

Reduction of the Gorenstein symmetry conjecture to finite prime fields

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Abstract

We prove that the Gorenstein symmetry conjecture for finite-dimensional algebras over arbitrary fields is equivalent to its restriction to the finite prime fields. A counterexample specializes to a finite field without changing its dimension or its finite self-injective dimension. In characteristic zero, a counterexample yields counterexamples of one fixed dimension over every prime field outside a finite set.

1 Introduction

All algebras in this paper are nonzero, associative, and unital. Coefficient rings are commutative and act centrally; ring maps preserve the identity. For a finite-dimensional algebra A over a field k , write ${}_A A$ and A_A for its left and right regular modules, and id for injective dimension in the corresponding category of all unital modules. The Gorenstein symmetry conjecture asserts

$$\text{id}({}_A A) < \infty \iff \text{id}(A_A) < \infty. \quad (1)$$

When both dimensions are finite, they are equal. We give the finite-dimensional argument for this fact in Section 2.

Qin and Han reduce (1) to 2-derived-simple algebras [2, Section 5]. Zhang proves invariance under excellent extensions, including scalar extension along finite separable field extensions [5, arXiv version, Example 2.2(3) and Theorem 3.2]. For the finitistic dimension conjecture, Igusa gives a reduction between the algebraic closure of a prime field and finite fields of the same characteristic, with a fixed bound on Loewy length [1, Theorem 5.1]. Here we reduce the Gorenstein symmetry conjecture over arbitrary fields to finite prime fields.

For a prime p , let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements.

Theorem 1.1. *The following statements are equivalent.*

1. *Every finite-dimensional algebra over every field satisfies the Gorenstein symmetry conjecture.*
2. *Every finite-dimensional algebra over every finite field satisfies the Gorenstein symmetry conjecture.*
3. *For every prime number p , every finite-dimensional $\mathbb{F}_p$ -algebra satisfies the Gorenstein symmetry conjecture.*

2020 *Mathematics Subject Classification.* Primary 16E10; Secondary 16E05, 16G10.

Keywords. Gorenstein symmetry conjecture, self-injective dimension, finite field, specialization, finite certificate.

For a fixed prime p , the conjecture for all fields of characteristic p is equivalent to the conjecture for $\mathbb{F}_p$.

The proof records a counterexample by finitely many matrices. A projective resolution of the right dual regular module gives a finite upper bound $d \geq 0$ on the left self-injective dimension. A nonzero Ext group in degree $d + 1$ for the left dual regular module forces the right self-injective dimension to be infinite. We construct a finite free coefficient model preserving both conditions and the exact value d at every field-valued fiber (Theorem 3.2). In characteristic zero this gives the following arithmetic refinement.

For a positive integer N , the ring $\mathbb{Z}[1/N] = \{a/N^j : a \in \mathbb{Z}, j \geq 0\}$ is viewed as a subring of $\mathbb{Q}$.

Theorem 1.2. *Suppose that k has characteristic zero and that a finite-dimensional k -algebra A satisfies*

$$\mathrm{id}({}_A A) = d < \infty, \quad \mathrm{id}(A_A) = \infty.$$

There exist positive integers m, N and an associative unital $\mathbb{Z}[1/N]$ -algebra $\mathcal{C}$, free of rank m , with the following property. Write $\mathcal{C}_{\mathbb{Q}} = \mathcal{C} \otimes_{\mathbb{Z}[1/N]} \mathbb{Q}$ and, for a prime $p \nmid N$, $\mathcal{C}_p = \mathcal{C} \otimes_{\mathbb{Z}[1/N]} \mathbb{F}_p$. Then

$$\begin{aligned} \mathrm{id}({}_{\mathcal{C}_{\mathbb{Q}}} \mathcal{C}_{\mathbb{Q}}) &= d, & \mathrm{id}((\mathcal{C}_{\mathbb{Q}})_{\mathcal{C}_{\mathbb{Q}}}) &= \infty, \\ \mathrm{id}({}_{\mathcal{C}_p} \mathcal{C}_p) &= d, & \mathrm{id}((\mathcal{C}_p)_{\mathcal{C}_p}) &= \infty \quad (p \nmid N). \end{aligned}$$

In particular, every $\mathcal{C}_p$ has the same dimension m over $\mathbb{F}_p$.

