

A reduction of the Gorenstein symmetry conjecture to gendo-symmetric algebras

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Abstract

We reduce the Gorenstein symmetry conjecture for Artin algebras to gendo-symmetric algebras relative to Artin duality.

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1 Introduction

The Gorenstein symmetry conjecture asserts that an Artin algebra has finite left self-injective dimension if and only if it has finite right self-injective dimension. It is recorded as Conjecture (13) on page 410 of Auslander, Reiten and Smalø's monograph [1]; see also the discussion in [6, Section 5]. Thus it asks whether one-sided finiteness makes the algebra Gorenstein, meaning that both regular modules have finite injective dimension.

For finite-dimensional algebras over a field, Han and Qin [6, Proposition 5] prove that Gorenstein symmetry passes from the two outer algebras of a 2-recollement to the middle algebra. A recollement expresses a triangulated category in terms of two others through fully faithful embeddings and adjoint quotient functors; a 2-recollement has four layers of functors, with every three consecutive layers forming a recollement. Their finite stratification argument reduces the conjecture to 2-derived-simple algebras, whose unbounded derived categories admit no such decomposition with two nonzero outer algebras [6, Corollary 8]. The same argument gives invariance of the conjecture under derived equivalence [6, Corollary 7].

Han, Qin and Wang [7, Theorem 3] extend this transfer principle to essentially surjective eventually homological isomorphisms between bounded derived categories of finite-dimensional algebras. These are triangle functors for which every target object is isomorphic to an image and for each pair of source objects the induced maps on morphism groups are isomorphisms in all sufficiently high degrees. Their proof identifies the subcategories of bounded projective and bounded injective complexes. The two possible inclusions between these subcategories encode the two one-sided finiteness conditions. We use this characterization below over a commutative Artin base.

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Fix a commutative Artin ring R . An *Artin R -algebra* is a nonzero associative unital ring, finite as an R -module, with a specified central unital action of R . All algebras below are Artin R -algebras, and modules are finitely generated unital right modules unless otherwise specified. The word finite applied to a module means finitely generated; a finite ring has finitely many elements. Let I be the direct sum of the injective envelopes of one representative of each simple R -module, and set $D = \text{Hom}_R(-, I)$. This is the usual Artin duality; its properties used here are proved in Section 2. For a finite R -module X , let $\ell_R X$ denote its composition length.

For a right module M over an algebra S , write $\text{pd}_S M$ and $\text{id}_S M$ for its projective and injective dimensions in the category of all right modules, allowing the value ∞ ; both dimensions of the zero module are taken to be zero. We display the side explicitly for regular modules. For right S -modules U, V and $n \geq 0$, $\text{Ext}_S^n(U, V)$ is the degree- n cohomology of $\text{Hom}_S(P_\bullet, V)$, where $P_\bullet \rightarrow U$ is a projective resolution. In degree zero it is $\text{Hom}_S(U, V)$. For a right S -module X , the left action on DX is $(s\lambda)(x) = \lambda(xs)$; for a left module it is $(\lambda s)(x) = \lambda(sx)$. On a homomorphism $g : X \rightarrow Y$, duality acts by precomposition, $Dg(\lambda) = \lambda \circ g$. For an algebra S , the dual bimodule action is

$$(s\varphi t)(x) = \varphi(txs) \quad (s, t, x \in S, \varphi \in DS).$$

In this notation, the conjecture asserts that every Artin algebra A satisfies

$$\text{id}({}_A A) < \infty \iff \text{id}(A_A) < \infty. \quad (1)$$

An algebra S is *symmetric relative to D* if $DS \cong S$ as bimodules. Write $\text{add } M$ for the direct summands of finite direct sums of a module M . A module M is a *generator* if $S \in \text{add } M$. In this paper, an algebra is *gendo-symmetric relative to D* if it is $\text{End}_S(M)$ for a generator over an algebra symmetric relative to D . For $R = k$ a field this is the usual definition. Endomorphism products are ordinary composition, $\alpha\beta = \alpha \circ \beta$.

Fang and Koenig [4, 5] develop the theory of endomorphism algebras of generators over symmetric algebras, now called gendo-symmetric algebras. Such an endomorphism algebra need not be self-injective, so symmetry of the base does not settle its Gorenstein symmetry. Chen and Koenig [3, Corollary 3.25] prove the conjecture for $\text{End}_S(S \oplus X)$ when S is a finite-dimensional self-injective algebra and X is indecomposable and nonprojective. Aihara, Chan and Honma [2] study generators over trivial extensions. Our generator consists of a trivial extension and its canonical quotient.

We prove that this construction preserves each one-sided finiteness condition for every Artin algebra. The matrix description of its endomorphism algebra gives a finite projective resolution on one side and an exact induction functor detecting the other side. The remaining implications follow by comparing the construction for the opposite algebra with the opposite endomorphism algebra. This comparison uses Hu and Xi's syzygy equivalence [8, Section 3]; its two-term complex proof is included in Section 6.

