

ADDITIVE ROUNDING FOR LP-TIGHT THREE-ITEM BIN PACKING

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ABSTRACT. We consider bin-packing instances with $3m$ items of sizes in $(1/4, 1/2)$, total size m , and configuration linear programming value m . Let d be the relative dimension of the polytope of zero-sum multiplicity patterns obtained by centering sizes at $1/3$. We determine the maximum additive integrality gap for every $d \leq 10$. It is zero for $d \leq 4$ and one for $5 \leq d \leq 10$. More generally, a representation of the multiplicity vector by at most five zero-sum patterns admits an integral packing, and a representation by at most eleven patterns requires at most one additional bin. The integral-packing result has a structural proof; the additive-one result combines structural restrictions with exact finite enumeration. For arbitrary d , we construct an LP-tight subinstance with at most $6d + 3$ items and no smaller additive gap. An integer representation preserving the signs of triple sums then yields a finite rational family attaining the worst gap, which is computable in time $2^{O((d+1)^2)}$. Finally, we construct a sharp family with simplex pattern polytopes and a linearly growing zero-sum matching deficit. Its eleven-dimensional member violates the zero-sum remainder condition used in the ten-dimensional proof. Thus the bound of eleven fractional patterns in that condition is sharp.

1. INTRODUCTION AND MAIN RESULTS

The configuration linear program is a fundamental relaxation for bin packing and cutting stock. It represents a fractional packing as a nonnegative combination of feasible configurations. The formulation originates in the work of Gilmore and Gomory [4]. Karmarkar and Karp [7] established a polylogarithmic additive approximation bound. Subsequent improvements by Rothvoß [9] and by Hoberg and Rothvoß [6] reduced the additive bound to logarithmic order. The integer round-up property requires $\text{OPT}(I) = \lceil \text{LP}(I) \rceil$; the modified integer round-up inequality allows one further bin.

We study the class of instances satisfying

$$(1.1) \quad |V| = 3m, \quad \frac{1}{4} < s_v < \frac{1}{2} \quad (v \in V), \quad \sum_{v \in V} s_v = \text{LP}(I) = m,$$

where m is a positive integer, V is the labelled item set, and s_v is the size of item v . The relaxation $\text{LP}(I)$ uses configurations that can be formed from the actual items. We write $\text{OPT}(I)$ for the minimum number of bins. Every pair fits in a bin, and no bin contains four items. Equality between the fractional optimum and the total size forces every positively weighted configuration in an optimum fractional solution to be a full triple.

Put $a_v = s_v - 1/3$. The full triples have centered sum zero and admit a fractional perfect matching. If $\nu_{\leq 0}(I)$ is the maximum number of disjoint nonpositive-sum triples, then

$$(1.2) \quad \text{OPT}(I) = m + \left\lceil \frac{m - \nu_{\leq 0}(I)}{2} \right\rceil.$$

Thus the inequality

$$(1.3) \quad \text{OPT}(I) \leq m + 1$$

is equivalent to $\nu_{\leq 0}(I) \geq m - 2$. We write $\nu_0(I)$ for the maximum number of disjoint zero-sum triples. A packing in exactly m bins consists entirely of such triples.

Group equal centered values into t classes, with distinct values $\beta_1, \dots, \beta_t$ and multiplicities $n \in \mathbb{Z}_{>0}^t$. Define

$$(1.4) \quad \mathcal{Z}_I = \{p \in \mathbb{Z}_{\geq 0}^t : \mathbf{1}^T p = 3, \beta^T p = 0, p \leq n\}, \quad \mathcal{P}_I = \text{conv } \mathcal{Z}_I.$$

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The fractional perfect matching condition is $n \in m\mathcal{P}_I$. Throughout, $\dim \mathcal{P}_I$ is the relative dimension of the complete zero-sum pattern polytope. For an integer $d \geq 0$, set

$$(1.5) \quad \Delta_d = \sup\{\text{OPT}(I) - \text{LP}(I) : I \text{ satisfies (1.1), } \dim \mathcal{P}_I \leq d\}.$$

The number of items is unrestricted in this definition.

1.1. Exact gaps through dimension ten.

Theorem 1.1. *Let I satisfy (1.1).*

- (i) *If n is a nonnegative real combination of at most five members of $\mathcal{Z}_I$, then $v_0(I) = m$ and $\text{OPT}(I) = m$.*
- (ii) *If n is a nonnegative real combination of at most eleven members of $\mathcal{Z}_I$, then*

$$v_{\leq 0}(I) \geq m - 2, \quad \text{OPT}(I) \leq m + 1.$$

In particular, $\text{OPT}(I) = m$ if $\dim \mathcal{P}_I \leq 4$, and $\text{OPT}(I) \leq m + 1$ if $\dim \mathcal{P}_I \leq 10$.

Corollary 1.2. *The exact maximum gaps through dimension ten are*

$$(1.6) \quad \Delta_d = \begin{cases} 0, & 0 \leq d \leq 4, \\ 1, & 5 \leq d \leq 10. \end{cases}$$

Thus dimension five is the smallest complete pattern dimension at which a positive additive gap occurs in (1.1).

The two assertions of Theorem 1.1 use a common integer remainder. An independent pattern representation permits removal of the integer parts of its coefficients. The fractional parts and their complements then determine two positive integer multiplicity vectors on the same support. Their coordinate sums bound the remainder size. Distinct zero-sum patterns overlap in at most one item, and patterns with a repeated value form disjoint directed paths on the value classes. For five patterns these facts give a complete structural argument. For eleven patterns they reduce the additive-one bound to seven finite matrix statements. We prove the largest-remainder statement directly and verify all seven by exact enumeration.

The dimension bounds also give statements in terms of the number of item types. An instance in (1.1) with at most six distinct sizes has an integral packing in m bins; one with at most twelve distinct sizes requires at most $m + 1$ bins. Independently of the number of patterns, the additive-one bound holds for $m \leq 7$, by Füredi's fractional matching theorem [3] and the rigidity of zero-sum Fano planes.

1.2. Finite extremal instances in arbitrary dimension.

Theorem 1.3. *Let I satisfy (1.1), and put $d = \dim \mathcal{P}_I$. There is a subinstance I' satisfying the same conditions, with $3m'$ items, such that*

$$(1.7) \quad m' \leq 2d + 1, \quad \dim \mathcal{P}_{I'} \leq d, \quad \text{OPT}(I) - m \leq \text{OPT}(I') - m'.$$

For every fixed d , the supremum Δ_d is attained by a rational instance with at most $6d + 3$ items and binary encoding length $O((d + 1)^2)$. It can be computed by a deterministic algorithm in time $2^{O((d+1)^2)}$.

The subinstance retains the original sizes and enough copies of every selected value to keep all selected patterns available. Its fractional representation therefore remains a fractional perfect matching on actual items. Replacing the centered values by bounded integers while preserving all triple signs gives a finite rational family depending only on d . The maximum gap over that family equals the maximum over the entire class in (1.5).

The quantifiers in this result concern all instances of a fixed pattern dimension. Goemans and Rothvoß [5] give a polynomial-time algorithm for a specified bin-packing instance with a fixed number of item types and binary-encoded data. Here the extremal instance itself is allowed to vary, and its size and encoding length are bounded solely in terms of the pattern dimension.

1.3. Sharp families and the twelve-pattern obstruction.

Theorem 1.4. *For every integer $B \geq 1$, there is a rational instance I_B satisfying (1.1) for which*

$$\begin{aligned} |I_B| &= 18B, & \text{LP}(I_B) &= 6B, & \text{OPT}(I_B) &= 6B + 1, \\ \dim \mathcal{P}_{I_B} &= 6B - 1, & \nu_0(I_B) &= 4B, & \nu_{\leq 0}(I_B) &= 6B - 1. \end{aligned}$$

The complete pattern polytope is a simplex. The integer remainder obtained from its unique fractional representation has $9B$ items and maximum zero-sum matching size B .

For $B = 1$, this theorem supplies the dimension-five example in Corollary 1.2. For $B = 2$, it gives an instance with complete pattern dimension eleven and twelve independent patterns. Its eighteen-item remainder has only two disjoint zero-sum triples. The condition used in the ten-dimensional proof would require three, followed by one nonpositive-sum triple from the remaining nine items. That condition holds for all basic remainders with at most eleven columns and at least nine items, and it fails at twelve columns even when those columns are the complete zero-sum pattern set. The family itself continues to satisfy the additive-one bound. Moreover, any matching attaining $\nu_{\leq 0} \geq m - 2$ in I_B uses at least $2B - 2$ strictly negative triples.

