

Reduction of the Gorenstein symmetry conjecture to finite rings

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Abstract

We prove that the Gorenstein symmetry conjecture for Artin algebras is equivalent to its restriction to finite rings of prime-power cardinality, preserving the finite self-injective dimension of a counterexample. For finite-dimensional algebras over fields, the conjecture reduces to finite prime fields. In characteristic zero, a counterexample yields counterexamples of one fixed dimension over every prime field outside a finite set.

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1 Introduction

All algebras in this paper are nonzero, associative, and unital. Coefficient rings are commutative and act centrally; ring maps preserve the identity. An *Artin algebra* is an algebra finite as a module over a commutative Artinian coefficient ring. A *finite ring* has a finite underlying set. For an Artin algebra A , write ${}_AA$ and A_A for its left and right regular modules, and id for injective dimension in the corresponding category of all unital modules. The Gorenstein symmetry conjecture asserts

$$\text{id}({}_AA) < \infty \iff \text{id}(A_A) < \infty. \quad (1)$$

When both dimensions are finite, they are equal. The argument in Section 2 applies to Artin algebras as explained in Section 7.

The question belongs to the study of left–right symmetry in the homological dimensions of Artinian rings. Hoshino’s 1994 paper [2, Introduction] formulates it in terms of the equality of the two self-injective dimensions and recalls Zaks’s result that they agree whenever both are finite. It also records the affirmative cases of self-injective dimension at most one and of finite representation type. The conjecture appears as Conjecture (13) in Auslander, Reiten, and Smalø’s list of homological conjectures [1, pp. 409–410]. That list places it alongside the finitistic dimension conjecture and records the implication from the latter. Thus the

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finiteness of one self-injective dimension, rather than the comparison of two already finite dimensions, is the central issue.

Subsequent work has studied the conjecture through decompositions and equivalences of module and derived categories. Qin and Han [5, Section 5] reduce it to 2-derived-simple algebras: these are finite-dimensional algebras whose derived categories admit no nontrivial 2-recollement by derived categories of algebras. Here a 2-recollement is a decomposition equipped with the successive adjoint functors specified in [5, arXiv version, Definition 2, p. 3]. Their reduction provides a smaller class on which to test the conjecture. Zhang [8, arXiv version, Example 2.2(3) and Theorem 3.2] proves invariance under excellent extensions; finite separable scalar extensions are among the examples.

More recently, Han, Qin, and Wang [3, Theorem 3] prove invariance under an essentially surjective eventually homological isomorphism between bounded derived categories of finite-dimensional algebras. Essential surjectivity means that every object of the target is isomorphic to an image. The eventual homological condition requires the functor-induced maps on morphisms between shifted objects to be isomorphisms in all sufficiently large degrees, with the bound allowed to depend on the pair of objects [3, Definition 4]. We use the version of 17 October 2026 for this statement. These results describe categorical operations under which the conjecture can be transferred.

A different reduction concerns the coefficient field. For the finitistic dimension conjecture, Igusa [4, Theorem 5.1] compares algebras over the algebraic closure of a prime field with algebras over finite fields of the same characteristic, keeping a bound on Loewy length. In the Gorenstein symmetry problem, finite homological data can also retain the failure of a finiteness condition. Indeed, a finite projective resolution bounds one self-injective dimension, and one suitable nonzero Ext group forces the other to be infinite. This permits specialization of a counterexample while preserving its finite self-injective dimension.

For a general Artin algebra, the coefficient ring may have mixed characteristic, so field specialization alone does not cover the problem. We treat this case using finite-length modules over a Cohen ring and obtain the following reduction.

Theorem 1.1. *The following statements are equivalent.*

1. *Every Artin algebra satisfies the Gorenstein symmetry conjecture.*
2. *Every finite ring satisfies the Gorenstein symmetry conjecture.*
3. *Every finite ring of prime-power cardinality satisfies the Gorenstein symmetry conjecture.*

More precisely, if an Artin algebra A satisfies $\text{id}({}_A A) = d < \infty$ and $\text{id}(A_A) = \infty$, then a finite ring A' of prime-power cardinality satisfies $\text{id}({}_{A'} A') = d$ and $\text{id}(A'_{A'}) = \infty$.

For a local center of residue characteristic $p > 0$, the finite ring can be chosen with cardinality a power of p and the same characteristic as the original algebra (Lemma 9.1). A coefficient field or a Cohen ring is constructed in Lemma 8.4. In the latter case, Smith matrices encode lengths in finite homological complexes, and successive square-zero lifts realize the required finite data over a finite coefficient ring (Sections 7–9).

For algebras over fields, the reduction takes the following form.

For a prime p , let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements.

Theorem 1.2. *The following statements are equivalent.*

1. *Every finite-dimensional algebra over every field satisfies the Gorenstein symmetry conjecture.*

2. Every finite-dimensional algebra over every finite field satisfies the Gorenstein symmetry conjecture.
3. For every prime number p , every finite-dimensional $\mathbb{F}_p$ -algebra satisfies the Gorenstein symmetry conjecture.

For a fixed prime p , the conjecture for all fields of characteristic p is equivalent to the conjecture for $\mathbb{F}_p$.

The proof records a counterexample by finitely many matrices. A projective resolution of the right dual regular module gives a finite upper bound $d \geq 0$ on the left self-injective dimension. A nonzero Ext group in degree $d + 1$ for the left dual regular module forces the right self-injective dimension to be infinite. We construct a finite free coefficient model preserving both conditions and the exact value d at every field-valued fiber (Theorem 3.2). In characteristic zero this gives the following arithmetic refinement.

For a positive integer N , the ring $\mathbb{Z}[1/N] = \{a/N^j : a \in \mathbb{Z}, j \geq 0\}$ is viewed as a subring of $\mathbb{Q}$.

Theorem 1.3. *Suppose that k has characteristic zero and that a finite-dimensional k -algebra A satisfies*

$$\mathrm{id}({}_A A) = d < \infty, \quad \mathrm{id}(A_A) = \infty.$$

There exist positive integers m, N and an associative unital $\mathbb{Z}[1/N]$ -algebra $\mathcal{C}$, free of rank m , with the following property. Write $\mathcal{C}_{\mathbb{Q}} = \mathcal{C} \otimes_{\mathbb{Z}[1/N]} \mathbb{Q}$ and, for a prime $p \nmid N$, $\mathcal{C}_p = \mathcal{C} \otimes_{\mathbb{Z}[1/N]} \mathbb{F}_p$. Then

$$\begin{aligned} \mathrm{id}({}_{\mathcal{C}_{\mathbb{Q}}} \mathcal{C}_{\mathbb{Q}}) &= d, & \mathrm{id}((\mathcal{C}_{\mathbb{Q}})_{\mathcal{C}_{\mathbb{Q}}}) &= \infty, \\ \mathrm{id}({}_{\mathcal{C}_p} \mathcal{C}_p) &= d, & \mathrm{id}((\mathcal{C}_p)_{\mathcal{C}_p}) &= \infty \quad (p \nmid N). \end{aligned}$$

In particular, every $\mathcal{C}_p$ has the same dimension m over $\mathbb{F}_p$.

The integer m can differ from $n = \dim_k A$, because restriction of scalars from a number field increases vector-space dimension. There is also a version retaining n , with finite extensions of $\mathbb{F}_p$ of uniformly bounded degree (Proposition 5.2). Theorems 1.2 and 1.3 are proved in Sections 4 and 5, respectively.

2 Finite homological certificates

In Sections 2–6, all modules used in constructions are finite-dimensional over the indicated field. We write $\mathrm{pd}_A M$ and $\mathrm{Ext}_A^j(M, S)$ for left A -modules; right modules are regarded as left modules over the opposite algebra A^{op} , whose multiplication is reversed. Projective and injective dimensions are the least lengths of projective and injective resolutions, respectively, or infinity if no finite resolution exists. We assign both dimensions zero to the zero module. For $j \geq 0$, $\mathrm{Ext}_A^j(M, S)$ is the degree- j cohomology of $\mathrm{Hom}_A(P_{\bullet}, S)$ for a projective resolution $P_{\bullet}$ of M ; in particular $\mathrm{Ext}_A^0(M, S) = \mathrm{Hom}_A(M, S)$.

For a finite-dimensional vector space V , set $D_k V = \mathrm{Hom}_k(V, k)$. For a left module X , its dual is a right module with $(\lambda a)(x) = \lambda(ax)$; for a right module X , its dual is a left module with $(a\lambda)(x) = \lambda(xa)$, where $a \in A$, $x \in X$, and $\lambda \in D_k X$. The dual of the regular module is an A -bimodule with action

$$(a\lambda b)(x) = \lambda(bxa) \quad (a, b, x \in A, \lambda \in D_k A). \quad (2)$$

For any left A -module X , evaluation at $1 \in A$ gives

$$\mathrm{Hom}_A(X, D_k A) \cong \mathrm{Hom}_k(X, k), \quad f \longmapsto (x \longmapsto f(x)(1)).$$

The inverse sends a linear functional $\ell : X \rightarrow k$ to the map $x \mapsto (a \mapsto \ell(ax))$. This map is A -linear by (2). Since vector-space duality is exact, $D_k A$ is injective in the category of all left modules. The right-handed statement follows by replacing A by A^{op} .