For an algebra A , its trivial extension $T(A) = A \ltimes DA$ is the R -module $A \oplus DA$ with multiplication, for $a, b \in A$ and $\lambda, \mu \in DA$,

$$(a, \lambda)(b, \mu) = (ab, a\mu + \lambda b)$$

and identity $(1, 0)$. The projection $\pi : T(A) \rightarrow A$, $(a, \lambda) \mapsto a$, makes A a right $T(A)$ -module by $x \cdot b = x\pi(b)$. Define

$$B = T(A), \quad E(A) = \text{End}_B(B \oplus A). \quad (2)$$

When A is fixed, we write $E = E(A)$.

Theorem 1.1. *For every Artin R -algebra A , the algebra $E = E(A)$ is gendo-symmetric relative to D , and*

$$\ell_R E = 5\ell_R A.$$

Furthermore,

$$\mathrm{id}({}_E E) < \infty \iff \mathrm{id}({}_A A) < \infty, \quad (3)$$

$$\mathrm{id}(E_E) < \infty \iff \mathrm{id}(A_A) < \infty. \quad (4)$$

If $d = \mathrm{id}({}_A A) < \infty$, then

$$\mathrm{id}({}_E E) \leq 2d + 2. \quad (5)$$

Moreover,

$$\mathrm{id}(A_A) \leq \mathrm{id}(E_E), \quad (6)$$

where the inequality is interpreted in $\{0, 1, 2, \dots\} \cup \{\infty\}$.

The proof is completed in Section 7. When $R = k$ is a field, the length formula reads $\dim_k E(A) = 5 \dim_k A$.

Corollary 1.2. *For each commutative Artin ring R , the Gorenstein symmetry conjecture for all Artin R -algebras is equivalent to its restriction to gendo-symmetric algebras relative to D . For each individual A , it holds for A if and only if it holds for $E(A)$.*

Proof. The two equivalences in Theorem 1.1 identify the finiteness of the corresponding dimensions for A and $E(A)$. Every $E(A)$ belongs to the stated subclass, and every algebra in that subclass is an Artin R -algebra. $\square$

2 Artin duality

Write $R = \prod_{j=1}^t R_j$, where $t \geq 1$ and each R_j is a commutative local Artin ring, and let k_j be its residue field. The socle of a module is the sum of its simple submodules; a submodule is essential if it meets every nonzero submodule nontrivially. For $1 \leq j \leq t$, the injective envelope I_j of k_j has socle k_j : every simple submodule intersects the essential submodule k_j , so is equal to it. Here $I = \bigoplus_{j=1}^t I_j$. Every map from k_j to I lands in its k_j -socle, so $\mathrm{Hom}_R(k_j, I) \cong \mathrm{End}_R(k_j) \cong k_j$. Since I is injective, D is exact. Applying D to a composition series shows that DX is finite and $\ell_R DX = \ell_R X$ for every finite X .

The evaluation map $X \rightarrow D^2 X$, $x \mapsto [\varphi \mapsto \varphi(x)]$, is injective. Indeed, for $x \neq 0$, the cyclic module Rx has a simple quotient in which its generator is nonzero; embed that quotient in I and extend the resulting map $Rx \rightarrow I$ to X by injectivity. Equality of lengths makes evaluation an isomorphism. Thus D is an exact duality.

For every left S -module X , finite or not, evaluation at 1 gives

$$\mathrm{Hom}_S(X, DS) \cong \mathrm{Hom}_R(X, I), \quad h \longmapsto [x \mapsto h(x)(1)].$$

The inverse sends λ to $x \mapsto [s \mapsto \lambda(sx)]$. Thus DS is injective; the right-module assertion follows with reversed actions. Dualizing a finite free surjection onto DX embeds a finite module X into a finite sum of copies of DS . Iteration gives an injective resolution with finite terms. Every finite injective is a summand of such a sum, so duality exchanges finite injectives and finite projectives.

If a finite module has injective dimension at most d , its d -th cosyzygy is injective: dimension shifting identifies its first Ext group with the original module's $(d+1)$ -st Ext group, which vanishes. Here the zeroth cosyzygy is the original module, and each successive

cosyzygy is the cokernel of the preceding embedding into an injective module. Dually, finite free surjections and their finite kernels construct projective resolutions with finite terms, terminating at any projective-dimension bound. Write S^{op} for the algebra with reversed multiplication; its right modules are left S -modules. Dualizing these resolutions gives

$$\text{id}({}_S S) = \text{pd}_S DS, \quad \text{id}(S_S) = \text{pd}_{S^{\text{op}}} DS. \quad (7)$$

3 The endomorphism algebra

Let $e, f \in E$ be the endomorphisms defined by $e(z, t) = (z, 0)$ and $f(z, t) = (0, t)$ for $z \in B, t \in A$. Thus $e + f = 1$, $ef = fe = 0$, and $e^2 = e, f^2 = f$. We identify DA with the square-zero ideal $\{(0, \lambda) : \lambda \in DA\}$ of B .

