0$, $0 < p \leq \infty$ and $\mathbf{a} \in l_p$ such that $\text{supp } \mathbf{a} \subset \{N, N+1, \dots\}$

$$\|\mathbf{a}\|_p \leq N^{-r} \|\mathbf{a}^{(r)}\|_p \text{ (Böhr–Favard's inequality);}$$

4) for all $r > 0$ and $\mathbf{a}$ such that $\text{supp } \mathbf{a} \subset \{N, N+1, \dots, 2N\}$

$$(2N)^{-r} \|\mathbf{a}^{(r)}\|_p \leq \|\Delta_{1/N}^r \mathbf{a}\|_p = \|\mathbf{a}\|_p \leq N^{-r} \|\mathbf{a}^{(r)}\|_p;$$

5) for all $\mathbf{a}$ such that $\text{supp } \mathbf{a} \subset \{1, 2, \dots, N\}$ and $0 < p < q \leq \infty$

$$\|\mathbf{a}\|_p \leq (\#\text{supp } \mathbf{a})^{1/p-1/q} \|\mathbf{a}\|_q \text{ (Nikolskii's inequality).}$$

Proof. Since

$$E_n(\mathbf{a})_p = \left(\sum_{k=n+1}^{\infty} |a_k|^p \right)^{1/p}, \quad 0 < p < \infty, \text{ and } E_n(\mathbf{a})_{\infty} = \sup_{k>n} |a_k|,$$

then for both $0 < p < \infty$ and $p = \infty$ we have $E_n(\mathbf{a})_p \leq \|\Delta_{1/n}^r \mathbf{a}\|_p$.

2) The evaluation is obvious.

- 3) Indeed, for positive r we have $\|(a_k)\|_p = \|(k^{-r} k^r a_k)\|_p \leq N^{-r} \|(a_k)\|_p$.
- 4) The required estimates follow from 3) and 2).
- 5) By Hölder's inequality with the exponent q/p

$$\left(\sum_{k \in \text{supp } \mathbf{a}} |a_k|^p \right)^{1/p} \leq (\#\text{supp } \mathbf{a})^{1/p-1/q} \|\mathbf{a}\|_q.$$

□

Remark 1. For $\mathbf{a} \in l_\infty$ the following statements are equivalent

- 1) $\mathbf{a} \in c_0$;
- 2) $\lim_{h \rightarrow 0} \|\Delta_h^r \mathbf{a}\|_\infty = 0$;
- 3) $E_n(\mathbf{a})_\infty \rightarrow 0$ ($n \rightarrow \infty$).

The assertion follows from the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1).

The introduced differences allow us to give a description of approximation sequence spaces considered in [10] in the framework of abstract approximation schemes.

Let $r > 0$, $0 < p < \infty$ and $0 < \theta \leq \infty$. Then the approximation space $A_\theta^r(l_p)$ consists of all elements $\mathbf{a} \in l_p$ such that $(k^{r-1/\theta} E_k(\mathbf{a}))_p \in l_\theta$. The (quasi)norm in $A_\theta^r(l_p)$ is defined as follows $\|\mathbf{a}\|_{A_\theta^r(l_p)} = \|(k^{r-1/\theta} E_k(\mathbf{a}))_p\|_\theta + \|\mathbf{a}\|_p$. These spaces can be considered as the discrete counterpart of the Nikolskii–Besov function spaces. But it is known that Nikolskii–Besov spaces primarily were defined through differences of higher orders [9]. Therefore, $B_\theta^r(l_p)$, $r > 0$, $0 < p, \theta \leq \infty$ denotes the Nikolskii–Besov sequence space of all l_p sequences for which the expression (which is a (quasi) norm on $B_\theta^r(l_p)$)

$$\|\mathbf{a}\|_{B_\theta^r(l_p)} = \left(\sum_{k=1}^{\infty} [k^r \|\Delta_{1/k}^s \mathbf{a}\|_p]^\theta \frac{1}{k} \right)^{1/\theta}$$

(when $\theta = \infty$ the sum is replaced by $\sup_{k \geq 1} (k^r \|\Delta_{1/k}^s \mathbf{a}\|_p)$) is finite for some $s > r$.