Every finite left module X embeds into a finite direct sum of copies of $D_k A$: dualize a finite free right-module surjection onto $D_k X$ and identify $D_k D_k X$ with X by evaluation. Repeating this construction on the finite cokernels gives an injective resolution with finite terms. A finite injective module is a direct summand of such a finite sum, and its dual is therefore projective. Conversely, the dual of a finite projective module is a direct summand of a finite sum of copies of $D_k A$, hence injective. If $\text{id}_A X \leq d$ in the full module category, dimension shifting along this resolution makes its d -th cokernel injective: its first Ext against any module is $\text{Ext}_A^{d+1}(-, X) = 0$. Here the zeroth cokernel means X itself. Dualizing the resulting finite resolution, and conversely dualizing finite projective resolutions, proves

$$\text{id}_A(A) = \text{pd}_{A^{\text{op}}}(D_k A), \quad \text{id}(A_A) = \text{pd}_A(D_k A). \quad (3)$$

Finite projective resolutions can be taken with finite terms: successive finite free surjections have finite kernels, and the kernel at the projective-dimension bound is projective by dimension shifting.

Lemma 2.1. *If a nonzero finite-dimensional left A -module M has $\text{pd}_A M = e < \infty$, then $\text{Ext}_A^e(M, A) \neq 0$.*

Proof. For $e = 0$, choose a positive integer t and embed the nonzero projective module M as a direct summand of A^t . At least one coordinate map $M \rightarrow A$ is nonzero. For $e > 0$, take a projective resolution of length e with finite terms, and write $u : P_e \rightarrow P_{e-1}$ for its first map. Choose a positive integer t and A -linear maps $\iota : P_e \rightarrow A^t$ and $\pi : A^t \rightarrow P_e$ with $\pi\iota = 1$. If $\text{Ext}_A^e(M, A) = 0$, every coordinate of ι extends across u . The extensions form a map $g : P_{e-1} \rightarrow A^t$ satisfying $gu = \iota$. Thus $(\pi g)u = 1$, so u is a split monomorphism. Write $P_{e-1} = u(P_e) \oplus Q$. The next differential vanishes on $u(P_e)$, and its restriction to the projective summand Q is injective onto the same image, by exactness. Replacing the first two terms by Q gives a resolution of length $e - 1$, a contradiction. $\square$

Lemma 2.2. *Let $d \geq 0$ be an integer. If $\text{id}_A(A) \leq d$, every finite-dimensional left A -module of finite projective dimension has projective dimension at most d . Consequently,*

$$\text{id}_A(A) \leq d < \text{pd}_A(D_k A) \implies \text{id}(A_A) = \infty.$$

Proof. For a nonzero module of finite projective dimension e , Lemma 2.1 gives a nonzero $\text{Ext}_A^e(M, A)$. The assumed injective dimension forces $e \leq d$. Apply this to $D_k A$ if its left projective dimension is finite, and then use (3). $\square$

In particular, if both regular injective dimensions are finite, Lemma 2.2 and its opposite-algebra version bound each by the other. This proves the equality used in the introduction.

Lemma 2.3. *Let $d \geq 0$ be an integer. If M is a finite-dimensional left A -module and $\text{pd}_A M > d$, there is a simple left A -module S with $\text{Ext}_A^{d+1}(M, S) \neq 0$. One can choose $\dim_k S \leq \dim_k A$.*

Proof. Starting from $\Omega^0 M = M$, choose finite free surjections $F_j \rightarrow \Omega^j M$ and define $\Omega^{j+1} M$ to be their kernels, for $0 \leq j < d$. Thus $0 \rightarrow \Omega^{j+1} M \rightarrow F_j \rightarrow \Omega^j M \rightarrow 0$ is exact. For any left A -module T , the long exact Ext sequence gives

$$\text{Ext}_A^i(\Omega^{j+1} M, T) \cong \text{Ext}_A^{i+1}(\Omega^j M, T) \quad (i \geq 1)$$

because F_j is projective. The module $\Omega^d M$ is nonprojective, since otherwise this sequence would give a projective resolution of M of length d . Choose a finite free surjection $F \rightarrow \Omega^d M$ with kernel T . It does not split, so $\text{Ext}_A^1(\Omega^d M, T) \neq 0$: in the long exact sequence obtained by applying $\text{Hom}_A(\Omega^d M, -)$, the connecting map sends the identity to zero exactly when it lifts to F . Iterating the displayed isomorphism gives $\text{Ext}_A^{d+1}(M, T) \neq 0$, also for $d = 0$.

The finite module T has a composition series, obtained by taking maximal proper submodules until the dimension reaches zero. If $\text{Ext}_A^{d+1}(M, S) = 0$ for every simple factor S of this series, the long exact sequence in the second variable, applied successively to its short exact sequences, would give $\text{Ext}_A^{d+1}(M, T) = 0$. Some simple factor therefore has the required nonvanishing. For any nonzero $s \in S$, the map $A \rightarrow S$, $a \mapsto as$, is surjective, proving the dimension bound. $\square$

For an integer $d \geq 0$, an algebra with left self-injective dimension at most d and right self-injective dimension infinite is characterized by two finite pieces of homological data:

$$\begin{aligned} 0 \longrightarrow P_d \longrightarrow \cdots \longrightarrow P_0 \longrightarrow (D_k A)_A \longrightarrow 0 \quad \text{with each } P_i \text{ projective,} \\ \text{Ext}_A^{d+1}(D_k A, S) \neq 0 \quad \text{for some finite-dimensional left module } S. \end{aligned} \quad (4)$$

Existence follows from (3) and Lemma 2.3; sufficiency follows from Lemma 2.2. To preserve an exact finite value $d > 0$, we will also retain

$$\text{Ext}_{A^{\text{op}}}^d(D_k A, A) \neq 0, \quad (5)$$

which follows from Lemma 2.1 applied over A^{op} .

3 Specialization of the certificates

We first describe the finite linear algebra used to preserve the Ext conditions. For finite-dimensional left modules M, S , the unnormalized bar resolution has terms

$$B_j = A \otimes_k A^{\otimes_k j} \otimes_k M \quad (j \geq 0), \quad B_{-1} = M.$$

The initial factor carries the left A -action, and $A^{\otimes_k 0} = k$. The augmentation $\partial_0 : B_0 \rightarrow M$ is $a_0 \otimes m \mapsto a_0 m$. For $j \geq 1$, $a_0, \dots, a_j \in A$, and $m \in M$, the differential $\partial_j : B_j \rightarrow B_{j-1}$ is

$$\begin{aligned} \partial_j(a_0 \otimes \cdots \otimes a_j \otimes m) &= \sum_{i=0}^{j-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_j \otimes m \\ &\quad + (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j m. \end{aligned}$$

For $j \geq 2$, the terms of $\partial_{j-1} \partial_j$ pair according to two original positions $i < \ell$, where $0 \leq i < \ell \leq j$. Multiplying first at ℓ and then at i has sign $(-1)^{i+\ell}$; multiplying first at i and then at $\ell-1$ has sign $(-1)^{i+\ell-1}$. The tensors agree by associativity, including the last module action, so they cancel. At the augmentation, $\partial_0 \partial_1(a_0 \otimes a_1 \otimes m) = (a_0 a_1)m - a_0(a_1 m) = 0$.

Define k -linear maps $h_j : B_j \rightarrow B_{j+1}$ by $h_j(x) = 1 \otimes x$, for $j \geq -1$. For $j \geq 0$, the first term of $\partial_{j+1}(1 \otimes x)$ is x . Each remaining term is 1 tensored with a term of $\partial_j x$, with its sign reversed. Hence

$$\partial_{j+1} h_j(x) = x - h_{j-1} \partial_j(x), \quad \partial_0 h_{-1}(m) = m.$$

These identities give exactness: if $\partial_j x = 0$, then $x = \partial_{j+1} h_j(x)$, and the augmentation is surjective. Each B_j is a finite free left A -module, with a basis coming from a k -basis of $A^{\otimes_k j} \otimes_k M$. Thus $B_\bullet$ is a projective resolution.

Applying $\text{Hom}_A(-, S)$ identifies its cochains with

$$C^j = \text{Hom}_k(A^{\otimes_k j} \otimes_k M, S) \quad (j \geq 0).$$

The identification sends an A -linear map to its restriction to $1 \otimes A^{\otimes_k j} \otimes_k M$; its inverse sends c to $a_0 \otimes v \mapsto a_0 c(v)$. For $c \in C^j$, its differential $\delta^j : C^j \rightarrow C^{j+1}$ is

$$\begin{aligned} (\delta^j c)(a_1, \dots, a_{j+1}, m) &= a_1 c(a_2, \dots, a_{j+1}, m) \\ &\quad + \sum_{i=1}^j (-1)^i c(a_1, \dots, a_i a_{i+1}, \dots, a_{j+1}, m) \\ &\quad + (-1)^{j+1} c(a_1, \dots, a_j, a_{j+1} m). \end{aligned} \quad (6)$$

For $j = 0$ the sum is empty, so $(\delta^0 c)(a, m) = ac(m) - c(am)$. For an integer $q \geq 0$, we use $H^q(C^\bullet) = \ker \delta^q / \text{im } \delta^{q-1}$, with $\delta^{-1} = 0$. It equals $\text{Ext}_A^q(M, S)$. In fixed bases, all these matrices have entries given by integer polynomials in the multiplication and action coefficients.