The elements $z \in B$ satisfying $zDA = 0$ are precisely those in DA . Indeed, if $(a, \lambda)(0, \mu) = 0$ for every $\mu \in DA$, then $a\mu = 0$ for all μ . Evaluating at 1 gives $\mu(a) = 0$ for every $\mu \in \text{Hom}_R(A, I)$, and hence $a = 0$ by evaluation into D^2A . A B -linear map $A \rightarrow B$ is determined by the image of 1, which must annihilate DA on the right. Conversely, each $u \in DA$ defines a B -linear map $t \mapsto ut$: for $t \in A$ and $b \in B$, one has $u(t\pi(b)) = (ut)b$. Evaluation at 1 therefore gives

$$\text{Hom}_B(A, B) = DA, \quad \text{Hom}_B(B, A) = A, \quad \text{End}_B(A) = A.$$

Together with $\text{End}_B(B) = B$, these identifications give

$$E = \begin{pmatrix} B & DA \\ A & A \end{pmatrix}. \quad (8)$$

Here $b \in B$ represents left multiplication on B , $u \in DA$ represents the map $A \rightarrow B, t \mapsto ut$, $v \in A$ represents $B \rightarrow A, z \mapsto v\pi(z)$, and the lower right entry $a \in A$ represents left multiplication by a on A .

In particular, if $b, b' \in B, u, u' \in DA$, and $v, v', a, a' \in A$, multiplication is

$$\begin{pmatrix} b & u \\ v & a \end{pmatrix} \begin{pmatrix} b' & u' \\ v' & a' \end{pmatrix} = \begin{pmatrix} bb' + uv' & bu' + ua' \\ v\pi(b') + av' & aa' \end{pmatrix}. \quad (9)$$

The B -actions on DA in this formula factor through π . The product uv' in the upper left block is included in the ideal $DA \subset B$. Every composite $A \rightarrow B \rightarrow A$ is zero, since the first map lands in DA and the second kills DA .

Consequently the map

$$q : E \longrightarrow A, \quad \begin{pmatrix} b & u \\ v & a \end{pmatrix} \longmapsto a \quad (10)$$

is an algebra epimorphism. For a right A -module X , denote by X_q the same R -module with right E -action $xr = xq(r)$.

Lemma 3.1. *The algebra B is symmetric relative to D . The algebra E is gendo-symmetric relative to D and has length $5\ell_{RA}$.*

Proof. Define the R -linear map $\tau : B \rightarrow I$ by $\tau(a, \lambda) = \lambda(1)$. Its associated bilinear form is

$$\tau((a, \lambda)(b, \mu)) = \mu(a) + \lambda(b).$$

This form is symmetric. The map $b \mapsto [c \mapsto \tau(cb)]$ identifies the two summands of $B = A \oplus DA$ with those of $DB = DA \oplus D^2A$, using evaluation $A \cong D^2A$; it is therefore an isomorphism. For $r, s, b, c \in B$, associativity and symmetry give

$$\tau((scr)b) = \tau(s(crb)) = \tau((crb)s) = \tau(c(rbs)).$$

By the dual action defined in Section 1, this is the bimodule identity for the map. Since $B \oplus A$ contains B , it is a generator, proving that E is gendo-symmetric. The upper left block in (8) has length $2\ell_RA$; each other block has length ℓ_RA , since D preserves length. Their sum is $5\ell_RA$. $\square$

For a right B -module X , set

$$H(X) = \text{Hom}_B(B \oplus A, X).$$

It is a right E -module by precomposition, $hr = h \circ r$. If X is a right A -module, its inflation along π has action $xb = x\pi(b)$ for $x \in X$, $b \in B$. A map $B \oplus A \rightarrow X$ then has the form $(z, t) \mapsto x\pi(z) + yt$ for unique $x, y \in X$. Precomposing with the block matrix in (9) gives

$$H(X) = X \oplus X, \quad (x, y) \begin{pmatrix} b & u \\ v & a \end{pmatrix} = (x\pi(b) + yv, ya). \quad (11)$$

For a homomorphism $g : X \rightarrow Y$ of inflated modules, $H(g)$ is $(x, y) \mapsto (g(x), g(y))$; hence this restriction of H is exact. The identifications $H(B) = eE$ and $H(A) = fE$ send a map into a summand of $B \oplus A$ to the corresponding row of an endomorphism. In particular, H takes projective A -modules to projective E -modules: it preserves finite direct sums and direct summands and sends A to fE .

4 The finite injective dimension

By (7), $\text{id}_A(A) = \text{pd}_A DA$ and $\text{id}_E(E) = \text{pd}_E DE$, with projective dimensions taken on the right. The ideal $DA \subseteq B$ has its action inflated along π .

