

# An explicit counterexample to the six-neighbor square tiling conjecture

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## Abstract

Luo, Sun and Wu conjectured that in a locally finite square tiling of the complex plane in which each square intersects exactly six others, all squares have the same size. We construct a counterexample by surrounding a ten-square core with alternating rectangular shells, each formed by six squares. The only point-only contacts occur at the three four-square points in the core; all subsequent contacts have positive length. A two-state induction shows that every new square has four contacts when inserted and acquires two more at the next shell. The nested rectangles exhaust the plane, so the construction is locally finite and gives a counterexample on the whole plane.

## 1 Introduction

In their work on discrete conformal geometry, Luo, Sun and Wu ended the paper by proposing Conjecture 7.4 [21, Conjecture 7.4, p. 985]. The rigidity of the two most regular planar patterns, the regular hexagonal circle packing and the regular hexagonal triangulation, underlies the convergence arguments for discrete conformal maps. Motivated by this rigidity, they ask whether the third pattern, the square tiling in which each square meets six others, has the same rigidity. We use the standard closed-tile interpretation of the word *intersects*: a meeting at a single corner counts as a neighbor, even though it has zero shared boundary length. The conjecture as published does not specify a contact convention.

On the circle packing side, the foundational results are the convergence theorem of Rodin and Sullivan [24], the uniformization and convergence work of He and Schramm [13, 14], and the infinite pattern rigidity theorem of He [12]. Circle pattern approximations by Bücking, Lan–Dai and Matthes [4, 17, 22, 5] extend the same approximation program. The classical analytic setting is presented in Ahlfors, Lehto–Virtanen and Pommerenke [1, 18, 23].

On the polyhedral side, work on discrete conformal variations, flows and uniformization [9, 19, 10, 11, 2] led to convergence and rigidity questions for infinite triangulations [26, 8, 21]. Harmonic-coordinate and electrical-network viewpoints [7, 15, 16] are also relevant here, as is the Koebe-related perspective of Luo and Wu [20].

Square dissections have a separate history: the rectangle dissections of Brooks–Smith–Stone–Tutte [3], Schramm’s prescribed-combinatorics square tilings [25], and the finite Riemann mapping perspective of Cannon–Floyd–Parry [6]. These methods help explain why square tilings are natural objects in discrete conformal geometry. The construction below is self-contained and uses no result from any of the works just cited.

The answer to Conjecture 7.4 is negative. Its exact scope matters: it assumes neither bounded side lengths nor the absence of four-square points. Our example contains three such points and squares of arbitrarily large side length.

The number of contacts does not determine the lengths of shared edges. At a four-square point, two diagonally opposite squares are neighbors although they share no edge segment.

Hence a finite patch can have all its interior tiles already meeting six neighbors while its six boundary tiles have fewer than six. We extend the construction indefinitely, avoiding extra singleton contacts and gaps. The two alternating rectangular shells below do this: each completes six boundary tiles and produces six new boundary tiles, while the side lengths grow without bound so that the shells cover the plane.

**Theorem 1.1** (Main theorem). *There exists a locally finite tiling  $\mathcal{T}$  of  $\mathbb{C} \cong \mathbb{R}^2$  by closed, axis-parallel Euclidean squares with pairwise disjoint interiors such that every square intersects exactly six distinct other squares, but the squares do not all have the same side length. Its only four-square points are three specified points in a finite core.*

The construction is as follows. Four unit squares form a central block. Six surrounding squares give each unit square six neighbors, while six boundary tiles have fewer than six neighbors. One type of shell completes these six tiles and produces six new boundary tiles; the second is obtained from the first by a quarter-turn. The two types alternate indefinitely. Figure 1 shows the core; all coordinates, including those of the shells, are given below.

The paper is organized as follows. In Section 2 we set up the precise contact convention and the core. In Section 3 we define the two shells and prove the interface count. Section 4 completes the proof of Theorem 1.1. Section 5 explains what the example does and does not show.

## 2 Contacts and the ten-square core

All squares in this paper are nondegenerate, closed and axis-parallel. Two different squares are *neighbors* when their intersection is nonempty. A positive-length side contact and a singleton corner contact (a point-only contact) each contribute *one* neighbor. A point where four squares meet is called a *four-square point*. A tiling consists of squares with pairwise disjoint interiors whose union is  $\mathbb{R}^2$ . Local finiteness means every compact set meets only finitely many tiles.

