

ALGEBRAICITY OF THE G_2 -INVARIANT TENSORS ON YUN'S EXCEPTIONAL MOTIVES

BENJAMIN SEVERN

ABSTRACT. Zhiwei Yun constructed, from a rigid Hecke eigensheaf for G_2 on $\mathbb{P}^1$ minus three points, a family of rank-seven motives whose monodromy is Zariski dense in G_2 . For each parameter x we construct a smooth projective variety $\overline{X}_x$ and an algebraic correspondence on it that acts on cohomology as an idempotent of rank seven with image Yun's rank-seven Hodge structure M_x . We show that the G_2 -invariant tensors of M_x , transported by Yun's tensor functor, are classes of algebraic cycles on the powers of $\overline{X}_x$, at every parameter; at Hodge-generic parameters these are all the rational Hodge classes in the tensor constructions of M_x . The cycles are top components of Satake convolution fibres, transported by Hecke correspondences. The projector is made algebraic by a weighted resolution of Yun's boundary divisor, built on the moduli stack of bundles from the stabiliser tori of its points, by a characteristic-zero census of those points, and by a cycle-class comparison on varieties with quotient singularities. We do not claim that the correspondence is idempotent in the Chow group. Yun announced that the invariant cubic is motivated in André's sense; we give a proof of that statement, and the step from motivated to algebraic is new.

1. INTRODUCTION

Let G_2 be the split exceptional group of that type over $\mathbb{Q}$, and let $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In [40] Yun constructs, for each complex point $x \in S$, a pure rational Hodge structure M_x of rank seven and weight six, whose Tate twist $N_x = M_x(3)$ is the Betti realisation of Yun's motive $M_{\chi,x}$. It occurs in the compactly supported cohomology of an explicit smooth variety T_x , a finite étale cover of an open subset of a six-dimensional Schubert cell attached to a Hecke modification at x (Section 2). The variation $x \mapsto M_x$ has monodromy Zariski dense in G_2 [40, Thm. 1.2.1, §§4.5.1–4.5.2], and M_x carries a nondegenerate invariant quadratic form b and an invariant alternating cubic form t : the images under Yun's tensor functor of the corresponding G_2 -invariants of the seven-dimensional representation V_7 .

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In [40, §1.2.3] Yun notes that the argument of Dettweiler and Reiter [12], which rests on André's theorem [3], shows that these motives have the expected motivic Galois group in André's sense, and omits the details. Our first result is a proof that the invariant cubic is a motivated class, by a route that does not use the normaliser argument of [12].

Theorem A. *For every $x \in S(\mathbb{C})$ the invariant cubic $t \in (M_x^{\otimes 3})(9)$ is a motivated class in the sense of André [3], given explicitly as a motivated projector applied to the class of an algebraic cycle obtained from the unique top-dimensional component of a Satake convolution fibre. If $x \in S(\mathbb{Q})$ it is motivated over $\mathbb{Q}$.*

Our main result upgrades this to algebraicity, in the following sense.

Theorem B. *For every $x \in S(\mathbb{C})$ there are a smooth projective variety $\overline{X}_x$ and an algebraic correspondence P_X on $\overline{X}_x$ whose action on $H^*(\overline{X}_x, \mathbb{Q})$ is an idempotent of rank seven with image isomorphic to M_x , such that:*

- (1) *for every n , the image of every G_2 -invariant tensor of $V_7^{\otimes n}$ under Yun's construction, viewed in $H^*(\overline{X}_x^n)$ through $P_X^{\otimes n}$, is the class of an algebraic cycle on $\overline{X}_x^n$;*
- (2) *if x is Hodge-generic for the variation M , the Mumford–Tate group of N_x is the stabiliser of the invariant pairing and cubic, a $\mathbb{Q}$ -form of G_2 , and every rational Hodge class in every tensor construction $T^{a,b}(N_x)$ (mixed tensor powers, their symmetric and exterior summands, and duals), viewed in this way, is the class of an algebraic cycle.*

So the Hodge conjecture holds for all tensor constructions of M_x at Hodge-generic x , relative to the projector P_X : M_x is the image on $H^*(\overline{X}_x)$ of an algebraic correspondence whose action on cohomology is idempotent of rank seven, and every Hodge class in $(M_x^{\otimes a} \otimes M_x^{\vee \otimes b})(c)$, viewed in $H^*(\overline{X}_x^{a+b})(c')$ through $P_X^{\otimes(a+b)}$, is algebraic. We do not show that $P_X \circ P_X = P_X$ in the Chow group, so no Chow motive is constructed (Remark 10.13), and nothing is claimed about the rest of $H^*(\overline{X}_x)$, about Hodge classes that may appear at special parameters where the Mumford–Tate group drops (Remark 11.1), about the family of Dettweiler and Reiter (Remark 11.2), or about the Hodge conjecture in general. The group G_2 is not of abelian type, so the usual tools for the Hodge conjecture (the Lefschetz $(1,1)$ theorem, abelian varieties, Kuga–Satake constructions) do not reach these classes.

The variety $\overline{X}_x$. Let Bun be Yun's moduli stack of G_2 -bundles with level structure (Section 2), $U \subset \text{Bun}$ its open point, and $\mathfrak{B} \subset \text{Bun}$ the open substack of points with stabiliser of dimension at most six. Section 6 constructs a sequence, determined by a choice of contracting cocharacters, of stack-theoretic weighted blow-ups $\mathfrak{B}_N \rightarrow \mathfrak{B}$ of the finitely many points of $\mathfrak{B} \setminus U$ of codimension at least two. Pulling it back along Yun's smooth map ω from the Schubert cell $Y = \text{Gr}_{x,\theta^\vee}$ (θ the highest root) to Bun , and

gluing with a neighbourhood of the exceptional divisor of the blow-up $\widehat{\text{Gr}}_x$ of the Schubert variety at its base point $\star$ (the unique point of the closed orbit), gives a smooth Deligne–Mumford stack $\mathcal{X}_x$, proper over $\widehat{\text{Gr}}_x$. Let X be its coarse space and R the normalisation of X in the function field of the Q_8 -torsor T_x . Then R is projective with quotient singularities, and $\overline{X}_x$ is a resolution of R by blow-ups with smooth centres.

Outline of the proof. The proof has five steps.

- (1) *Satake cycles* (Section 3). Geometric Satake identifies the G_2 -invariants of $V_7^{\otimes n}$ with the top Borel–Moore homology of the convolution fibre $m_n^{-1}(e)$. Because the quasi-minuscule Schubert variety of G_2 is rationally smooth, this homology is spanned by the top-dimensional irreducible components, which are algebraic cycles.
- (2) *Hecke transport* (Section 3). Hecke correspondences, made finite by framing the endpoints of a chain of modifications, carry these components to cycles W_u on the n -th power of the Hecke fibre T_x , and Yun’s evaluation maps compute their classes: a fixed nonzero multiple of the transported tensor.
- (3) *The projector* (Sections 5–8). The rank-seven piece lives in the compactly supported cohomology of the open variety T_x . Yun’s cleanness theorem identifies its compactly supported and ordinary cohomology on the relevant block. On an arbitrary smooth compactification this gives a projector onto the clean piece that is motivated (Theorem A) but a priori not algebraic, because it needs inverse Lefschetz operators on the boundary. On the weighted model above, the odd cohomology of R is the clean piece: the boundary contributes nothing (Theorem 7.1), so the projector is a finite-group idempotent composed with an explicit cellular correction. The hypotheses of the weighted construction are verified by a census of the points of $\mathfrak{B}$ (Section 8).
- (4) *Closure* (Section 9). The cycle W_u is closed up in R^n ; the projected class of the closure does not depend on the closure.
- (5) *Transfer* (Section 10). Because R has quotient singularities, Vistoli’s intersection theory composes correspondences through R , and the cycle class is compatible with this composition. This transfers the projector and the cycles from R to the smooth $\overline{X}_x$.

The weighted resolution is essential. On the tacnode, a clean local system has zero boundary contribution on the weighted blow-up and a nonzero one on the ordinary resolution (Example 5.3), and one extra point blow-up of the weighted model of (Y, D_x) already makes the boundary contribution nonzero (Example 7.9).

Relation to earlier work. Dettweiler and Reiter [12] constructed the first motives for motivated cycles with motivic Galois group G_2 , from a rigid local system obtained by middle convolution; their statements for Grothendieck

motives are conditional on the standard conjectures. Yun [40] constructed the family studied here, and later stated that his G_2 -local system is the same as that of Dettweiler and Reiter [41, Thm. 6.1.5 and the paragraph after Thm. 6.2.3]; equality of the local systems does not by itself give an algebraic correspondence between the two families of motives. Færgeman [16] gives motivic realisations in Katz’s sense. We know of no earlier algebraicity statement for the invariant cubic of a G_2 -motive.