Lemma 3.1. *Let $q \geq 1$ be an integer, let R be a subring of a field k , and let $C_R^{q-1} \rightarrow C_R^q \rightarrow C_R^{q+1}$ be a complex of finite free R -modules. If its degree- q cohomology after tensoring with k is nonzero, the same holds after tensoring with any field F along a unital map $R \rightarrow F$.*

Proof. Let r, s be the ranks over k of the incoming and outgoing differentials. Every minor of size $r + 1$, respectively $s + 1$, is zero in k , and therefore zero in R . The corresponding ranks over F are at most r, s . Here $\text{rank}_R C_R^q$ is the number of elements in a free basis of the middle term. The complex identity remains valid over F , so the incoming image lies in the outgoing kernel. Rank-nullity gives

$$\begin{aligned} \dim_F H^q(C_R^\bullet \otimes_R F) &\geq \text{rank}_R C_R^q - r - s \\ &= \dim_k H^q(C_R^\bullet \otimes_R k) > 0. \end{aligned} \quad \square$$

Theorem 3.2. *Let A be an n -dimensional algebra over a field k , and suppose*

$$\text{id}(A) = d < \infty, \quad \text{id}(A_A) = \infty.$$

There exist a subring $R \subset k$, finitely generated as a unital ring, and an R -algebra A_R , free of rank n , such that $A_R \otimes_R k \cong A$ and, for every unital map $R \rightarrow F$ to a field, the fiber $A_F = A_R \otimes_R F$ satisfies

$$\dim_F A_F = n, \quad \text{id}(A_F) = d, \quad \text{id}((A_F)_{A_F}) = \infty.$$

Proof. Fix a basis $a_1 = 1, a_2, \dots, a_n$ of A . Choose the resolution and the module S in (4), and fix vector-space bases for the P_i and S , and use the dual basis to $a_1, \dots, a_n$ for $D_k A$. For every $0 \leq i \leq d$, choose a nonnegative integer t_i and right A -linear maps

$$P_i \xrightarrow{\iota_i} A^{t_i} \xrightarrow{\pi_i} P_i, \quad \pi_i \iota_i = 1_{P_i}. \quad (7)$$

Regard the augmented resolution as a finite complex of vector spaces. Write $P_{-1} = D_k A$ and set $P_i = 0$ for $i < -1$ or $i > d$. Let $\partial_i : P_i \rightarrow P_{i-1}$ be its maps for $0 \leq i \leq d$, and set $\partial_{-1} = 0$ and $\partial_{d+1} = 0$. Its exactness gives k -linear maps $h_i : P_i \rightarrow P_{i+1}$ for $-1 \leq i < d$ such that

$$\partial_{i+1} h_i + h_{i-1} \partial_i = 1_{P_i} \quad (-1 \leq i \leq d). \quad (8)$$

The maps outside the specified range are zero. To construct h_i , choose a complement to $\ker \partial_i$ in P_i . Exactness identifies this complement, by ∂_i , with $\ker \partial_{i-1}$. Define h_{i-1} on that

kernel by the inverse and set it to zero on the chosen complement. On each of the two summands of P_i , one of the two terms in (8) is the identity and the other is zero. At the endpoints this reads $\partial_0 h_{-1} = 1_{D_k A}$ and $h_{d-1} \partial_d = 1_{P_d}$.

Let R be the unital subring of k generated by the following finite set of entries: the multiplication table of A , the action matrices of the P_i and S , and the matrices of $\partial_i, \iota_i, \pi_i, h_i$. On the free R -module with basis $a_1, \dots, a_n$, define A_R using this multiplication table. Replace each chosen vector space by the free R -module on its basis, with actions and maps given by the selected matrices. Associativity, unit laws, module laws, module linearity of $\partial_i, \iota_i, \pi_i$, and the equations $\partial_i \partial_{i+1} = 0$, (7), and (8) are polynomial identities in these entries with integer coefficients. They hold over R , since $R \hookrightarrow k$ is injective. The first basis vector of A_R is its identity, and every field fiber is nonzero of dimension n .

Define $D_R A_R = \text{Hom}_R(A_R, R)$ with the action (2). Its action matrices come from the multiplication table. The finite free dual basis gives, for every $R \rightarrow F$, an isomorphism of A_F -bimodules

$$(D_R A_R) \otimes_R F \cong D_F(A_R \otimes_R F). \quad (9)$$

This isomorphism sends $\lambda \otimes u$ to $a \otimes v \mapsto uv\lambda(a)$, where $\lambda(a)$ is mapped from R to F . The dual bases show that it is bijective, and (2) shows compatibility with both actions. Use $D_R A_R$ as the augmentation term of the model. Equations (7) make each specialized P_i a direct summand of a finite free right A_F -module. Equations (8) make the specialized augmented complex exact as a complex of F -vector spaces, hence also as a complex of A_F -modules. Therefore

$$\text{pd}_{A_F^{\text{op}}}(D_F A_F) \leq d. \quad (10)$$

Let S_R be the free R -module with the fixed basis and action matrices of S . Apply the bar construction over R to the left A_R -modules $D_R A_R, S_R$. Each cochain term is finite free over R . After tensoring with any field, it is the bar cochain complex of the corresponding fiber modules, by the fixed bases and (9). Lemma 3.1 in degree $d+1$ gives

$$\text{Ext}_{A_F}^{d+1}(D_F A_F, S_R \otimes_R F) \neq 0.$$

Together with (10), Lemma 2.2 implies $\text{id}((A_F)_{A_F}) = \infty$.

If $d > 0$, apply the bar construction over A_R^{op} with source $D_R A_R$ and target A_R . Equation (5) and Lemma 3.1 imply $\text{Ext}_{A_F^{\text{op}}}^d(D_F A_F, A_F) \neq 0$. Thus (10) is an equality. If $d = 0$, it is already an equality because the dual module is nonzero. Finally (3) gives the asserted left injective dimension. $\square$

4 Finite fields and prime fields

The first assertion below is the weak Nullstellensatz of Zariski [7, Section 1, assertion H3]. An element is integral over a commutative ring if it satisfies a monic polynomial with coefficients in that ring.

Lemma 4.1. *A field finitely generated as an algebra over a field is a finite field extension. A field finitely generated as a unital ring is finite.*

Proof. First, if a field is integral over a subring, that subring is a field. Indeed, for a nonzero element x of the subring, multiply a monic relation of degree $e \geq 1$ for x^{-1} by x^{e-1} . The resulting equation expresses x^{-1} in the subring.

Let k be a field and $E = k[c_1, \dots, c_t]$ a field generated by $t \geq 0$ elements as a k -algebra. After reordering the generators, let $c_1, \dots, c_r$ be a maximal algebraically independent subset over k , where $0 \leq r \leq t$. Here $k(c_1, \dots, c_r)$ is the fraction field of the polynomial ring

$k[c_1, \dots, c_r]$; both rings mean k when $r = 0$. Each remaining generator is algebraic over this fraction field: a polynomial dependence on $c_1, \dots, c_r, c_i$ gives a nonzero polynomial in c_i , since the first r elements are independent. Clearing the denominators of their monic equations gives a nonzero $f \in k[c_1, \dots, c_r]$ such that these generators are integral over $k[c_1, \dots, c_r, 1/f]$. If these equations have degrees $e_i \geq 1$, for $r < i \leq t$, their monomials $\prod_{i=r+1}^t c_i^{a_i}$, with $0 \leq a_i < e_i$, span the resulting algebra over that subring. An empty product is 1. Thus $E = k[c_1, \dots, c_r, 1/f][c_{r+1}, \dots, c_t]$ is a finite module over that subring. To check integrality, take module generators $u_1 = 1, u_2, \dots, u_s$, where $s \geq 1$, and $x \in E$. Write $xu_i = \sum_{j=1}^s b_{ij}u_j$, with $b_{ij} \in k[c_1, \dots, c_r, 1/f]$. Let $B = (b_{ij})_{1 \leq i, j \leq s}$ and let I_s be the identity matrix. Then $(xI_s - B)(u_1, \dots, u_s)^T = 0$, where the superscript denotes transpose. Multiplication by the adjugate gives $\det(xI_s - B)u_i = 0$ for every i . Since $u_1 = 1$, the monic polynomial $\det(TI_s - B)$ in the indeterminate T vanishes at x . The first paragraph therefore makes $k[c_1, \dots, c_r, 1/f]$ a field.

If $r > 0$, choose a monic irreducible polynomial $g \in k[c_1]$ which does not divide f . Such a polynomial exists: $k[c_1]$ has infinitely many monic irreducibles, since a product of any finite list plus 1 has an irreducible factor outside the list; only finitely many can divide a nonzero coefficient of f when it is written as a polynomial in $c_2, \dots, c_r$. The ideal generated by g in $k[c_1, \dots, c_r]$ is prime, because its quotient is a polynomial ring over the field $k[c_1]/(g)$. If g were invertible after adjoining $1/f$, then $gh = f^j$ for some $h \in k[c_1, \dots, c_r]$ and $j \geq 0$, contrary to primality and $g \nmid f$. Thus $r = 0$. The finitely many algebraic generators now give $[E : k] < \infty$.

For the second assertion, in positive characteristic p the field is a finite extension of $\mathbb{F}_p$. In characteristic zero, write such a field as $E = \mathbb{Z}[c_1, \dots, c_t]$. It is finitely generated over $\mathbb{Q}$, hence a number field. Choose $N \geq 1$ so that the coefficients of the monic minimal polynomials of all the c_i lie in $\mathbb{Z}[1/N]$. Then $E = \mathbb{Z}[1/N][c_1, \dots, c_t]$ is integral over $\mathbb{Z}[1/N]$. The first paragraph would make $\mathbb{Z}[1/N]$ a field, whereas a prime not dividing N is not invertible in this ring. $\square$

Thus every maximal-ideal residue field of a nonzero finitely generated unital commutative ring is finite.

Proposition 4.2. *Suppose $\dim_k A = n$, $\text{id}({}_A A) = d < \infty$, and $\text{id}(A_A) = \infty$. There is an n -dimensional algebra B over a finite field F such that*

$$\text{id}({}_B B) = d, \quad \text{id}(B_B) = \infty.$$

If k has positive characteristic p , F can be chosen of characteristic p .