Rietz, Scheithauer, and Terno [8] constructed families failing the integer round-up property. Caprara et al. [1] obtained small numerical examples from the Petersen graph. The family above specifies the complete pattern polytope and both matching numbers simultaneously. Eisenbrand, Pálvölgyi, and Rothvoß [2] relate three-item bin packing to permutation discrepancy and prove restrictions on rounding methods. Our obstruction concerns the explicit zero-sum containment condition for the integer remainder.

Section 2 proves the matching and pattern formulations. Section 3 develops the integer remainder and its structural properties. Sections 4 and 5 prove the two assertions of Theorem 1.1. Section 6 treats the bound from the number of items. Section 7 proves the finite extremal reduction, and Section 8 constructs the sharp family and the twelve-pattern obstruction. The finite enumeration is specified in Appendix A.

2. ZERO-SUM TRIPLES AND THE CONFIGURATION RELAXATION

2.1. Configuration linear programming and LP tightness. For a positive integer t , write $[t] = \{1, \dots, t\}$. Vector inequalities are coordinatewise, e_u is a standard unit vector, and $\text{supp } x = \{u : x_u \neq 0\}$. A configuration of a labelled instance $I = (V, s)$ is a subset $C \subseteq V$ with $\sum_{v \in C} s_v \leq 1$. Write $\mathcal{C}(I)$ for the family of configurations. The configuration linear program is

$$(2.1) \quad \begin{aligned} \text{LP}(I) &= \min \sum_{C \in \mathcal{C}(I)} x_C \\ \text{subject to} \quad &\sum_{C \ni v} x_C \geq 1 \quad (v \in V), \\ &x_C \geq 0 \quad (C \in \mathcal{C}(I)). \end{aligned}$$

A packing is a partition of V into configurations. Deleting unused items from configurations shows that the corresponding integer covering problem has the same optimum $\text{OPT}(I)$.

For a real assignment $a : V \rightarrow \mathbb{R}$, put $a(U) = \sum_{v \in U} a_v$ and define the three-uniform hypergraphs

$$H_0(a) = \{e \in \binom{V}{3} : a(e) = 0\}, \quad H_{\leq 0}(a) = \{e \in \binom{V}{3} : a(e) \leq 0\}.$$

A matching is a family of disjoint edges; its maximum size is denoted by ν . A fractional matching assigns nonnegative weights to the edges so that the incident weights sum to at most one at each vertex. It is perfect when every such sum equals one. Its maximum total edge weight is ν^* . We use $\nu_0(a) = \nu(H_0(a))$ and $\nu_{\leq 0}(a) = \nu(H_{\leq 0}(a))$, or the argument I when $a_v = s_v - 1/3$.

Lemma 2.1. *Suppose $|V| = 3m$, $1/4 < s_v < 1/2$, and $\sum_v s_v = m$. Then $\text{LP}(I) = m$ if and only if $H_0(a)$ has a fractional perfect matching, where $a_v = s_v - 1/3$. Every optimum solution of (2.1) of value m has tight covering constraints and is supported on triples of total size one.*

Proof. For every feasible x ,

$$m = \sum_v s_v \leq \sum_v s_v \sum_{C \ni v} x_C = \sum_C x_C \sum_{v \in C} s_v \leq \sum_C x_C.$$

At objective value m , equality holds throughout. Positivity of every s_v makes each covering constraint tight. Every configuration of positive weight has total size one. Such a configuration has exactly three items, since pairs have total size less than one and four items have total size greater than one. Its centered sum is zero. Conversely, the vertex equations of a fractional perfect matching give $3 \sum_e x_e = |V| = 3m$. Its weights are feasible in (2.1) and attain the volume lower bound. $\square$

2.2. The integer optimum.

Proposition 2.2. *Under the size and total-volume assumptions of Lemma 2.1,*

$$(2.2) \quad \text{OPT}(I) = m + \left\lceil \frac{m - v_{\leq 0}(I)}{2} \right\rceil.$$

Thus $\text{OPT}(I) \leq m + K$ is equivalent to $v_{\leq 0}(I) \geq m - 2K$ for every integer $K \geq 0$.

Proof. Let $q = v_{\leq 0}(I)$. A matching of size q , followed by pairing the remaining items, uses

$$q + \left\lceil \frac{3m - 3q}{2} \right\rceil = m + \left\lceil \frac{m - q}{2} \right\rceil$$

bins. Conversely, a packing with T triples has $T \leq q$. Its other $3m - 3T$ items require at least $\lceil (3m - 3T)/2 \rceil$ bins. The resulting bound $m + \lceil (m - T)/2 \rceil$ is at least the right-hand side of (2.2). The final equivalence follows by taking ceilings of integers divided by two. $\square$

Lemma 2.3. *If $|U| \geq 3$ and $a(U) = 0$, then U contains a triple of nonpositive sum.*

Proof. Each vertex belongs to $\binom{|U|-1}{2}$ triples, so

$$\sum_{T \in \binom{U}{3}} a(T) = \binom{|U|-1}{2} a(U) = 0.$$

At least one summand is nonpositive. $\square$

Multiplying a finite real assignment by a sufficiently small positive constant places all its values in $(-1/12, 1/6)$ without changing H_0 or $H_{\leq 0}$. Adding $1/3$ then gives legitimate item sizes. Hence all arguments about centered assignments below apply to the bin-packing class (1.1).

2.3. Multiplicity patterns and their polytope. Let $\beta = (\beta_1, \dots, \beta_t)^T$ list the distinct centered values, and let n_u be their multiplicities. A pattern is a nonnegative integer vector with coordinate sum three. It represents an actual triple precisely when it is bounded by n . We use $\mathcal{Z}_I$ and $\mathcal{P}_I$ from (1.4).

Lemma 2.4. *The hypergraph $H_0(a)$ has a fractional perfect matching if and only if*

$$(2.3) \quad n = \sum_{p \in \mathcal{Z}_I} y_p p, \quad y_p \geq 0.$$

Every such representation has $\sum_p y_p = m$. Equivalently, $n \in m\mathcal{P}_I$.

Proof. Aggregate the weights of a labelled fractional perfect matching by their patterns. This gives (2.3). Conversely, pattern $p \leq n$ has $N(p) = \prod_u \binom{n_u}{p_u} > 0$ labelled realizations. Distribute its weight y_p uniformly among them. A fixed item of class u occurs in the proportion p_u/n_u of these realizations, including proportion zero if $p_u = 0$. Its resulting degree is $\sum_p y_p p_u/n_u = 1$. Summing coordinates in (2.3) gives $3m = 3 \sum_p y_p$. $\square$

For an integer multiplicity vector $b \geq 0$, let

$$Q(b, \beta) = \{q \in \mathbb{Z}_{\geq 0}^t : \mathbf{1}^T q = 3, \beta^T q \leq 0, q \leq b\}.$$

Then

$$(2.4) \quad v_{\leq 0}(b, \beta) = \max \left\{ \sum_q z_q : \sum_q z_q q \leq b, \quad z_q \in \mathbb{Z}_{\geq 0}, \quad q \in Q(b, \beta) \right\}.$$

Indeed, a labelled matching supplies the counts z_q . Conversely, the coordinate inequalities allow distinct labels to be assigned to all positions of each type. This observation will also be used when b is an integer remainder.

3. INTEGER REMAINDERS OF FRACTIONAL REPRESENTATIONS

3.1. Independent patterns and complementary multiplicities.

Lemma 3.1. *If $n \in m\mathcal{P}_I$ and $d = \dim \mathcal{P}_I$, then*

$$n = \sum_{j=1}^{\ell} \alpha_j p_j, \quad \alpha_j > 0, \quad \sum_j \alpha_j = m, \quad \ell \leq d + 1,$$

for affinely independent $p_j \in \mathcal{Z}_I$. These columns are also linearly independent.

Proof. Choose a convex representation of n/m with the fewest positive coefficients. An affine dependence among its patterns would give real numbers z_j , not all zero, with $\sum_j z_j p_j = 0$ and $\sum_j z_j = 0$. Subtracting a suitable positive multiple of z from the coefficient vector preserves nonnegativity and makes at least one coefficient zero. This contradicts minimality. An affinely independent set in a d -dimensional affine space has at most $d + 1$ members. Finally, every linear dependence among these patterns has coefficient sum zero, since each column has coordinate sum three. It is therefore an affine dependence. $\square$

Take such a representation and put $b_j = \lfloor \alpha_j \rfloor$. The inequality $Pb \leq n$ allows b_j copies of each pattern to be assigned disjoint actual items. Remove them. Discard the columns with integral coefficients, and write the remaining system as

$$(3.1) \quad P = (p_1 \cdots p_k), \quad r = P\lambda, \quad 0 < \lambda_j < 1, \quad h = \mathbf{1}^T \lambda \in \mathbb{Z}.$$

Thus $3h$ items remain, and $m - h$ zero-sum triples have already been packed. The case $k = 0$ is an integral packing. Restrict all vectors to the rows used by P , and define

$$(3.2) \quad \mu = \mathbf{1} - \lambda, \quad \kappa = P\mu = P\mathbf{1} - r.$$

We call κ the complementary multiplicity vector.