Place four unit squares around the origin:

$$\begin{aligned} Q_{--} &= [-1, 0] \times [-1, 0], & Q_{+-} &= [0, 1] \times [-1, 0], \\ Q_{-+} &= [-1, 0] \times [0, 1], & Q_{++} &= [0, 1] \times [0, 1]. \end{aligned}$$

Add

$$\begin{aligned} L &= [-3, -1] \times [-1, 1], & R &= [1, 3] \times [-1, 1], \\ U_L &= [-3, 0] \times [1, 4], & U_R &= [0, 3] \times [1, 4], \\ D_L &= [-3, 0] \times [-4, -1], & D_R &= [0, 3] \times [-4, -1]. \end{aligned}$$

The ten squares tile

$$A_0 = [-3, 3] \times [-4, 4]. \tag{2.1}$$

Their displayed coordinate intervals give disjoint interiors.

**Lemma 2.1** (Core contacts). *Each of the four unit squares has exactly six neighbors among the ten core squares. Each of  $L, R, U_L, U_R, D_L, D_R$  has exactly four. The only singleton contacts in the core are the two diagonal pairs at each of  $(0, -1), (0, 0), (0, 1)$ ; these are also its only four-square points.*

*Proof.* Consider  $Q_{--}$ . It has positive-length contacts with  $Q_{+-}, Q_{-+}, L, D_L$ , but it meets  $Q_{++}$  and  $D_R$  only at a point. The other three unit squares follow by reflection. Among the

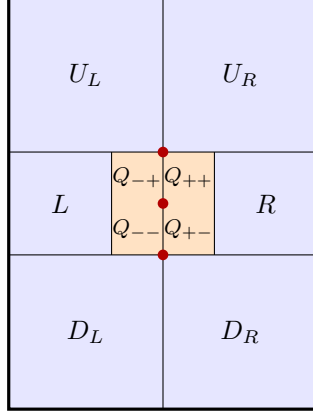


Figure 1: The ten-square core. The red points are its three four-square points. The six blue boundary squares have four neighbors at this stage; the four central unit squares already have six.

outer squares,  $L$  meets  $Q_{--}, Q_{-+}, U_L, D_L$ , while  $U_L$  meets  $Q_{-+}, Q_{++}, L, U_R$ ; the remaining four follow by reflection.

It remains to locate the singleton contacts. At any core corner other than the three points  $(0, -1), (0, 0), (0, 1)$ , at most three tiles meet, and every pair there already shares a side segment. The coordinate intervals exclude any further singleton intersection. Hence these three points are the only points at which four squares meet, and at each of them the two diagonal pairs are the only point-only contacts.  $\square$

The boundary of  $A_0$  has a six-tile pattern: the top and bottom are each split into two side-3 squares at  $x = 0$ , whereas each vertical side is divided into three segments of lengths 3, 2, 3, with the central side-2 square centered at  $y = 0$ . This is the initial  $A$  state.

### 3 Two alternating rectangular shells

We describe the two boundary states in coordinates, so that the induction does not rely on a figure. The construction alternates between the two states described below, called the  $A$  state and the  $B$  state. An  $A$  state with  $a > b > 0$  has outer rectangle

$$A(a, b) = [-a, a] \times [-a - b/2, a + b/2]. \quad (3.1)$$

Its six boundary tiles, each currently having four neighbors, are the two top squares (left and right) of side  $a$ ,

$$[-a, 0] \times [b/2, a + b/2], \quad [0, a] \times [b/2, a + b/2]$$

together with their reflections in the horizontal axis, and the two middle squares (left and right) of side  $b$ ,

$$[-a, -a + b] \times [-b/2, b/2], \quad [a - b, a] \times [-b/2, b/2].$$

All other tiles of the finite stage lie in the interior of  $A(a, b)$ . The core is  $A(3, 2)$ .

Given an  $A$  state with parameters  $(a, b)$ , set  $u = 3a + b/2$ . The next state, the  $B$  state, has outer rectangle

$$B(a, u) = [-a - u, a + u] \times [-u, u]. \quad (3.2)$$

Its six boundary tiles have four neighbors each. The top and bottom edges are each divided into three segments of lengths  $u, 2a, u$ , where the middle segment has length  $u - (a + b/2) = 2a$ ; the left and right edges are each divided into two segments of length  $u$ , meeting at  $y = 0$ .