Proof. Take R, A_R from Theorem 3.2 and a maximal ideal $\mathfrak{m}$ of R . The ring R is a nonzero finitely generated $\mathbb{Z}$ -algebra, so $F = R/\mathfrak{m}$ is finite. The fiber $B = A_R \otimes_R F$ has the asserted dimensions. In characteristic p , R is an $\mathbb{F}_p$ -algebra, and its residue fields have characteristic p . $\square$

Proof of Theorem 1.2. Statement (1) implies (2) and (3). If (1) fails, replace a counterexample by its opposite algebra if necessary to put its finite injective dimension on the left. Proposition 4.2 then produces a counterexample over a finite field, proving that (2) implies (1).

If $F/\mathbb{F}_p$ is a finite field extension and B is a finite-dimensional F -algebra, regard B as an $\mathbb{F}_p$ -algebra by restricting its central scalar action. The underlying unital ring, its regular modules, and their injective dimensions are unchanged, whereas

$$\dim_{\mathbb{F}_p} B = [F : \mathbb{F}_p] \dim_F B.$$

Thus (3) implies (2). For a fixed p , use the characteristic-preserving clause of Proposition 4.2 and the same restriction of scalars. This proves the last assertion. $\square$

5 Characteristic zero

Lemma 5.1. *Under the hypotheses of Theorem 1.3, with $n = \dim_k A$, there is a finite field extension $L/\mathbb{Q}$ and an n -dimensional L -algebra B with $\text{id}(B) = d$ and $\text{id}(B_B) = \infty$. The same ring, viewed as a $\mathbb{Q}$ -algebra, has dimension $n[L : \mathbb{Q}]$ and the same two injective dimensions.*

Proof. The coefficient ring R from Theorem 3.2 has characteristic zero. Its localization $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is nonzero and finitely generated over $\mathbb{Q}$. A maximal ideal gives a residue field L finite over $\mathbb{Q}$ by the weak Nullstellensatz. Apply Theorem 3.2 to $R \rightarrow L$. Restriction of scalars to $\mathbb{Q}$ changes dimension by the factor $[L : \mathbb{Q}]$ and leaves the ring and its regular modules unchanged. $\square$

Proof of Theorem 1.3. Let C be the $\mathbb{Q}$ -algebra obtained in Lemma 5.1 and put $m = \dim_{\mathbb{Q}} C$. Apply the construction of Theorem 3.2 anew to C , using $D_{\mathbb{Q}}C$. This gives a finite collection of rational matrix entries for its multiplication, modules, resolution, splitting maps, and vector-space contraction. Choose a positive integer N divisible by every denominator. All these entries lie in $\mathbb{Z}[1/N]$.

The same matrices define a free rank- m algebra $\mathcal{C}$ over $\mathbb{Z}[1/N]$. The splitting and contraction identities hold in this ring, since it embeds in $\mathbb{Q}$. All minors above the ranks of the relevant bar differentials also vanish in this ring. The proof of Theorem 3.2 therefore applies to each map $\mathbb{Z}[1/N] \rightarrow \mathbb{F}_p$, for $p \nmid N$, preserving both the finite value d and the infinite opposite dimension. The rational fiber recovers C . $\square$

The original dimension can instead be retained over finite extensions of the prime fields. The next statement also bounds their degrees.

Proposition 5.2. *Under the hypotheses of Theorem 1.3, let $n = \dim_k A$. There exist positive integers r, N such that for every prime $p \nmid N$ there are a finite extension $F/\mathbb{F}_p$ with $[F : \mathbb{F}_p] \leq r$ and an n -dimensional F -algebra B_p satisfying*

$$\text{id}(B_p) = d, \quad \text{id}((B_p)_{B_p}) = \infty.$$

Proof. Take L, B as in Lemma 5.1 and set $r = [L : \mathbb{Q}]$. Select all the finite matrix data for B used in the proof of Theorem 3.2. Choose a $\mathbb{Q}$ -basis $b_1 = 1, b_2, \dots, b_r$ of L . There is a positive integer N such that the multiplication coefficients of this basis and the coordinates of every selected matrix entry belong to $\mathbb{Z}[1/N]$. Consequently

$$O = \bigoplus_{i=1}^r \mathbb{Z}[1/N]b_i \subset L$$

is a subring containing every selected entry and free of rank r over $\mathbb{Z}[1/N]$. The identities and vanishing minors used in Theorem 3.2 hold in O .

For $p \nmid N$, the commutative algebra O/pO is nonzero of dimension r over $\mathbb{F}_p$. A maximal ideal of this finite algebra has residue field F with $[F : \mathbb{F}_p] \leq r$. Specializing the selected O -model of B along $O \rightarrow F$ gives the required n -dimensional algebra. $\square$

Corollary 5.3. *If the Gorenstein symmetry conjecture holds for every finite-dimensional $\mathbb{F}_p$ -algebra for infinitely many primes p , then it holds for every finite-dimensional algebra over every field of characteristic zero.*

Proof. A characteristic-zero counterexample, after taking the opposite algebra if necessary, would give a counterexample over every $\mathbb{F}_p$ with $p \nmid N$, by Theorem 1.3. An infinite set of primes contains one not dividing N . $\square$

6 Finite detection over fields

The field reduction also has a computability consequence. A semidecision procedure for failure means an algorithm which halts with a counterexample if the conjecture fails, and never outputs a counterexample if it holds. It need not halt in the latter case.

Proposition 6.1. *Failure of the Gorenstein symmetry conjecture for finite-dimensional algebras over arbitrary fields is semidecidable.*

Proof. Enumerate primes p , positive integers n , and nonnegative integers d , in such a way that every triple is eventually reached. For each triple, enumerate the finitely many multiplication tables on $\mathbb{F}_p^n$ with the first basis vector prescribed as identity. Associativity and the unit laws are finite equations, so their validity is decidable.

For each resulting algebra A , form the two actions on $D_{\mathbb{F}_p}A$ using (2). For either side, compute d successive kernels of finite free surjections. More explicitly, for a finite left module M with vector-space basis $m_1, \dots, m_t$, use $A^t \rightarrow M$, $(a_i) \mapsto \sum_i a_i m_i$. Its kernel and its action matrices are obtained by linear algebra. Use the right-handed version for right modules. Set $\Omega^0 M = M$, and let $\Omega^{j+1} M$ denote the kernel at the next step, for $0 \leq j < d$.

For a left module, $\text{pd}_A M \leq d$ is equivalent to projectivity of $\Omega^d M$. Indeed dimension shifting identifies $\text{Ext}_A^1(\Omega^d M, X)$ with $\text{Ext}_A^{d+1}(M, X)$; vanishing for all X characterizes projectivity, and the converse follows by truncating the resolution. To test a finite module K for projectivity, take the above finite free surjection $q : A^{\dim_{\mathbb{F}_p} K} \rightarrow K$. It splits precisely when there is an A -linear map $s : K \rightarrow A^{\dim_{\mathbb{F}_p} K}$ with $qs = 1_K$. Both module linearity and this last equation are finite linear equations in the entries of s , decidable over $\mathbb{F}_p$. For the zero module the zero section suffices. Apply the same test over A^{op} for right modules.

Halt if exactly one of the two dual modules has projective dimension at most d . Lemma 2.2, on the appropriate side, certifies that the other injective dimension is infinite. If a counterexample exists over any field, Theorem 1.2 provides one over a prime finite field; the enumeration reaches its multiplication table and a suitable d . $\square$

7 Finite-length modules over a discrete valuation ring

Fix a prime p , a discrete valuation ring O of characteristic zero with maximal ideal pO , and an integer $N \geq 1$. Write $T = O/(p^N)$. For an O -module of finite length, ℓ_O denotes its composition length; its simple factors are copies of O/pO . A finitely generated T -module has a decomposition

$$L = \bigoplus_{j=1}^r O/(p^{e_j}), \quad 1 \leq e_j \leq N. \quad (11)$$

The empty sum represents the zero module, and $\ell_O(L) = \sum_{j=1}^r e_j$. These are the elementary divisors over the principal ideal domain O . They will allow us to retain lengths after changing coefficients.

Lemma 7.1. *The functor $D_T = \text{Hom}_T(-, T)$ is an exact duality on finitely generated T -modules. If A is a nonzero T -algebra finite as a T -module, then*

$$\text{id}_A(A) = \text{pd}_{A^{\text{op}}}(D_T A), \quad \text{id}(A_A) = \text{pd}_A(D_T A).$$

Proof. Every ideal of T is $p^a T$ for $0 \leq a \leq N$. A T -linear map $p^a T \rightarrow T$ sends p^a to an element annihilated by p^{N-a} , hence to $p^a b$ for some $b \in T$. It extends to multiplication by b on T . Baer's criterion therefore makes T injective over itself, so D_T is exact on all T -modules.

For elements $x = (x_j)$ and $y = (y_j)$ of the module in (11), consider the pairing

$$\langle x, y \rangle = \sum_{j=1}^r p^{N-e_j} x_j y_j \pmod{p^N} \quad (12)$$

It identifies a second copy of L with $D_T L$. Indeed the image of 1 under a homomorphism $O/(p^e) \rightarrow T$ is an arbitrary element of $p^{N-e} T$, which is uniquely $p^{N-e} y$ for $y \in O/(p^e)$. This also proves that evaluation $L \rightarrow D_T D_T L$ is an isomorphism. Thus D_T is a duality.