Since $\mathbb{L}$ is clean along D_x , the extensions by zero and the full direct image of $\mathbb{L}$ to $\widehat{\mathrm{Gr}}_x$ agree, hence agree with the intermediate extension; so the clean piece $H_c^*(\widehat{\mathrm{Gr}}_x \setminus D_x, \mathbb{L})$ is an intersection cohomology group of $\widehat{\mathrm{Gr}}_x$. Algebraic and motivic projectors onto pieces of the decomposition theorem have been studied for Chow motives: for semismall maps by de Cataldo and Migliorini [11], by Corti and Hanamura [10], and, as intersection motives, by Wildeshaus [38]. We do not use these results. What we add for Yun’s family is a model, the weighted resolution, on which the clean piece is cut out by a finite-group idempotent and an explicit cellular correction (Theorem 7.1 and Corollary 7.8).

Theorem B(1) for the cubic implies the first assertion of Theorem A, since algebraic classes are motivated. Section 4 is kept because its proof is short, works on any smooth compactification, and gives the statement over $\mathbb{Q}$ for $x \in S(\mathbb{Q})$.

What is computed. Three finite computations enter, all exact over $\mathbb{Q}$: the enumeration of the bundle types of codimension at most six (Lemma 8.1), the orbit classification and stabilisers at the base point (Proposition 8.5), and, at each of the 46 points of $\mathfrak{B}$ of positive codimension, an explicit contracting cocharacter (Proposition 8.7). Appendix B states the algorithms and prints every row.

Conventions. Cohomology has rational coefficients unless stated otherwise, and varieties are over $\mathbb{C}$. The group G_2 has simple roots α_1 (short) and α_2 (long); we write roots as $a\alpha_1 + b\alpha_2$ and cocharacters as $m_1\alpha_1^\vee + m_2\alpha_2^\vee$, or (a, b) and (m_1, m_2) . “Hodge-generic” is relative to the variation M over S .

Coefficients. The group $\tilde{A} = Q_8$ (Section 2) has no two-dimensional rational representation, and we never use one. The odd block of $\mathbb{Q}[\tilde{A}]$ is the quaternion division algebra $D = e_- \mathbb{Q}[\tilde{A}] = (-1, -1)_{\mathbb{Q}}$, $e_- = (1 - z)/2$, where z is the central element. For a $\mathbb{Q}[\tilde{A} \times \tilde{A}]$ -module H put $H_{\mathrm{odd}} = e_-^{(1)} e_-^{(2)} H$, a module over $D \otimes D^{\mathrm{op}} \cong M_4(\mathbb{Q})$. Yun’s two commuting actions are the *target* action $\tilde{A}(2)$ and the *source* action $\tilde{A}(1)$ [40, §3.2.5]; on D the target acts by $x \mapsto gx$ and the source by $x \mapsto xg^{-1}$. On every space used below $e_-^{(1)} H = H_{\mathrm{odd}}$: for $H_c^6(T_x)$ by [40, (3.16)], and for the pieces over the base point because the fibre there is an $\tilde{A} \times \tilde{A}$ -bitorsor. The two-dimensional representation V_χ appears only after extension of scalars to $\mathbb{Q}(i)$, where $D \otimes \mathbb{Q}(i) \cong \mathrm{End}_{\mathbb{Q}(i)}(V_\chi)$, and only in comparisons with [40].

Main notation

$\text{Bun}, U, \mathfrak{B}$	Yun's moduli stack; its open point $B\tilde{A}$; the open sub-stack of stabiliser dimension ≤ 6
$\tilde{A}, z, D, e_-$	Q_8 ; its central element; the odd block $(-1, -1)_{\mathbb{Q}}$; the odd central idempotent
ε	the primitive idempotent of $D \otimes D^{\text{op}}$ cutting out M_x (Section 2)
λ, V_7	the quasi-minuscule coweight $\theta^\vee$, θ the highest root; the representation V_λ
Y, D_x, ω	the Schubert cell $\text{Gr}_{x,\lambda}$; Yun's divisor; the source map $Y \rightarrow \text{Bun}$
T_x	Yun's $\tilde{A}$ -torsor over $Y \setminus D_x$ (his $\tilde{Y}_x$)
$\widehat{\text{Gr}}_x, E_\star$	the blow-up of $\text{Gr}_{x,\leq\lambda}$ at the base point $\star$; its exceptional divisor
η_y, T_y	a contracting cocharacter and a maximal torus of the stabiliser at a point y of $\mathfrak{B}$
$\mathcal{X}_x, X, R$	the good model; its coarse space; the normalisation in the total ring of fractions of T_x
$P_\star, P_{\text{clean}}$	the cellular projector at $E_\star$; the clean projector
$\bar{X}_x, q$	a resolution of R ; the resolution map

2. YUN'S CONSTRUCTION

We recall what we use from [40], with Yun's notation; statements are cited in the numbering of arXiv:1112.2434v1.

2.1. The moduli stack. Let $G = G_2$, let $P_0 \subset L_0G$ be the parahoric subgroup attached by Gross–Reeder–Yu to $-1 \in W$, and let P_∞ be the parahoric of the same type containing the opposite Iwahori subgroup at ∞ [40, §§2.1.3–2.1.4, 2.2.1]. Their reductive quotients are both the symmetric subgroup $K \subset G$, of type $A_1 \times A_1$. Yun works with a double-cover variant

$$\text{Bun} = \text{Bun}_G(\tilde{P}_0, I_1, P_\infty)$$

[40, §2.2]: the parahoric $\tilde{P}_0$ at 0 is the variant of [40, §2.2.4], built from the canonical double cover $\tilde{K}$ of K [40, §2.1.6], and there is Iwahori level I_1 at 1 [40, §2.2.5]. The stack Bun is smooth of dimension zero. It contains an open point $U \cong B\tilde{A}$, the classifying stack of a finite group $\tilde{A}$; for G_2 the group $\tilde{A}$ is the quaternion group Q_8 , whose centre $\langle z \rangle$ has order two [40, §§2.2.5–2.3]. The base point $u_0 \in U$ is that of [40, §2.2.6]. Let χ be the odd character of $\tilde{A}$ (the one with $\chi(z) = -1$). The *odd sheaf* $\mathbb{F}$ on $U = B\tilde{A}$ is the rank-four rational local system attached to the left $\tilde{A}$ -module D ; right multiplication makes D act on $\mathbb{F}$ by endomorphisms. Over $\mathbb{Q}(i)$, $\mathbb{F} \otimes \mathbb{Q}(i) \cong F_\chi \otimes_{\mathbb{Q}(i)} V_\chi^*$, where V_χ is the two-dimensional irreducible representation on which z acts by -1 and F_χ is Yun's odd sheaf.

Theorem 2.1 (Cleanness, [40, Thm. 2.4.2]). *For $j : U \hookrightarrow \text{Bun}$, $j_!\mathbb{F} \cong Rj_*\mathbb{F}$.*

Yun proves $j_! F_\chi \cong Rj_* F_\chi$ for ℓ -adic sheaves; the cone of $j_! \mathbb{F} \rightarrow Rj_* \mathbb{F}$ vanishes after $\otimes_{\mathbb{Q}} \mathbb{Q}(i)$, hence vanishes. The Betti form of this statement, on finite-type smooth charts of Bun and on Y , is Lemma A.3.