Lemma 3.2. *For a nonempty remainder whose centered values are not all zero,*

$$(3.3) \quad r, \kappa \in \mathbb{Z}_{>0}^t, \quad \text{rank } P = k, \quad \beta^T P = 0, \quad \beta^T r = \beta^T \kappa = 0,$$

$$(3.4) \quad \mathbf{1}^T r = 3h, \quad \mathbf{1}^T \kappa = 3(k - h), \quad k + 1 \leq t \leq \min\{3h, 3(k - h)\}, \quad 3h \leq 2k - 1.$$

Proof. Both vectors are integer because $r = n - Pb$ and $\kappa = P\mathbf{1} - r$. Each retained row contains a positive entry, so $P\lambda$ and $P\mu$ are strictly positive there. Deleting unused rows does not alter column rank. All remaining identities follow from the column sums and the zero-sum equations.

The nonzero vector β belongs to $\ker P^T$, giving $t \geq k + 1$. Positive integrality bounds t by each of $\mathbf{1}^T r$ and $\mathbf{1}^T \kappa$. In particular, $k + 1 \leq 3(k - h)$, which is equivalent to $3h \leq 2k - 1$. $\square$

The all-zero remainder can be partitioned into zero-sum triples directly. For a general remainder, an integer vector $z \geq 0$ specifies available pattern copies exactly when $Pz \leq r$. In particular, if $J \subseteq \{1, \dots, k\}$ satisfies

$$(3.5) \quad |J| = h - 3, \quad \sum_{j \in J} p_j \leq r,$$

then these patterns give $h - 3$ disjoint zero-sum triples. The remaining nine items have total centered value zero, so Lemma 2.3 gives one further nonpositive-sum triple. This yields $h - 2$ triples in the remainder and $m - 2$ in the original instance. Repeated entries in a pattern make the coordinate inequalities in (3.5) essential.

3.2. Intersections and repeated values.

Lemma 3.3. *Distinct zero-sum patterns on distinct value classes satisfy*

$$(3.6) \quad \sum_u \min\{p_{ui}, p_{uj}\} \leq 1 \quad (i \neq j).$$

Proof. If the sum were at least two, choose an integer vector q of coordinate sum two with $q \leq p_i, p_j$. Write $p_i - q = e_u$ and $p_j - q = e_v$. Their zero-sum equations give $\beta_u = \beta_v$, hence $u = v$ and $p_i = p_j$. $\square$

In particular, two distinct columns can be positive together in at most one row. At most one entry equal to two can occur in a given row. A row containing an entry three has no entry two. A pure pattern $3e_u$ requires $\beta_u = 0$, so there is at most one pure pattern.

For each pattern $2e_u + e_v$, with $u \neq v$, draw the directed edge $u \rightarrow v$ on the value classes.

Lemma 3.4. *These directed edges form vertex-disjoint directed paths. They do not meet the row of a pure pattern.*

Proof. The equation for such a pattern is $\beta_v = -2\beta_u$. Distinct values make both the successor and the predecessor of any vertex unique. A directed cycle of length j would give $\beta_u = (-2)^j \beta_u$; every value on it would be zero, contrary to distinctness. If an edge met the zero-value row, its other endpoint would also have value zero. Thus the components containing edges are directed paths disjoint from the zero-value row. $\square$

Lemma 3.5. *Every nonempty remainder in (3.1), with distinct centered values, contains a selected pattern using three distinct value classes. In particular, this pattern is bounded by r .*

Proof. Suppose every selected pattern has a repeated value or is pure. If a repeated-value pattern occurs, choose a terminal vertex v of a path in Lemma 3.4. Its row appears only as the single entry of the last edge, say column j . Thus $r_v = \lambda_j \in (0, 1)$, contrary to integrality. If no repeated-value pattern occurs, there is only the pure zero pattern. Its single coefficient cannot have an integer sum while lying in $(0, 1)$. Hence a column with three distinct value classes occurs. Each of its entries is at most one, so positive integrality of r makes it available. $\square$

Lemma 3.6. *If the remainder in (3.1) has $h \leq 4$, it contains at least $\max\{0, h-2\}$ disjoint nonpositive-sum triples. If $h \leq 2$, all its items can be partitioned into zero-sum triples.*

Proof. For $h = 1$, the three remaining items already form a zero-sum triple. For $h = 2$, take the available zero-sum triple from Lemma 3.5; its three-item complement also has sum zero. For $h = 3$, apply Lemma 2.3. For $h = 4$, take one available zero-sum triple and apply Lemma 2.3 to the other nine items. The empty remainder needs no further packing. $\square$

4. INTEGRAL PACKINGS THROUGH DIMENSION FOUR

The support bounds reduce the integral-packing problem for five patterns to a six-value configuration. The next two lemmas settle that configuration using only its zero-sum equations.

Lemma 4.1. *On six distinct real values of total sum zero, the triples using three distinct values and having sum zero form either an empty set or a single pair of complementary triples.*

Proof. The complement of any such triple is another such triple. By Lemma 3.3, two distinct triples intersect in at most one value. If a further triple met one member of a complementary pair in exactly one value, it would meet the other member in exactly two, again contradicting Lemma 3.3. Thus there is no further triple. $\square$

Lemma 4.2. *Let $\beta \in \mathbb{R}^6$ have distinct coordinates and $\mathbf{1}^T \beta = 0$. There is no matrix $P \in \mathbb{Z}_{\geq 0}^{6 \times 5}$ of rank five, with column sums three and $\beta^T P = 0$, such that*

$$(4.1) \quad P\theta = \mathbf{1}_6, \quad 0 < \theta_j < 1 \quad (1 \leq j \leq 5).$$

Proof. Summing rows in (4.1) gives $\mathbf{1}^T \theta = 2$. By Lemma 3.5, at least one selected column uses three distinct values. By Lemma 4.1, at most two such columns exist. If both were selected, their sum would be $\mathbf{1}_6$. The rank assumption would then make the unique solution of (4.1) the vector having coefficients one on those two columns and zero on the others. Thus exactly one selected column uses three distinct values. Denote its triple by S , and its coefficient by σ .

Let $\eta \in \{0, 1\}$ indicate whether a pure zero column is selected. The other $4 - \eta$ columns are repeated-value patterns. Their directed graph is a union of paths by Lemma 3.4. Every nonzero value lies on one of these paths. Indeed, a nonzero row outside the paths would appear only in S , forcing $\sigma = 1$. A zero row, if present, cannot lie on a repeated-value path. It must have a pure column, since otherwise the same argument would force $\sigma = 1$. Consequently the paths cover exactly $6 - \eta$ vertices and have $4 - \eta$ edges,

so there are exactly two paths. Each terminal vertex belongs to S , because otherwise its row equation would force the coefficient of its last edge to be one.

Along a path starting at a value b , the values are $b, -2b, 4b, \dots$. The starts of both paths are nonzero. We now consider their possible lengths.

Suppose $\eta = 0$. The lengths are either $(2, 2)$ or $(1, 3)$. For lengths $(2, 2)$, the values are

$$a, -2a, 4a, \quad b, -2b, 4b.$$

Their total sum gives $b = -a$. The two terminal values in S therefore sum to zero. Its third value would have to be zero, although all six path values are nonzero.

For lengths $(1, 3)$, write the values as

$$a, -2a, \quad b, -2b, 4b, -8b.$$

The total sum gives $a = -5b$. The terminal values $-2a$ and $-8b$ sum to $2b$, so $S = \{-2a, -8b, -2b\}$. Let u be the coefficient of the short-path column, and let v, w, z be the successive coefficients on the long path. The rows for $a, -2a, b, -2b$ give

$$2u = 1, \quad u + \sigma = 1, \quad 2v = 1, \quad v + 2w + \sigma = 1.$$

Hence $u = v = \sigma = 1/2$ and $w = 0$, contrary to (4.1).