Give $D_T A$ the bimodule action in (2), with values in T . For every left A -module X , the same evaluation and inverse maps as in Section 2 give $\text{Hom}_A(X, D_T A) \cong \text{Hom}_T(X, T)$. Hence $D_T A$ is injective in the full module category. Dualizing a finite free surjection onto $D_T X$ embeds each finitely generated X into a finite sum of copies of $D_T A$. The cokernel is again finitely generated, since A is finite over the Artinian ring T . Iteration gives an injective resolution with finitely generated terms. Finitely generated injectives split off such sums, so their duals are projective; duals of finitely generated projectives are injective. If the injective dimension is at most d , dimension shifting makes the d -th cokernel injective, as its first Ext against every module vanishes. Dualizing the truncated resolution and, conversely, dualizing finite projective resolutions proves both asserted equalities, including infinite values. $\square$

The proofs of Lemmas 2.1–2.3 also apply to finitely generated modules over an Artin algebra: finite free surjections have finitely generated kernels, projectives split off finite free modules, and composition series have finite length. In Lemma 2.3, the last dimension bound is specific to field algebras. The Ext nonvanishing assertion uses only the composition series. Consequently, for a finite T -algebra,

$$\text{id}(A_A) \leq d < \text{pd}_A(D_T A) \implies \text{id}(A_A) = \infty. \quad (13)$$

Here and below a finite T -algebra means finite as a module over T ; its underlying set can be infinite.

7.1 Matrices and lengths

Choose decompositions (11) for each module in a finite collection. An O -linear map

$$f : \bigoplus_{j=1}^s O/(p^{e_j}) \longrightarrow \bigoplus_{i=1}^r O/(p^{b_i})$$

is represented by a matrix $F = (F_{ij})$ over O , with entries in row i read modulo p^{b_i} . It is well defined precisely when

$$p^{e_j} F_{ij} = p^{b_i} c_{ij} \quad (1 \leq i \leq r, 1 \leq j \leq s) \quad (14)$$

for suitable $c_{ij} \in O$. Likewise an equality of maps can be expressed by requiring each row of their matrix difference to be a multiple of its target relation p^{b_i} . Recording the quotients turns these conditions into polynomial equalities over O .

The matrix from O^{r+s} to O^r obtained by adjoining the target relations to F is

$$Q_f = [\text{diag}(p^{b_1}, \dots, p^{b_r}) \mid F]. \quad (15)$$

Its cokernel is the cokernel of f . The first r columns have full rank over the fraction field of O . Smith reduction therefore gives invertible matrices U of size r and V of size $r + s$ such that

$$UQ_fV = [\text{diag}(p^{a_1}, \dots, p^{a_r}) \mid 0], \quad 0 \leq a_i \leq N. \quad (16)$$

The bound follows because the cokernel is killed by p^N . One obtains this diagonal form by choosing a nonzero entry of least valuation, moving it to the first position, and clearing its row and column: it divides every entry over a valuation ring. Repeat on the remaining submatrix, and absorb unit factors by multiplying rows by units. The resulting row and column operations are invertible.

Record U, V , their inverses, and (16). Suppose that these coefficients and identities are realized over $T' = O'/(p^N)$, where O' is another discrete valuation ring with uniformizer p . Replacing each cyclic summand by $O'/(p^{e_j})$ defines the new modules and maps. Then

$$\ell_{O'}(\text{coker } f') = \sum_i a_i, \quad \ell_{O'}(\text{im } f') = \sum_i b_i - \sum_i a_i. \quad (17)$$

Indeed the recorded inverses give invertible row and column operations over T' , and the cokernel of multiplication by p^{a_i} on T' is $O'/(p^{a_i})$. At $a_i = 0$ this summand is zero; at $a_i = N$ it is T' . If the target is zero, both lengths are zero and the matrices are empty.

For consecutive maps $L_1 \xrightarrow{f} L_2 \xrightarrow{g} L_3$ with $gf = 0$, length additivity gives

$$\ell_O(\ker g / \text{im } f) = \ell_O(L_2) - \ell_O(\text{im } g) - \ell_O(\text{im } f). \quad (18)$$

Thus the identities and Smith matrices for the two maps retain the length of this cohomology after realization over T' . In particular they retain exactness and nonzero cohomology. An initial zero map tests injectivity; a final zero map tests surjectivity.

7.2 Algebras, duals, and Hom complexes

A multiplication on a module (11) is determined by the products of its chosen cyclic generators. Well-definedness in each argument, associativity, and the identity element are finitely many congruences in these coefficients. Module actions, commuting left and right actions, and linearity of specified maps are expressed in the same way. Include witnesses as in (14) for every congruence. These equations then remain valid after realization over T' .

Use the cyclic decomposition of A and (12) for $D_T A$. Record its two actions together with the equations

$$\langle a\lambda b, x \rangle = \langle \lambda, bxa \rangle \quad (a, b, x \in A, \lambda \in D_T A).$$

It suffices to impose these equations on the cyclic generators. The pairing remains perfect over T' , by the cyclic calculation in Lemma 7.1. The recorded actions consequently identify the resulting module with $D_{T'} A'$ as a bimodule.

For a finitely generated projective left module P_i , choose an integer $m_i \geq 0$ and A -linear maps

$$P_i \xrightarrow{\iota_i} A^{m_i} \xrightarrow{\pi_i} P_i, \quad \pi_i \iota_i = 1.$$

Their equations preserve projectivity. To preserve Ext we also describe the actual Hom terms. Put $E_i = \iota_i \pi_i$, an idempotent endomorphism of A^{m_i} . For any left module X , identify $\text{Hom}_A(A^{m_i}, X)$ with X^{m_i} by evaluation on the free generators. For an A -linear map F between free modules, write F^* for precomposition on these Hom modules. Explicitly, take

integers $a, b \geq 0$, and let $e_1, \dots, e_b$ be the target free generators of $F : A^a \rightarrow A^b$. Write its value on the j -th source generator as $\sum_{i=1}^b f_{ij}e_i$, with $f_{ij} \in A$. Then

$$F^* : X^b \longrightarrow X^a, \quad (F^*(x))_j = \sum_{i=1}^b f_{ij}x_i \quad (1 \leq j \leq a).$$

Multiplication denotes the given left action on X .

The map $h \mapsto h\pi_i$ identifies $\text{Hom}_A(P_i, X)$ with $\text{im } E_i^* \subset X^{m_i}$; its inverse is restriction along ι_i . Choose a cyclic presentation H_i for this image and maps

$$H_i \xrightarrow{u_i} X^{m_i} \xrightarrow{v_i} H_i, \quad v_i u_i = 1, \quad u_i v_i = E_i^*. \quad (19)$$

If $\partial_i : P_i \rightarrow P_{i-1}$ is a differential, define the map between the corresponding free modules by

$$F_i = \iota_{i-1} \partial_i \pi_i : A^{m_i} \longrightarrow A^{m_{i-1}}.$$

The Hom differential $\delta_i : H_{i-1} \rightarrow H_i$ is

$$\delta_i = v_i F_i^* u_{i-1}, \quad u_i \delta_i = F_i^* u_{i-1}. \quad (20)$$

For verification, if $h : P_{i-1} \rightarrow X$, then

$$(h\pi_{i-1})F_i = h\partial_i \pi_i.$$

Moreover $F_i E_i = F_i$, so $F_i^* u_{i-1}$ lies in $\text{im } E_i^*$, proving the second identity in (20). Record (19)–(20) as well as the Smith data for the required differentials. They identify the realized complex with the Hom complex over A' and preserve its specified cohomology lengths. For right modules use A^{op} .

8 Coefficient rings and finite specialization

The auxiliary rings and algebras in Lemmas 8.1–8.7 are commutative. For a prime p , write $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : a, b \in \mathbb{Z}, p \nmid b\}$. An algebra S over a ring R is *formally smooth* if every R -algebra map $S \rightarrow B/J$, for an R -algebra B and a nilpotent ideal J , lifts to B . It suffices to treat square-zero ideals: apply that case successively to $B/J^{i+1} \rightarrow B/J^i$, for $i \geq 1$. The coefficient-ring and lifting constructions used here belong to classical commutative algebra; see [6, Sections 10.160 and 16.5].

Lemma 8.1. *Every field is formally smooth over its prime field. Every finitely generated field extension of $\mathbb{F}_p$ has a separating transcendence basis: it is finite separable over a purely transcendental subfield.*

Proof. First consider a field k of characteristic p . Write $k^{p^e} = \{a^{p^e} : a \in k\}$ for $e \geq 1$. A p -basis is a subset $\mathcal{B} \subset k$ whose monomials $\prod_{b \in \mathcal{B}} b^{i_b}$, with finite support and $0 \leq i_b < p$, form a basis of k over k^p . To obtain one, take a maximal subset with these monomials linearly independent over k^p . Such a subset exists by the maximality principle, since every linear relation involves finitely many elements. The span of its monomials is the field $k^p(\mathcal{B})$: each finite part is a finite-dimensional domain over k^p , hence a field. If $a \notin k^p(\mathcal{B})$, its minimal polynomial there is $X^p - a^p$, of degree p , so adjoining a preserves independence, a contradiction. Here the degree is p because $X^p - a^p$ has a single root in an algebraic closure and is reducible only if that root belongs to the coefficient field. Iterating this basis expansion shows that the same monomials with $0 \leq i_b < p^e$ form a basis over k^{p^e} : write

each exponent in base p and apply the Frobenius map $a \mapsto a^p$ to the preceding expansion. Independence follows at each step by the original basis.

Let $k \rightarrow B/J$ be a homomorphism of $\mathbb{F}_p$ -algebras, with J nilpotent. Choose $e \geq 1$ with $J^{p^e} = 0$. For $a \in k$, the p^e -th power of any lift of its image is independent of the lift. Addition and multiplication commute with these powers, so this defines a homomorphism $k^{p^e} \rightarrow B$. Choose arbitrary lifts in B of the images of the elements of $\mathcal{B}$. Their p^e -th powers have the prescribed values. The basis just proved gives the presentation

$$k \cong k^{p^e} [X_b \mid b \in \mathcal{B}] / (X_b^{p^e} - b^{p^e} \mid b \in \mathcal{B}).$$

Indeed, reduction by these relations gives precisely the basis monomials. Sending the generators to the chosen lifts therefore defines the desired map $k \rightarrow B$.