**Lemma 3.1** (A-state to B-state). *Set  $u = 3a + b/2$ . Add the six squares*

$$\begin{aligned} N &= [-a, a] \times [a + b/2, u], & S &= [-a, a] \times [-u, -a - b/2], \\ W_N &= [-a - u, -a] \times [0, u], & W_S &= [-a - u, -a] \times [-u, 0], \\ E_N &= [a, a + u] \times [0, u], & E_S &= [a, a + u] \times [-u, 0]. \end{aligned}$$

*They fill  $B(a, u) \setminus A(a, b)$  with pairwise disjoint interiors, so the resulting finite tiling is in the B state. Each old boundary tile gains exactly two neighbors. Each of the six new squares has exactly four neighbors in the enlarged finite tiling. No new singleton contact or four-square point occurs.*

*Proof.* We first verify coverage. Since  $u - (a + b/2) = 2a$ , the rectangles  $N$  and  $S$  are squares of side  $2a$ . The other four are squares of side  $u$ ; the pair  $W_N, W_S$  partitions the left strip  $[-a - u, -a] \times [-u, u]$ , and the pair  $E_N, E_S$  partitions the right strip. Hence these six squares fill  $B(a, u) \setminus A(a, b)$  with pairwise disjoint interiors.

It remains to count contacts. Table 1 lists *every* contact involving a new square. Superscript  $o$  denotes an old boundary tile; for example  $N_W^o$  is the old top-left tile and  $W^o$  the old middle-left tile. The four tiles in each row are distinct. For example,  $W_N$  meets the old middle-left

| New square | Old neighbors  | New neighbors |
|------------|----------------|---------------|
| $N$        | $N_W^o, N_E^o$ | $W_N, E_N$    |
| $S$        | $S_W^o, S_E^o$ | $W_S, E_S$    |
| $W_N$      | $N_W^o, W^o$   | $N, W_S$      |
| $W_S$      | $S_W^o, W^o$   | $S, W_N$      |
| $E_N$      | $N_E^o, E^o$   | $N, E_S$      |
| $E_S$      | $S_E^o, E^o$   | $S, E_N$      |

Table 1: Complete A-to-B contact table.

tile over a segment of length  $b/2 > 0$ , the old top-left tile over a segment of length  $a > 0$ ,  $N$  over length  $2a > 0$ , and  $W_S$  over length  $u > 0$ . By symmetry, the same computation applies to every entry of the table, so each listed contact has positive length. It remains to check that no contact is omitted. A new top or bottom tile meets the old rectangle only on its corresponding horizontal side, and a new left or right tile only on its corresponding vertical side; the subdivisions of the old sides shown above then give exactly the old tiles in the table. Among new tiles, the six coordinate rectangles give precisely the listed adjacent pairs. For any unlisted pair, the intervals are separated by a positive gap in at least one coordinate, except at an old rectangle corner, where the three tiles present already meet along the positive-length side contacts listed in the table. Thus no unlisted pair meets, even at a point. Finally, if four axis-parallel squares with pairwise disjoint interiors meet at a point, they must occupy the four quadrants there, so opposite pairs would have point-only contacts; no new four-square point occurs. Reading the old columns, we see that each of the six old boundary tiles occurs exactly twice.

Thus the six new squares form the boundary of the B state described above, and every earlier tile lies strictly inside  $B(a, u)$ .  $\square$

**Lemma 3.2** (B-state to A-state). *From the B state of Lemma 3.1, put  $v = a + 3u$  and add*

$$\begin{aligned} W &= [-v, -a - u] \times [-u, u], & E &= [a + u, v] \times [-u, u], \\ N_W &= [-v, 0] \times [u, u + v], & N_E &= [0, v] \times [u, u + v], \\ S_W &= [-v, 0] \times [-u - v, -u], & S_E &= [0, v] \times [-u - v, -u]. \end{aligned}$$

These six squares fill  $A(v, 2u) \setminus B(a, u)$  with pairwise disjoint interiors. Each old  $B$ -boundary tile gains exactly two neighbors, and each new square has exactly four; no singleton contact or four-square point occurs. The resulting finite tiling is in an  $A$  state, with new parameters  $(a', b') = (v, 2u)$ .

*Proof.* As in Lemma 3.1, we first verify coverage. The left and right squares  $W, E$  have side  $v - (a + u) = 2u$ , and the other four have side  $v$ ; the coordinate intervals show that these six squares fill  $A(v, 2u) \setminus B(a, u)$  with pairwise disjoint interiors.

Table 2 lists all contacts involving new squares. Here  $N^o, S^o$  are the old central top and bottom tiles, and  $W_N^o, W_S^o, E_N^o, E_S^o$  are the four old side tiles. For example,  $N_W$  meets the

| New square | Old neighbors  | New neighbors |
|------------|----------------|---------------|
| $W$        | $W_N^o, W_S^o$ | $N_W, S_W$    |
| $E$        | $E_N^o, E_S^o$ | $N_E, S_E$    |
| $N_W$      | $W_N^o, N^o$   | $W, N_E$      |
| $N_E$      | $E_N^o, N^o$   | $E, N_W$      |
| $S_W$      | $W_S^o, S^o$   | $W, S_E$      |
| $S_E$      | $E_S^o, S^o$   | $E, S_W$      |

Table 2: Complete B-to-A contact table.

old upper-left side tile over a segment of length  $u > 0$ , the old central top tile over a segment of length  $a > 0$ ,  $W$  over length  $2u > 0$ , and  $N_E$  over length  $v > 0$ . By symmetry, all listed contacts have positive length. The subdivisions of the old sides and the six coordinate rectangles exclude every unlisted pair, including point-only contacts, by the same argument as in Lemma 3.1; reading the old columns, we see that each old boundary tile appears in exactly two rows.