Finally, suppose $\eta = 1$. The path lengths are $(1, 2)$, and the six values are

$$0, \quad a, -2a, \quad b, -2b, 4b.$$

Their total sum gives $a = 3b$. The terminal values sum to $-2b$, so the third value of S must be $2b$. The remaining values are $0, 3b, b, -2b$, none of which equals $2b$, since $b \neq 0$. This exhausts the possibilities. $\square$

Proof of the first assertion of Theorem 1.1. Choose an independent representation supported on at most five of the given patterns, and form its integer remainder. An all-zero remainder can be partitioned directly. Otherwise (3.4) gives $3h \leq 2k - 1 \leq 9$, so $h \leq 3$. The cases $h \leq 2$ have integral packings by Lemma 3.6.

If $h = 3$, the same bounds force $k = 5$ and $t = 6$. The positive integer vector κ has coordinate sum $3(k - h) = 6$, hence $\kappa = \mathbf{1}_6$. It follows that $\mathbf{1}^T \beta = 0$ and $P\mu = \mathbf{1}_6$ with $0 < \mu < 1$, contrary to Lemma 4.2. Therefore $h = 3$ cannot occur. The integral remainder packing, together with the triples removed initially, uses exactly m zero-sum triples. The volume lower bound gives $\text{OPT}(I) = m$. If $\dim \mathcal{P}_I \leq 4$, Lemma 3.1 supplies the required five-pattern representation. $\square$

5. THE ADDITIVE-ONE THEOREM THROUGH DIMENSION TEN

5.1. A finite family of matrix statements. For integers $5 \leq h < k$, let $\mathcal{A}_{k,h}$ consist of the pairs (P, κ) with the following properties. The matrix $P \in \mathbb{Z}_{\geq 0}^{t \times k}$ has column sums three and rank k . The vectors κ and $r = P\mathbf{1} - \kappa$ are strictly positive and integral. There is a solution

$$P\mu = \kappa, \quad 0 < \mu_j < 1, \quad \mathbf{1}^T \mu = k - h.$$

The space $\ker P^T$ contains a vector with distinct coordinates. Finally, every $(h - 3)$ -subset J of columns satisfies $\sum_{j \in J} p_j \not\leq r$. Write $E(k, h)$ for the assertion $\mathcal{A}_{k,h} = \emptyset$.

Proposition 5.1. *For representations supported on at most eleven patterns, it suffices to prove $E(k, h)$ for*

(5.1) $(k, h) \in \{(8, 5), (9, 5), (10, 5), (10, 6), (11, 5), (11, 6), (11, 7)\}.$

Proof. Choose an independent positive representation using a subset of the given patterns, and form (3.1). The cases $h \leq 4$ are settled by Lemma 3.6. If $h \geq 5$, (3.4) and $k \leq 11$ give exactly (5.1). The corresponding exclusion supplies (3.5), which gives $h - 2$ nonpositive-sum triples as above. $\square$

5.2. Rows supported on one column. Call a row *private* if it is positive in exactly one column. Its positive entry and its two multiplicities obey

$$(5.2) \quad (\kappa_u, r_u; p_{uj}) \in \{(1, 1; 2), (1, 2; 3), (2, 1; 3)\}.$$

Indeed, $p_{uj}\mu_j = \kappa_u$ and $p_{uj}(1 - \mu_j) = r_u$ are positive integers whose sum is at most three. Let a be the number of private rows of the first kind, and let p_1, p_2 count the other two kinds. Then $p_1 + p_2 \leq 1$. These $c = a + p_1 + p_2$ rows lie in c distinct columns, each unavailable in the remainder.

A private double column has one additional entry, in a nonprivate row. We call it the attachment row of that column. By Lemma 3.4, these a attachment rows are distinct. Consequently,

$$(5.3) \quad c + a \leq t.$$

Only subsets of the other $k - c$ columns need to be considered in the noncontainment condition defining $\mathcal{A}_{k,h}$.

5.3. An extremal inequality for unit complementary multiplicities. For $q \geq 1$, set

$$(5.4) \quad \tau_q(N) = \sum_{i=1}^q \binom{N_i}{2}, \quad \sum_i N_i = N, \quad |N_i - N_j| \leq 1.$$

The complementary form of Turán's theorem [10] states that a graph on N vertices with independence number at most q has at least $\tau_q(N)$ edges. Equality holds precisely for a union of q complete graphs whose orders differ by at most one; empty parts are permitted. We use both the inequality and its equality case.

Lemma 5.2. *Suppose $(P, \kappa) \in \mathcal{A}_{k,h}$ and $\kappa = \mathbf{1}$. Let a count the private double columns, let $p \in \{0, 1\}$ count the private pure column, and put $c = a + p$. Then*

$$(5.5) \quad t \geq 2c - p + \tau_{h-4}(k - c).$$

Proof. Since $r_u = \sum_j p_{uj} - 1$, a row violates $\sum_{j \in J} p_j \leq r$ exactly when its entire support is contained in J . Restrict attention to subsets of the $k - c$ nonprivate columns. The private rows and the a attachment rows cannot give such a violation. Any remaining row whose support lies in these columns has support size at least two. Choose a pair of columns in each of these supports and form a simple graph on the $k - c$ columns.

An independent set of size $h - 3$ contains no chosen pair and hence no complete row support. It would therefore satisfy all capacity inequalities, a contradiction. The graph has independence number at most $h - 4$ and at most $t - c - a$ edges. The extremal bound gives $t \geq c + a + \tau_{h-4}(k - c)$, as claimed. $\square$

For example, $k = 10, h = 6, t = 12$ forces $\kappa = \mathbf{1}$. The minimum of the right-hand side of (5.5), over the allowed integers c, p , is thirteen. Thus this case is excluded directly.

5.4. Eleven columns with remainder size twenty-one.

Proposition 5.3. *The assertion E(11, 7) holds.*

Proof. Suppose $(P, \kappa) \in \mathcal{A}_{11,7}$. The bounds (3.4) imply $t = 12$ and $\kappa = \mathbf{1}$. Substitution in (5.5) leaves the four rows of Table 1. Each attains equality.

TABLE 1. The equality possibilities for $k = 11, h = 7$.

p	a	$k - c$	Graph on nonprivate columns	Edges
1	1	9	$K_3 \sqcup K_3 \sqcup K_3$	9
1	2	8	$K_3 \sqcup K_3 \sqcup K_2$	7
1	3	7	$K_3 \sqcup K_2 \sqcup K_2$	5
1	4	6	$K_2 \sqcup K_2 \sqcup K_2$	3

For completeness, when $p = 0$ the expression $2c + \tau_3(11 - c)$ is at least thirteen. When $p = 1$, the inequality $2c - 1 + \tau_3(11 - c) \leq 12$ holds exactly for $c = 2, 3, 4, 5$, with equality in each case.

Equality in Lemma 5.2 means that all rows other than private and attachment rows contribute distinct edges of the displayed extremal graph. Each such row has support exactly two. To see this last claim, suppose a row also contains a third column. Replace its chosen edge by another pair from that support. If the third column belongs to the same complete component, the replacement duplicates an existing edge and reduces the number of distinct edges below the extremal value. If it belongs to a different component, remove the original edge xy and use an edge xz between components. Choose x, y , one vertex from the component of z other than z , and one from the third component. These form an independent set of size four. Both possibilities contradict the construction in Lemma 5.2.

Every private double column has coefficient $1/2$. Its attachment row has right-hand side one, so the other columns contribute exactly $1/2$ there. Hence

$$(5.6) \quad \mu_j \leq \frac{1}{2} \quad \text{for every nonprivate column meeting an attachment row.}$$

Consider a component K_2 . Each of its two columns meets one graph row and must use an attachment row to complete its column sum. If its entry in the graph row were two, its remaining single entry would be in an attachment row. This would give two repeated-value columns with the same terminal row, contradicting Lemma 3.4. Both entries in the graph row are therefore one. Its equation is $\mu_i + \mu_j = 1$, so (5.6) forces both coefficients to be $1/2$.

In a component K_3 , each column meets two graph rows. Either it has entry one in each and a third entry in an attachment row, or it has entries two and one in the graph rows and no other entry. In the latter case it defines a directed edge between those two graph rows. If all three columns have this form, the degree restrictions in Lemma 3.4 force a directed cycle. If exactly one column has this form, the three row equations can be written

$$x + y = 1, \quad x + 2z = 1, \quad y + z = 1.$$

Their solution is $(x, y, z) = (1/3, 2/3, 1/3)$. The coefficient y belongs to a column meeting an attachment row and contradicts (5.6). If exactly two columns have repeated entries, the corresponding edges form a directed path. The equations become

$$a' + 2b' = 1, \quad b' + 2c' = 1, \quad a' + c' = 1,$$

with solution $(a', b', c') = (3/5, 1/5, 2/5)$. The first column meets an attachment row, again contradicting (5.6). Thus every graph-row entry is one, and all three coefficients are $1/2$.