The integer m can differ from $n = \dim_k A$, because restriction of scalars from a number field increases vector-space dimension. There is also a version retaining n , with finite extensions of $\mathbb{F}_p$ of uniformly bounded degree (Proposition 5.2). Theorems 1.1 and 1.2 are proved in Sections 4 and 5, respectively.

AI contributions. ChatGPT and Codex generated and revised the research arguments and performed AI proof review. Codex prepared the manuscript and checked the proof dependencies and references.

2 Finite homological certificates

All modules used in constructions below are finite-dimensional over the indicated field. We write $\mathrm{pd}_A M$ and $\mathrm{Ext}_A^j(M, S)$ for left A -modules; right modules are regarded as left modules over the opposite algebra A^{op} , whose multiplication is reversed. Projective and injective dimensions are the least lengths of projective and injective resolutions, respectively, or infinity if no finite resolution exists. We assign both dimensions zero to the zero module. For $j \geq 0$, $\mathrm{Ext}_A^j(M, S)$ is the degree- j cohomology of $\mathrm{Hom}_A(P_{\bullet}, S)$ for a projective resolution $P_{\bullet}$ of M ; in particular $\mathrm{Ext}_A^0(M, S) = \mathrm{Hom}_A(M, S)$.

For a finite-dimensional vector space V , set $D_k V = \mathrm{Hom}_k(V, k)$. For a left module X , its dual is a right module with $(\lambda a)(x) = \lambda(ax)$; for a right module X , its dual is a left module with $(a\lambda)(x) = \lambda(xa)$, where $a \in A$, $x \in X$, and $\lambda \in D_k X$. The dual of the regular module is an A -bimodule with action

$$(a\lambda b)(x) = \lambda(bxa) \quad (a, b, x \in A, \lambda \in D_k A). \quad (2)$$

For any left A -module X , evaluation at $1 \in A$ gives

$$\mathrm{Hom}_A(X, D_k A) \cong \mathrm{Hom}_k(X, k), \quad f \longmapsto (x \longmapsto f(x)(1)).$$

The inverse sends a linear functional $\ell : X \rightarrow k$ to the map $x \mapsto (a \mapsto \ell(ax))$. This map is A -linear by (2). Since vector-space duality is exact, $D_k A$ is injective in the category of all left modules. The right-handed statement follows by replacing A by A^{op} .

Every finite left module X embeds into a finite direct sum of copies of $D_k A$: dualize a finite free right-module surjection onto $D_k X$ and identify $D_k D_k X$ with X by evaluation. Repeating this construction on the finite cokernels gives an injective resolution with finite terms. A finite injective module is a direct summand of such a finite sum, and its dual is therefore projective. Conversely, the dual of a finite projective module is a direct summand of a finite sum of copies of $D_k A$, hence injective. If $\text{id}_A X \leq d$ in the full module category, dimension shifting along this resolution makes its d -th cokernel injective: its first Ext against any module is $\text{Ext}_A^{d+1}(-, X) = 0$. Here the zeroth cokernel means X itself. Dualizing the resulting finite resolution, and conversely dualizing finite projective resolutions, proves

$$\text{id}_A(A) = \text{pd}_{A^{\text{op}}}(D_k A), \quad \text{id}(A_A) = \text{pd}_A(D_k A). \quad (3)$$

Finite projective resolutions can be taken with finite terms: successive finite free surjections have finite kernels, and the kernel at the projective-dimension bound is projective by dimension shifting.

Lemma 2.1. *If a nonzero finite-dimensional left A -module M has $\text{pd}_A M = e < \infty$, then $\text{Ext}_A^e(M, A) \neq 0$.*

Proof. For $e = 0$, choose a positive integer t and embed the nonzero projective module M as a direct summand of A^t . At least one coordinate map $M \rightarrow A$ is nonzero. For $e > 0$, take a projective resolution of length e with finite terms, and write $u : P_e \rightarrow P_{e-1}$ for its first map. Choose a positive integer t and A -linear maps $\iota : P_e \rightarrow A^t$ and $\pi : A^t \rightarrow P_e$ with $\pi\iota = 1$. If $\text{Ext}_A^e(M, A) = 0$, every coordinate of ι extends across u . The extensions form a map $g : P_{e-1} \rightarrow A^t$ satisfying $gu = \iota$. Thus $(\pi g)u = 1$, so u is a split monomorphism. Write $P_{e-1} = u(P_e) \oplus Q$. The next differential vanishes on $u(P_e)$, and its restriction to the projective summand Q is injective onto the same image, by exactness. Replacing the first two terms by Q gives a resolution of length $e - 1$, a contradiction. $\square$