Theorem 2.1. *Spaces $A_\theta^r(l_p)$ and $B_\theta^r(l_p)$ coincide with the equivalence of (quasi)norms.*

Proof. The estimate $\|\mathbf{a}\|_{A_\theta^r(l_p)} \leq \|\mathbf{a}\|_{B_\theta^r(l_p)}$ follows from Jackson's inequality. The opposite inequality can be obtained by using the technique from [9] and we will not dwell on its details. □

The following statement is a counterpart of the relation obtained in [5]. Let $t > 0$ and

$$K(\mathbf{a}, t; l_p, W^r(l_p)) = \inf_{\mathbf{d} \in W^r(l_p)} (\|\mathbf{a} - \mathbf{d}\|_p + t^r \|\mathbf{d}\|_{W^r(l_p)})$$

be a K-functional of the element $\mathbf{a} \in l_p$. Then $C_1 \|\Delta_t^r \mathbf{a}\|_p \leq K(\mathbf{a}, t; l_p, W^r(l_p)) \leq C_2 \|\Delta_t^r \mathbf{a}\|_p$ with constants independent of $\mathbf{a}$ and t . A direct proof of these estimates is effortless.

The differences allow us to reformulate compactness criterion in the l_p spaces.

Definition 2.2. The sequences of a set $A \subset l_p$ are called equicontinuous if for some $r > 0$ for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $0 < h < \delta$, the relation $\|\Delta_h^r \mathbf{a}\|_p < \varepsilon$ holds for every sequence $\mathbf{a} \in A$.

Theorem 2.2. *Let $D \subset l_p$, $1 \leq p < \infty$, or $D \subset c_0$ be a closed set. Then D is compact if and only if D is bounded and the sequences $\mathbf{a}$ from the set D are equicontinuous.*

Proof. This assertion follows from the criterion of compactness in l_p [7] and the second statement of 2.1. □

3. APPROXIMATIONS IN MORREY SEQUENCE SPACES

The main properties of Morrey sequence spaces are given in [3], for Morrey function spaces see e.g. [12]. Here we consider approximations similar to those in 2, namely by sequences with supports in segments of integers symmetric about zero.

The symbols $\mathbb{N}$ and $\mathbb{Z}$ as usual denote the sets of natural and integer numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $m \in \mathbb{Z}$, $B(m, N) = \{m - N, m - N + 1, \dots, m + N\}$ be a ball centered at m and radius N , $|B(m, N)| = 2N + 1$. For any set $E \subset \mathbb{Z}$, by definition $\mathbf{a}\chi_E = (b_k)$, where $b_k = a_k$, if $k \in E$ and $b_k = 0$, if $k \notin E$.

The Morrey sequence space $m^{q,p}$, $0 < p \leq q < \infty$, is the collection of all complex valued sequences $\mathbf{a} = (a_k)$, $k \in \mathbb{Z}$, for which

$$\|\mathbf{a}\|_{m^{q,p}} = \sup_{m \in \mathbb{Z}, N \in \mathbb{N}_0} |B(m, N)|^{1/q-1/p} \left(\sum_{k \in B(m, N)} |a_k|^p \right)^{1/p} = \sup_{m \in \mathbb{Z}, N \in \mathbb{N}_0} (2N + 1)^{1/q-1/p} \|\mathbf{a}\chi_{B(m, N)}\|_p < \infty.$$

These spaces are Banach (quasi) normed spaces [3]. Note that for $p=q$ the spaces $m^{q,p}$ coincide with l_q .

For sequences $\mathbf{a} = (a_k)_{k \in \mathbb{Z}}$, the differences of order $r > 0$ with step $0 \leq h$ and the r -th derivative of order r are defined by $\Delta_h^r \mathbf{a} = (\min(1, |\tilde{k}h|^r) a_k)_{k \in \mathbb{Z}}$, $\mathbf{a}^{(r)} = (|\tilde{k}|^r a_k)_{k \in \mathbb{Z}}$, where $\tilde{k} = \tilde{k}(k) = 1$, if $k = 0$, and $\tilde{k} = k$, if $k \neq 0$.