It follows that all ten nonpure columns have coefficient $1/2$. The pure column has coefficient $1/3$. Their sum is $5 + 1/3 = 16/3$, whereas $\mathbf{1}^T \mu = 11 - 7 = 4$. This contradiction proves the assertion. $\square$

5.5. The finite exclusions and the main theorem.

Proposition 5.4. *All seven assertions in (5.1) hold.*

Proof. For each pair (k, h) , enumerate all lengths in (3.4), all positive integer vectors κ, r with the specified sums, and all nonnegative integer matrices with the required column and row sums. The private-row description (5.2), the intersection restriction (3.6), and the directed-path restriction reduce this finite enumeration. Appendix A proves that the row and column orderings retain a representative of every possible system and justifies the further pruning rules.

Every matrix reaching the terminal calculation is excluded by an integer identity proving inconsistency, deficient column rank, or a uniquely forced coefficient outside $(0, 1)$. The complete counts appear in Table 2. The data contain the identity for each terminal matrix. Independent integer arithmetic checks these identities; a second enumeration of the initial row data checks the range of the completed calculations. No terminal matrix survives.

The last row of parameter values also has the direct proof in Proposition 5.3. $\square$

Proof of the second assertion of Theorem 1.1. By Propositions 5.1 and 5.4, every remainder arising from a representation on at most eleven patterns contains at least $h - 2$ nonpositive-sum triples when $h \geq 2$. Together with the $m - h$ triples removed initially, these give at least $m - 2$ triples. The cases $h = 0, 1$ are immediate from the already removed triples. Apply Proposition 2.2.

If $\dim \mathcal{P}_I \leq 10$, Lemma 3.1 gives a representation with at most eleven patterns. For the assertions about item types in the introduction, if all centered values are zero, integral packing is immediate. Otherwise $\mathbf{1}$ and β are independent: a nonzero constant β admits no zero-sum pattern. All patterns lie in the two

independent affine hyperplanes $\mathbf{1}^T p = 3$ and $\beta^T p = 0$. Thus $\dim \mathcal{P}_I \leq t - 2$. The two parts of the theorem apply when $t \leq 6$ and $t \leq 12$, respectively. $\square$

Corollary 5.5. *Suppose $n = \sum_j \alpha_j p_j$ is any representation by available zero-sum patterns with $\alpha_j \geq 0$. If at most eleven coefficients are nonintegral, then $\text{OPT}(I) \leq m + 1$.*

Proof. Remove all integer parts. If the remaining columns are dependent, take a vertex of the nonempty bounded polytope $\{x \in [0, 1]^k : Px = r\}$. Its columns with coordinates strictly between zero and one are linearly independent, since a dependence would allow a two-sided perturbation. Remove the patterns with coordinate one and discard the columns with coordinate zero. At most eleven independent fractional columns remain, so the remainder proof of Theorem 1.1 applies. Every removed pattern is available because its total contribution is bounded by the current integer multiplicities. $\square$

6. INSTANCES WITH AT MOST TWENTY-ONE ITEMS

The number of items gives a second route to the additive bound, independent of the number of patterns. We use the following fractional matching theorem of Füredi [3].

Theorem 6.1 (Füredi). *For every finite three-uniform hypergraph H ,*

$$v^*(H) \leq \frac{7}{3}v(H).$$

If H contains no copy of the Fano plane, then $v^*(H) \leq 2v(H)$.

Here a Fano plane consists of seven vertices and seven triples, with each pair of vertices in exactly one triple. Its seven triples are called lines. Containing a copy means containing its lines as edges; additional edges are allowed.

Lemma 6.2. *If all seven lines of a Fano plane have sum zero under a real assignment, then all seven vertex values are zero.*

Proof. Let S be the sum of the seven values. Summing the line equations counts each vertex three times, so $3S = 0$. Fix a vertex v . There are four lines not containing v , and each other vertex lies on two of them. Their equations give $2(S - a_v) = 0$. Thus $a_v = S = 0$ for every v . $\square$

Let z be the number of zero-valued items and let H' be obtained from H_0 by deleting all triples made entirely of zero-valued items. Every edge of H' contains at most one zero-valued item, since two zero values in a zero-sum triple force the third to be zero. By Lemma 6.2, H' contains no Fano plane. If x is a fractional perfect matching of H_0 , its restriction to H' covers each nonzero-valued item with total weight one. Consequently,

$$3m - z = \sum_{e \in E(H')} x_e |e \setminus \{v : a_v = 0\}| \leq 3 \sum_{e \in E(H')} x_e.$$

Applying Theorem 6.1 gives the useful estimate

$$(6.1) \quad v(H') \geq \left\lceil \frac{3m - z}{6} \right\rceil.$$

Theorem 6.3. *Every instance satisfying (1.1) with $m \leq 7$ has $v_{\leq 0}(I) \geq m - 2$, and hence $\text{OPT}(I) \leq m + 1$.*

Proof. A fractional perfect matching of H_0 has weight m . For $m \leq 5$, the first bound in Theorem 6.1 yields $v(H_0) \geq \lceil 3m/7 \rceil \geq m - 2$.

Let $m = 6$. It suffices to find three disjoint zero-sum triples, then use Lemma 2.3 on the other nine items; a matching of four zero-sum triples already meets the target. If $z \leq 6$, (6.1) gives three edges of H' . If $z = 7$, it gives two edges using at most two zero-valued items; at least five zero-valued items remain, so a further all-zero triple is available. If $z = 8$, the same two edges leave at least six zero-valued items and can be extended by two all-zero triples. If $z \geq 9$, three all-zero triples are available directly. These cases prove $v_{\leq 0} \geq 4$.

Let $m = 7$. Four disjoint zero-sum triples suffice, because the other nine items have total zero. If $z \leq 6$, (6.1) gives four edges of H' . For $z = 7$, it gives three edges, which leave at least four zero-valued items, and hence one all-zero triple. For $8 \leq z \leq 11$, it gives two edges; at least $z - 2 \geq 6$ zero-valued items remain, so two all-zero triples can be added. Finally, $z \geq 12$ supplies four all-zero triples directly. Thus $v_{\leq 0} \geq 5$. $\square$

7. FINITE EXTREMAL INSTANCES IN ARBITRARY PATTERN DIMENSION

The rounding used for Theorem 1.1 minimizes the size of the integer remainder. A different rounding preserves the availability of every selected pattern and therefore preserves fractional perfect matchability. The following lemma holds for zero-sum patterns of any fixed cardinality.

7.1. Retaining all selected patterns.

Lemma 7.1. *Let $R \geq 3$, let $\beta \in \mathbb{R}^t$ have distinct coordinates, and let $p_1, \dots, p_k \in \mathbb{Z}_{\geq 0}^t$ satisfy*

$$\mathbf{1}^T p_j = R, \quad \beta^T p_j = 0, \quad p_j \leq n.$$

Suppose $n \in \mathbb{Z}_{>0}^t$ has a representation

$$n = \sum_{j=1}^k \alpha_j p_j, \quad \alpha_j > 0, \quad m = \sum_j \alpha_j \in \mathbb{Z}.$$

Then there are integers $q_j \geq 0$ and real numbers $\gamma_j > 0$ such that

$$(7.1) \quad r = n - Pq = P\gamma \in \mathbb{Z}_{>0}^t, \quad p_j \leq r \leq n \quad (1 \leq j \leq k),$$

$$(7.2) \quad h = \mathbf{1}^T \gamma = m - \mathbf{1}^T q \in \mathbb{Z}, \quad h \leq (R-1)k - 1.$$

The removed items can be partitioned into $\mathbf{1}^T q$ selected zero-sum patterns.

Proof. Set

$$(7.3) \quad q_j^0 = \max\{0, \lfloor \alpha_j \rfloor - (R-2)\}, \quad \gamma_j^0 = \alpha_j - q_j^0, \quad r^0 = P\gamma^0.$$

Then $0 < \gamma_j^0 < R-1$; whenever $q_j^0 > 0$, also $\gamma_j^0 \geq R-2$. The vector $r^0 = n - Pq^0$ is positive and integral, and $r^0 \leq n$. Its coefficient sum $h_0 = m - \mathbf{1}^T q^0$ is an integer less than $(R-1)k$.