Lemma 2.2. *Let $d \geq 0$ be an integer. If $\text{id}_A(A) \leq d$, every finite-dimensional left A -module of finite projective dimension has projective dimension at most d . Consequently,*

$$\text{id}_A(A) \leq d < \text{pd}_A(D_k A) \implies \text{id}(A_A) = \infty.$$

Proof. For a nonzero module of finite projective dimension e , Lemma 2.1 gives a nonzero $\text{Ext}_A^e(M, A)$. The assumed injective dimension forces $e \leq d$. Apply this to $D_k A$ if its left projective dimension is finite, and then use (3). $\square$

In particular, if both regular injective dimensions are finite, Lemma 2.2 and its opposite-algebra version bound each by the other. This proves the equality used in the introduction.

Lemma 2.3. *Let $d \geq 0$ be an integer. If M is a finite-dimensional left A -module and $\text{pd}_A M > d$, there is a simple left A -module S with $\text{Ext}_A^{d+1}(M, S) \neq 0$. One can choose $\dim_k S \leq \dim_k A$.*

Proof. Starting from $\Omega^0 M = M$, choose finite free surjections $F_j \rightarrow \Omega^j M$ and define $\Omega^{j+1} M$ to be their kernels, for $0 \leq j < d$. Thus $0 \rightarrow \Omega^{j+1} M \rightarrow F_j \rightarrow \Omega^j M \rightarrow 0$ is exact. For any left A -module T , the long exact Ext sequence gives

$$\text{Ext}_A^i(\Omega^{j+1} M, T) \cong \text{Ext}_A^{i+1}(\Omega^j M, T) \quad (i \geq 1)$$

because F_j is projective. The module $\Omega^d M$ is nonprojective, since otherwise this sequence would give a projective resolution of M of length d . Choose a finite free surjection $F \rightarrow \Omega^d M$ with kernel T . It does not split, so $\text{Ext}_A^1(\Omega^d M, T) \neq 0$: in the long exact sequence obtained by applying $\text{Hom}_A(\Omega^d M, -)$, the connecting map sends the identity to zero exactly when it lifts to F . Iterating the displayed isomorphism gives $\text{Ext}_A^{d+1}(M, T) \neq 0$, also for $d = 0$.

The finite module T has a composition series, obtained by taking maximal proper submodules until the dimension reaches zero. If $\text{Ext}_A^{d+1}(M, S) = 0$ for every simple factor S of this series, the long exact sequence in the second variable, applied successively to its short exact sequences, would give $\text{Ext}_A^{d+1}(M, T) = 0$. Some simple factor therefore has the required nonvanishing. For any nonzero $s \in S$, the map $A \rightarrow S$, $a \mapsto as$, is surjective, proving the dimension bound. $\square$

For an integer $d \geq 0$, an algebra with left self-injective dimension at most d and right self-injective dimension infinite is characterized by two finite pieces of homological data:

$$\begin{aligned} 0 \longrightarrow P_d \longrightarrow \cdots \longrightarrow P_0 \longrightarrow (D_k A)_A \longrightarrow 0 \quad \text{with each } P_i \text{ projective,} \\ \text{Ext}_A^{d+1}(D_k A, S) \neq 0 \quad \text{for some finite-dimensional left module } S. \end{aligned} \quad (4)$$

Existence follows from (3) and Lemma 2.3; sufficiency follows from Lemma 2.2. To preserve an exact finite value $d > 0$, we will also retain

$$\text{Ext}_{A^{\text{op}}}^d(D_k A, A) \neq 0, \quad (5)$$

which follows from Lemma 2.1 applied over A^{op} .