Lemma 4.1. *Suppose $d = \text{pd}_A DA < \infty$. Then*

$$\text{pd}_E H(DA) \leq d, \quad \text{pd}_E A_q \leq d + 2, \quad \text{pd}_E (DA)_q \leq 2d + 2.$$

If K is the kernel of $H(A) \rightarrow A_q$, $(x, y) \mapsto y$, then $\text{pd}_E K \leq d + 1$.

Proof. Choose an exact sequence

$$0 \longrightarrow P_d \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow DA \longrightarrow 0$$

of right A -modules with each P_i projective. For $d = 0$, this means $P_0 \cong DA$. Applying H to the modules inflated along π gives an exact sequence with E -projective terms $H(P_i)$, proving $\text{pd}_E H(DA) \leq d$.

Left exactness of H , applied to $0 \rightarrow DA \rightarrow B \xrightarrow{\pi} A \rightarrow 0$, together with the following cokernel calculation gives

$$0 \longrightarrow H(DA) \longrightarrow eE \xrightarrow{H(\pi)} fE \longrightarrow A_q \longrightarrow 0. \quad (12)$$

We use row coordinates $eE = B \oplus DA$ and $fE = A \oplus A$ from (8). The map $H(\pi)$ sends (b, u) to $(\pi(b), 0)$. Its image is $K = \{(x, 0) : x \in A\}$, and projection to the second coordinate

identifies its cokernel with A_q . Splicing the resolution of $H(DA)$ into (12) gives $\text{pd}_E K \leq d+1$ and $\text{pd}_E A_q \leq d+2$.

Inflate the same length- d resolution of DA along q . Every term is a direct summand of a finite sum of copies of A_q , and thus has E -projective dimension at most $d+2$. For an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of right E -modules, the Ext long exact sequence gives $\text{pd}_E N \leq \max\{\text{pd}_E M, \text{pd}_E L + 1\}$. To apply this bound explicitly, put $Z_0 = (DA)_q$ and, for $0 \leq i < d$, let Z_{i+1} be the kernel of the induced epimorphism $(P_i)_q \rightarrow Z_i$. Then $Z_d \cong (P_d)_q$, and the short exact sequences

$$0 \longrightarrow Z_{i+1} \longrightarrow (P_i)_q \longrightarrow Z_i \longrightarrow 0 \quad (0 \leq i < d)$$

give, by descending induction, $\text{pd}_E Z_i \leq d+2+d-i$. Taking $i=0$ proves the last bound. If $d=0$, it follows directly from $(DA)_q \cong (P_0)_q$. $\square$

Proposition 4.2. *If $d = \text{id}({}_A A) < \infty$, then $\text{id}({}_E E) \leq 2d+2$.*

Proof. The decomposition ${}_E E = Ee \oplus Ef$ gives $DE = D(Ee) \oplus D(Ef)$ as right E -modules. For either left module Ee or Ef , its dual has right action $(\varphi r)(s) = \varphi(rs)$, where $r \in E$, s belongs to that left module, and φ is an R -linear map from it to I .

First, $D(Ee) \cong eE$. Use row coordinates $eE = B \oplus DA$ and column coordinates $Ee = B \oplus A$. For $c, z \in B$, $\lambda \in DA$, and $t \in A$, define

$$\langle (c, \lambda), (z, t) \rangle = \tau(cz) + \lambda(t). \quad (13)$$

It gives an isomorphism $B \oplus DA \rightarrow D(B \oplus A)$ by the trace isomorphism $B \cong DB$ and the identity on DA . If $r = \begin{pmatrix} b & u \\ v & a \end{pmatrix}$, then

$$(c, \lambda)r = (cb + \lambda v, cu + \lambda a), \quad r(z, t) = (bz + ut, v\pi(z) + at).$$

The identities

$$\tau(\lambda v z) = \lambda(v\pi(z)), \quad \tau(cut) = (cu)(t), \quad (\lambda a)(t) = \lambda(at)$$

give

$$\begin{aligned} \langle (c, \lambda)r, (z, t) \rangle &= \tau(cbz) + \lambda(v\pi(z)) + (cu)(t) + \lambda(at) \\ &= \langle (c, \lambda), r(z, t) \rangle. \end{aligned}$$

Thus (13) identifies eE with $D(Ee)$ as a right E -module, and this summand is projective.