Fix a coordinate u . If no removed pattern uses u , then $r_u^0 = n_u$, which accommodates every selected pattern. Otherwise choose i with $q_i^0 > 0$ and $p_{ui} > 0$. Its contribution gives $r_u^0 \geq R-2$.

Suppose first that $\beta_u \neq 0$. A zero-sum pattern uses this value at most $R-1$ times. Entries at most $R-2$ are already accommodated. If $p_{uj} = R-1$, then $i = j$ gives $r_u^0 \geq (R-2)(R-1) \geq R-1$. If $i \neq j$, the positive contribution of column j , together with the contribution of column i , gives $r_u^0 > R-2$. Integrality again yields $r_u^0 \geq R-1$.

There is at most one zero-value coordinate u_0 . A zero-sum pattern cannot use exactly $R-1$ copies of zero, since its last value would also be zero. Thus its entry at u_0 is at most $R-2$, or it is the pure pattern Re_{u_0} . The former case is already covered. If the pure pattern also fits, take $q = q^0$ and $\gamma = \gamma^0$.

It remains to consider $r_{u_0}^0 < R$ when a pure pattern is selected. Since it was available in n , some removed pattern uses u_0 . Removing a pure pattern would leave its coefficient at least $R-2$ and therefore at least R zero-valued items. Hence the removed pattern is mixed. It contributes at least $R-2$ to $r_{u_0}^0$, and the pure pattern contributes positively. Consequently, $r_{u_0}^0 = R-1$. Undo one removal of a mixed pattern using u_0 . This increases that coordinate by at least one and decreases no coordinate, so all selected patterns become available.

To check (7.2) after this change, let γ_*^0 be the coefficient of the pure pattern. The failed capacity inequality implies

$$R\gamma_*^0 + (R-2) \leq R-1, \quad \gamma_*^0 \leq 1/R.$$

Hence

$$h_0 < (R-1)(k-1) + 1/R, \quad h_0 \leq (R-1)(k-1).$$

After undoing one removal,

$$h = h_0 + 1 \leq (R-1)(k-1) + 1 \leq (R-1)k - 1,$$

where the last inequality uses $R \geq 3$. Finally, $Pq \leq n$ allows the removed pattern positions to be filled with distinct actual items of each type. $\square$

Proposition 7.2. *For every instance I in (1.1), with $d = \dim \mathcal{P}_I$, there is a subinstance I' satisfying (1.1), with $3m'$ items, such that*

$$(7.4) \quad m' \leq 2d + 1, \quad \dim \mathcal{P}_{I'} \leq d, \quad \text{OPT}(I) - m \leq \text{OPT}(I') - m'.$$

Proof. Choose the representation in Lemma 3.1 and apply Lemma 7.1 with $R = 3$. The remaining multiplicities have $m' \leq 2k - 1 \leq 2d + 1$. All selected patterns are available on these multiplicities, and their positive representation gives a fractional perfect matching by Lemma 2.4. Their centered total is zero, so the total size and the configuration optimum of the subinstance both equal m' .

Every zero-sum pattern of I' is also a zero-sum pattern of I , giving the dimension inequality. The removed items occupy exactly $m - m'$ bins. Combining those bins with an optimum packing of I' gives $\text{OPT}(I) \leq m - m' + \text{OPT}(I')$, as required. $\square$

Thus any counterexample to an additive bound in a fixed pattern dimension has a counterexample with at most $6d + 3$ items in the same class. The argument preserves the item sizes of the chosen subinstance.

The corresponding statement for R -item bins follows in the same way. If an instance has Rm items with sizes in $(1/(R+1), 1/(R-1))$, total size and configuration value both m , then every bin has at most R items and every $(R-1)$ -set fits. Writing $v_{\leq 0}^{(R)}$ for the maximum matching of feasible R -sets, the counting proof of Proposition 2.2 gives

$$\text{OPT}(I) = m + \left\lceil \frac{m - v_{\leq 0}^{(R)}}{R - 1} \right\rceil.$$

Moreover, Lemma 7.1 produces an LP-tight subinstance with

$$m' \leq (R - 1)(d + 1) - 1$$

and additive gap at least that of the original instance.

7.2. Integer values preserving all triple signs.

Lemma 7.3. *Let $R \geq 3$, and let $\beta \in \mathbb{R}^t$ have distinct coordinates. There is a vector $w \in \mathbb{Z}^t$ with distinct coordinates such that*

$$(7.5) \quad \text{sgn}(p^T w) = \text{sgn}(p^T \beta) \quad \text{whenever } p \in \mathbb{Z}_{\geq 0}^t, \mathbf{1}^T p = R, \quad \|w\|_\infty \leq tR^{t-1}.$$

The order of all coordinates is preserved as well.

Proof. Consider the finitely many linear forms $p^T x$ with $\mathbf{1}^T p = R$, together with $(e_u - e_v)^T x$ for $u < v$. For a form that vanishes at β , impose equality to zero. For a positive form impose $L(x) \geq 1$, and for a negative form impose $-L(x) \geq 1$. A sufficiently large positive multiple of β satisfies this system, so its polyhedron S is nonempty.

The forms $Re_u^T x$ restrict each coordinate to the sign of β_u , or fix it to zero. Hence $f(x) = \sum_u \text{sgn}(\beta_u)x_u$ equals $\sum_u |x_u|$ on S , and all its nonempty bounded sublevel sets in S are compact. The minimum is attained on a compact face. An extreme point of that face is a vertex of S .

Choose t independent active equations at a vertex and write them as $Bx = b$. The rows of the integer matrix B have Euclidean norm at most R , and each coordinate of b is zero or one. By Cramer's rule, $w = |\det B|x$ is an integer vector. Expanding the determinant with one column replaced by b expresses each w_u , up to sign, as a sum of at most t minors of order $t - 1$. Hadamard's determinant inequality bounds each minor by R^{t-1} . Thus $|w_u| \leq tR^{t-1}$. Positive scaling preserves all imposed equalities and signs. The comparison forms preserve distinctness and coordinate order. $\square$

7.3. The largest gap at a fixed dimension. We now show that Δ_d , defined in (1.5), equals a maximum over an explicitly bounded finite family.

Theorem 7.4. *The supremum Δ_d is attained by an instance having*

$$(7.6) \quad |I| \leq 6d + 3, \quad t \leq 3d + 3, \quad s_u = \frac{1}{3} + \frac{w_u}{12H + 1}, \quad |w_u| \leq t3^{t-1}, \quad H = \max_u |w_u|.$$

Its binary encoding length is $O((d+1)^2)$. The value Δ_d can be computed by a deterministic algorithm in $2^{O((d+1)^2)}$ time.

Proof. Apply Proposition 7.2 to an arbitrary instance of dimension at most d . The retained representation uses at most $d + 1$ patterns, each supported on at most three classes. These patterns cover all positive multiplicities, so $t \leq 3d + 3$.

Apply Lemma 7.3 with $R = 3$. All zero-sum and nonpositive-sum patterns remain unchanged. The multiplicity vector is a positive combination of the retained zero-sum patterns, so its total centered value remains zero. Consequently, fractional perfect matchability, the complete pattern dimension, and both matching numbers are preserved. The sizes in (7.6) belong to $(1/4, 1/2)$, including the case $H = 0$. They therefore give an instance in the same class with the same gap as the subinstance.

There are finitely many choices in (7.6). Each coordinate w_u uses $O(t)$ bits, and the multiplicities, whose total is $O(d + 1)$, use $O((d + 1) \log(d + 2))$ bits. The common denominator has $O(t)$ bits. Thus the total encoding length is $O((d + 1)^2)$, and the number of candidates is at most $2^{O((d+1)^2)}$.

Each candidate has $O(t^3)$ triple patterns. Fractional perfect matchability can be tested by enumerating independent subsets of at most t zero-sum columns and solving their linear systems exactly. The number of systems is $2^{O((d+1) \log(d+2))}$. Rational matrix rank gives the complete pattern dimension. On $N = O(d + 1)$ labelled items, maximum matching can be computed by the recurrence

$$M(S) = \max \{M(S \setminus \{v\}), 1 + M(S \setminus e) : e \ni v, e \subseteq S, a(e) \leq 0\},$$

where $v \in S$ is fixed at each nonempty state and $M(\emptyset) = 0$. There are 2^N states and a polynomial number of transitions per state. All arithmetic and per-candidate work fit in the asserted running time.

Every original gap is bounded above by the gap of a candidate satisfying the same dimension bound. Conversely, every retained candidate belongs to the original class. Therefore the supremum in (1.5) equals the maximum over this finite family. $\square$

Proposition 7.2 and Theorem 7.4 prove Theorem 1.3. In particular, for every integer $K \geq 0$, the inequality $\text{OPT}(I) - \text{LP}(I) \leq K$ holds for all instances of pattern dimension at most d if and only if it holds for the finite family in (7.6).