3 Specialization of the certificates

We first describe the finite linear algebra used to preserve the Ext conditions. For finite-dimensional left modules M, S , the unnormalized bar resolution has terms

$$B_j = A \otimes_k A^{\otimes_k j} \otimes_k M \quad (j \geq 0), \quad B_{-1} = M.$$

The initial factor carries the left A -action, and $A^{\otimes_k 0} = k$. The augmentation $\partial_0 : B_0 \rightarrow M$ is $a_0 \otimes m \mapsto a_0 m$. For $j \geq 1$, $a_0, \dots, a_j \in A$, and $m \in M$, the differential $\partial_j : B_j \rightarrow B_{j-1}$ is

$$\begin{aligned} \partial_j(a_0 \otimes \cdots \otimes a_j \otimes m) &= \sum_{i=0}^{j-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_j \otimes m \\ &\quad + (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j m. \end{aligned}$$

For $j \geq 2$, the terms of $\partial_{j-1} \partial_j$ pair according to two original positions $i < \ell$, where $0 \leq i < \ell \leq j$. Multiplying first at ℓ and then at i has sign $(-1)^{i+\ell}$; multiplying first at i and then at $\ell-1$ has sign $(-1)^{i+\ell-1}$. The tensors agree by associativity, including the last module action, so they cancel. At the augmentation, $\partial_0 \partial_1(a_0 \otimes a_1 \otimes m) = (a_0 a_1)m - a_0(a_1 m) = 0$.

Define k -linear maps $h_j : B_j \rightarrow B_{j+1}$ by $h_j(x) = 1 \otimes x$, for $j \geq -1$. For $j \geq 0$, the first term of $\partial_{j+1}(1 \otimes x)$ is x . Each remaining term is 1 tensored with a term of $\partial_j x$, with its sign reversed. Hence

$$\partial_{j+1} h_j(x) = x - h_{j-1} \partial_j(x), \quad \partial_0 h_{-1}(m) = m.$$

These identities give exactness: if $\partial_j x = 0$, then $x = \partial_{j+1} h_j(x)$, and the augmentation is surjective. Each B_j is a finite free left A -module, with a basis coming from a k -basis of $A^{\otimes_k j} \otimes_k M$. Thus $B_\bullet$ is a projective resolution.

Applying $\text{Hom}_A(-, S)$ identifies its cochains with

$$C^j = \text{Hom}_k(A^{\otimes_k j} \otimes_k M, S) \quad (j \geq 0).$$

The identification sends an A -linear map to its restriction to $1 \otimes A^{\otimes_k j} \otimes_k M$; its inverse sends c to $a_0 \otimes v \mapsto a_0 c(v)$. For $c \in C^j$, its differential $\delta^j : C^j \rightarrow C^{j+1}$ is

$$\begin{aligned} (\delta^j c)(a_1, \dots, a_{j+1}, m) &= a_1 c(a_2, \dots, a_{j+1}, m) \\ &\quad + \sum_{i=1}^j (-1)^i c(a_1, \dots, a_i a_{i+1}, \dots, a_{j+1}, m) \\ &\quad + (-1)^{j+1} c(a_1, \dots, a_j, a_{j+1} m). \end{aligned} \quad (6)$$

For $j = 0$ the sum is empty, so $(\delta^0 c)(a, m) = ac(m) - c(am)$. For an integer $q \geq 0$, we use $H^q(C^\bullet) = \ker \delta^q / \text{im } \delta^{q-1}$, with $\delta^{-1} = 0$. It equals $\text{Ext}_A^q(M, S)$. In fixed bases, all these matrices have entries given by integer polynomials in the multiplication and action coefficients.

Lemma 3.1. *Let $q \geq 1$ be an integer, let R be a subring of a field k , and let $C_R^{q-1} \rightarrow C_R^q \rightarrow C_R^{q+1}$ be a complex of finite free R -modules. If its degree- q cohomology after tensoring with k is nonzero, the same holds after tensoring with any field F along a unital map $R \rightarrow F$.*

Proof. Let r, s be the ranks over k of the incoming and outgoing differentials. Every minor of size $r + 1$, respectively $s + 1$, is zero in k , and therefore zero in R . The corresponding ranks over F are at most r, s . Here $\text{rank}_R C_R^q$ is the number of elements in a free basis of the middle term. The complex identity remains valid over F , so the incoming image lies in the outgoing kernel. Rank-nullity gives

$$\begin{aligned} \dim_F H^q(C_R^\bullet \otimes_R F) &\geq \text{rank}_R C_R^q - r - s \\ &= \dim_k H^q(C_R^\bullet \otimes_R k) > 0. \end{aligned} \quad \square$$