Now let k have characteristic zero and let $k \rightarrow B/J$ be a homomorphism of $\mathbb{Q}$ -algebras with J nilpotent. Choose a transcendence basis of $k/\mathbb{Q}$. Arbitrary lifts of its elements define a map from its polynomial ring; every nonzero polynomial maps to a unit modulo J , hence to a unit in B , so this extends to the rational function field. The remaining extension is algebraic separable. Over each finite subextension, a primitive element has a separable minimal polynomial. A lift u of its image can be corrected to a root: for a square-zero kernel replace u by $u - f(u)/f'(u)$, where f is its minimal polynomial, evaluated using the lifted coefficient map, and f' is its formal derivative. The derivative is a unit, and Taylor expansion proves the assertion. Iteration handles a nilpotent kernel. The root lifting a given root is unique, since the difference of two such roots times a unit is zero. Consequently the maps on finite subextensions agree on overlaps and give a map on k .

Finally let $L/\mathbb{F}_p$ be finitely generated, of transcendence degree $r \geq 0$. Choose any transcendence basis $u_1, \dots, u_r$. Frobenius identifies the finite degrees $[L : \mathbb{F}_p(u_1, \dots, u_r)]$ and $[L^p : \mathbb{F}_p(u_1^p, \dots, u_r^p)]$. The tower formula gives

$$[L : L^p] [L^p : \mathbb{F}_p(u_1^p, \dots, u_r^p)] = p^r [L : \mathbb{F}_p(u_1, \dots, u_r)],$$

so $[L : L^p] = p^r$. Thus a p -basis of L consists of r elements $t_1, \dots, t_r$. They are algebraically independent over $\mathbb{F}_p$. Indeed, a nonzero polynomial relation of least total degree can be grouped by its exponents modulo p as

$$\sum_{0 \leq i_1, \dots, i_r < p} t_1^{i_1} \cdots t_r^{i_r} g_{i_1, \dots, i_r}(t_1, \dots, t_r)^p = 0,$$

where the $g_{i_1, \dots, i_r}$ are polynomials in $\mathbb{F}_p[X_1, \dots, X_r]$, at least one of them nonzero. Each nonzero such polynomial has smaller total degree than the original relation. Independence over L^p forces all their values to vanish. A nonzero constant cannot vanish; any other nonzero polynomial gives a relation of smaller degree. Both alternatives are contradictions. Set $F = \mathbb{F}_p(t_1, \dots, t_r)$. The finite extension L/F satisfies $L = FL^p$. Here FL^{p^e} denotes the subfield of L generated by F and L^{p^e} , for $e \geq 1$. Taking powers and adjoining F inductively gives $L = FL^{p^e}$ for every $e \geq 1$. For some e , all p^e -th powers of a finite set of field generators are separable over F : remove the successive p -powers in their irreducible polynomials. Their powers generate FL^{p^e} , which is therefore separable over F . Since this field is L , the basis is separating. The empty basis means L is finite separable over $\mathbb{F}_p$. $\square$

Lemma 8.2. *Let $R \rightarrow S$ be flat, and let $I \subset R$ satisfy $I^2 = 0$. If S/IS is formally smooth over R/I , then S is formally smooth over R .*

Proof. Choose a polynomial algebra $P = R[x_t \mid t \in T]$ surjecting onto S , and denote its kernel by J . The index set T may be infinite; every polynomial uses finitely many variables.

Let $E = \bigoplus_{t \in T} S dx_t$ be the free S -module with indicated basis. Formal differentiation gives an S -linear map

$$d : J/J^2 \longrightarrow E, \quad f \bmod J^2 \longmapsto \sum_{t \in T} (\partial f / \partial x_t \bmod J) dx_t.$$

It is well defined by the product rule.

We first verify that S is formally smooth exactly when d has an S -linear left inverse. If S is formally smooth, lift the identity to a section $S \rightarrow P/J^2$. The difference between $P \rightarrow P/J^2$ and its composite through this section is an R -derivation with values in J/J^2 . It sends $f \in J$ to $f \bmod J^2$; its values on the variables define a left inverse of d . Conversely, given a left inverse q and a map $S \rightarrow B/K$ with $K^2 = 0$, lift all variables to obtain $P \rightarrow B$. The images of relations define an S -linear map $\eta : J/J^2 \rightarrow K$, where S acts on K through $S \rightarrow B/K$. Replace the image of each x_t by that image minus $\eta(q(dx_t))$. Taylor expansion modulo K^2 shows that the new image of every $f \in J$ is $\eta(f \bmod J^2) - \eta(q(d(f \bmod J^2))) = 0$. The corrected map factors through S , proving the criterion.

Flatness of S gives $J \cap IP = IJ$: tensor $0 \rightarrow J \rightarrow P \rightarrow S \rightarrow 0$ with R/I ; its left exactness follows from flatness of S , and its kernel identification is precisely this equality. Consequently the relation module for $P/IP \rightarrow S/IS$ is $J/(IJ + J^2)$. The criterion gives a left inverse to the reduction of d modulo I . Lift its values on the free basis of E to obtain an S -linear map $q_0 : E \rightarrow J/J^2$. The endomorphism $\Delta = 1 - q_0 d$ of J/J^2 has image in $I(J/J^2)$, so $\Delta^2 = 0$. Hence

$$((1 + \Delta)q_0)d = (1 + \Delta)(1 - \Delta) = 1.$$

The criterion applied again proves the assertion. $\square$

A *Cohen ring* for a field k of characteristic $p > 0$ is a complete discrete valuation ring of characteristic zero with maximal ideal generated by p and residue field k .

Lemma 8.3. *Every field of positive characteristic has a Cohen ring.*

Proof. Let the field be k of characteristic p . Begin with $\mathbb{Z}_p = \varprojlim_{m \geq 1} \mathbb{Z}/p^m \mathbb{Z}$, the ring of p -adic integers. Its nonzero elements are a power of p times a unit, as seen from the first nonzero digit; it is a complete discrete valuation ring with residue field $\mathbb{F}_p$. We construct a flat local $\mathbb{Z}_p$ -algebra C whose maximal ideal is pC and whose residue field is k . Well-order a set of generators of $k/\mathbb{F}_p$. Suppose a local algebra with residue field $K \subset k$ has been constructed. For a next generator transcendental over K , take a polynomial algebra in one variable t and localize at the prime ideal generated by p ; its residue field is $K(t)$. For an algebraic generator, lift its monic minimal polynomial to the current ring and take the quotient by this lift. The quotient is finite free, and is local with maximal ideal generated by p : it is integral over the old local ring and its quotient modulo p is the required field. At a limit stage take the union, or equivalently the direct limit, of the preceding rings. These extensions are flat local maps. They are injective because a flat local map is faithfully flat. Thus the construction preserves flatness over $\mathbb{Z}_p$, and gives the asserted C . In particular, multiplication by p is injective on C .

Set $O = \varprojlim_{m \geq 1} C/p^m C$. For each $n \geq 1$, projection onto $C/p^n C$ is surjective: a representative in C gives a compatible sequence with the prescribed component. Its kernel is $p^n O$: if a compatible sequence vanishes modulo p^n , divide its component modulo p^{m+n} by p^n to obtain a unique class modulo p^m . Injectivity of multiplication by p on C ensures uniqueness and compatibility. Thus $O/p^n O = C/p^n C$, and O is complete and separated for powers of p . If $px = 0$ in O , the component of x modulo p^{m+1} belongs to $p^m C/p^{m+1} C$, again by cancellation in C . Its component modulo p^m is zero for every m , so $x = 0$.

An element of O nonzero modulo p is a unit: lift its inverse modulo p and invert the remaining factor $1 - py$ by the convergent series $\sum_{j \geq 0} p^j y^j$. Every nonzero element therefore has the form $p^a u$, for a unique integer $a \geq 0$ and a unit u . Separation makes a finite. Since p is a nonzerodivisor, these factorizations show that a product of nonzero elements is nonzero. A nonzero ideal is generated by an element of least such exponent, so every nonzero ideal is a power of pO . It follows that O is a discrete valuation ring. It has characteristic zero because p is a nonzerodivisor and every integer prime to p is a unit. Its residue field is k . $\square$

Lemma 8.4. *Let R be a commutative Artinian local ring with residue field k . Equal characteristic means $\text{char } R = \text{char } k$; otherwise the characteristics are unequal. In equal characteristic R contains a copy of k and is finite-dimensional over it. In unequal characteristic, there is a Cohen ring O with residue field k and a homomorphism $O \rightarrow R$ inducing the identity on residue fields; R is finite as an O -module.*

Proof. Let $\mathfrak{m}$ be the maximal ideal and choose $h \geq 1$ with $\mathfrak{m}^h = 0$. In equal characteristic the prime field embeds in R . Starting from the identity of $k = R/\mathfrak{m}$, Lemma 8.1 successively lifts it to $k \rightarrow R/\mathfrak{m}^j$, for $2 \leq j \leq h$. Each kernel is square-zero, since $2(j-1) \geq j$. The last map is injective and is a section of the residue map.