Next, in column coordinates $Ef = DA \oplus A$, an element (w, t) , with $w \in DA$ and $t \in A$, has action

$$\begin{pmatrix} b & u \\ v & a \end{pmatrix} (w, t) = (bw + ut, at).$$

For $x \in A$ and $\lambda \in DA$, the pairing

$$((x, \lambda), (w, t)) \longmapsto w(x) + \lambda(t)$$

identifies $D(Ef)$ with $A \oplus DA$. Its dual action is determined by

$$(bw + ut)(x) + \lambda(at) = w(x\pi(b)) + u(tx) + \lambda(at) = w(x\pi(b)) + (xu + \lambda a)(t).$$

Thus its right action is

$$(x, \lambda) \begin{pmatrix} b & u \\ v & a \end{pmatrix} = (x\pi(b), xu + \lambda a). \quad (14)$$

Comparison with (11) shows that

$$D(Ef) \longrightarrow H(A), \quad (x, \lambda) \longmapsto (x, 0)$$

is E -linear, with kernel $(DA)_q$ and image K from Lemma 4.1. We obtain a short exact sequence

$$0 \longrightarrow (DA)_q \longrightarrow D(Ef) \longrightarrow K \longrightarrow 0.$$

For a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$, the terms $\text{Ext}_E^n(N, V)$ and $\text{Ext}_E^n(L, V)$ adjacent to $\text{Ext}_E^n(M, V)$ vanish when $n > \max\{\text{pd}_E L, \text{pd}_E N\}$, for every right E -module V . Hence $\text{pd}_E M \leq \max\{\text{pd}_E L, \text{pd}_E N\}$. Lemma 4.1 therefore gives

$$\text{pd}_E D(Ef) \leq \max\{2d + 2, d + 1\} = 2d + 2.$$

Together with the projectivity of $D(Ee)$, this proves

$$\text{id}_E(E) = \text{pd}_E DE \leq 2d + 2. \quad \square$$

5 Descent along the quotient summand

The lower right block identifies the corner algebra fEf with A . Left multiplication by this corner makes fE an A - E -bimodule. In row coordinates $fE = A \oplus A$, its left action is $a(x, y) = (ax, ay)$, so ${}_A fE$ is free of rank two. Write $\text{mod-}S$ for the category of finitely generated right modules over an algebra S , and define

$$F : \text{mod-}A \longrightarrow \text{mod-}E, \quad F(U) = U \otimes_A fE.$$

This functor is exact because ${}_A fE$ is free. It preserves projectives because $F(A) \cong fE$ is a direct summand of E_E . For a right E -module N , the R -submodule $Nf = \{nf : n \in N\}$ is a right fEf -module, identified with a right A -module, and tensor–Hom adjunction gives

$$\text{Hom}_E(F(U), N) \cong \text{Hom}_A(U, Nf). \quad (15)$$

Explicitly, an E -linear map $\varphi : F(U) \rightarrow N$ is sent to $u \mapsto \varphi(u \otimes f)$, whose image lies in Nf . Conversely, an A -linear map $\psi : U \rightarrow Nf$ gives

$$u \otimes fr \longmapsto \psi(u)r \quad (u \in U, r \in E).$$

This is well-defined: $\psi(u) = \psi(u)f$ makes the value depend only on fr , and $\psi(ua)r = \psi(u)ar$ gives the tensor balancing relation for $a \in A = fEf$. These constructions are inverse and natural. Right multiplication by f is an idempotent projection on fE , with image $fEf = A$. It follows that

$$(F(U))f \cong U \otimes_A fEf \cong U, \quad u \otimes a \longmapsto ua.$$

The inverse of the last map is $u \mapsto u \otimes f$.

Proposition 5.1. *For every finitely generated right A -module U ,*

$$\text{pd}_E F(U) = \text{pd}_A U,$$

including the value ∞ . Moreover, $\text{id}(A_A) \leq \text{id}(E_E)$.

Proof. Applying F to a projective resolution of U gives an E -projective resolution of $F(U)$. Applying (15) term by term consequently gives, for every right E -module N and every integer $n \geq 0$,

$$\mathrm{Ext}_E^n(F(U), N) \cong \mathrm{Ext}_A^n(U, Nf). \quad (16)$$

The resolution already gives $\mathrm{pd}_E F(U) \leq \mathrm{pd}_A U$. If $\mathrm{pd}_E F(U) \leq r$ for a nonnegative integer r , take $N = F(V)$ for an arbitrary finite right A -module V . Then (16) implies $\mathrm{Ext}_A^n(U, V) = 0$ for all $n > r$, so $\mathrm{pd}_A U \leq r$. Indeed, for $r > 0$ let $\Omega^r U$ be defined by an exact sequence $0 \rightarrow \Omega^r U \rightarrow P_{r-1} \rightarrow \cdots \rightarrow P_0 \rightarrow U \rightarrow 0$ with finite projective terms, and set $\Omega^0 U = U$. Choose an epimorphism $P \rightarrow \Omega^r U$ with P finite projective and kernel L . Dimension shifting gives

$$\mathrm{Ext}_A^1(\Omega^r U, L) \cong \mathrm{Ext}_A^{r+1}(U, L) = 0.$$

Thus $0 \rightarrow L \rightarrow P \rightarrow \Omega^r U \rightarrow 0$ splits, so $\Omega^r U$ is projective. This proves the equality, also when either dimension is infinite.