2.2. The Birkhoff decomposition. By [40, §2.2.2], the points of $\text{Bun}_G(P_0, P_\infty)$ are bundles $\mathcal{P}_{\tilde{w}}$ indexed by the double cosets $\tilde{w} \in W_K \backslash \widehat{W} / W_K$ of the affine Weyl group, with $\text{Aut}(\mathcal{P}_{\tilde{w}}) = \Gamma_0 \cap \text{Ad}(\tilde{w})P_0$. For $\tilde{w} = t^\mu w$ the Lie algebra of $\text{Aut}(\mathcal{P}_{\tilde{w}})$ is $\mathfrak{t} \oplus \bigoplus_\alpha \bigoplus_{n=\ell_\alpha}^{h_\alpha} t^n E_\alpha$, the sum running over the roots α and the powers t^n of the coordinate at 0 for which the loop root vector $t^n E_\alpha$ lies in Aut , with

$$(1) \quad \ell_\alpha = \lceil -\text{ht}(\alpha)/2 \rceil, \quad h_\alpha = \langle \alpha, \mu \rangle - \lceil \text{ht}(w^{-1}\alpha)/2 \rceil.$$

Write $N_\alpha = \max(h_\alpha - \ell_\alpha + 1, 0)$ for the number of loop root vectors in direction α . The constant torus T is a maximal torus of $\text{Aut}(\mathcal{P}_{\tilde{w}})$. A point of Bun over $\mathcal{P}_{\tilde{w}}$ also records a Borel subgroup $b \in G/B$ (the level structure at 1), and $\text{Aut}(\mathcal{P}_{\tilde{w}})$ acts on G/B through evaluation at 1,

$$\text{ev}_1 : \text{Aut}(\mathcal{P}_{\tilde{w}}) \rightarrow G.$$

2.3. The Hecke fibre and Yun's divisor. Let $\theta = 3\alpha_1 + 2\alpha_2$ be the highest root and $\lambda = \theta^\vee = \alpha_1^\vee + 2\alpha_2^\vee$ the quasi-minuscule coweight (the highest short root of the dual group), so that $V_\lambda = V_7$. For $x \in S$, the Schubert variety $\text{Gr}_{x, \leq \lambda}$ in the affine Grassmannian at x has dimension $\langle 2\rho, \lambda \rangle = 6$ and consists of the base point $\star$ and the open orbit $Y = \text{Gr}_{x, \lambda}$ [40, §3.4], [20, Lemma 5.22]. Modifying the bundle u_0 at x by a point of Y defines

$$\omega : Y \longrightarrow \text{Bun},$$

and Yun's divisor is $D_x = \omega^{-1}(\text{Bun} \setminus U)$, so that $\text{Gr}_{x, \leq \lambda}^U = \text{Gr}_{x, \leq \lambda} \setminus D_x$ [40, §3.4]. Here u_0 is the fixed *target* of the modification and ω is the *source* map [40, (3.3), §3.2.3]. Over $Y \setminus D_x$ the framing of the source $\omega(y) \in U$ gives a finite étale $\tilde{A}$ -torsor $T_x \rightarrow Y \setminus D_x$, whose deck group is the source action $\tilde{A}(1)$ and on which the target action $\tilde{A}(2) = \text{Aut}(u_0)$ acts compatibly [40, (3.5)]; this is Yun's $\tilde{Y}_x$ [40, §§3.3.2–3.3.3]. Let $\mathbb{L} = e_{-p*} \mathbb{Q}_{T_x}$, a rank-four local system with fibre D (monodromy by left multiplication; D acts on the right through $\tilde{A}(1)$).

Lemma 2.2. *The map ω is smooth, and $j'_! \mathbb{L} \cong Rj'_! \mathbb{L}$ for $j' : Y \setminus D_x \hookrightarrow Y$.*

Proof. On the orbit of relative position λ either Hecke projection is smooth with fibre Gr_λ ; pulling one endpoint back to u_0 is the base change along the étale map $\text{pt} \rightarrow B\tilde{A} \rightarrow \text{Bun}$, so the other endpoint map ω is smooth. Smooth base change transports Theorem 2.1 to Y for Yun's ℓ -adic sheaves, and Lemma A.3 gives the Betti statement. $\square$

2.4. A dictionary. We use the following objects of [40, §§3.1–3.4].

- Hk is the Hecke stack over $\text{Bun} \times \text{Bun} \times S$, parametrising two bundles and a modification between them at a point of S ; GR is its base change along $u_0 \times \text{id}_S$ in the target, which is Cartesian by definition [40, §3.2.3]. GR_x^U is the fibre over x with source in U , and $\widetilde{\text{GR}}_x^U$ its $\tilde{A}$ -torsor of frames of the source [40, (3.3)–(3.7)]. We call $\widetilde{\text{GR}}_x^U$ the *one-step space*; it carries the source action $\tilde{A}(1)$ and the target action $\tilde{A}(2)$.
- For a Satake kernel $\mathcal{K}$ (a perverse sheaf on the affine Grassmannian equivariant for the arc group), $T(\mathcal{K}, -)$ is Yun's Hecke functor: pull back along the source map, tensor with $\mathcal{K}$, push forward with compact supports along the target map [40, (3.3)].
- $E(\mathcal{K})_x$ is the eigenvalue at x : the multiplicity space of $j_! \mathbb{F}$ in $T(\mathcal{K}, j_! \mathbb{F})$, at the fibre over x [40, (3.18)]. Rationally we put $E(\mathcal{K})_x = \varepsilon H_c^*(\widetilde{\text{GR}}_x^U, \mathcal{K})$ (Section 3.2).
- Yun's tensor isomorphisms $\varphi_{\mathcal{K}_1, \mathcal{K}_2} : E(\mathcal{K}_1) \otimes E(\mathcal{K}_2) \rightarrow E(\mathcal{K}_1 * \mathcal{K}_2)$ are [40, (3.19)].

2.5. The rank-seven piece. Over $\mathbb{Q}(i)$, Yun's eigenvalue is $E_\chi(V_7)_x = (V_\chi^* \otimes H_c^*(T_x, \mathbb{Q}(i)))^{\tilde{A}}$ in the relevant degree [40, (3.12), (3.18)]. The base point $\star$ lies in the open set $\text{Gr}_{x, \leq \lambda}^U$ [40, (3.13)]: it is not a boundary point. The group $\tilde{A}(2)$ acts on Y and on T_x , commuting with $\tilde{A}(1)$; under $V_\chi^* \boxtimes V_\chi$ of [40, (3.9)], $\tilde{A}(1)$ acts on V_χ^* and $\tilde{A}(2)$ on V_χ . Put

$$\varepsilon = \frac{1}{32} \sum_{a_1, a_2 \in \tilde{A}} \phi_\chi(a_1 a_2^{-1}) a_1 \otimes a_2 \in \mathbb{Q}[\tilde{A} \times \tilde{A}], \quad M_x = \varepsilon H_c^6(T_x, \mathbb{Q}),$$

with ϕ_χ the rational character of V_χ . Up to a scalar this is Yun's idempotent e_χ [40, §3.3.2]. The normalisation is forced: with $\Phi = \sum_b \phi_\chi(b)(b, 1)$ and $\Delta = \sum_a (a, a)$, which commute, $\Delta^2 = 8\Delta$ and $\Phi^2 = 4\Phi$, so $(c\Phi\Delta)^2 = 32c \cdot c\Phi\Delta$, and $c\Phi\Delta$ is idempotent exactly for $c = 1/32$. On D , ε acts by $x \mapsto \frac{1}{2} \text{trd}(x)$, the projection onto the identity line $\mathbb{Q} \cdot 1$, and it kills the even block. As a correspondence, ε has coefficient $\frac{1}{16}$ on the graphs of the pairs with $a_1 = a_2$, $-\frac{1}{16}$ on those with $a_1 = za_2$, and 0 on the others. By [40, (3.16)], $H_c^6(T_x)_{\text{odd}} \otimes \mathbb{Q}(i) \cong \text{End}(V_\chi) \otimes_{\mathbb{Q}(i)} E_\chi(V_7)_x(-3)$ has dimension 28. As ε is a primitive idempotent of $D \otimes D^{\text{op}} \cong M_4(\mathbb{Q})$, Morita theory gives $H_c^6(T_x)_{\text{odd}} \cong D \otimes_{\mathbb{Q}} M_x$, so $\dim_{\mathbb{Q}} M_x = 7$ and $M_x(3) \otimes \mathbb{Q}(i) = E_\chi(V_7)_x$. The odd part is pure of weight six by Lemma 2.2, since compactly supported cohomology has weights at most six and ordinary cohomology at least six; so $M_x(3)$ is the Betti realisation of Yun's $M_{\chi, x}$ [40, Prop. 3.3.3]. As x varies, $N_x = M_x(3)$ is the fibre of a rational local system N_S with $N_S \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong E_\chi(V_7)$ (Lemma A.6).