In the following, the sums over empty sets are assumed to be zero.

Let $\mathbf{a} \in m^{q,p}$. Then for any $\mathbf{b}$ with $\text{supp } \mathbf{b} \subset B(0, L)$

$$\begin{aligned} \|(\mathbf{a} - \mathbf{b})\chi_{B(m, N)}\|_p &= \left(\sum_{k \in B(m, N), |k| \leq L} |a_k - b_k|^p + \sum_{k \in B(m, N), |k| > L} |a_k|^p \right)^{1/p} \geq \\ &\left(\sum_{k \in B(m, N), |k| > L} |a_k|^p \right)^{1/p} = \|\mathbf{a}\chi_{B(m, N) \cap (\mathbb{Z} \setminus B(0, L))}\|_p. \end{aligned}$$

Hence,

$$E_L(\mathbf{a})_{m^{q,p}} = \inf_{\mathbf{b}, \text{supp } \mathbf{b} \subset B(0, L)} \|\mathbf{a} - \mathbf{b}\|_{m^{q,p}} = \|\mathbf{a}\chi_{\mathbb{Z} \setminus B(0, L)}\|_{m^{q,p}}.$$

Below, for any $r > 0$

$$\begin{aligned} \left(\sum_{k \in B(m, N), |k| > L} |a_k|^p \right)^{1/p} &= \left(\sum_{k \in B(m, N), |k| > L} (\min(1, |k/L|^r) |a_k|)^p \right)^{1/p} \leq \\ &\|\Delta_{1/L}^r \mathbf{a}\chi_{B(m, N)}\|_p \end{aligned}$$

From this follows Jackson's inequality

$$E_L(\mathbf{a})_{m^{q,p}} \leq \|\Delta_{1/L}^r \mathbf{a}\|_{m^{q,p}}.$$

It is known [3] that $\mathbf{d} = (1/|k|^\alpha)_{k \in \mathbb{Z}} \in m^{q,p}$ if $q = 1/\alpha$, $0 < p < q < \infty$ (for certainty we assume that $d_0 = 1$). Let us show that $\|\mathbf{d} - \mathbf{d}\chi_{B(0, k)}\|_{m^{q,p}} \rightarrow 0$ ($k \rightarrow \infty$).

Indeed,

$$\|\mathbf{d} - \mathbf{d}\chi_{B(0, 2^j)}\|_{m^{q,p}} \geq (2^{j+2} + 1)^{1/q-1/p} \|\mathbf{d}\chi_{\mathbb{Z} \setminus B(0, 2^j)}\chi_{B(0, 2^{j+1})}\|_p \geq$$

$$2^{(j+2)(1/q-1/p)} \left(\sum_{|k|=2^j+1}^{2^{j+1}} \frac{1}{|k|^{\alpha p}} \right)^{1/p} \geq 2^{1+\alpha-2/p}.$$

Theorem 3.1. $E_L(\mathbf{a})_{m^q,p} \rightarrow 0$ ($L \rightarrow \infty$) if and only if $\|\Delta_h^r \mathbf{a}\|_{m^q,p} \rightarrow 0$ ($h \rightarrow 0$).

Proof. Sufficiency follows from Jackson's inequality. Let us proceed to the proof of necessity. By condition $E_L(\mathbf{a})_{m^q,p} \rightarrow 0$ ($L \rightarrow \infty$), i.e. $\|\mathbf{a}\chi_{\mathbb{Z} \setminus B(0,L)}\|_{m^q,p} \rightarrow 0$. Hence, for a given $\varepsilon > 0$ there exists $L_\varepsilon \in \mathbb{N}$ such that $\|\mathbf{a}\chi_{\mathbb{Z} \setminus B(0,L_\varepsilon-1)}\|_{m^q,p} < \varepsilon$ and $1/L_\varepsilon^r < \varepsilon$. Let $|h| < 1/L_\varepsilon^2$. For

$$I = \sum_{k \in B(m,N)} (\min(1, |\tilde{k}h|^r) |a_k|)^p = \sum_{k \in B(m,N), |k| < L_\varepsilon} (\min(1, |\tilde{k}h|^r) |a_k|)^p + \sum_{k \in B(m,N), |k| \geq L_\varepsilon} (\min(1, |\tilde{k}h|^r) |a_k|)^p$$