Theorem 3.2. *Let A be an n -dimensional algebra over a field k , and suppose*

$$\text{id}(A) = d < \infty, \quad \text{id}(A_A) = \infty.$$

There exist a subring $R \subset k$, finitely generated as a unital ring, and an R -algebra A_R , free of rank n , such that $A_R \otimes_R k \cong A$ and, for every unital map $R \rightarrow F$ to a field, the fiber $A_F = A_R \otimes_R F$ satisfies

$$\dim_F A_F = n, \quad \text{id}(A_F) = d, \quad \text{id}((A_F)_{A_F}) = \infty.$$

Proof. Fix a basis $a_1 = 1, a_2, \dots, a_n$ of A . Choose the resolution and the module S in (4), and fix vector-space bases for the P_i and S , and use the dual basis to $a_1, \dots, a_n$ for $D_k A$. For every $0 \leq i \leq d$, choose a nonnegative integer t_i and right A -linear maps

$$P_i \xrightarrow{\iota_i} A^{t_i} \xrightarrow{\pi_i} P_i, \quad \pi_i \iota_i = 1_{P_i}. \quad (7)$$

Regard the augmented resolution as a finite complex of vector spaces. Write $P_{-1} = D_k A$ and set $P_i = 0$ for $i < -1$ or $i > d$. Let $\partial_i : P_i \rightarrow P_{i-1}$ be its maps for $0 \leq i \leq d$, and set $\partial_{-1} = 0$ and $\partial_{d+1} = 0$. Its exactness gives k -linear maps $h_i : P_i \rightarrow P_{i+1}$ for $-1 \leq i < d$ such that

$$\partial_{i+1} h_i + h_{i-1} \partial_i = 1_{P_i} \quad (-1 \leq i \leq d). \quad (8)$$

The maps outside the specified range are zero. To construct h_i , choose a complement to $\ker \partial_i$ in P_i . Exactness identifies this complement, by ∂_i , with $\ker \partial_{i-1}$. Define h_{i-1} on that

kernel by the inverse and set it to zero on the chosen complement. On each of the two summands of P_i , one of the two terms in (8) is the identity and the other is zero. At the endpoints this reads $\partial_0 h_{-1} = 1_{D_k A}$ and $h_{d-1} \partial_d = 1_{P_d}$.

Let R be the unital subring of k generated by the following finite set of entries: the multiplication table of A , the action matrices of the P_i and S , and the matrices of $\partial_i, \iota_i, \pi_i, h_i$. On the free R -module with basis $a_1, \dots, a_n$, define A_R using this multiplication table. Replace each chosen vector space by the free R -module on its basis, with actions and maps given by the selected matrices. Associativity, unit laws, module laws, module linearity of $\partial_i, \iota_i, \pi_i$, and the equations $\partial_i \partial_{i+1} = 0$, (7), and (8) are polynomial identities in these entries with integer coefficients. They hold over R , since $R \hookrightarrow k$ is injective. The first basis vector of A_R is its identity, and every field fiber is nonzero of dimension n .

Define $D_R A_R = \text{Hom}_R(A_R, R)$ with the action (2). Its action matrices come from the multiplication table. The finite free dual basis gives, for every $R \rightarrow F$, an isomorphism of A_F -bimodules

$$(D_R A_R) \otimes_R F \cong D_F(A_R \otimes_R F). \quad (9)$$

This isomorphism sends $\lambda \otimes u$ to $a \otimes v \mapsto uv\lambda(a)$, where $\lambda(a)$ is mapped from R to F . The dual bases show that it is bijective, and (2) shows compatibility with both actions. Use $D_R A_R$ as the augmentation term of the model. Equations (7) make each specialized P_i a direct summand of a finite free right A_F -module. Equations (8) make the specialized augmented complex exact as a complex of F -vector spaces, hence also as a complex of A_F -modules. Therefore

$$\text{pd}_{A_F^{\text{op}}}(D_F A_F) \leq d. \quad (10)$$

Let S_R be the free R -module with the fixed basis and action matrices of S . Apply the bar construction over R to the left A_R -modules $D_R A_R, S_R$. Each cochain term is finite free over R . After tensoring with any field, it is the bar cochain complex of the corresponding fiber modules, by the fixed bases and (9). Lemma 3.1 in degree $d+1$ gives

$$\text{Ext}_{A_F}^{d+1}(D_F A_F, S_R \otimes_R F) \neq 0.$$

Together with (10), Lemma 2.2 implies $\text{id}((A_F)_{A_F}) = \infty$.