Lemma 2.3 (Rational smoothness). *$\text{Gr}_{x, \leq \lambda}$ is rationally smooth: its intersection complex is $\mathbb{Q}[6]$.*

Proof. In characteristic zero the opposite-cell chart of $\mathrm{Gr}_{\leq \lambda}$ through $\star$ is the minimal nilpotent cone, and blowing up the vertex gives the negative line bundle over the adjoint variety $E = \mathrm{G}_2/P$ [20, (5.21), Lemma 5.22]. The variety E has one Schubert class in each even degree $0, \dots, 10$, and $h^5 \neq 0$ for the ample class h , so the maps $H^{2i}(E) \rightarrow H^{2i+2}(E)$ given by h are isomorphisms between one-dimensional spaces. The Gysin sequence of the punctured cone then shows that the link of $\star$ is a rational homology sphere. $\square$

3. SATAKE CYCLES AND THEIR HECKE TRANSPORT

Fix $x \in S(\mathbb{C})$, $\lambda = \theta^\vee$, $d = \langle 2\rho, \lambda \rangle = 6$ and an integer $n \geq 1$. Coefficients are $\mathbb{Q}$, and sheaves are constructible sheaves on complex points. Yun's statements are ℓ -adic; the comparison of the objects and maps used here with his is Appendix A, and we state its results where they are used.

3.1. Top components of convolution fibres. Let

$$C_n = \mathrm{Gr}_{x, \leq \lambda} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{x, \leq \lambda} \quad (n \text{ factors})$$

be the space of chains $b_n \rightarrow b_{n-1} \rightarrow \cdots \rightarrow b_1 \rightarrow b_0 = u_0$ of modifications at x , each of relative position at most λ , with fixed target u_0 . Step i runs from the source b_i to the target b_{i-1} ; the i -th convolution coordinate is the position of b_i relative to b_{i-1} , so coordinate 1 is adjacent to the fixed target, and b_n is the variable source. Write $\omega_i : C_n \rightarrow \mathrm{Bun}$ for the bundle b_i and $m_n : C_n \rightarrow \mathrm{Gr}_{x, \leq n\lambda}$ for the composite relative position, and put $\mathbf{F} = m_n^{-1}(e)$, the chains whose composite is the identity modification of u_0 . Put

$$C_n^{(U)} = \omega_n^{-1}(U), \quad \Omega = C_n^{U^n} = \bigcap_{i=1}^n \omega_i^{-1}(U) \subset C_n^{(U)} \quad (\text{open}).$$

Since $\omega_n(\mathbf{F}) = u_0 \in U$, the compact set $\mathbf{F}$ lies in $C_n^{(U)}$.

Lemma 3.1. (a) $\dim \mathbf{F} \leq 3n$. (b) C_n is rationally smooth of dimension nd , and $m_{n*} \mathbb{Q}_{C_n}[nd](nd/2) \cong \mathrm{IC}_\lambda^{*n}$. (c) There is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Sat}}(\mathrm{IC}_0, \mathrm{IC}_\lambda^{*n}) \cong H_{6n}^{BM}(\mathbf{F}, \mathbb{Q})(-3n),$$

and the right side has the basis $\{[Z]\}$, Z running over the $3n$ -dimensional irreducible components of $\mathbf{F}$ with their reduced structure and complex orientation.

Proof. (a) is stratified semismallness of convolution [24, Lemma 4.4]. (b) C_n is an iterated locally trivial fibration with fibre the rationally smooth $\mathrm{Gr}_{\leq \lambda}$ (Lemma 2.3), so $\mathrm{IC}(C_n) = \mathbb{Q}[nd]$, and IC_λ^{*n} is by definition m_{n*} of the twisted external product of the $\mathrm{IC}_\lambda = \mathbb{Q}[d](d/2)$. (c) m_n is proper, so with $i : \mathbf{F} \hookrightarrow C_n$ and base change, $\mathrm{Hom}(\delta_e, m_{n*} \mathbb{Q}[nd](nd/2)) = H^0(\mathbf{F}, i^! \mathbb{Q}_{C_n}[nd](nd/2))$. Rational smoothness gives $\mathbb{Q}_{C_n}[2nd](nd) \cong \omega_{C_n}$, hence $i^! \mathbb{Q}_{C_n}[nd](nd/2) = \omega_{\mathbf{F}}[-nd](-nd/2)$, and the Hom is $H_{nd}^{BM}(\mathbf{F})(-nd/2)$, with $nd = 2 \cdot 3n$. By (a) this is the

top Borel–Moore group, freely spanned by the fundamental classes of the $3n$ -dimensional components. Both sides of (c) are Homs in the abelian category of perverse sheaves, because IC_λ^{*n} is perverse [24, Prop. 4.2] and semisimple by the decomposition theorem [5]. $\square$

For $u \in (V_7^{\otimes n})^{G_2}$ let $s_u : \mathrm{IC}_0 \rightarrow \mathrm{IC}_\lambda^{*n}$ be the morphism attached to u by the Satake equivalence [24, Thm. 7.3], and $Z_u = \sum_Z c_Z(u) Z$ the rational cycle with $[Z_u] = s_u$. For $n = 3$ the space is one-dimensional ($V_7 \otimes V_7 = \mathbf{1} \oplus V_7 \oplus V_{14} \oplus V_{27}$), so there is exactly one nine-dimensional component Z_9 of $\mathbb{F}$, and $Z_t = c Z_9$ with $c \in \mathbb{Q}^\times$.

3.2. Framing. The one-step space $\widetilde{\mathrm{GR}}_x^U$ of Section 2.4 carries the source action $\tilde{A}(1)$ and the target action $\tilde{A}(2)$. By [40, (3.9)],

$$H_c^*(\widetilde{\mathrm{GR}}_x^U)_{\mathrm{odd}} \otimes \mathbb{Q}(i) \cong V_\chi^* \boxtimes V_\chi \otimes E_\chi \cong \mathrm{Hom}(V_{\mathrm{source}}, V_{\mathrm{target}}) \otimes E_\chi,$$

with row index the target and column index the source; rationally $\tilde{A}(2)$ acts on D by $x \mapsto gx$ and $\tilde{A}(1)$ by $x \mapsto xg^{-1}$. Framing the source b_n of a chain in Ω , and then the intermediate bundles $b_1, \dots, b_{n-1}$, gives a tower of finite étale torsors

$$P = (\widetilde{\mathrm{GR}}_x^U)^n \xrightarrow{\mathrm{fr}_{\mathrm{int}}} Q \xrightarrow{\mathrm{fr}_{\mathrm{end}}} \Omega, \quad \deg \mathrm{fr}_{\mathrm{int}} = 8^{n-1}, \quad \deg \mathrm{fr}_{\mathrm{end}} = 8,$$

where factor i of P is step i of the chain (factor 1 is adjacent to u_0), and the internal group $\tilde{A}^{n-1}$ acts at b_i by $x_i \mapsto x_i g^{-1}$, $x_{i+1} \mapsto g x_{i+1}$. The endpoints are bookkept as follows.

endpoint	Yun's action	on coefficients	slot
$b_0 = u_0$	$\tilde{A}(2)$ of step 1	$g x_1$	target, row of slot 1
$b_i, 0 < i < n$	$\tilde{A}(1)$ of step $i = \tilde{A}(2)$ of step $i + 1$	$x_i g^{-1} \otimes g x_{i+1}$	internal pair $i, i + 1$
b_n	$\tilde{A}(1)$ of step n	$x_n g^{-1}$	source, column of slot n

The coefficient system is $\mathbb{G} = \omega_n^* \mathbb{F}$, with fibre D at b_n . It is retained: $\mathrm{fr}_{\mathrm{end}}^*$ identifies $H_c^*(\Omega, \mathbb{G})$ with $(D \otimes H_c^*(Q))^{\tilde{A}}$, and likewise for ordinary cohomology, by pullback onto invariants. All maps are equivariant for the external actions and for the internal group; the factorwise ε is applied only after the outer legs are contracted. We use the monoidal structure of [40, §3.1.3] in the form

$$T(\mathcal{K}_1 * \mathcal{K}_2, \mathcal{F}) = T(\mathcal{K}_1, T(\mathcal{K}_2, \mathcal{F})),$$

with $\mathcal{K}_1$ on the convolution coordinate adjacent to u_0 (proper base change on the bounded iterated Hecke correspondence); this fixes the order in [40, (3.19)]: the first factor of $\varphi_{\mathcal{K}_1, \mathcal{K}_2}$ is factor 1 of P .

