

A covering-strip and moment certificate for Korenblum's constant

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Abstract

We give an explicit certificate for $c_2 \geq 54/125 = 0.432$, where c_2 is Korenblum's constant for the Bergman space. This is an unrefereed preprint. The construction combines an elementary two-zero Schwarz estimate on the covering strip of an annulus, the classical hyperbolic Jung inequality, and Wikström's moment-duality criterion. A rational twelve-atom measure with a Lebesgue tail satisfies the required inequalities for every moment index. Two independently implemented exact checkers, one using Arb and one using only integer interval arithmetic, verify the quantitative hypotheses. The best public certified lower bound located in the accompanying priority audit is Wikström's 0.4263. Neither optimality nor a new version of Jung's theorem is claimed.

1 Statement and analytic reduction

Korenblum introduced the maximum principle in [1]; Hayman [2] proved that a positive admissible radius exists. We give a lower bound, not a determination of the optimal constant.

Normalize Bergman area by $dA = dx dy/\pi$, so $\|\sum a_k z^k\|_2^2 = \sum |a_k|^2/(k+1)$. Korenblum's constant c_2 is the supremum of those $c \in (0, 1)$ for which $|f| \leq |g|$ on $c < |z| < 1$ implies $\|f\|_2 \leq \|g\|_2$, for all $f, g \in A^2(\mathbb{D})$.

Theorem 1. *Korenblum's constant satisfies $c_2 \geq 54/125$.*

We retain the moment framework of Wikström [3], but use a separately derived coefficient estimate. Put $w = -\log c$,

$$X = \sinh^2 \frac{\pi^2}{2w}, \quad K(t) = \frac{X}{X + \sin^2\left(\frac{\pi \log t}{2w}\right)}, \quad G(t) = \frac{1}{2} \left(1 + \sqrt{\frac{3 + K(t)^2}{3(1 - K(t)^2)}} \right) \quad (c^2 < t < 1).$$

Write $\beta(t) = 1 - 1/G(t)$, with $\beta(c^2) = \beta(1) = 1$ by limits. The K here is a covering-strip bound; it is not the annular product appearing in Wang's estimate as quoted in [3].

Lemma 1. *If $\omega : A(c, 1) \rightarrow \mathbb{D}$ is holomorphic, then its image of $|z| = r$ has pseudohyperbolic diameter at most $K(r^2)$.*

Proof. Lift to $h(\zeta) = \omega(e^\zeta)$ in $\log c < \operatorname{Re} \zeta < 0$. The map $\exp(i\pi(\zeta - \log c)/w)$ maps this strip biholomorphically onto the upper half-plane. After a rotation of the annulus, choose lifts $a = \log r$ and $a + i\theta$, $0 \leq \theta \leq 2\pi$, of the two points. Postcompose h with a disk automorphism making its value at a zero. Periodicity gives another zero at $a + 2\pi i$. Dividing successively by the two

Blaschke factors and using Schwarz's lemma bounds the value at the other point by the product of its pseudohyperbolic distances to these zeros.

Set $A = \pi^2/w$, $\phi = \pi \log(r/c)/w$ and $v = \pi\theta/w$. The distances in question are $\mathbf{p}(v)$ and $\mathbf{p}(2A - v)$, where

$$\mathbf{p}(v) = \frac{\sinh(v/2)}{\sqrt{\sinh^2(v/2) + \sin^2 \phi}}.$$

For $u = v/2 > 0$, the derivative of $\log \mathbf{p}(2u)$ is

$$\frac{\sin^2 \phi \coth u}{\sinh^2 u + \sin^2 \phi}.$$

This is positive and decreasing, because its positive numerator decreases and its positive denominator increases. Thus $\log \mathbf{p}$ is concave, and $\mathbf{p}(v)\mathbf{p}(2A - v) \leq \mathbf{p}(A)^2 = K(r^2)$. Endpoints follow by continuity. $\square$

Lemma 2. *Suppose $f = \sum a_k z^k$, $g = \sum b_k z^k$ are holomorphic in $\mathbb{D}$ and $|f| \leq |g|$ on $A(c, 1)$. For $c < r < 1$, let*

$$Y(r^2) = \sum_{\substack{k \geq 1 \\ |a_k| < |b_k|}} (|b_k|^2 - |a_k|^2) r^{2k}, \quad M_h(r)^2 = \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^2 d\theta.$$

Then $Y(r^2) \leq G(r^2)(M_g(r)^2 - M_f(r)^2)$.