For the second assertion we may assume $m = \mathrm{id}(E_E) < \infty$. Set $N = fE$ in (16). Since fE is a direct summand of E_E , we obtain

$$\mathrm{Ext}_A^n(U, A) \cong \mathrm{Ext}_E^n(F(U), fE) = 0 \quad (n > m)$$

for every finite right A -module U . Take an injective resolution of A_A with finite terms, and let C be its m -th cosyzygy: for $m > 0$, it occurs in the exact sequence

$$0 \rightarrow A \rightarrow I^0 \rightarrow \cdots \rightarrow I^{m-1} \rightarrow C \rightarrow 0;$$

for $m = 0$, set $C = A$. For every right ideal $J \subseteq A$, dimension shifting gives

$$\mathrm{Ext}_A^1(A/J, C) \cong \mathrm{Ext}_A^{m+1}(A/J, A) = 0.$$

The module A/J is finite, so the preceding vanishing applies. The exact sequence obtained by applying $\mathrm{Hom}_A(-, C)$ to $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ shows that every map $J \rightarrow C$ extends to $A \rightarrow C$. By Baer's criterion, C is injective, and hence $\mathrm{id}(A_A) \leq m$. $\square$

6 Syzygies and derived equivalences

For an algebra S , let $D^b(\mathrm{mod}\text{-}S)$ be the bounded derived category of finitely generated right modules. Complexes have differentials $d^i : X^i \rightarrow X^{i+1}$, with $d^{i+1}d^i = 0$. Their cohomology is $H^i(X) = \ker d^i / \mathrm{im} d^{i-1}$, for $i \in \mathbb{Z}$. The bounded derived category is obtained from the bounded homotopy category by inverting chain maps inducing isomorphisms on all cohomology modules. The shift is $X[n]^i = X^{i+n}$, with differential $(-1)^n d^{i+n}$. The category $K^b(\mathcal{C})$ consists of bounded complexes with terms in an additive category $\mathcal{C}$, with chain maps modulo chain homotopy. Write $\mathrm{proj}\text{-}S$ and $\mathrm{inj}\text{-}S$ for the full subcategories of finite projective and finite injective right S -modules. A derived equivalence means a triangle equivalence between the bounded derived categories. For a module X , the notation $X[0]$ denotes the complex concentrated in degree zero. For chain maps $u, v : X^\bullet \rightarrow Y^\bullet$, a chain homotopy from u to v consists of maps $h^i : X^i \rightarrow Y^{i-1}$ satisfying $u^i - v^i = d_Y^{i-1}h^i + h^{i+1}d_X^i$. A thick subcategory is a full triangulated subcategory closed under direct summands. A tilting complex is an object T of $K^b(\mathrm{proj}\text{-}S)$ such that $\mathrm{Hom}_{K^b(\mathrm{proj}\text{-}S)}(T, T[n]) = 0$ for $n \neq 0$ and the thick subcategory generated by T is all of $K^b(\mathrm{proj}\text{-}S)$. The tilting-complex theorem gives a derived equivalence with $\mathrm{End}_{K^b(\mathrm{proj}\text{-}S)}(T)$; we use the Artin-algebra version recalled in [8, Section 2].

Lemma 6.1. *Let S be a self-injective Artin algebra, and let*

$$0 \longrightarrow N \xrightarrow{j} P \xrightarrow{p} M \longrightarrow 0$$

be exact, with $P \in \text{add } S$. Then $\text{End}_S(S \oplus M)$ and $\text{End}_S(S \oplus N)$ are derived equivalent.

Proof. Put $V = S \oplus M$. The functor $\text{Hom}_S(V, -)$ identifies $\text{add } V$ with the finite projective right modules over $\text{End}_S(V)$: on finite sums of V this is the matrix description of homomorphisms, and restricting the corresponding idempotents gives the assertion for summands. It therefore also identifies their bounded homotopy categories.

In $K^b(\text{add } V)$, take $C = [P \xrightarrow{p} M]$, in degrees zero and one, and $S[0] \oplus C$. For nonzero shifts, only 1 and -1 can have overlapping terms; $S[0]$ has no maps to its own nonzero shifts. A map $S[0] \rightarrow C[1]$ is a map $S \rightarrow M$; write it as $p\ell$, with $\ell : S \rightarrow P$, by projectivity of S . The homotopy with component $-\ell$ makes the map zero, since the differential of $C[1]$ is $-p$. A map $C \rightarrow S[-1]$ is a map $h : M \rightarrow S$ with $hp = 0$, hence is zero. The other two cross terms have no overlapping degrees. A map $C \rightarrow C[1]$ is represented by $u : P \rightarrow M$. Write $u = pa$ by projectivity of P ; using $-a$ in degree zero gives a null homotopy with our shift convention. A map $C \rightarrow C[-1]$ is represented by $h : M \rightarrow P$ with $hp = 0$, so is zero. Thus all nonzero-shift self maps vanish.

The degree-one subcomplex $M[-1]$ of C gives a degree-wise split exact sequence of complexes

$$0 \longrightarrow M[-1] \longrightarrow C \longrightarrow P[0] \longrightarrow 0.$$

Its triangle and $P \in \text{add } S$ show that $S[0] \oplus C$ generates $V[0]$, hence all of $K^b(\text{add } V)$.