The eigenvalue at x . By [40, (3.3), (3.7)] and proper base change,

$$\begin{aligned} (u_0 \times \mathrm{id})^* T(\mathcal{K}, j_! \mathbb{F})_x &= H_c^*(\mathrm{GR}_x^U, \omega^* \mathbb{F} \otimes \mathcal{K}) \\ &= (D \otimes H_c^*(\widetilde{\mathrm{GR}}_x^U, \mathcal{K}))^{\tilde{A}} \cong e_- H_c^*(\widetilde{\mathrm{GR}}_x^U, \mathcal{K}), \end{aligned}$$

and we put $E(\mathcal{K})_x = \varepsilon H_c^*(\widetilde{\mathrm{GR}}_x^U, \mathcal{K})$, so that $H_c^*(\widetilde{\mathrm{GR}}_x^U, \mathcal{K})_{\mathrm{odd}} \cong D \otimes E(\mathcal{K})_x$ and $E(\mathcal{K})_x \otimes_{\mathbb{Q}} \mathbb{Q}(i) = E_{\chi}(\mathcal{K})_x$ of [40, (3.18)]. For $\mathcal{K} = \mathrm{IC}_{\lambda}^{*n}$ the space is replaced, through m_n , by $C_n^{(U)}$, and $E(\mathrm{IC}_{\lambda}^{*n})_x = \varepsilon H_c^{nd}(C_n^{(U)}, \omega_n^* \mathbb{F})(nd/2)$, all other degrees vanishing [40, (3.12)]. In particular $E(V_7)_x = M_x(3)$. Yun's tensor unit is $1_{\mathrm{Yun}} = \mathrm{id}_V \in E_{\chi}(\mathrm{IC}_0)_x = \mathrm{End}_{\tilde{A}}(V_{\chi})$.

Lemma 3.2 (Intermediate bundles in U). *Extension by zero $H_c^*(\Omega, \mathbb{G}) \rightarrow H_c^*(C_n^{(U)}, \mathbb{G})$ is an isomorphism.*

Proof. Induction on n , the case $n = 1$ being empty. By the monoidal structure, $T(\mathcal{K}_1 * \cdots * \mathcal{K}_n, j_! \mathbb{F}) = T(\mathcal{K}_1, T(\mathcal{K}_2 * \cdots * \mathcal{K}_n, j_! \mathbb{F}))$, and the inner term is $j_! \mathbb{F} \boxtimes E(\mathcal{K}_2 * \cdots * \mathcal{K}_n)$ by [40, Lemma 2.4.8, §3.4.4] and Theorem 2.1. Its stalks vanish off U , so in the outer pull-push, a compactly supported cohomology of $C_n^{(U)}$ by proper base change, chains with $b_1 \notin U$ contribute zero. Iterating on the inner term removes $b_2, \dots, b_{n-1}$. $\square$

3.3. Yun's evaluation maps.

Lemma 3.3 (Framing, Künneth and evaluation). *Work over $\mathbb{Q}(i)$, with $V = V_{\chi}$, a basis (e_a) , dual basis (e^a) and $E_{ab} = e_a e^b$. For one step put $H = H_c^6(\widetilde{\mathrm{GR}}_x^U(3)_{\mathrm{odd}}$ and $E = E_{\chi}(V_7)_x = \mathrm{Hom}_{\tilde{A}(2)}(V, (V \otimes H)^{\tilde{A}(1)})$. Yun's evaluation maps [40, Lemma 2.4.8, (3.8)–(3.9)] give an isomorphism*

$$\Psi : \mathrm{End}(V) \otimes E \xrightarrow{\sim} H, \quad \Psi(vf \otimes e) = (f \otimes 1)e(v),$$

and we define the multiplicity embedding

$$\iota(e) = \Psi(I \otimes e) = \sum_a (e^a \otimes 1) e(e_a) \in \varepsilon H,$$

a partial trace; rationally it is the partial reduced trace, half the rational partial trace on the rank-four module D . With $D \otimes \varepsilon H \cong H$ normalised by $1 \otimes m \mapsto m$,

$$(*) \quad e(e_a) = \sum_b e_b \otimes E_{ab} \otimes \iota(e).$$

Under $\mathrm{fr}_{\mathrm{int}}^*$, $\mathrm{fr}_{\mathrm{end}}^*$ and Künneth, the iterated isomorphism [40, (3.19)] sends $e_1 \otimes \cdots \otimes e_n$ to the $\tilde{A}$ -map

$$e_a \mapsto \sum_{a_2, \dots, a_{n+1}} e_{a_{n+1}} \otimes E_{aa_2} \otimes E_{a_2 a_3} \otimes \cdots \otimes E_{a_n a_{n+1}} \otimes \iota(e_1) \otimes \cdots \otimes \iota(e_n)$$

in the factor order of P . For each outer pair (a, a_{n+1}) this is invariant under the internal group before any ε is applied. Contracting the outer legs by the trace Θ on V (rationally, the partial reduced trace) sets $a_{n+1} = a$:

$$\Theta(\varphi_n(e_1 \otimes \cdots \otimes e_n)) = J_n \otimes \iota(e_1) \otimes \cdots \otimes \iota(e_n),$$

$$J_n = \sum e_{a_1 a_2} \otimes e_{a_2 a_3} \otimes \cdots \otimes e_{a_n a_1}.$$

Rationally J_n is the adjoint of n -fold multiplication for the invariant form $h(x, y) = \mathrm{trd}(x\bar{y})$; for $n = 3$, $J_3 = \frac{1}{4} \sum_{s,t} b_s \otimes b_s^{-1} b_t \otimes b_t^{-1}$ with $(b_s) = (1, i, j, k)$. Moreover $\varepsilon^{\otimes n} J_n = 2^{1-n} 1^{\otimes n}$.

Proof. fr_{int} and fr_{end} are finite étale torsors, so pullback is an isomorphism onto invariants. Künneth for H_c applies to the product, and all classes are in even degree $d = 6$ per factor [40, (3.12)], so there are no Koszul signs. By [40, (3.8)], $V_{\text{source}}^* \otimes (V_{\text{source}} \otimes H)^{\hat{A}(1)} \rightarrow H$, $f \otimes \sum v_j \otimes h_j \mapsto \sum f(v_j)h_j$, is an isomorphism; with [40, Lemma 2.4.8] it gives Ψ , and inverting the evaluation formula gives $(*)$. The operator of the term $e_b \otimes E_{ab}$ sends the source vector e_b to the target vector e_a . By $T(\mathcal{K}_1 * \mathcal{K}_2) = T(\mathcal{K}_1) \circ T(\mathcal{K}_2)$ the iterated isomorphism applies $T(\mathcal{K}_1, \text{ev}_2)$ and then the next evaluation: the source leg of step i is fed into the evaluation of step $i + 1$, which gives the displayed formula (Lemma 3.4). Internal invariance: $\sum_b (E_{ab} g^{-1}) \otimes (g E_{bc}) = \sum_u E_{au} \otimes E_{uc}$, and likewise for the other internal pairs. The identification of J_3 and the value of $\varepsilon^{\otimes n} J_n$ are direct computations in $D \otimes \mathbb{Q}(i) \cong M_2(\mathbb{Q}(i))$ (with $i \mapsto \text{diag}(i, -i)$, $j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$): ε kills the off-diagonal units and sends each E_{aa} to $\frac{1}{2}$, so only the two terms with all indices equal survive. $\square$

Lemma 3.4 (Evaluation naturality). *Write $\mathcal{T}_i = T(\mathcal{K}_i, -)$, $F_0 = j_! \mathbb{F}_V$, $\mathcal{K}' = \mathcal{K}_2 * \dots * \mathcal{K}_n$, $\mathcal{G}' = T(\mathcal{K}', F_0)$ and $E' = E(\mathcal{K}')$, and let $\text{ev}' : F_0 \otimes E' \rightarrow \mathcal{G}'$ be Yun's evaluation [40, §3.4.4, Lemma 2.4.8]. The square*

$$\begin{array}{ccc} V \otimes E(\mathcal{K}_1) \otimes E' & \xrightarrow{\text{ev}_1 \otimes 1} & (\mathcal{T}_1 F_0)_{u_0} \otimes E' \\ \downarrow 1 \otimes \varphi_{\mathcal{K}_1, \mathcal{K}'} & & \downarrow \rho \\ V \otimes E(\mathcal{K}_1 * \mathcal{K}') & \xrightarrow{\text{ev}_1 \dots \text{ev}_n} & (\mathcal{T}_1 \mathcal{G}')_{u_0} \end{array}$$

commutes, where ρ is the projection formula followed by $(\mathcal{T}_1 \text{ev}')_{u_0}$, and the lower right term is identified with $T(\mathcal{K}_1 * \mathcal{K}', F_0)_{u_0}$ by the convolution base-change map. Its passage to $\widetilde{\text{GR}}_x^U \times (\widetilde{\text{GR}}_x^U)^{n-1}$ (stalk by $!$ -base change, pull-back of the finite framing torsors, then Künneth) contains the same pulled-back morphism $\omega_1^* \text{ev}'$ on both paths and commutes with extension by zero.