Proof. The quotient f/g has removable singularities in the annulus. The constant and boundary cases are immediate, so assume it maps into $\mathbb{D}$. Equip $\mathbb{D}$ with hyperbolic distance twice the inverse hyperbolic tangent of the pseudohyperbolic distance. By Lemma 1, the compact image of the circle has hyperbolic diameter at most $D = 2 \operatorname{artanh} K(r^2)$. The classical hyperbolic Jung inequality [8] places it in a disk of hyperbolic radius R_h with

$$\sinh R_h \leq \frac{2}{\sqrt{3}} \sinh(D/2).$$

For completeness, consider a smallest enclosing hyperbolic disk. At its center the directions to its boundary contacts cannot lie in an open semicircle, since a small motion would shrink the radius. Two opposite contacts, or three contacts whose directions contain zero in their convex hull, give a pair separated by an angle at least $2\pi/3$. The hyperbolic cosine rule gives $\cosh D \geq 1 + \frac{3}{2} \sinh^2 R_h$, proving the inequality.

The enclosing disk has pseudohyperbolic center $\alpha \in \mathbb{D}$ and radius $\rho = \tanh(R_h/2)$, and hence

$$\frac{1}{1 - \rho^2} = \frac{1 + \cosh R_h}{2} \leq G(r^2).$$

Set $F = f - \alpha g$ and $H = g - \bar{\alpha} f$. On the circle, $|F| \leq \rho |H|$, and coefficientwise

$$|H_k|^2 - |F_k|^2 = (1 - |\alpha|^2)(|b_k|^2 - |a_k|^2).$$

Dropping the negative $|F_k|^2$ contributions in the indices defining Y and adding nonnegative $|H_k|^2$ terms gives

$$Y(r^2) \leq \frac{M_H(r)^2}{1 - |\alpha|^2} \leq \frac{M_g(r)^2 - M_f(r)^2}{1 - \rho^2} \leq G(r^2)(M_g(r)^2 - M_f(r)^2).$$

The center can be complex and may depend on r . $\square$

The following is Wikström's moment-duality argument [3, Lemma 3.1], with the coefficient estimate just proved substituted for its analytic input.

Lemma 3. *If a probability measure μ on $[c^2, 1]$, with no atom at 1, satisfies*

$$\int t^k d\mu(t) \geq \frac{1}{k+1} \quad (k \geq 0), \quad \int \beta(t)t^k d\mu(t) \leq \frac{1}{k+1} \quad (k \geq 1),$$

then c is admissible in the definition of c_2 .

Proof. Write $x_k = (|a_k|^2 - |b_k|^2)_+$, $y_k = (|b_k|^2 - |a_k|^2)_+$, $X(t) = \sum_{k \geq 0} x_k t^k$ and $Y(t) = \sum_{k \geq 1} y_k t^k$. Lemma 2 rearranges to $X(t) \leq y_0 + \beta(t)Y(t)$. Integrate and apply the two moment inequalities. Nonnegative-series monotone convergence gives $\sum x_k/(k+1) \leq y_0 + \sum_{k \geq 1} y_k/(k+1)$, which is the norm comparison. Limits give the endpoint convention at c^2 . $\square$

2 The rational certificate

Take $c = 54/125$, $R = 999/1000$ and

$$d\mu(t) = \sum_{j=1}^{12} w_j d\delta_{t_j}(t) + \mathbf{1}_{[R,1]}(t) dt.$$

The entries below are exact terminating decimals, not rounded displays. They have $t_1 = c^2$, $t_{12} = R$, positive weights summing exactly to R , and no atom at 1.

j	t_j	w_j
1	0.186624	0.401004850339305473
2	0.587760864611916678	0.334589004564615711
3	0.8229718555012977	0.152356299682819746
4	0.924010443899645262	0.062924486526793319
5	0.965990583618882126	0.026544759438047238
6	0.984044289625064295	0.011714716766935399
7	0.992173624211215355	0.00540693721877966
8	0.995987383331351062	0.002581897617100729
9	0.997819789755709063	0.001239794794409609
10	0.998676970469441039	0.000541241719003444
11	0.998976894402894056	0.000064073011221399
12	0.999	0.000031938320968273

Let $g(t) = 1/G(t)$. A simple global linear lower bound on the Lebesgue tail suffices. Define $\Delta = 1 - R$, $U = -\pi \log R/(2w)$ and

$$a = \frac{(\pi/(2w))^2(1 - U^2/6)^2}{X}, \quad \ell = \frac{2563746486913}{10^{14}}.$$

Both checkers prove $U^2 < 6$ and

$$0 < \ell < \frac{2\sqrt{6a}}{\sqrt{3a\Delta^2(2 + a\Delta^2)} + \sqrt{4 + 6a\Delta^2 + 3a^2\Delta^4}}. \quad (1)$$