A chain endomorphism of C is a pair (a, b) with $a : P \rightarrow P$, $b : M \rightarrow M$, and $pa = bp$. Its restriction to N defines an endomorphism of N . For any $c : N \rightarrow N$, injectivity of P extends jc to $a : P \rightarrow P$; since pa kills N , it induces the required b . If the restriction is zero, then $a = hp$ for some $h : M \rightarrow P$, and surjectivity of p gives $b = ph$. These are exactly the null-homotopic pairs. Hence

$$\text{End}_{K^b(\text{add } V)}(C) \cong \text{End}_S(N).$$

A chain map $S[0] \rightarrow C$ is a map $S \rightarrow P$ killed by p , so factors uniquely through j ; there are no nonzero homotopies for this pair. A chain map $C \rightarrow S[0]$ is a map $P \rightarrow S$, modulo maps of the form hp . Restriction to N is onto by injectivity of S , and has precisely this kernel. We obtain

$$\text{Hom}_{K^b(\text{add } V)}(S[0], C) \cong \text{Hom}_S(S, N), \quad \text{Hom}_{K^b(\text{add } V)}(C, S[0]) \cong \text{Hom}_S(N, S).$$

Restriction preserves composition within C and with either cross term. The composite $S \rightarrow C \rightarrow S$ is the composite $S \rightarrow N \rightarrow S$, and the restriction of $C \rightarrow S \rightarrow C$ is $N \rightarrow S \rightarrow N$. Thus these identifications give the entire endomorphism algebra $\text{End}_S(S \oplus N)$. The tilting-complex theorem applies. This is also the special case of [8, Example (a), Theorem 3.5] associated with the displayed short exact sequence. $\square$

Lemma 6.2. *A derived equivalence between Artin algebras preserves the finiteness of the left self-injective dimension and, separately, that of the right self-injective dimension.*

Proof. Let S be an Artin algebra. The following characterizations hold in $D^b(\text{mod-}S)$:

$$\begin{aligned} X \in K^b(\text{proj-}S) &\iff \text{for every } Y, \text{ Hom}(X, Y[n]) = 0 \text{ for all sufficiently large } n, \\ X \in K^b(\text{inj-}S) &\iff \text{for every } Y, \text{ Hom}(Y, X[n]) = 0 \text{ for all sufficiently large } n. \end{aligned}$$

For a bounded projective complex, or a bounded injective complex, the respective Hom groups are computed by maps of complexes; disjoint degree ranges give necessity.

For sufficiency in the first line, represent X by a bounded complex $X^\bullet$, zero in degrees greater than b . A bounded-above projective resolution can be constructed as follows. Start with $P^i = 0$ for $i > b$. Once $P^{i+1}, P^{i+2}, \dots$ and a chain map f to $X^\bullet$ have been constructed, choose a finite free module P^i surjecting onto

$$\{(x, z) \in X^i \oplus P^{i+1} : d_X^i x = f^{i+1}(z), d_P^{i+1} z = 0\}.$$

Use its two components as f^i and d_P^i . This module of pairs is finite since the algebra is Artin. The equations give $d_P^{i+1} d_P^i = 0$ and $d_X^i f^i = f^{i+1} d_P^i$. Every cycle $x \in X^i$ lifts using the pair $(x, 0)$. If a cycle $z \in P^i$ has $f^i(z) = d_X^{i-1} x$, the pair (x, z) lifts from P^{i-1} , so z is a boundary. Thus $f : P^\bullet \rightarrow X^\bullet$ is a quasi-isomorphism, meaning that it induces isomorphisms on cohomology.

Choose an integer $a < b$ with $H^i(X) = 0$ for all $i \leq a$, and set $M = \text{coker}(P^{a-1} \rightarrow P^a)$. The complex

$$0 \longrightarrow M \longrightarrow P^{a+1} \longrightarrow P^{a+2} \longrightarrow \dots \longrightarrow P^b \longrightarrow 0,$$

with M in degree a , is quasi-isomorphic to $P^\bullet$. Its subcomplex in degrees greater than a is bounded projective, and its quotient is $M[-a]$. The resulting triangle and the necessary vanishing already proved transfer the hypothesis from X to $M[-a]$.

There are finitely many isomorphism classes of simple right S -modules. Applying the hypothesis to representatives of these classes and taking the largest of their bounds gives an integer $d \geq 0$ such that $\text{Ext}_S^{d+1}(M, Y) = 0$ for every simple Y . For a short exact sequence $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$, the long exact sequence places $\text{Ext}_S^{d+1}(M, Y)$ between $\text{Ext}_S^{d+1}(M, Y')$ and $\text{Ext}_S^{d+1}(M, Y'')$. Induction on composition length gives vanishing for every finite Y . A finite free surjection onto the d -th syzygy of M , defined as in Proposition 5.1, has a finite kernel L . Its extension class vanishes by

$$\text{Ext}_S^1(\Omega^d M, L) \cong \text{Ext}_S^{d+1}(M, L) = 0.$$

It splits, so $\Omega^d M$ is projective. Hence M , and then X , is represented by a bounded projective complex.