Proof. By Lemma A.5, the square is the fibre at x of the restriction along $u_0 \times \text{id}_S$ of the evaluation square that specifies Yun's map [40, (3.19)], and by Lemma A.4 its corners and arrows are operations on a finite diagram of varieties. It commutes by the construction of [40, (3.19)] from the monoidal Hecke functor and the canonical decomposition of [40, Lemma 2.4.8]: applying the multiplicity Hom to the right-hand composite is $\varphi_{\mathcal{K}_1, \mathcal{K}'}$. The comparison maps of the passage to the framed product are natural in the coefficient morphism ev' , so both paths contain $\omega_1^* \text{ev}'$ before the outer pushforward. Restriction to $b_1 \in U$ commutes with it, since the evaluation identifies both objects with their extension by zero; repeat on the tail. Iterating, a framed first step with source coefficient V is sent to the matrix-unit expansion of Lemma 3.3: at one step this is the inverse of Ψ , and the square substitutes the expansion of the tail for each source vector. $\square$

3.4. The transported class.

Proposition 3.5 (The transported class). *Let $u \in (V_7^{\otimes n})^{G_2}$, $Z_u^\circ = Z_u \cap \Omega$, $\tilde{Z}_u \subset Q$ the sheet over Z_u° on which the frame of b_n is the canonical one (on*

$\mathbb{F}$ the source b_n is canonically u_0), and $W_u = \mathrm{fr}_{\mathrm{int}}^{-1}(\tilde{Z}_u) \subset P$, with reduced pullback multiplicities. Put

$$\tau_u = \iota^{\otimes n} \varphi_n^{-1}(E(s_u)_x(1_{Y_{\mathrm{un}}})) \in N_x^{\otimes n},$$

with φ_n the iterated [40, (3.19)] and ι as in Lemma 3.3. Let $i_{\mathbb{F}} : \mathbb{F} \hookrightarrow C_n^{(U)}$ and $i_{\mathbb{F}}^{\circ} : \mathbb{F} \cap \Omega \hookrightarrow \Omega$ be the inclusions, ext extension by zero from Ω and fgt forget-supports. Then:

- (1) $E(s_u)_x(1_{Y_{\mathrm{un}}})$ is the $\tilde{A}$ -map $v \mapsto \alpha_v = \mathrm{PD}_c(i_{\mathbb{F}*}[Z_u \otimes v]) \in H_c^{nd}(C_n^{(U)}, \mathbb{G})(nd/2)$;
- (2) with $\beta_v = \mathrm{ext}^{-1}\alpha_v$ (Lemma 3.2), $\mathrm{fgt}_{\Omega}\beta_v = \mathrm{PD}_{\Omega}(i_{\mathbb{F}*}^{\circ}[Z_u^{\circ} \otimes v])$ in ordinary cohomology $H^{nd}(\Omega, \mathbb{G})(nd/2)$;
- (3) in ordinary cohomology $H^{nd}(P)(nd/2)$, on the all-odd block,

$$\Theta \mathrm{fr}_{\mathrm{int}}^* \mathrm{fr}_{\mathrm{end}}^* \mathrm{fgt}_{\Omega}\beta = 4e_{-}^{(n)} \mathrm{cl}(W_u) = \mathrm{fgt}_P(J_n \otimes \tau_u),$$

where $e_{-}^{(n)}$ is the odd projector of the source action on factor n .

All these identities are between rational classes.

Proof. (a) By Lemma 3.1(c), proper base change identifies s_u with the fundamental-class morphism of Z_u into $i_{\mathbb{F}}^! \mathbb{Q}_{C_n}[nd](nd/2) = \omega_{\mathbb{F}}[-nd](-nd/2)$, followed by the counit $i_* i^! \rightarrow \mathrm{id}$. By [40, (3.3), (3.7)] and proper base change, the stalk of $T(\mathrm{IC}_{\lambda}^{*n}, j_! \mathbb{F})$ at u_0 is $R\Gamma_c(C_n^{(U)}, \mathbb{G})[nd](nd/2)$. On $\mathbb{F}$ the composite is the identity modification, which identifies b_n with u_0 and trivialises $\mathbb{G}$. Tensoring the supported counit with $\mathbb{G}$, evaluating at v and applying $R\Gamma_c$ gives α_v , and $E(\mathrm{IC}_0)_x(1_{Y_{\mathrm{un}}})$ corresponds to id_V . (b) Open/closed base change, with $\mathrm{fgt}_{\Omega} = \mathrm{res}_{\Omega \subset C} \mathrm{fgt}_C \mathrm{ext}$. The cycle Z_u° need not be proper, so its class is an ordinary one. (c) Pullback along the endpoint cover gives $\mathrm{fr}_{\mathrm{end}}^*[Z_u^{\circ} \otimes v] = \sum_{g \in \tilde{A}} g v \otimes [g \tilde{Z}_u]$, g the source action. The partial reduced trace of this map in v is $\sum_g \chi(g)[g \tilde{Z}_u] = 2[\tilde{Z}_u] - 2[z \tilde{Z}_u] = 4e_{-}^{(n)}[\tilde{Z}_u]$, and flat pullback along $\mathrm{fr}_{\mathrm{int}}$ gives $4e_{-}^{(n)} \mathrm{cl}(W_u)$. On the other side, the iterated evaluation is fixed by Lemma 3.4; the comparison of Lemma 3.6 identifies the compactly supported framed trace with $J_n \otimes \tau_u$, and forgetting supports gives the equality in ordinary cohomology. Pullback, extension by zero, proper pushforward and finite étale pullback commute with cycle classes. For rationality, use the coefficient $\mathbb{F}$ (fibre D) with its commuting right D -action and $\mathrm{Hom}_{\tilde{A}(2), D^{\mathrm{op}}}(D, -)$; the contraction is $\frac{1}{2} \mathrm{Tr}_{\mathbb{Q}}$, which becomes the trace on V after splitting, and faithful scalar extension proves the identities over $\mathbb{Q}$. $\square$

The compactly supported side of (c) is the following comparison, a computation with Yun's specified maps.

Lemma 3.6 (Supported and framed comparison). *Over $\mathbb{Q}(i)$ write $\mathbb{F} \otimes_{\mathbb{Q}} \mathbb{Q}(i) \cong \mathbb{F}_V \otimes_{\mathbb{Q}(i)} V^*$ and $\mathbb{G}_V = \omega_n^* \mathbb{F}_V$, and for $Y' \in \{C_n^{(U)}, \Omega\}$ put $\mathcal{M}_c(Y') =$*

$\mathrm{Hom}_{\tilde{A}(2)}(V, H_c^{nd}(Y', \mathbb{G}_V)(3n))$ and $\mathcal{M}(Y') = \mathrm{Hom}_{\tilde{A}(2)}(V, H^{nd}(Y', \mathbb{G}_V)(3n))$. The trivialisation on $\mathbf{F}$ and the supported counit define $a_c : H_{nd}^{BM}(\mathbf{F})(-3n) \rightarrow \mathcal{M}_c(C_n^{(U)})$ and $a_o : H_{nd}^{BM}(\mathbf{F} \cap \Omega)(-3n) \rightarrow \mathcal{M}(\Omega)$. Let $B_c = \Theta \mathrm{fr}_{\mathrm{int}}^* \mathrm{fr}_{\mathrm{end}}^*$ and $B_o = \Theta \mathrm{fr}_{\mathrm{int}}^* \mathrm{fr}_{\mathrm{end}}^*$ on the compactly supported and ordinary multiplicity spaces. Then the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathrm{Sat}}(\mathrm{IC}_0, \mathrm{IC}_{\lambda}^{*n}) & \xrightarrow{E(-)_x(1_{Y_{\mathrm{un}}})} & \mathcal{M}_c(C_n^{(U)}) & & \\
\downarrow \cong & & \downarrow \mathrm{ext}^{-1} & & \\
H_{nd}^{BM}(\mathbf{F})(-3n) & \xrightarrow{\mathrm{ext}^{-1} a_c} & \mathcal{M}_c(\Omega) & \xrightarrow{B_c} & H_c^{nd}(P, \mathbb{Q}(i))(3n) \\
\downarrow \mathrm{res} & & \downarrow \mathrm{fgt} & & \downarrow \mathrm{fgt} \\
H_{nd}^{BM}(\mathbf{F} \cap \Omega)(-3n) & \xrightarrow{a_o} & \mathcal{M}(\Omega) & \xrightarrow{B_o} & H^{nd}(P, \mathbb{Q}(i))(3n)
\end{array}$$

commutes, the upper left identification using $i_{\mathbf{F}}^!$. If $\tilde{Z} \subset Q$ is the canonical endpoint sheet over $Z \cap \Omega$ and $W = \mathrm{fr}_{\mathrm{int}}^{-1} \tilde{Z}$, then

$$B_o a_o[Z \cap \Omega] = 4e_-^{(n)} \mathrm{cl}(W),$$

$$B_c \mathrm{ext}^{-1} \varphi_n(e_1 \otimes \cdots \otimes e_n) = J_n \otimes \iota(e_1) \otimes \cdots \otimes \iota(e_n).$$

Consequently, with $\tau_Z = \iota^{\otimes n} \varphi_n^{-1} E(s_Z)_x(1_{Y_{\mathrm{un}}})$,

$$4e_-^{(n)} \mathrm{cl}(W) = \mathrm{fgt}_P(J_n \otimes \tau_Z), \quad \varepsilon^{\otimes n} \mathrm{cl}(W) = 2^{-n-1} \mathrm{fgt}_P(\tau_Z),$$

and both identities hold over $\mathbb{Q}$.