Indeed the latter right side is greater than 0.0256374648691356266. For $t \in [R, 1]$, put $\delta = 1 - t$ and $y = \sin^2(\pi \log t / (2w)) / X$. Since $-\log t \geq \delta$ and $\sin v \geq v - v^3/6$, one has $y \geq a\delta^2$. Direct algebra gives

$$g(t) = \frac{2\sqrt{3y(2+y)}}{\sqrt{3y(2+y)} + \sqrt{4+6y+3y^2}}.$$

This expression increases with y . Substitute $a\delta^2$ for y , bound its numerator below by $2\sqrt{6a}\delta$ and its denominator above by the denominator in (1). Thus $g(t) \geq \ell(1-t)$ on $[R, 1]$. In particular,

$$\int_R^1 g(t)t^k dt \geq \ell \left(\frac{1-R^{k+1}}{k+1} - \frac{1-R^{k+2}}{k+2} \right).$$

The two desired moment inequalities reduce respectively to

$$A_k = \sum_j w_j t_j^k - \frac{R^{k+1}}{k+1} \geq 0,$$

$$B_k = \int_R^1 g(t)t^k dt + \frac{R^{k+1}}{k+1} - \sum_j w_j \beta(t_j)t_j^k \geq 0.$$

Here $A_0 = 0$ by exact mass balance. Both checkers verify $A_k > 0$ through $k = 31278$. For all $k \geq 31279$, the last atom alone suffices, since $w_{12}(k+1) \geq R$.

Both checkers verify $B_k > 0$ for $1 \leq k \leq 23999$ using the displayed linear integral lower bound and upper enclosures of the atomic β . For the rest, set $t_0 = 0.9998$, $t_1 = 0.9999$. The exact comparison

$$\ell(1-t_1)(t_1-t_0)t_0^{24000} > \sum_j w_j t_j^{24000}$$

has a certified gap greater than $2.1063 \cdot 10^{-12}$. Since every $t_j \leq R < t_0$, it persists at every larger exponent. As $\beta \leq 1$, it proves the second infinite tail. The verified finite relative margins, normalized by $R^{k+1}/(k+1)$, exceed $2.0961 \cdot 10^{-6}$ for A_k and $2.2847 \cdot 10^{-7}$ for B_k . Lemma 3 proves Theorem 1.

3 Independent checks, provenance and limitations

The supplied `certificate-jung.json` is exact rational input. `verify_jung_arb.py` uses Arb at 75 decimal digits. `verify_jung_rational.py` uses only standard-library integers, fractions and integer square roots, rounding every operation outwards on a 2^{200} denominator. It implements Machin's formula for π , the positive atanh series for $-\log t$, sine with a Lagrange remainder, and exponential with a geometric remainder after argument reduction. Neither checker imports the search or the other checker. Seventeen independent arithmetic enclosure controls against 100-digit Arb pass. Both reject mass-preserving bad moments, an unsafe linear envelope, a false tail cutoff, a wrong analytic model and execution with assertions disabled.

The frozen certificate SHA-256 is

80a056f2cec4cfd933944fa51100bc5a1a9657f1a1de509e3a51401ff282002b.

Its numerical hypotheses, rather than the analytic proof itself, are checked by the programs. No proof-assistant formalization is claimed. The complete proof of the substituted analytic estimate is given above for mathematical review. A separate retained sixteen-atom certificate gives 0.42638465

using Wang’s original estimate [5, 6] and Wikström’s original moment lemma unchanged. The exact 0.4263 public baseline was reproduced before optimization, including its original published software [4].

Credit goes to Wang and Schuster for the annular-pseudodistance approach, to the classical Schwarz and hyperbolic Jung inequalities, and to Wikström for moment duality, the atomic-plus-Lebesgue construction and its LP/refinement strategy. The computations widen the atomic support and tail cutoff, then replace the analytic estimate by the covering-strip/enclosing-disk argument. Larger unsuccessful numerical trials do not prove a barrier. No optimality of the measure, analytic estimate or constant is asserted. The analytic proof and its computational hypotheses remain subject to independent review.

AI use and author review. Analysis and proof drafting by AI agents under my direction; I reviewed the results at summary level. AI agents also prepared the certificate, checking software, and publication materials. This disclosure does not claim a line-by-line human verification of the proof or software.

Availability. The certificates, checkers, verification receipts, and source are available at <https://github.com/rpinho/korenblum-c68-certificate> under the MIT license.

References

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