

A reduction of the Gorenstein symmetry conjecture to gendo-symmetric algebras

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Abstract

Over any field, the Gorenstein symmetry conjecture for finite-dimensional algebras is equivalent to its restriction to gendo-symmetric algebras.

1 Introduction

Let k be a field. All algebras in this paper are nonzero finite-dimensional associative k -algebras with identity, and all modules are finite-dimensional and unital. Modules are right modules unless a left action is specified. Write $D = \text{Hom}_k(-, k)$ for linear duality. For a right module M over an algebra R , write $\text{pd}_R M$ and $\text{id}_R M$ for its projective and injective dimensions; these dimensions are allowed to be infinite. We display the side explicitly for regular modules. For right R -modules U, V , the group $\text{Ext}_R^n(U, V)$, with integer $n \geq 0$, is the degree- n cohomology of $\text{Hom}_R(P_\bullet, V)$, where $P_\bullet \rightarrow U$ is a projective resolution indexed in nonnegative degrees. In particular, $\text{Ext}_R^0(U, V) = \text{Hom}_R(U, V)$.

For any algebra R , the bimodule structure on DR is $(r\varphi s)(t) = \varphi(str)$ for $r, s, t \in R$ and $\varphi \in DR$. The Gorenstein symmetry conjecture asserts that every such algebra A satisfies

$$\text{id}({}_A A) < \infty \iff \text{id}(A_A) < \infty. \quad (1)$$

A finite-dimensional algebra S is *symmetric* if $DS \cong S$ as S -bimodules. A right S -module M is a *generator* if $S \in \text{add } M$, where $\text{add } M$ consists of direct summands of finite direct sums of copies of M . An algebra is *gendo-symmetric* if it is isomorphic to $\text{End}_S(M)$ for a generator M over a symmetric algebra S . Products in endomorphism algebras are composition: $\alpha\beta = \alpha \circ \beta$.

Gendo-symmetric algebras were studied by Fang and Koenig [3, 4]. Aihara, Chan and Honma [1] consider endomorphism algebras of generators over trivial extensions. For a self-injective algebra S and an indecomposable nonprojective S -module X , Chen and Koenig [2, Corollary 3.25] prove Gorenstein symmetry for $\text{End}_S(S \oplus X)$. Han and Qin [5, Corollary 8] give a reduction of the conjecture through recollements of derived categories. The reduction here uses the canonical quotient of a trivial extension and applies to every base algebra.

For an algebra A , its trivial extension $T(A) = A \ltimes DA$ is the vector space $A \oplus DA$ with multiplication, for $a, b \in A$ and $\lambda, \mu \in DA$,

$$(a, \lambda)(b, \mu) = (ab, a\mu + \lambda b)$$

and identity $(1, 0)$. The projection $\pi : T(A) \rightarrow A$, $(a, \lambda) \mapsto a$, makes A a right $T(A)$ -module by $x \cdot b = x\pi(b)$. Define

$$B = T(A), \quad E(A) = \text{End}_B(B \oplus A). \quad (2)$$

When A is fixed, we write $E = E(A)$.

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Theorem 1.1. *For every algebra A , the algebra $E = E(A)$ is gendo-symmetric and*

$$\dim_k E = 5 \dim_k A.$$

If $d = \text{id}(A_A) < \infty$, then

$$\text{id}(E E) \leq 2d + 2. \quad (3)$$

Moreover,

$$\text{id}(A_A) \leq \text{id}(E E), \quad (4)$$

where the inequality is interpreted in $\{0, 1, 2, \dots\} \cup \{\infty\}$.

The proof is completed in Section 4.