Proof. Rational smoothness and proper base change give $i_{\mathbf{F}}^! \mathbb{Q}_{C_n}[nd](3n) = \omega_{\mathbf{F}}[-nd](-3n)$, and the supported fundamental-class morphism followed by its counit is the Satake morphism. All maps in the diagram are, by Lemmas A.4 and A.5, restrictions at the fixed parameter of the specified constructions, and Lemma A.1 compares them stage by stage. The extension from Ω is an isomorphism by Lemma 3.2. Open base change of the counit and $\mathrm{fgt}_{\Omega} = j^* \mathrm{fgt}_C \mathrm{ext}$ give the lower left square; finite étale pullback and partial trace commute with forget-supports, giving the right square. On the canonical endpoint sheet the pulled-back coefficient map is $v \mapsto \sum_{g \in \tilde{A}} gv \otimes [g\tilde{Z}]$; its trace is $2[\tilde{Z}] - 2[z\tilde{Z}] = 4e_-^{(n)}[\tilde{Z}]$, and flat pullback along $\mathrm{fr}_{\mathrm{int}}$ gives $4e_-^{(n)} \mathrm{cl}(W)$. Lemma 3.4, followed by framed base change and Künneth, inductively gives the open-legged matrix-unit formula of Lemma 3.3, and its outer trace is $J_n \otimes \iota(e_1) \otimes \cdots \otimes \iota(e_n)$. Only the two terms with all indices equal survive ε , so $\varepsilon^{\otimes n} J_n = 2^{1-n} 1^{\otimes n}$, and $\varepsilon^{\otimes n}$ absorbs $e_-^{(n)}$. All geometric shifts and degrees are even, and the fundamental class and the duality orientation are both the complex orientation, so the coefficient is positive. Descent to $\mathbb{Q}$ is as in Proposition 3.5. $\square$

Proposition 3.7 (The value). *In ordinary cohomology $H^{nd}(P, \mathbb{Q})(nd/2)$,*

$$\varepsilon^{\otimes n} \mathrm{cl}(W_u) = 2^{-n-1} \mathrm{fgt}_P(\tau_u),$$

and $\tau_u \neq 0$ for $u \neq 0$. For $n = 3$ and $u = t$ the coefficient is $\frac{1}{16}$.

Proof. Apply $\varepsilon^{\otimes n}$ to Proposition 3.5(c). Rescaling Z_u rescales s_u and W_u together and does not change the ratio. For example, take the finite set $\tilde{A}$ in place of the one-step space, $n = 3$, $Q = \tilde{A}$ via multiplication and $\tilde{Z} = \{1\}$: then $\sum_{abc=1} a \otimes b \otimes c = 16J_3$ in $D^{\otimes 3}$, so $\varepsilon^{\otimes 3} \text{cl}(W) = 4$, while $\iota(1_{Y_{\text{un}}})^{\otimes 3} = 4^3$ and $4^3/16 = 4$.

Nonvanishing: E_x , the stalk at x of Yun's eigenvalue functor, is an exact monoidal functor from the semisimple rigid monoidal category Sat , with simple unit and $\text{End}(\mathbf{1}) = \mathbb{Q}$, to finite-dimensional vector spaces, by [40, (3.19), §3.1.3]. Such a functor is faithful: if $f \neq 0$, its mate $\tilde{f} : \mathbf{1} \rightarrow Y \otimes X^\vee$ under rigidity is a nonzero map out of the simple unit, hence a monomorphism, so $E(\tilde{f})$ is a monomorphism out of $E(\mathbf{1}) \neq 0$; monoidal functors preserve duals, so $E(f) \neq 0$. $\square$

Remark 3.8 (Where fusion is used). The construction of Z_u , Lemmas 3.2 and 3.3, Proposition 3.5 and the nonvanishing in Proposition 3.7 use only the monoidal structure of $T(-, j_! \mathbb{F})$ and the compatibility of cycle classes with geometric operations. The commutativity constraint of geometric Satake enters only through the Satake equivalence itself, which supplies the bijection $u \mapsto s_u$ and identifies the Tannakian group with G_2 . That τ_t is the invariant *alternating* cubic is proved without it, by monodromy (Lemma A.10).

Remark 3.9 (Removing the base point). The cycle W_u lives on T_x^n , where no step is the trivial modification. The odd block of H_c^d is unchanged by removing $\tilde{A} \cdot \star$ from each factor. Indeed $\star$ is interior, so forget-supports is an isomorphism there by Lemma 2.2, and the local cohomology of the rationally smooth $\text{Gr}_{\leq \lambda}$ at $\star$ is concentrated in degree $2d = 12$, so restriction $H^d(\text{Gr}_{x, \leq \lambda}^U)_{\text{odd}} \rightarrow H^d(T_x)_{\text{odd}}$ is an isomorphism in degree $d = 6$. The n -fold statement follows by Künneth.

Remark 3.10 (The case $n = 3$). Here $C_3^{(U)}$ has dimension 18, $Z_t = c Z_9$, and on Z_9 the source b_3 is u_0 , so $Z_9 \cap \Omega = \{b_1, b_2 \in U\}$ is a dense open subset of Z_9 . It is nonempty: otherwise $i_*[Z_9]$ would be supported on $C_3^{(U)} \setminus \Omega$ and vanish, contradicting $E(s_t)_x \neq 0$. The tower $P \rightarrow Q \rightarrow \Omega$ has degrees 64 and 8, and Proposition 3.7 gives $\varepsilon^{\otimes 3} \text{cl}(W_t) = \frac{1}{16} \text{fgt}_P(\tau_t)$.

4. THE MOTIVATED CUBIC

4.1. The clean projector. Let $\widehat{\text{Gr}}_x = \text{Bl}_\star \text{Gr}_{x, \leq \lambda}$, a projective variety containing the open set $\widehat{\text{Gr}}_x \setminus D_x$, and let $V^\circ \rightarrow \widehat{\text{Gr}}_x \setminus D_x$ be the finite étale cover extending T_x ; the exceptional locus of V° over $\star$ is a disjoint union of copies of $E_\star \cong G_2/P$, interior to V° . By Lemma 2.2, on the odd block $H_c^k(V^\circ) \rightarrow H^k(V^\circ)$ is an isomorphism, and both sides are pure of weight k . Choose an equivariant smooth projective compactification $\bar{V} \supset V^\circ$ with normal crossing boundary, and let $s : H_c^k(V^\circ) \rightarrow H^k(\bar{V})$ be extension by zero and $r : H^k(\bar{V}) \rightarrow H^k(V^\circ)$ restriction. Then rs is an isomorphism on

the odd block, and

$$H^k(\bar{V})_{\text{odd}} = \text{im}(s)_{\text{odd}} \oplus \ker(r)_{\text{odd}}.$$

By weights, $\text{im}(s)$ is the kernel of the restrictions to the smooth boundary components and $\ker(r)$ is the sum of their Gysin images, so both are sub-objects in André's category, and semisimplicity [3] gives a motivated projector

$P_{\text{clean}} = s(rs)^{-1}r$ on the odd block, extended by zero on its complement,

onto $\text{im}(s)_{\text{odd}}$ along $\ker(r)_{\text{odd}}$. (On all of $H^k(\bar{V})$ the two subspaces need not be complementary: both contain the top class of each component.)