by virtue of $|h| < 1/L_\varepsilon^2$ we have

$$I \leq \frac{1}{L_\varepsilon^{rp}} \sum_{k \in B(m,N), |k| < L_\varepsilon} |a_k|^p + \sum_{k \in B(m,N), |k| \geq L_\varepsilon} |a_k|^p \leq \varepsilon^p \sum_{k \in B(m,N)} |a_k|^p + \sum_{k \in B(m,N), |k| \geq L_\varepsilon} |a_k|^p.$$

Thus, for all $|h| < 1/L_\varepsilon^2$ we have $\|\Delta_h^r \mathbf{a}\|_{m^q,p} \leq C(\varepsilon \|\mathbf{a}\|_{m^q,p} + \varepsilon)$. $\square$

Thus, from the point of view of the approximations under consideration, the subspaces of $m^{q,p}$ for whose elements the differences tend to zero are of interest. Such subspaces for the Morrey function spaces were introduced in [14].

Proposition 3.1. *The following inequalities hold*

1) if $\text{supp } \mathbf{b} \subset B(0, n_0)$, $n_0 \in \mathbb{N}$, then for all $r > 0$

$$\|\mathbf{b}^{(r)}\|_{m^q,p} \leq n_0^r \|\mathbf{b}\|_{m^q,p} \text{ (Bernstein's inequality);}$$

2) if $\text{supp } \mathbf{b} \subset \mathbb{Z} \setminus B(0, n_0 - 1)$, $n_0 \in \mathbb{N}$, then for all $r > 0$

$$\|\mathbf{b}\|_{m^q,p} \leq n_0^{-r} \|\mathbf{b}^{(r)}\|_{m^q,p} \text{ (Böhr-Favard's inequality);}$$

3) if $\text{supp } \mathbf{b} \subset B(0, 2n_0) \setminus B(0, n_0 - 1)$, $n_0 \in \mathbb{N}$, then for all $r > 0$

$$(2n_0)^{-r} \|\mathbf{b}^{(r)}\|_{m^q,p} \leq \|\Delta_{1/n_0}^r \mathbf{b}\|_{m^q,p} = \|\mathbf{b}\|_p \leq n_0^{-r} \|\mathbf{b}^{(r)}\|_{m^q,p};$$

4) if $\text{supp } \mathbf{b} \subset B(0, n_0)$, $n_0 \in \mathbb{N}$ and $0 < p_1 < p_2 \leq q < \infty$ then

$$\|\mathbf{b}\|_{m^q,p_1} \leq (\#\text{supp } \mathbf{b})^{1/p_1-1/p_2} \|\mathbf{b}\|_{m^q,p_2} \text{ (Nikolskii's inequality)}.$$

Proof. 1) From $\|\mathbf{b}^{(r)}\chi_{B(m,N)}\|_p \leq n_0^r \|\mathbf{b}\chi_{B(m,N)}\|_p$ follows the validity of the statement.

2) For positive r

$$\begin{aligned} \|\mathbf{b}\chi_{B(m,N)}\|_p^p &= \sum_{k \in B(m,N), |k| \geq n_0} |b_k|^p \leq \\ n_0^{-rp} \sum_{k \in B(m,N), |k| \geq n_0} |k^r b_k|^p &\leq n_0^{-rp} \|\mathbf{b}^{(r)}\chi_{B(m,N)}\|_p^p \end{aligned}$$

from which the inequality to be proved follows.

3) This inequality follows from 1), 2) and 2.1.

4) By Hölder's inequality with the exponent p_2/p_1

$$\|\mathbf{b}\chi_{B(m,N)}\|_{p_1} \leq (\#\text{supp}\mathbf{b})^{1/p_1-1/p_2} \|\mathbf{b}\chi_{B(m,N)}\|_{p_2}.$$

Then the required inequality follows from the obvious estimate

$$(2N+1)^{1/q-1/p_1} \leq (2N+1)^{1/q-1/p_2}.$$

□

The results of this note were partly announced in [6].

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