Corollary 1.2. *For each field k , the following statements are equivalent:*

- (i) *Every finite-dimensional k -algebra satisfies (1).*
- (ii) *Every finite-dimensional gendo-symmetric k -algebra satisfies (1).*
- (iii) *The algebra $E(A)$ satisfies (1) for every finite-dimensional k -algebra A .*

Proof. Theorem 1.1 gives the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Assume (iii), and let $\text{id}(A_A) = d < \infty$. Then (3) and (iii) give $\text{id}(E(A)_{E(A)}) < \infty$, and (4) gives $\text{id}(A_A) < \infty$. Applying this implication to A^{op} , the algebra with reversed multiplication, proves the converse direction of (1). $\square$

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2 The endomorphism algebra

Let $e, f \in E$ be the endomorphisms defined by $e(z, t) = (z, 0)$ and $f(z, t) = (0, t)$ for $z \in B, t \in A$. Thus $e + f = 1$, $ef = fe = 0$, and $e^2 = e, f^2 = f$. We identify DA with the square-zero ideal $\{(0, \lambda) : \lambda \in DA\}$ of B .

The elements $z \in B$ satisfying $zDA = 0$ are precisely those in DA . Indeed, if $(a, \lambda)(0, \mu) = 0$ for every $\mu \in DA$, then $a\mu = 0$ for all μ . Evaluating at 1 gives $\mu(a) = 0$ for every linear functional μ , and hence $a = 0$. A B -linear map $A \rightarrow B$ is determined by the image of 1, which must annihilate DA on the right. Conversely, each $u \in DA$ defines a B -linear map $t \mapsto ut$: for $t \in A$ and $b \in B$, one has $u(t\pi(b)) = (ut)b$. Evaluation at 1 therefore gives

$$\text{Hom}_B(A, B) = DA, \quad \text{Hom}_B(B, A) = A, \quad \text{End}_B(A) = A.$$

Together with $\text{End}_B(B) = B$, these identifications give

$$E = \begin{pmatrix} B & DA \\ A & A \end{pmatrix}. \quad (5)$$

Here $b \in B$ represents left multiplication on B , $u \in DA$ represents the map $A \rightarrow B, t \mapsto ut$, $v \in A$ represents $B \rightarrow A, z \mapsto v\pi(z)$, and the lower right entry $a \in A$ represents left multiplication by a on A .

In particular, if $b, b' \in B$, $u, u' \in DA$, and $v, v', a, a' \in A$, multiplication is

$$\begin{pmatrix} b & u \\ v & a \end{pmatrix} \begin{pmatrix} b' & u' \\ v' & a' \end{pmatrix} = \begin{pmatrix} bb' + uv' & bu' + ua' \\ v\pi(b') + av' & aa' \end{pmatrix}. \quad (6)$$

The B -actions on DA in this formula factor through π . The product uv' in the upper left block is included in the ideal $DA \subset B$. Every composite $A \rightarrow B \rightarrow A$ is zero, since the first map lands in DA and the second kills DA .

Consequently the map

$$q : E \longrightarrow A, \quad \begin{pmatrix} b & u \\ v & a \end{pmatrix} \longmapsto a \quad (7)$$

is an algebra epimorphism. For a right A -module X , denote by X_q the same vector space with right E -action $xr = xq(r)$.

Lemma 2.1. *The algebra B is symmetric. The algebra E is gendo-symmetric and has dimension $5 \dim_k A$.*

Proof. Define the linear functional $\tau : B \rightarrow k$ by $\tau(a, \lambda) = \lambda(1)$. Its associated bilinear form is

$$\tau((a, \lambda)(b, \mu)) = \mu(a) + \lambda(b).$$

This form is symmetric. If it vanishes for all (b, μ) , setting $b = 0$ gives $a = 0$, and setting $\mu = 0$ gives $\lambda = 0$; hence it is nondegenerate. Thus $b \mapsto [c \mapsto \tau(cb)]$ is a bijection $B \rightarrow DB$. For $r, s, b, c \in B$, associativity and symmetry give

$$\tau((scr)b) = \tau(s(crb)) = \tau((crb)s) = \tau(c(rbs)).$$

By the dual action defined in Section 1, this is the bimodule identity for the map. Since $B \oplus A$ contains B , it is a generator, proving that E is gendo-symmetric. The upper left block in (5) has dimension $2 \dim_k A$; each of the other three blocks has dimension $\dim_k A$. Their sum is $5 \dim_k A$. $\square$