If $d > 0$, apply the bar construction over A_R^{op} with source $D_R A_R$ and target A_R . Equation (5) and Lemma 3.1 imply $\text{Ext}_{A_F^{\text{op}}}^d(D_F A_F, A_F) \neq 0$. Thus (10) is an equality. If $d = 0$, it is already an equality because the dual module is nonzero. Finally (3) gives the asserted left injective dimension. $\square$

4 Finite fields and prime fields

The first assertion below is the weak Nullstellensatz of Zariski [4, Section 1, assertion H3]; see also [3, Tag 00FV]. An element is integral over a commutative ring if it satisfies a monic polynomial with coefficients in that ring.

Lemma 4.1. *A field finitely generated as an algebra over a field is a finite field extension. A field finitely generated as a unital ring is finite.*

Proof. First, if a field is integral over a subring, that subring is a field. Indeed, for a nonzero element x of the subring, multiply a monic relation of degree $e \geq 1$ for x^{-1} by x^{e-1} . The resulting equation expresses x^{-1} in the subring.

Let k be a field and $E = k[c_1, \dots, c_t]$ a field generated by $t \geq 0$ elements as a k -algebra. After reordering the generators, let $c_1, \dots, c_r$ be a maximal algebraically independent subset over k , where $0 \leq r \leq t$. Here $k(c_1, \dots, c_r)$ is the fraction field of the polynomial ring

$k[c_1, \dots, c_r]$; both rings mean k when $r = 0$. Each remaining generator is algebraic over this fraction field: a polynomial dependence on $c_1, \dots, c_r, c_i$ gives a nonzero polynomial in c_i , since the first r elements are independent. Clearing the denominators of their monic equations gives a nonzero $f \in k[c_1, \dots, c_r]$ such that these generators are integral over $k[c_1, \dots, c_r, 1/f]$. If these equations have degrees $e_i \geq 1$, for $r < i \leq t$, their monomials $\prod_{i=r+1}^t c_i^{a_i}$, with $0 \leq a_i < e_i$, span the resulting algebra over that subring. An empty product is 1. Thus $E = k[c_1, \dots, c_r, 1/f][c_{r+1}, \dots, c_t]$ is a finite module over that subring. To check integrality, take module generators $u_1 = 1, u_2, \dots, u_s$, where $s \geq 1$, and $x \in E$. Write $xu_i = \sum_{j=1}^s b_{ij}u_j$, with $b_{ij} \in k[c_1, \dots, c_r, 1/f]$. Let $B = (b_{ij})_{1 \leq i, j \leq s}$ and let I_s be the identity matrix. Then $(xI_s - B)(u_1, \dots, u_s)^T = 0$, where the superscript denotes transpose. Multiplication by the adjugate gives $\det(xI_s - B)u_i = 0$ for every i . Since $u_1 = 1$, the monic polynomial $\det(TI_s - B)$ in the indeterminate T vanishes at x . The first paragraph therefore makes $k[c_1, \dots, c_r, 1/f]$ a field.

If $r > 0$, choose a monic irreducible polynomial $g \in k[c_1]$ which does not divide f . Such a polynomial exists: $k[c_1]$ has infinitely many monic irreducibles, since a product of any finite list plus 1 has an irreducible factor outside the list; only finitely many can divide a nonzero coefficient of f when it is written as a polynomial in $c_2, \dots, c_r$. The ideal generated by g in $k[c_1, \dots, c_r]$ is prime, because its quotient is a polynomial ring over the field $k[c_1]/(g)$. If g were invertible after adjoining $1/f$, then $gh = f^j$ for some $h \in k[c_1, \dots, c_r]$ and $j \geq 0$, contrary to primality and $g \nmid f$. Thus $r = 0$. The finitely many algebraic generators now give $[E : k] < \infty$.