8. SHARP FAMILIES AND THE ELEVEN-DIMENSIONAL REMAINDER OBSTRUCTION

8.1. An eighteen-item instance. Consider the distinct centered values and multiplicities

$$(8.1) \quad \beta = (-34, -28, -20, 7, 10, 13, 14, 17, 21)^T, \quad n = (2, 2, 2, 1, 3, 1, 3, 3, 1)^T.$$

Their total multiplicity is eighteen and their weighted sum is zero. Multiplying the values by $1/1000$ and adding $1/3$ gives sizes in $(1/4, 1/2)$. The complete zero-sum patterns are

$$(8.2) \quad \begin{aligned} p_1 &= (-20, 10, 10), & p_2 &= (-20, 7, 13), \\ p_3 &= (-28, 14, 14), & p_4 &= (-28, 7, 21), \\ p_5 &= (-34, 17, 17), & p_6 &= (-34, 13, 21). \end{aligned}$$

Here a triple of values denotes its multiplicity vector in the row order of (8.1). Each zero-sum triple has one negative and two positive values: two negative values have sum at most -40 , which the largest positive value cannot offset. The positive pairs with sums twenty, twenty-eight, and thirty-four are exactly those listed in (8.2).

The pattern matrix is

$$(8.3) \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Its columns are independent. The rows for values ten, fourteen, and seventeen first determine the coefficients of p_1, p_3, p_5 ; the three negative-value rows then determine the remaining coefficients. Moreover,

$$(8.4) \quad A\mathbf{1} = \mathbf{21}, \quad n = A\alpha, \quad \alpha = (3/2, 1/2, 3/2, 1/2, 3/2, 1/2)^T.$$

Thus $\text{LP} = 6$, and the complete pattern polytope is a five-dimensional simplex with the displayed unique representation.

Proposition 8.1. *The instance (8.1) has*

$$v_0 = 4, \quad v_{\leq 0} = 5, \quad \text{OPT} = 7.$$

Proof. On the positive-labelled items, draw an edge for each pair that can occur in a zero-sum triple. This graph is the disjoint union of four triangles: one on the three copies of ten, one on the three copies of fourteen, one on the three copies of seventeen, and one on the values seven, thirteen, and twenty-one. Every zero-sum matching gives a matching in this graph, so it has at most four edges. Taking p_1, p_3, p_5, p_2 realizes four disjoint zero-sum triples.

There are five disjoint nonpositive-sum triples

$$(8.5) \quad (-20, 10, 10), \quad (-20, 7, 13), \quad (-28, 14, 14), \quad (-28, 10, 14), \quad (-34, 17, 17).$$

Their sums are 0, 0, 0, -4, 0. A matching of six nonpositive-sum triples would cover all items. Since the total sum is zero, every one of its triples would be zero-sum, contradicting $v_0 = 4$. The optimum follows from Proposition 2.2. $\square$

Proof of Corollary 1.2. The first assertion of Theorem 1.1 gives $\Delta_d = 0$ for $d \leq 4$. The second gives $\Delta_d \leq 1$ for $d \leq 10$. The instance in Proposition 8.1 has complete pattern dimension five and additive gap one. By monotonicity of Δ_d , it gives $\Delta_d \geq 1$ for every $d \geq 5$. $\square$

This example also distinguishes two kinds of integer remainder. Removing $\lfloor \alpha \rfloor = (1, 0, 1, 0, 1, 0)^T$ leaves

$$(8.6) \quad r = \frac{1}{2}A\mathbf{1} = \mathbf{1}_9.$$

Only p_2, p_4, p_6 are available, and these patterns pairwise intersect. The values ten, fourteen, and seventeen lie in no available zero-sum triple. Thus this remainder has no zero-sum fractional perfect matching. The retained-pattern construction of Lemma 7.1 keeps the capacities needed for a fractional perfect matching.

8.2. Separating the scales. For $B \geq 1$, take B copies of (8.1), and multiply the centered values of copy j by $C_j = 100^j$, where $0 \leq j < B$. Set

$$(8.7) \quad s_{j,u} = \frac{1}{3} + \frac{C_j \beta_u}{1000C_{B-1}}, \quad n_{j,u} = n_u.$$

These are rational sizes in $(1/4, 1/2)$. Distinct copies have distinct centered values, because their absolute-value intervals are disjoint.

Lemma 8.2. *Every zero-sum triple in (8.7) belongs to a single copy.*

Proof. Work with the unnormalized integer values $C_j \beta_u$. The basic values are nonzero integers of absolute value between seven and thirty-four, and no pair of basic values has sum zero. Consider a triple using more than one scale and let C be its largest scale. If exactly one value has this scale, its absolute value is at least $7C$, while the other two have total absolute value at most $68C/100$. If two values have scale C , their sum is a nonzero integer multiple of C , with absolute value at least C . The third value has absolute value at most $34C/100$. Neither case permits sum zero. $\square$

Proof of Theorem 1.4. By Lemma 8.2, the complete zero-sum pattern matrix is $P_B = \text{diag}(A, \dots, A)$. It has $6B$ independent columns, so its convex hull is a simplex of dimension $6B - 1$. The fractional representations in (8.4) combine to give $\text{LP}(I_B) = 6B$. Zero-sum edges stay within copies; hence $v_0(I_B) = 4B$ by Proposition 8.1.

Choose the five triples in (8.5) in every copy. They leave the three items $(-34C_j, 17C_j, 21C_j)$. For $1 \leq j < B$, add the triple

$$(8.8) \quad (-34C_j, 17C_j, 21C_{j-1}).$$

These triples are pairwise disjoint and avoid the five triples already chosen in every copy. Their sums are $-17C_j + 21C_{j-1} < 0$. We obtain $5B + (B - 1) = 6B - 1$ nonpositive-sum triples. A matching of $6B$ such triples would have to be entirely zero-sum, since it would cover every item of a zero-total assignment. This contradicts $v_0(I_B) = 4B$. Thus $v_{\leq 0}(I_B) = 6B - 1$, and $\text{OPT}(I_B) = 6B + 1$ by Proposition 2.2.

The unique fractional representation has coefficients α in each copy. Its integer remainder has multiplicity one in all $9B$ value classes. Within each copy only the three mixed patterns p_2, p_4, p_6 are

available, and they pairwise intersect. By Lemma 8.2, no additional zero-sum triple joins different copies. The maximum zero-sum matching of the remainder therefore has size exactly B . $\square$

Corollary 8.3. *In the instance I_B , every matching of at least $6B - 2$ nonpositive-sum triples contains at least $2B - 2$ triples of strictly negative sum.*

Proof. At most $v_0(I_B) = 4B$ of its triples can have sum zero. $\square$

8.3. The first failure of the selected-pattern condition.

Proposition 8.4. *For every $B \geq 2$, the assertion $E(6B, 3B)$ is false. In particular, $E(12, 6)$ is false even when the selected columns are all the vertices of the complete pattern polytope.*

Proof. Use P_B and put

$$\mu = \frac{1}{2}\mathbf{1}_{6B}, \quad \kappa = r = \mathbf{1}_{9B}.$$

The column sums, rank, and positive integer conditions follow from Equations (8.3) and (8.4). The scaled vector $(C_j\beta_u)_{j,u}$ has distinct coordinates and lies in $\ker P_B^T$. Also $\mathbf{1}^T\mu = 3B = k - h$. The remainder contains at most B disjoint zero-sum triples. Since $B < 3B - 3 = h - 3$ for $B \geq 2$, no required column subset is accommodated. This gives a member of $\mathcal{A}_{6B,3B}$. $\square$

For $B = 2$, the original instance has thirty-six items, pattern dimension eleven, fractional optimum twelve, and integer optimum thirteen. Its eighteen-item remainder contains only two disjoint zero-sum triples, whereas (3.5) would require three. Because the complete zero-sum pattern polytope is already a simplex, adding further zero-sum columns cannot change this conclusion. The strictly negative triples in Equations (8.5) and (8.8) supply the additional packing flexibility.

9. FURTHER QUESTIONS

The general additive statement for the class in (1.1) can be written in matching form.

Conjecture 9.1. *Every real zero-total assignment on $3m$ labelled items whose zero-sum triple hypergraph has a fractional perfect matching satisfies*

$$v_{\leq 0} \geq m - 2.$$

The exact values in Corollary 1.2 locate the first positive additive gap at pattern dimension five. The next undetermined value is Δ_{11} . In every fixed dimension, Theorem 1.3 replaces the supremum over unrestricted instances by a maximum over a bounded rational family.