For a right B -module X , set

$$H(X) = \text{Hom}_B(B \oplus A, X).$$

It is a right E -module by precomposition, $hr = h \circ r$. If X is a right A -module, its inflation along π has action $xb = x\pi(b)$ for $x \in X$, $b \in B$. A map $B \oplus A \rightarrow X$ then has the form $(z, t) \mapsto x\pi(z) + yt$ for unique $x, y \in X$. Precomposing with the block matrix in (6) gives

$$H(X) = X \oplus X, \quad (x, y) \begin{pmatrix} b & u \\ v & a \end{pmatrix} = (x\pi(b) + yv, ya). \quad (8)$$

For a homomorphism $g : X \rightarrow Y$ of inflated modules, $H(g)$ is $(x, y) \mapsto (g(x), g(y))$; hence this restriction of H is exact. The identifications $H(B) = eE$ and $H(A) = fE$ send a map into a summand of $B \oplus A$ to the corresponding row of an endomorphism. In particular, H takes projective A -modules to projective E -modules: it preserves finite direct sums and direct summands and sends A to fE .

3 The finite injective dimension

Linear duality is exact and exchanges finite projective modules with finite injective modules on the opposite side. Dualizing resolutions therefore gives

$$\text{id}({}_A A) = \text{pd}_A DA, \quad \text{id}({}_E E) = \text{pd}_E DE.$$

Here both projective dimensions are those of right modules, and DA , regarded as an ideal of B , has the action inflated along π .

Lemma 3.1. *Suppose $d = \text{pd}_A DA < \infty$. Then*

$$\text{pd}_E H(DA) \leq d, \quad \text{pd}_E A_q \leq d + 2, \quad \text{pd}_E (DA)_q \leq 2d + 2.$$

If K is the kernel of $H(A) \rightarrow A_q$, $(x, y) \mapsto y$, then $\text{pd}_E K \leq d + 1$.

Proof. Choose an exact sequence

$$0 \longrightarrow P_d \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow DA \longrightarrow 0$$

of right A -modules with each P_i projective. For $d = 0$, this means $P_0 \cong DA$. Applying H to the modules inflated along π gives an exact sequence with E -projective terms $H(P_i)$, proving $\text{pd}_E H(DA) \leq d$.

Left exactness of H , applied to $0 \rightarrow DA \rightarrow B \xrightarrow{\pi} A \rightarrow 0$, together with the following cokernel calculation gives

$$0 \longrightarrow H(DA) \longrightarrow eE \xrightarrow{H(\pi)} fE \longrightarrow A_q \longrightarrow 0. \quad (9)$$

We use row coordinates $eE = B \oplus DA$ and $fE = A \oplus A$ from (5). The map $H(\pi)$ sends (b, u) to $(\pi(b), 0)$. Its image is $K = \{(x, 0) : x \in A\}$, and projection to the second coordinate identifies its cokernel with A_q . Splicing the resolution of $H(DA)$ into (9) gives $\text{pd}_E K \leq d + 1$ and $\text{pd}_E A_q \leq d + 2$.

Inflate the same length- d resolution of DA along q . Every term is a direct summand of a finite sum of copies of A_q , and thus has E -projective dimension at most $d + 2$. For an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of right E -modules, the Ext long exact sequence gives $\text{pd}_E N \leq \max\{\text{pd}_E M, \text{pd}_E L + 1\}$. To apply this bound explicitly, put $Z_0 = (DA)_q$ and, for $0 \leq i < d$, let Z_{i+1} be the kernel of the induced epimorphism $(P_i)_q \rightarrow Z_i$. Then $Z_d \cong (P_d)_q$, and the short exact sequences

$$0 \longrightarrow Z_{i+1} \longrightarrow (P_i)_q \longrightarrow Z_i \longrightarrow 0 \quad (0 \leq i < d)$$

give, by descending induction, $\text{pd}_E Z_i \leq d + 2 + d - i$. Taking $i = 0$ proves the last bound. If $d = 0$, it follows directly from $(DA)_q \cong (P_0)_q$. $\square$

Proposition 3.2. *If $d = \text{id}({}_A A) < \infty$, then $\text{id}({}_E E) \leq 2d + 2$.*

Proof. The decomposition ${}_E E = Ee \oplus Ef$ gives $DE = D(Ee) \oplus D(Ef)$ as right E -modules. For either left module Ee or Ef , its dual has right action $(\varphi r)(s) = \varphi(rs)$, where $r \in E$, s belongs to that left module, and φ is a linear functional on it.