For the second assertion, in positive characteristic p the field is a finite extension of $\mathbb{F}_p$. In characteristic zero, write such a field as $E = \mathbb{Z}[c_1, \dots, c_t]$. It is finitely generated over $\mathbb{Q}$, hence a number field. Choose $N \geq 1$ so that the coefficients of the monic minimal polynomials of all the c_i lie in $\mathbb{Z}[1/N]$. Then $E = \mathbb{Z}[1/N][c_1, \dots, c_t]$ is integral over $\mathbb{Z}[1/N]$. The first paragraph would make $\mathbb{Z}[1/N]$ a field, whereas a prime not dividing N is not invertible in this ring. $\square$

Thus every maximal-ideal residue field of a nonzero finitely generated unital commutative ring is finite.

Proposition 4.2. *Suppose $\dim_k A = n$, $\text{id}(A_A) = d < \infty$, and $\text{id}(A_A) = \infty$. There is an n -dimensional algebra B over a finite field F such that*

$$\text{id}(B_B) = d, \quad \text{id}(B_B) = \infty.$$

If k has positive characteristic p , F can be chosen of characteristic p .

Proof. Take R, A_R from Theorem 3.2 and a maximal ideal $\mathfrak{m}$ of R . The ring R is a nonzero finitely generated $\mathbb{Z}$ -algebra, so $F = R/\mathfrak{m}$ is finite. The fiber $B = A_R \otimes_R F$ has the asserted dimensions. In characteristic p , R is an $\mathbb{F}_p$ -algebra, and its residue fields have characteristic p . $\square$

Proof of Theorem 1.1. Statement (1) implies (2) and (3). If (1) fails, replace a counterexample by its opposite algebra if necessary to put its finite injective dimension on the left. Proposition 4.2 then produces a counterexample over a finite field, proving that (2) implies (1).

If $F/\mathbb{F}_p$ is a finite field extension and B is a finite-dimensional F -algebra, regard B as an $\mathbb{F}_p$ -algebra by restricting its central scalar action. The underlying unital ring, its regular modules, and their injective dimensions are unchanged, whereas

$$\dim_{\mathbb{F}_p} B = [F : \mathbb{F}_p] \dim_F B.$$

Thus (3) implies (2). For a fixed p , use the characteristic-preserving clause of Proposition 4.2 and the same restriction of scalars. This proves the last assertion. $\square$

5 Characteristic zero

Lemma 5.1. *Under the hypotheses of Theorem 1.2, with $n = \dim_k A$, there is a finite field extension $L/\mathbb{Q}$ and an n -dimensional L -algebra B with $\text{id}_B(B) = d$ and $\text{id}(B_B) = \infty$. The same ring, viewed as a $\mathbb{Q}$ -algebra, has dimension $n[L : \mathbb{Q}]$ and the same two injective dimensions.*

Proof. The coefficient ring R from Theorem 3.2 has characteristic zero. Its localization $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is nonzero and finitely generated over $\mathbb{Q}$. A maximal ideal gives a residue field L finite over $\mathbb{Q}$ by the weak Nullstellensatz. Apply Theorem 3.2 to $R \rightarrow L$. Restriction of scalars to $\mathbb{Q}$ changes dimension by the factor $[L : \mathbb{Q}]$ and leaves the ring and its regular modules unchanged. $\square$

Proof of Theorem 1.2. Let C be the $\mathbb{Q}$ -algebra obtained in Lemma 5.1 and put $m = \dim_{\mathbb{Q}} C$. Apply the construction of Theorem 3.2 anew to C , using $D_{\mathbb{Q}}C$. This gives a finite collection of rational matrix entries for its multiplication, modules, resolution, splitting maps, and vector-space contraction. Choose a positive integer N divisible by every denominator. All these entries lie in $\mathbb{Z}[1/N]$.

The same matrices define a free rank- m algebra $\mathcal{C}$ over $\mathbb{Z}[1/N]$. The splitting and contraction identities hold in this ring, since it embeds in $\mathbb{Q}$. All minors above the ranks of the relevant bar differentials also vanish in this ring. The proof of Theorem 3.2 therefore applies to each map $\mathbb{Z}[1/N] \rightarrow \mathbb{F}_p$, for $p \nmid N$, preserving both the finite value d and the infinite opposite dimension. The rational fiber recovers C . $\square$

The original dimension can instead be retained over finite extensions of the prime fields. The next statement also bounds their degrees.