For a complex $X^\bullet$, set $(DX)^i = D(X^{-i})$ with differential $(-1)^{i+1} D(d_X^{-i-1})$. Exactness of D gives a contravariant equivalence of bounded derived categories, exchanging projective and injective complexes. Under the natural identification $D(X[n]) \cong (DX)[-n]$, it identifies $\text{Hom}(Y, X[n])$ with $\text{Hom}(DX, DY[n])$. Applying the first characterization over S^{op} proves the second. Thus both subcategories are intrinsic and preserved by a derived equivalence.

Finally,

$$\begin{aligned} \text{id}(S_S) < \infty &\iff K^b(\text{proj-}S) \subseteq K^b(\text{inj-}S), \\ \text{id}({}_S S) < \infty &\iff K^b(\text{inj-}S) \subseteq K^b(\text{proj-}S). \end{aligned}$$

For the first equivalence, S_S generates the projective subcategory under shifts, cones and summands. For the second, $D({}_S S)$ generates the injective subcategory and has projective dimension $\text{id}({}_S S)$ by (7). Finite injective and projective resolutions give the forward inclusions; taking these respective generators gives the reverse implications. Both inclusions are preserved by the equivalence. $\square$

7 The opposite algebra and the reduction

Proof of Theorem 1.1. Lemma 3.1 gives the structure and length formula. Propositions 4.2 and 5.1 prove (5) and (6), and hence one direction of each finiteness equivalence.

The identity on the underlying direct sums gives $T(A)^{\text{op}} \cong T(A^{\text{op}})$. Indeed, the dual bimodule actions for A^{op} interchange the two actions for A , so both products are $(ba, b\lambda + \mu a)$ on $(a, \lambda), (b, \mu)$. Applying D to endomorphisms reverses composition. The bimodule isomorphism $DB \cong B$ therefore gives

$$E(A)^{\text{op}} \cong \text{End}_{T(A^{\text{op}})}(T(A^{\text{op}}) \oplus D(A^{\text{op}})).$$

Here the dual of the right B -module A is read as a right B^{op} -module. For $a^{\text{op}} \in A^{\text{op}}$, its action is $(\lambda a^{\text{op}})(x) = \lambda(xa)$, exactly the right action on the dual ideal of $T(A^{\text{op}})$.

The algebra $T(A^{\text{op}})$ is symmetric, hence self-injective by Section 2, and has the exact sequence of right modules

$$0 \longrightarrow D(A^{\text{op}}) \longrightarrow T(A^{\text{op}}) \longrightarrow A^{\text{op}} \longrightarrow 0.$$

Lemma 6.1 now gives a derived equivalence between $E(A)^{\text{op}}$ and $E(A^{\text{op}})$.

If $\text{id}(A_A) < \infty$, apply Proposition 4.2 to A^{op} . The left self-injective dimension of $E(A^{\text{op}})$ is finite. Lemma 6.2 gives the same for $E(A)^{\text{op}}$, so $\text{id}(E(A)_{E(A)}) < \infty$. If $\text{id}_{(E(A)E(A))} < \infty$, the right self-injective dimension of $E(A)^{\text{op}}$ is finite. The same lemma transfers this to $E(A^{\text{op}})$. Proposition 5.1, applied to A^{op} , then gives $\text{id}({}_A A) < \infty$. These are the remaining directions of (3) and (4). $\square$

Corollary 7.1. *Let p be prime and let A be a nonzero finite ring whose additive group is a p -group. Write p^N for its characteristic, with $N \geq 1$, and take $R = \mathbb{Z}/p^N\mathbb{Z}$. For a finite set X , write $|X|$ for its cardinality. Then $E(A)$ is a finite ring with*

$$|E(A)| = |A|^5,$$

and the two finiteness equivalences of Theorem 1.1 hold for this same A .

Proof. The ring R is its own minimal injective cogenerator. To check injectivity by Baer's criterion, for $0 \leq j \leq N$, a homomorphism from its ideal $p^j R$ to R sends p^j to an element killed by p^{N-j} , hence to $p^j y$ for some $y \in R$. Multiplication by y extends the map. Its socle $p^{N-1}R$ is simple and essential, proving the cogenerator assertion. Thus $D = \text{Hom}_R(-, R)$. On each cyclic summand $R/(p^j)$, with $1 \leq j \leq N$, this dual is $p^{N-j}R$, of cardinality p^j , so $|DA| = |A|$. The four blocks in (8) give $|E(A)| = |A|^2 |DA|^2 |A| = |A|^5$. Apply Theorem 1.1 over this R . $\square$

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