First, $D(Ee) \cong eE$. Use row coordinates $eE = B \oplus DA$ and column coordinates $Ee = B \oplus A$. For $c, z \in B$, $\lambda \in DA$, and $t \in A$, define

$$\langle (c, \lambda), (z, t) \rangle = \tau(cz) + \lambda(t). \quad (10)$$

It is nondegenerate because the trace pairing on B and the evaluation pairing between DA and A are nondegenerate. If $r = \begin{pmatrix} b & u \\ v & a \end{pmatrix}$, then

$$(c, \lambda)r = (cb + \lambda v, cu + \lambda a), \quad r(z, t) = (bz + ut, v\pi(z) + at).$$

The identities

$$\tau(\lambda vz) = \lambda(v\pi(z)), \quad \tau(cut) = (cu)(t), \quad (\lambda a)(t) = \lambda(at)$$

give

$$\begin{aligned} \langle (c, \lambda)r, (z, t) \rangle &= \tau(cbz) + \lambda(v\pi(z)) + (cu)(t) + \lambda(at) \\ &= \langle (c, \lambda), r(z, t) \rangle. \end{aligned}$$

Thus (10) identifies eE with $D(Ee)$ as a right E -module, and this summand is projective.

Next, in column coordinates $Ef = DA \oplus A$, an element (w, t) , with $w \in DA$ and $t \in A$, has action

$$\begin{pmatrix} b & u \\ v & a \end{pmatrix} (w, t) = (bw + ut, at).$$

For $x \in A$ and $\lambda \in DA$, the pairing

$$((x, \lambda), (w, t)) \mapsto w(x) + \lambda(t)$$

identifies $D(Ef)$ with $A \oplus DA$. Its dual action is determined by

$$(bw + ut)(x) + \lambda(at) = w(x\pi(b)) + u(tx) + \lambda(at) = w(x\pi(b)) + (xu + \lambda a)(t).$$

Thus its right action is

$$(x, \lambda) \begin{pmatrix} b & u \\ v & a \end{pmatrix} = (x\pi(b), xu + \lambda a). \quad (11)$$

Comparison with (8) shows that

$$D(Ef) \longrightarrow H(A), \quad (x, \lambda) \mapsto (x, 0)$$

is E -linear, with kernel $(DA)_q$ and image K from Lemma 3.1. We obtain a short exact sequence

$$0 \longrightarrow (DA)_q \longrightarrow D(Ef) \longrightarrow K \longrightarrow 0.$$

For a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$, the terms $\text{Ext}_E^n(N, V)$ and $\text{Ext}_E^n(L, V)$ adjacent to $\text{Ext}_E^n(M, V)$ vanish when $n > \max\{\text{pd}_E L, \text{pd}_E N\}$, for every V . Hence $\text{pd}_E M \leq \max\{\text{pd}_E L, \text{pd}_E N\}$. Lemma 3.1 therefore gives

$$\text{pd}_E D(Ef) \leq \max\{2d + 2, d + 1\} = 2d + 2.$$

Together with the projectivity of $D(Ee)$, this proves

$$\text{id}_E(E) = \text{pd}_E DE \leq 2d + 2. \quad \square$$

4 Descent along the quotient summand

The lower right block identifies the corner algebra fEf with A . Left multiplication by this corner makes fE an A - E -bimodule. In row coordinates $fE = A \oplus A$, its left action is $a(x, y) = (ax, ay)$, so ${}_A fE$ is free of rank two. Write $\text{mod-}R$ for the category of finite-dimensional right modules over an algebra R , and define

$$F : \text{mod-}A \longrightarrow \text{mod-}E, \quad F(U) = U \otimes_A fE.$$

This functor is exact because ${}_A fE$ is free. It preserves projectives because $F(A) \cong fE$ is a direct summand of E_E . For a right E -module N , the subspace Nf is a right $A = fEf$ -module, and tensor–Hom adjunction gives