Proposition 5.2. *Under the hypotheses of Theorem 1.2, let $n = \dim_k A$. There exist positive integers r, N such that for every prime $p \nmid N$ there are a finite extension $F/\mathbb{F}_p$ with $[F : \mathbb{F}_p] \leq r$ and an n -dimensional F -algebra B_p satisfying*

$$\text{id}_{(B_p)_{B_p}} = d, \quad \text{id}((B_p)_{B_p}) = \infty.$$

Proof. Take L, B as in Lemma 5.1 and set $r = [L : \mathbb{Q}]$. Select all the finite matrix data for B used in the proof of Theorem 3.2. Choose a $\mathbb{Q}$ -basis $b_1 = 1, b_2, \dots, b_r$ of L . There is a positive integer N such that the multiplication coefficients of this basis and the coordinates of every selected matrix entry belong to $\mathbb{Z}[1/N]$. Consequently

$$O = \bigoplus_{i=1}^r \mathbb{Z}[1/N]b_i \subset L$$

is a subring containing every selected entry and free of rank r over $\mathbb{Z}[1/N]$. The identities and vanishing minors used in Theorem 3.2 hold in O .

For $p \nmid N$, the commutative algebra O/pO is nonzero of dimension r over $\mathbb{F}_p$. A maximal ideal of this finite algebra has residue field F with $[F : \mathbb{F}_p] \leq r$. Specializing the selected O -model of B along $O \rightarrow F$ gives the required n -dimensional algebra. $\square$

Corollary 5.3. *If the Gorenstein symmetry conjecture holds for every finite-dimensional $\mathbb{F}_p$ -algebra for infinitely many primes p , then it holds for every finite-dimensional algebra over every field of characteristic zero.*

Proof. A characteristic-zero counterexample, after taking the opposite algebra if necessary, would give a counterexample over every $\mathbb{F}_p$ with $p \nmid N$, by Theorem 1.2. An infinite set of primes contains one not dividing N . $\square$

6 Finite detection of failure

The field reduction also has a computability consequence. A semidecision procedure for failure means an algorithm which halts with a counterexample if the conjecture fails, and never outputs a counterexample if it holds. It need not halt in the latter case.

Proposition 6.1. *Failure of the Gorenstein symmetry conjecture for finite-dimensional algebras over arbitrary fields is semidecidable.*

Proof. Enumerate primes p , positive integers n , and nonnegative integers d , in such a way that every triple is eventually reached. For each triple, enumerate the finitely many multiplication tables on $\mathbb{F}_p^n$ with the first basis vector prescribed as identity. Associativity and the unit laws are finite equations, so their validity is decidable.

For each resulting algebra A , form the two actions on $D_{\mathbb{F}_p}A$ using (2). For either side, compute d successive kernels of finite free surjections. More explicitly, for a finite left module M with vector-space basis $m_1, \dots, m_t$, use $A^t \rightarrow M$, $(a_i) \mapsto \sum_i a_i m_i$. Its kernel and its action matrices are obtained by linear algebra. Use the right-handed version for right modules. Set $\Omega^0 M = M$, and let $\Omega^{j+1} M$ denote the kernel at the next step, for $0 \leq j < d$.

For a left module, $\text{pd}_A M \leq d$ is equivalent to projectivity of $\Omega^d M$. Indeed dimension shifting identifies $\text{Ext}_A^1(\Omega^d M, X)$ with $\text{Ext}_A^{d+1}(M, X)$; vanishing for all X characterizes projectivity, and the converse follows by truncating the resolution. To test a finite module K for projectivity, take the above finite free surjection $q : A^{\dim_{\mathbb{F}_p} K} \rightarrow K$. It splits precisely when there is an A -linear map $s : K \rightarrow A^{\dim_{\mathbb{F}_p} K}$ with $qs = 1_K$. Both module linearity and this last equation are finite linear equations in the entries of s , decidable over $\mathbb{F}_p$. For the zero module the zero section suffices. Apply the same test over A^{op} for right modules.

Halt if exactly one of the two dual modules has projective dimension at most d . Lemma 2.2, on the appropriate side, certifies that the other injective dimension is infinite. If a counterexample exists over any field, Theorem 1.1 provides one over a prime finite field; the enumeration reaches its multiplication table and a suitable d . $\square$

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