For basic remainders with at least nine items and at most eleven independent columns, the proof of Theorem 1.1 obtains the required matching from (3.5) and the nine-item averaging argument. Proposition 8.4 gives a twelve-column remainder in which (3.5) fails, although the original instance has gap one. The family in Theorem 1.4 further shows that matchings of the required size can need a number of strictly negative triples growing linearly with the pattern dimension. These examples suggest studying matchings that combine zero-sum triples with exchanges between their components.

APPENDIX A. THE FINITE ENUMERATION

This appendix specifies the enumeration used in Proposition 5.4. All data and program names are relative to the accompanying directory. The C++ source is `code/enumerate.cpp`. The Python programs `code/generate_cases.py`, `code/check_coverage.py`, and `code/check_certificates.py` generate the initial cases and perform independent checks. Completed calculations and their integer identities are in `data/enumeration/`.

A.1. Initial row data. Fix (k, h) from (5.1). Enumerate each integer

$$k + 1 \leq t \leq \min\{3h, 3(k - h)\}.$$

Enumerate the positive integer partitions of $3(k - h)$ and $3h$ into t parts, representing κ and r . For each pair of partitions, fix the ordering of κ and enumerate every distinct alignment of the entries of r . Within rows having the same κ entry, order the r entries increasingly. Thus the data record all multisets of pairs (κ_u, r_u) with the prescribed sums.

Next choose a, p_1, p_2 in (5.2), subject to the available numbers of the three row types, $p_1 + p_2 \leq 1$, and (5.3). Place the private double columns first and the pure column, if any, immediately afterwards. By simultaneously permuting private rows and their columns, their entries can be fixed to a diagonal block of twos followed by an optional three. All remaining rows are nonprivate. Their coordinate sums are $\kappa_u + r_u$, and they are generated by all distributions of this sum among the columns with each final column sum bounded by three. Private double columns have remaining capacity one, the pure column has capacity zero, and the other columns have capacity three.

This describes every possible pair (P, κ) up to row and column permutations. The independent program `code/check_coverage.py` generates the same initial set by a different recursion. It constructs the sorted sequence of pairs (κ_u, r_u) directly, without separate partitions or alignments. It checks equality of the two sets of initial cases.

A.2. Ordering rows and columns. Let $\mathcal{J}$ be the family of all $(h - 3)$ -subsets of the nonprivate columns. For a row vector v of type (κ_u, r_u) , define

$$(A.1) \quad C_u(v) = \left\{ J \in \mathcal{J} : \sum_{j \in J} v_j > r_u \right\}.$$

The integer $|C_u(v)|$ is unchanged by permuting nonprivate columns. Row types are arranged in a fixed order. Among rows of the same type, order first by decreasing $|C_u(v)|$, and then by increasing lexicographic order of the row vector. Order the nonprivate columns lexicographically by their full column vectors. During construction, this last rule is imposed only on adjacent columns whose prefixes are identical.

These row and column rules are compatible. In a given orbit under the allowed row and column permutations, choose a row-major lexicographically least matrix after sorting row types and their invariant counts $|C_u(v)|$. Its rows with the same type and count are already lexicographically sorted. If two adjacent nonprivate columns first differ in the wrong order, exchanging them strictly decreases the matrix at that first difference. Reordering rows within the same type and count can only further decrease it. This contradicts minimality. Hence every orbit has a representative satisfying both ordering rules.

The private double columns are left in their initially fixed order. The resulting repeated representatives do not affect coverage of the finite set.

A.3. Pruning partial matrices. The partial matrix is extended one row at a time. Every proposed row is subject to the remaining column capacities and has support of size at least two. The following restrictions are consequences of Lemmas 3.3 and 3.4.

Each pair of columns can meet in at most one row. At most one entry two occurs in a row, and a row cannot contain both an entry two and an entry three. At most one pure column occurs in the entire matrix. A row cannot be the attachment row of two private double columns. As soon as a column has total sum three, its repeated-value edge, if present, is determined. The directed edges of all completed columns must have indegree and outdegree at most one, contain no directed cycle, and avoid the pure zero row. Their endpoints cannot change when further rows are added, so these tests are valid before the matrix is complete.

The row sets $C_u(v)$ must eventually cover $\mathcal{J}$. For each unused row, compute an upper bound on the number of sets that one of its permitted distributions can cover. The sum of these bounds is an upper bound on the number of currently uncovered sets that the unused rows could cover. A branch is discarded when this sum is too small. Counting the same set more than once only enlarges this upper bound and therefore preserves every possible completion. Rows are otherwise restricted solely by the column capacities and the ordering rules above. When $\kappa = \mathbf{1}$, Lemma 5.2 is also applied before the row enumeration.

Upon reaching a terminal matrix, the program verifies all column sums and the full cover of $\mathcal{J}$. It then performs exact elimination on $P\mu = \kappa$. For $k \leq 11$ and $h \leq 7$, there are at most $\binom{11}{4} = 330$ target subsets and at most $\binom{11}{2} = 55$ column pairs. The implementation uses masks of lengths 512 and 128, respectively. Matrix entries and capacities are at most three, and the support bounds give $t \leq 15$, within the allocated limit of 21. Rational elimination uses arbitrary precision integer numerators and denominators.

A.4. Integer identities at terminal matrices. Every terminal rejection is accompanied by one of the following integer identities. These can be checked using additions and multiplications, independently of the elimination used to find them.

An inconsistent system supplies $y \in \mathbb{Z}^t$ with

$$(A.2) \quad y^T P = 0, \quad y^T \kappa \neq 0.$$

A rank-deficient matrix supplies $z \in \mathbb{Z}^k \setminus \{0\}$ with $Pz = 0$. If the unique solution has a coefficient outside $(0, 1)$, the data contain integers $D > 0$, $z \in \mathbb{Z}^k$ with $Pz = D\kappa$ and some $z_j \notin (0, D)$, together with an integer matrix B and a positive diagonal matrix D' satisfying

$$(A.3) \quad BP = D'.$$

This proves full column rank and the uniqueness of $\mu = z/D$.

The final possible restriction is that $\ker P^T$ contain a vector with distinct coordinates. This fails precisely when some pair of coordinates agrees throughout the kernel. Indeed, if no such pair exists, each equality cuts out a proper hyperplane in $\ker P^T$, and a finite union of proper real hyperplanes cannot cover that vector space. A forced equality can be demonstrated by

$$Pz = q(e_u - e_v), \quad z \in \mathbb{Z}^k, \quad q \in \mathbb{Z}_{>0}.$$

The seven complete calculations require only (A.2), rank deficiency, and (A.3). Their totals are respectively 113 139, 72, and 45.

TABLE 2. Complete finite exclusions. The second column counts initial row-data assignments, including the choices of private rows. The third counts terminal matrices with an integer exclusion.

(k, h)	Initial cases	Terminal matrices	Surviving matrices
(8, 5)	88	0	0
(9, 5)	692	0	0
(10, 5)	1,889	90	0
(10, 6)	648	0	0
(11, 5)	3,035	18,231	0
(11, 6)	3,035	93,381	0
(11, 7)	332	1,554	0
Total	9,719	113,256	0

A.5. Completion and reproduction. Each initial case is assigned an integer identifier. Its terminal matrices carry that identifier, and the search writes a completion line for the case. The program records a final completion line only after its entire assigned set of cases has been exhausted. For (11, 6), the initial cases are divided into four disjoint sets by their identifiers modulo four. All other parameter pairs are calculated in a single set. The coverage checker verifies that every initial identifier occurs in exactly one completed calculation. The integer checker verifies each terminal identity and compares the rejection counts with the corresponding completion record.

From the root of the accompanying directory,

```
python3 code/check_all.py
```

checks the supplied integer identities, the initial-case coverage, the exact examples, and the small remainders from Section 4. The latter check independently lists the three possible nonempty remainder windows for at most five columns and verifies zero-sum decompositions. Its program is `code/check_low_dimensions.py`. The command

```
python3 code/reproduce.py --output reproduction --workers 4
```

regenerates all seven calculations and performs the independent checks. It requires a C++17 compiler, the Boost headers, and Python 3.10 or later. Checking the supplied data requires only Python's standard library.

The representative matrix for $E(12, 6)$ in Proposition 8.4, its distinct integer values, and its fractional coefficients are included in `data/examples/`. The example program checks the complete zero-sum pattern list, rank, coefficient equations, capacity inequalities, and the displayed matchings.

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