Thus the six new squares form the boundary of the  $A$  state  $A(v, 2u)$ : the new top and bottom tiles have side  $v$ , while the left and right middle tiles have side  $2u$ . Since  $v - 2u = a + u > 0$ , the condition  $a' > b' > 0$  is preserved.  $\square$

## 4 Proof of the main theorem

*Proof of Theorem 1.1.* We verify in turn that the shells can be iterated indefinitely, that the resulting tiles cover the plane and form a locally finite tiling, and that every tile has exactly six distinct neighbors. Start with the ten-square  $A$  state in Section 2. Apply Lemma 3.1, then Lemma 3.2, and repeat. The parameter update over two shells is

$$(a, b) \mapsto \left(10a + \frac{3}{2}b, 6a + b\right). \quad (4.1)$$

For every  $a > b > 0$ , the new parameters again satisfy  $a' > b' > 0$ ; indeed  $a' - b' = 4a + b/2 > 0$ . Hence both shell lemmas apply at every stage. Starting from  $(a, b) = (3, 2)$ , we obtain  $u = 10$  and then  $(a', b') = (33, 20)$ ; all square sides are positive.

Next, we verify coverage and local finiteness. Each shell fills exactly the region between two successive nested rectangles, with interiors disjoint from those of all earlier tiles. The first rectangle is centered at the origin, and both shell formulas preserve that center. In an  $A$ -to- $B$  step the half-width grows from  $a$  to  $a + u$  and the half-height from  $a + b/2$  to  $u$ ; in the next  $B$ -to- $A$  step the half-width becomes  $v = a + 3u$  and the half-height  $u + v$ . Each enlargement strictly increases both the width and the height, so the old closed rectangle lies in the interior of the new one, and no shell can touch a tile from two stages earlier. In particular  $a' > 10a$  by

(4.1), so both half-widths and half-heights tend to infinity. The nested rectangles therefore exhaust  $\mathbb{R}^2$ , and their square dissections combine into a tiling of the entire plane. Every compact set lies in the interior of a sufficiently large finite-stage rectangle, and no later shell meets that interior, so it meets only finitely many tiles; the tiling is locally finite.

It remains to count contacts. Lemma 2.1 gives each of the four unit squares six neighbors and each of the six outer core tiles four. Each outer core tile receives exactly two new contacts from the first shell. In either shell lemma, every newly inserted tile has exactly four contacts and receives exactly two from the following shell. After that shell it lies in the interior of the new outer rectangle, so no later shell can touch it. Thus *every* tile has exactly six distinct neighbors in the limit tiling. Lemmas 3.1 and 3.2 exclude extra singleton contacts at every shell, so the only ones are the six diagonal pairs at the core's three four-square points. Finally, the core already contains squares of sides 1, 2, 3, so the squares are not all of the same size. This proves every assertion.  $\square$

## 5 Consequences and scope

Under the standard closed-tile interpretation of intersection, the construction refutes the published statement of [21, Conjecture 7.4]. It does not address a strengthened version that imposes a uniform upper bound on square sizes, forbids four-square points, or counts only positive-length side contacts. Those are different hypotheses. In our example the point-only contacts are confined to the finite core, while the side lengths diverge outward under (4.1). The contact tables in Section 3 hold for arbitrary parameters, so the induction covers every shell.

*Remark.* The key invariant is that each shell consists of six squares, the same number as the degree of every tile in the completed tiling. At a finite stage the newest six tiles have only four contacts; the next shell supplies their two missing neighbors, after which they lie permanently in the interior. Omitting the infinite continuation would leave an incomplete finite patch rather than a counterexample.

## Acknowledgements

The first author is supported by NSFC (No. 12571055), NSFHN (No. 2022JJ10059) and Scientific Research Program of NUDT (No. JS2023-01).

## Declaration on the use of generative AI

The author used a generative AI assistant for language editing and typographical consistency checks during the preparation of this manuscript. All mathematical results, including the construction, proofs, and their verification, were produced by the author. The author has reviewed the final manuscript in its entirety and takes full responsibility for its content.

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