$$\text{Hom}_E(F(U), N) \cong \text{Hom}_A(U, Nf). \quad (12)$$

Explicitly, an E -linear map $\varphi : F(U) \rightarrow N$ is sent to $u \mapsto \varphi(u \otimes f)$, whose image lies in Nf . Conversely, an A -linear map $\psi : U \rightarrow Nf$ gives

$$u \otimes fr \longmapsto \psi(u)r \quad (u \in U, r \in E).$$

This is well-defined: $\psi(u) = \psi(u)f$ makes the value depend only on fr , and $\psi(ua)r = \psi(u)ar$ gives the tensor balancing relation for $a \in A = fEf$. These constructions are inverse and natural. Right multiplication by f is an idempotent projection on fE , with image $fEf = A$. It follows that

$$(F(U))f \cong U \otimes_A fEf \cong U, \quad u \otimes a \longmapsto ua.$$

The inverse of the last map is $u \mapsto u \otimes f$; also $(fE)f = A$.

Proposition 4.1. *For every finite-dimensional right A -module U ,*

$$\text{pd}_E F(U) = \text{pd}_A U,$$

including the value ∞ . Moreover, $\text{id}(A_A) \leq \text{id}(E_E)$.

Proof. Applying F to a projective resolution of U gives an E -projective resolution of $F(U)$. Applying (12) term by term consequently gives, for every right E -module N and every integer $n \geq 0$,

$$\text{Ext}_E^n(F(U), N) \cong \text{Ext}_A^n(U, Nf). \quad (13)$$

The resolution already gives $\text{pd}_E F(U) \leq \text{pd}_A U$. If $\text{pd}_E F(U) \leq r$ for a nonnegative integer r , take $N = F(V)$ for an arbitrary finite right A -module V . Then (13) implies $\text{Ext}_A^n(U, V) = 0$ for all $n > r$, so $\text{pd}_A U \leq r$. Indeed, for $r > 0$ let $\Omega^r U$ be defined by an exact sequence $0 \rightarrow \Omega^r U \rightarrow P_{r-1} \rightarrow \cdots \rightarrow P_0 \rightarrow U \rightarrow 0$ with finite projective terms, and set $\Omega^0 U = U$. Choose an epimorphism $P \rightarrow \Omega^r U$ with P finite projective and kernel L . Dimension shifting gives

$$\text{Ext}_A^1(\Omega^r U, L) \cong \text{Ext}_A^{r+1}(U, L) = 0.$$

Thus $0 \rightarrow L \rightarrow P \rightarrow \Omega^r U \rightarrow 0$ splits, so $\Omega^r U$ is projective. This proves the equality, also when either dimension is infinite.

For the second assertion we may assume $m = \text{id}(E_E) < \infty$. Set $N = fE$ in (13). Since fE is a direct summand of E_E , we obtain

$$\text{Ext}_A^n(U, A) \cong \text{Ext}_E^n(F(U), fE) = 0 \quad (n > m)$$

for every finite right A -module U . Take an injective resolution of A_A with finite terms, and let C be its m -th cosyzygy: for $m > 0$, it occurs in the exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{m-1} \longrightarrow C \longrightarrow 0;$$

for $m = 0$, set $C = A$. For every right ideal $I \subseteq A$, dimension shifting gives

$$\mathrm{Ext}_A^1(A/I, C) \cong \mathrm{Ext}_A^{m+1}(A/I, A) = 0.$$

The module A/I is finite, so the preceding vanishing applies. The exact sequence obtained by applying $\mathrm{Hom}_A(-, C)$ to $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ shows that every map $I \rightarrow C$ extends to $A \rightarrow C$. By Baer's criterion, C is injective, and hence $\mathrm{id}(A_A) \leq m$. $\square$

Proof of Theorem 1.1. Lemma 2.1 proves the structural and dimension assertions. Proposition 3.2 proves (3), and Proposition 4.1 proves (4). $\square$

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