In unequal characteristic, k has characteristic $p > 0$. Choose O by Lemma 8.3. For every $j \geq 1$, $O/p^j O$ is flat over $\mathbb{Z}/p^j \mathbb{Z}$: O is torsion-free, hence flat, over the principal ideal domain $\mathbb{Z}_{(p)}$, and passage to these quotients is base change. Lemmas 8.1 and 8.2, applied successively with the square-zero ideal (p^{j-1}) , show that $O/p^j O$ is formally smooth over $\mathbb{Z}/p^j \mathbb{Z}$. Since $p \in \mathfrak{m}$, the ring $R/\mathfrak{m}^j$ is an algebra over $\mathbb{Z}/p^j \mathbb{Z}$. Starting from the residue-field isomorphism, lift the compatible maps $O/p^j O \rightarrow R/\mathfrak{m}^j$ one step at a time. At step j use formal smoothness of $O/p^j O$ and compose the previous map with $O/p^j O \rightarrow O/p^{j-1} O$. At $j = h$ this gives $O \rightarrow R$.

In either case, denote the coefficient field or ring by C . Choose generators $x_1, \dots, x_s$ of $\mathfrak{m}$. Every element of R is a coefficient from C plus an element of $\mathfrak{m}$. Expand the coefficients of the latter in the same way, and repeat h times. The final remainder belongs to $\mathfrak{m}^h = 0$. Hence the monomials $x_1^{a_1} \cdots x_s^{a_s}$ with $a_i \geq 0$ and $\sum_i a_i < h$ span R over C . If $s = 0$, the single monomial is 1. This proves finiteness. $\square$

For a ring R , a *standard smooth presentation* means

$$S = R[x_1, \dots, x_m]/(f_1, \dots, f_c), \quad 0 \leq c \leq m,$$

where f_i are polynomials and the determinant of $(\partial f_i / \partial x_j)_{1 \leq i, j \leq c}$ is a unit in S . The derivatives are formal polynomial derivatives; for $c = 0$ the determinant is 1. An algebra with such a presentation will be called standard smooth.

Lemma 8.5. *A standard smooth algebra is formally smooth. Polynomial extensions and localizations at one element of a standard smooth algebra are standard smooth. A standard smooth algebra over R/I , for a nilpotent ideal $I \subset R$, is the reduction of a standard smooth R -algebra.*

Proof. For formal smoothness, first let the kernel J of a target surjection satisfy $J^2 = 0$. Choose lifts of the variable images. The values of the f_i lie in J . The displayed derivative matrix is invertible at these lifts, because its determinant is a unit modulo J . Correct the first c lifted variables by the negative of the inverse matrix times this vector of errors, and keep the other variables fixed. Taylor expansion has no terms of order two in the corrections, so all f_i now vanish. Successive square-zero lifts handle a nilpotent kernel.

Adjoining polynomial variables leaves the same invertible minor. For a localization S_g , represent g by a polynomial and adjoin a variable y with relation $gy - 1$. Use the original c

variables and y for the new minor. Its determinant is the old determinant times g , which is a unit in S_g . Reordering these variables first gives the required presentation.

For the last assertion, lift the defining polynomials from R/I to R , take their quotient algebra, and invert the determinant of the lifted minor. Its reduction is the original algebra, since the original determinant was already a unit. Adjoining an inverse by the preceding construction makes this a standard smooth presentation over R . $\square$

Lemma 8.6. *Let p be prime and $n \geq 1$. Let Λ be a flat commutative local $\mathbb{Z}/p^n\mathbb{Z}$ -algebra such that $\Lambda/p\Lambda$ is a field. Every homomorphism from a finitely presented commutative $\mathbb{Z}/p^n\mathbb{Z}$ -algebra to Λ factors through a standard smooth $\mathbb{Z}/p^n\mathbb{Z}$ -algebra.*

Proof. We prove the assertion by induction on n . For $n = 1$, the images of finitely many algebra generators lie in a finitely generated subfield L of Λ . By Lemma 8.1, there are algebraically independent elements $t_1, \dots, t_r$, with $r \geq 0$, such that $L/\mathbb{F}_p(t_1, \dots, t_r)$ is finite separable. The empty list gives $\mathbb{F}_p(t_1, \dots, t_r) = \mathbb{F}_p$. Choose a primitive element u and its monic minimal polynomial. After inverting a nonzero polynomial in the t_i , its coefficients lie in the resulting polynomial algebra. Adjoin u and invert the derivative of its minimal polynomial. The quotient embeds in L : division by the monic polynomial reduces any element to degree below its degree, and linear independence over $\mathbb{F}_p(t_1, \dots, t_r)$ makes the kernel zero. The invertible derivative gives a standard smooth subalgebra of L . Inverting finitely many additional nonzero elements includes all the specified images. Their relations hold in this subalgebra because it embeds in L . Thus the given map factors through it.

Suppose $n \geq 2$, and put $R = \mathbb{Z}/p^n\mathbb{Z}$ and $I = p^{n-1}R$, so $I^2 = 0$. The quotient Λ/IA is flat over R/I , by base change, and has the same residue field. We first show that every map $A \rightarrow \Lambda$, with A finitely presented over R , factors as

$$A \longrightarrow B/J \longrightarrow \Lambda, \quad B \text{ standard smooth over } R, \quad J \subseteq IB.$$

All ideals appearing in finite type R -algebras are finitely generated, since R is Noetherian.

By induction, $A/IA \rightarrow \Lambda/IA$ factors through a standard smooth R/I -algebra $\bar{B}$. Choose nonnegative integers s, t and a finite presentation

$$\bar{B} = (A/IA)[z_1, \dots, z_s]/(g_1, \dots, g_t).$$

Lift the g_j to $A[z_1, \dots, z_s]$, and lift the images of the z_i to Λ . The evaluated relations have the form $p^{n-1}\mu_j$, with $\mu_j \in \Lambda$. Consequently

$$A_1 = A[z_1, \dots, z_s, v_1, \dots, v_t]/(g_j - p^{n-1}v_j \mid 1 \leq j \leq t)$$

maps to Λ , with $v_j \mapsto \mu_j$, and $A_1/IA_1 = \bar{B}[v_1, \dots, v_t]$. Lemma 8.5 lifts this last standard smooth algebra to a standard smooth R -algebra B . Formal smoothness gives a map $B \rightarrow A_1$ inducing the chosen isomorphism modulo I . It is surjective: $A_1 = \text{im}(B) + IA_1$, and substitution of this equality into its last term, using $I^2 = 0$, gives $A_1 = \text{im}(B)$. Its kernel J is contained in IB , as required.

We now remove the relations in J . Let $\varphi : B \rightarrow \Lambda$ be the resulting map, and first consider one generator $h = p^{n-1}b$ of J , with $b \in B$. Flatness identifies the kernel of multiplication by p^{n-1} on Λ with $p\Lambda$, since that kernel on R is pR . Hence $\varphi(b) = p\lambda$ for some $\lambda \in \Lambda$. The algebra $B[x]/(b - px)$ maps to Λ by $x \mapsto \lambda$. Apply the preceding construction to factor this map through B'/J' , where B' is standard smooth and $J' \subseteq IB'$. Formal smoothness lifts $B \rightarrow B'/J'$ to $\alpha : B \rightarrow B'$. The composite with $B' \rightarrow \Lambda$ is exactly φ , because the latter map factors through B'/J' . If $\xi \in B'$ lifts the image of x , then $\alpha(b) - p\xi \in J'$. Therefore

$$\alpha(h) = p^{n-1}\alpha(b) \in p^n B' + p^{n-1} J' = 0.$$

The last equality uses $p^n = 0$ and $J' \subseteq p^{n-1}B'$. Apply this procedure successively to the finitely many generators of J . Previously killed generators remain zero, and the remaining images still belong to I times the current standard smooth algebra. The final map kills J , so it factors through B/J . Composing with $A \rightarrow B/J$ completes the induction. $\square$

Lemma 8.7. *Let O be a Cohen ring of residue characteristic p , and fix $N \geq 1$. Every finite collection of elements of O , together with finitely many polynomial equalities over $\mathbb{Z}_{(p)}$ that they satisfy, can be realized in $O'/(p^N)$ for a Cohen ring O' with finite residue field.*

Proof. The module O is torsion-free over the principal ideal domain $\mathbb{Z}_{(p)}$, hence flat. Base change shows that $O/(p^N)$ is flat over $\mathbb{Z}/p^N\mathbb{Z}$. Introduce one variable for each specified element and impose the specified polynomial equalities, with coefficients reduced modulo p^N . The resulting finitely presented $\mathbb{Z}/p^N\mathbb{Z}$ -algebra maps to $O/(p^N)$. Lemma 8.6 factors this map through a standard smooth $\mathbb{Z}/p^N\mathbb{Z}$ -algebra S .

The map $S \rightarrow O/pO$ shows that $S/pS \neq 0$. It is a finitely generated $\mathbb{F}_p$ -algebra, so a maximal ideal has finite residue field F , by Lemma 4.1. Choose a Cohen ring O' with residue field F , using Lemma 8.3. Formal smoothness over $\mathbb{Z}/p^N\mathbb{Z}$ lifts $S \rightarrow F$ successively to $S \rightarrow O'/(p^j)$, for $2 \leq j \leq N$. The kernel of $O'/(p^j) \rightarrow O'/(p^{j-1})$ is square-zero, since $2(j-1) \geq j$. For $N = 1$, the map to F itself is the required map. The last lift realizes all the specified equalities. The ring $O'/(p^N)$ is finite: its successive quotients $p^j O' / p^{j+1} O'$, for $0 \leq j < N$, are one-dimensional vector spaces over F . $\square$

Proposition 8.8. *Let O be a Cohen ring of residue characteristic p , and let A be a nonzero unital O -algebra of finite length as an O -module. Suppose*

$$\mathrm{id}_A(A) = d < \infty, \quad \mathrm{id}(A_A) = \infty.$$

There is a finite ring A' whose cardinality is a power of p such that

$$\mathrm{id}_{A'}(A') = d, \quad \mathrm{id}(A'_A) = \infty, \quad \mathrm{char} A' = \mathrm{char} A.$$

Proof. Let $N \geq 1$ be the least integer with $p^N A = 0$. Such an integer exists by finite length. Then $\mathrm{char} A = p^N$, and the cyclic decomposition of A contains a summand $O/(p^N)$. Put $T = O/(p^N)$. Every finitely generated A -module has finite length over O .

By Lemma 7.1, choose a right projective resolution

$$0 \longrightarrow P_d \longrightarrow \cdots \longrightarrow P_0 \longrightarrow (D_T A)_A \longrightarrow 0$$

with finitely generated terms. Choose the projective splittings from Section 7, and Smith matrices for all its maps, including the augmentation. Together with zero compositions, their lengths encode exactness at every term, including injectivity at P_d . If $d > 0$, Lemma 2.1 over A^{op} gives

$$\mathrm{Ext}_{A^{\mathrm{op}}}^d(D_T A, A) \neq 0.$$

Record the Hom terms, differentials, and their Smith matrices that compute this group from the same resolution; the outgoing map at degree d is zero.

On the left, $\mathrm{pd}_A(D_T A) = \infty$. The argument of Lemma 2.3 gives a finitely generated left module X such that $\mathrm{Ext}_A^{d+1}(D_T A, X) \neq 0$. Take a projective resolution of ${}_A(D_T A)$ through degree $d + 2$, with finitely generated terms. Retain exactness at the augmentation and degrees $0, \dots, d + 1$, using Smith matrices. Such an exact prefix computes Ext in degree $d + 1$: it extends to a projective resolution by taking finite free covers of its leftmost kernel. Record the Hom terms in degrees $d, d + 1, d + 2$, their two differentials, and Smith matrices.

Choose cyclic decompositions for all these modules. Record the multiplication, unit, actions, projective splittings, dual pairing, and Hom identities as described in Section 7. Choose representatives in O for their coefficients. Include the division witnesses for every congruence and the inverse matrices in every Smith identity. This is a finite set of elements satisfying finitely many polynomial equalities over $\mathbb{Z}_{(p)}$.

Apply Lemma 8.7. Set $T' = O'/(p^N)$ for its resulting Cohen ring. Replacing each $O/(p^e)$ by $O'/(p^e)$ and using the realized multiplication and unit gives an associative unital algebra A' . If the exponents in A are $e_1, \dots, e_r$, then

$$|A'| = |O'/pO'|^{\sum_{j=1}^r e_j}.$$

The dual pairing identifies the realized dual module with $D_{T'}A'$. Projective splittings and (18) give a right projective resolution of this module of length d . Hence $\text{id}_{(A')A'} \leq d$.

Equations (19)–(20) identify the realized Hom complexes with those computing Ext over A' . Let X' be the realized left module. Their cohomology lengths are unchanged. Therefore

$$\text{Ext}_{A'}^{d+1}(D_{T'}A', X') \neq 0,$$

Equation (13) gives $\text{id}(A'_{A'}) = \infty$. For $d > 0$, the nonzero degree- d right Ext group also survives, proving $\text{id}_{(A')A'} = d$. For $d = 0$, the upper bound already gives equality.

The summand with exponent N gives $p^{N-1}A' \neq 0$, while $p^N A' = 0$. Since A' is unital, these statements give $\text{char } A' = p^N$. $\square$

9 Artin algebras and finite rings

An Artin algebra is finite over its center, which is a commutative Artinian ring. To see this from the definition, choose a commutative Artinian coefficient ring R over which A is finite. The center $Z(A)$ is an R -submodule of A , hence has finite R -length and satisfies the descending chain condition on ideals. The same finite generating set for A over R generates it over $Z(A)$.

Lemma 9.1. *Suppose A is an Artin algebra whose center is local, with $\text{id}_{(A)A} = d < \infty$ and $\text{id}(A_A) = \infty$. There exists a finite ring A' of prime-power cardinality with these same two self-injective dimensions. If the residue field of $Z(A)$ has characteristic $p > 0$, the cardinality of A' can be chosen to be a power of p , and $\text{char } A' = \text{char } A$.*

Proof. Let $R = Z(A)$, let $\mathfrak{m}$ be its maximal ideal, and put $k = R/\mathfrak{m}$. Lemma 8.4 supplies a coefficient field $\Lambda = k$ in equal characteristic or a Cohen ring Λ in unequal characteristic, with R finite over Λ . Since A is finite over R , it is finite over Λ . In equal characteristic, A is a finite-dimensional k -algebra, and Proposition 4.2 applies. If k has characteristic p , its finite residue-field output has the same characteristic p .

In unequal characteristic the residue characteristic is $p > 0$. The image of p lies in $\mathfrak{m}$, so some p^N kills A . The finite Λ -module A is therefore of finite length. Proposition 8.8 gives the result and preserves its characteristic. $\square$

Proof of Theorem 1.1. Decompose the commutative Artinian ring $Z(A)$ into its local factors. Its central orthogonal idempotents give $A = A_1 \times \dots \times A_s$, with each $Z(A_i)$ local. Every module decomposes by these idempotents; Hom and exact sequences decompose componentwise. Applying this to injective resolutions and to their components gives

$$\text{id}_{(A)A} = \max_i \text{id}_{(A_i)A_i}, \quad \text{id}(A_A) = \max_i \text{id}((A_i)_{A_i}).$$

If the first is $d < \infty$ and the second is infinite, choose a factor A_i with infinite right self-injective dimension. Its left dimension is an integer $d_i \leq d$. Lemma 9.1 produces a finite ring A'_i of cardinality a power of a prime p , with left dimension d_i and infinite right dimension.

To retain d , consider the finite-dimensional $\mathbb{F}_p$ -algebra $B_d = \mathbb{F}_p Q_d / J^2$, where Q_d is the directed path $1 \rightarrow 2 \rightarrow \cdots \rightarrow d+1$, $\mathbb{F}_p Q_d$ is its path algebra (products concatenate paths from left to right), and J is the ideal generated by arrows. For $d = 0$ this means $B_0 = \mathbb{F}_p$. The ideal $\text{rad } B_d$, generated by the images of the arrows, has square zero and quotient $\mathbb{F}_p^{d+1}$. It is the Jacobson radical: every nilpotent ideal annihilates simple modules, and the quotient is a product of fields. Let e_j be the trivial path at vertex j , and let $S_j = e_j B_d / e_j \text{rad } B_d$ be the corresponding simple right module, for $1 \leq j \leq d+1$. For $j \leq d$, the module $e_j \text{rad } B_d$ is S_{j+1} , whereas $e_{d+1} B_d = S_{d+1}$. Thus

$$0 \longrightarrow e_{d+1} B_d \longrightarrow \cdots \longrightarrow e_j B_d \longrightarrow S_j \longrightarrow 0$$

is a projective resolution of length $d+1-j$. The map from $e_{t+1} B_d$ to $e_t B_d$ sends e_{t+1} to the arrow $t \rightarrow t+1$; its image is $e_t \text{rad } B_d$, and its kernel is $e_{t+1} \text{rad } B_d$. The augmentation is the quotient map. For $j = d+1$ it is an isomorphism. Its differentials have image in the radicals of their targets; applying $\text{Hom}_{B_d^{\text{op}}}(-, S_{d+1})$ makes the differentials zero and leaves a nonzero term in degree $d+1-j$. Hence $\text{pd}_{B_d^{\text{op}}} S_j = d+1-j$. Every finitely generated right module has a composition series, so the long exact Ext sequence bounds its projective dimension by d . Reversing the path gives the identical bound and a simple module of projective dimension d on the left. In particular the two dual regular modules have projective dimension at most d , so both self-injective dimensions are at most d . Lemma 2.1, applied to the simple module of projective dimension d on each side, gives the reverse bounds. Thus both self-injective dimensions of B_d equal d .

The ring $A' = A'_i \times B_d$ has cardinality a power of p , left self-injective dimension d , and infinite right self-injective dimension. This proves the quantitative assertion. Taking opposite rings handles the other orientation of a failure. Every finite ring is an Artin algebra over its finite prime subring. Thus (1) implies (2), and (2) implies (3). The construction just given shows that failure of (1) implies failure of (3), completing the equivalence. $\square$

Corollary 9.2. *Failure of the Gorenstein symmetry conjecture for Artin algebras is semidecidable.*

Proof. Enumerate primes p , positive integers r , and nonnegative integers d , so that every triple is eventually reached. On a set of p^r elements, enumerate addition and multiplication tables and distinguished zero and identity elements. The ring axioms and $0 \neq 1$ are finite tests. For each ring A obtained, its characteristic is p^N for some $N \geq 1$. Set $T = \mathbb{Z}/p^N \mathbb{Z}$ and compute $D_T A = \text{Hom}_T(A, T)$ by enumerating maps of finite sets and testing additivity and scalar linearity. Its two actions are given by (2).

For any finite module M , a free surjection is obtained by taking one free generator for every element of M and sending it to that element. Its kernel is a finite set and can be computed exactly. Iterate d times on each dual regular module. The last kernel is projective precisely when its finite free surjection has a module-linear section; enumeration of all maps between the two finite sets decides this. At $d = 0$, test the dual module itself. At a zero kernel, the zero section suffices. Thus both inequalities $\text{pd}(D_T A) \leq d$ are decidable.

Halt when exactly one holds. Lemma 7.1 and (13) give the finite and infinite sides on that same ring. If an Artin counterexample exists, Theorem 1.1 supplies a finite ring of prime-power cardinality with the same orientation, and the enumeration reaches its tables and a suitable d . $\square$

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