Definition 4.1. Let $\iota: \bigsqcup E_\star \rightarrow \bar{V}$ be the inclusion of the copies of $E_\star \cong G_2/P$, and let $H_E \subset H^*(\bar{V})$ be the span of the Gysin images $\iota_* H^{2i}(E_\star)$, $0 \leq i \leq 4$, over all copies; these lie in degrees 2 to 10. Let Π_E be the projector onto H_E orthogonal for the intersection form (Lemma 4.2), and put

$$P_\star = [\Delta] - \Pi_E, \quad P_A = \varepsilon P_\star P_{\text{clean}}.$$

Lemma 4.2. *The intersection form of $\bar{V}$ is nondegenerate on H_E . So Π_E is defined, and it is the action of the algebraic cycle $\sum_{a,b} g^{ab}[Z_a \times Z_b]$, where (Z_a) are the Gysin images of the Schubert cycles of the copies of $E_\star$ in codimensions 0 to 4 and (g^{ab}) is the inverse of their intersection matrix $(Z_a \cdot Z_b)$. It commutes with the action of $\tilde{A}$, and its image consists of classes supported on $E_\star$.*

Proof. The copies are disjoint, so the form is block diagonal over them. On one copy $E = E_\star$, $\iota_* a \cdot \iota_* b = \int_E ab c_1(N_E)$ for $a \in H^{2i}(E)$, $b \in H^{2j}(E)$, which is zero unless $i + j = 4$. The variety $E = G_2/P$ has dimension five and one Schubert class in each even degree $0, \dots, 10$, and for the ample generator ℓ of $H^2(E)$ one has $\ell^5 \neq 0$, so ℓ^k spans $H^{2k}(E)$ for $0 \leq k \leq 5$. The normal bundle of the exceptional divisor of the blow-up of the point $\star$ is negative, so $c_1(N_E) = c\ell$ with $c \neq 0$. Hence for $i + j = 4$ the pairing $H^{2i}(E) \times H^{2j}(E) \rightarrow \mathbb{Q}$, $(a, b) \mapsto \int_E ab c_1(N_E)$, is a nonzero pairing of one-dimensional spaces, and the form on $\iota_* \bigoplus_{i=0}^4 H^{2i}(E)$ is nondegenerate. The formula for Π_E is the usual one for an orthogonal projector in a basis of algebraic classes. The group $\tilde{A}$ permutes the copies and preserves the form and H_E , so it commutes with the orthogonal projector. $\square$

The point classes $\iota_* H^{10}(E_\star)$, in degree 12, pair to zero with all of H_E and are not in it; they restrict to zero on the noncompact V° , so they lie in $\ker(r)$. As subspaces, the Tate summands in degrees 2 to 10 (Lemma 4.3(a) below, and Theorem 5.2 on R) are the odd parts of the $\iota_* H^{2i}(E_\star)$, $0 \leq i \leq 4$, so $P_\star$ removes them. The classes of $H_c^6(T_x)$ extended by zero are orthogonal to H_E , since their supports are disjoint from $E_\star$, so $P_\star$ fixes them.

Lemma 4.3. (a) $P_A H^6(\bar{V}) = M_x$, and restriction $H^6(\bar{V}) \rightarrow H^6(T_x)$ is injective on it. (b) For every closed cycle $\bar{W} \subset \bar{V}^n$ with $\bar{W} \cap T_x^n = W$, the class $P_A^{\otimes n} \text{cl}(\bar{W})$ depends only on W , and is the unique element of $M_x^{\otimes n}$ whose restriction to T_x^n is $\varepsilon^{\otimes n} \text{cl}(W)$.

Proof. (a) On the odd block, $\text{im}(s)$ in degree six is $H_c^6(V^\circ)_{\text{odd}}$, which by Remark 3.9 and the sequence for $\tilde{A} \cdot \star$ is $D \otimes M_x$ together with the Tate summand $D \otimes H^6(E_\star)$; $P_\star$ removes the latter and ε has rank one on D . Restriction to T_x is injective on $D \otimes M_x$ by Lemma 2.2 and Remark 3.9. (b) Two closures differ by a cycle supported on $\bigcup_k \bar{V}^{k-1} \times (\bar{V} \setminus T_x) \times \bar{V}^{n-k}$, whose class, by Künneth, lies in the sum of $H^*(\bar{V})^{\otimes k-1} \otimes \ker(H^*(\bar{V}) \rightarrow H^*(T_x)) \otimes H^*(\bar{V})^{\otimes n-k}$. That kernel is $\ker(r)$ plus the classes supported on $E_\star$. The latter lie in $\ker(r)$ in degree 12 and in $H_E \cap \text{im}(s)$ in degrees 2 to 10. On the odd block P_{clean} kills $\ker(r)$ and fixes $\text{im}(s)$, and $P_\star$ kills H_E , so $P_A^{\otimes n}$ kills the difference. The last assertion follows from (a) and Künneth for T_x^n . $\square$

Lemma 4.4 (Rationality). For $x \in S(\mathbb{Q})$ choose the compactification and its boundary resolution over $\mathbb{Q}$. The canonical-sheet cycle W_t , its closure, ε and $P_\star$ are defined over $\mathbb{Q}$, and the ℓ -adic realisation of $P_A^{\otimes 3} \text{cl}(\bar{W}_t)$ in $H^{18}(\bar{V}_{\mathbb{Q}}^3, \mathbb{Q}_\ell(9))$ is fixed by $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Proof. $u_0 \in U(\mathbb{Q})$ [40, §2.2.6], and ε is a $\mathbb{Q}$ -correspondence: e_χ is Galois-invariant [40, §3.3.2], and the Galois-invariant sum of graphs descends even if the finite group scheme is not constant. At rational x the chain and framing spaces are defined over $\mathbb{Q}$; the unique top component Z_9 is Galois-stable, hence defined over $\mathbb{Q}$; the canonical sheet is defined by the isomorphism of the source with the fixed target given by the identity composite, which descends; so W_t is defined over $\mathbb{Q}$. The compactification and boundary resolution exist equivariantly over $\mathbb{Q}$, with boundary components possibly permuted by Galois, and the closure of W_t is defined over $\mathbb{Q}$. The span of the classes supported on the exceptional divisors is Galois-stable, and its Gram-inverse projector does not depend on the cellular basis, so $P_\star$ is defined over $\mathbb{Q}$. The maps s_ℓ, r_ℓ are Galois-equivariant, hence so is $P_{\text{clean}, \ell} = s_\ell(r_\ell s_\ell)^{-1} r_\ell$; the motivated projector is the unique projector onto $\text{im } s$ along $\ker r$, so it has this realisation, and its Betti counterpart corresponds under the comparison of Lemmas A.1 and A.2. $\square$

Proof of Theorem A. The class $P_A^{\otimes 3} \text{cl}(\bar{W}_t)$ is a motivated projector applied to an algebraic class, hence motivated [3, §§1–2]. By Lemma 4.3(b) it does not depend on the closure, and by Proposition 3.7 it is $\frac{1}{16} \tau_t$, which is nonzero; by Lemma A.10 τ_t spans the line of the invariant alternating cubic. Let $x \in S(\mathbb{Q})$. The class is motivated over $\bar{\mathbb{Q}}$ by the above: by [3, Prop. 2.3 and §2.4] its Betti and ℓ -adic incarnations are one motivated cycle, and by the first assertion of [3, §2.5 a), Scolie, p. 17] (with $L = \mathbb{C}$, $K = \mathbb{Q}$), $A_{\text{mot}}(\bar{V}_{\mathbb{C}}^3) = A_{\text{mot}}(\bar{V}_{\mathbb{Q}}^3)$. By Lemma 4.4 this element of $A_{\text{mot}}(\bar{V}_{\mathbb{Q}}^3) \subset H^{18}(\bar{V}_{\mathbb{Q}}^3, \mathbb{Q}_\ell(9))$

is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -invariant, and the second assertion of the same Scolie (the motivated classes over $\mathbb{Q}$ are the invariants) shows that it is motivated over $\mathbb{Q}$. $\square$

5. GOOD MODELS AND THE CLEAN PIECE

The projector P_{clean} of Section 4 is motivated, not known to be algebraic. This section isolates the property of a model of (Y, D_x) that replaces it by a finite-group idempotent.

Definition 5.1. A *good model* of (Y, D_x) is a proper morphism $\pi_{\mathcal{X}} : \mathcal{X} \rightarrow \widehat{\text{Gr}}_x$ from a smooth Deligne–Mumford stack, an isomorphism over $\widehat{\text{Gr}}_x \setminus D_x$, such that $\pi_{\mathcal{X}}^{-1}(D_x)$ is a divisor whose components are smooth, individually labelled Cartier divisors, meeting as coordinate hyperplanes in smooth charts, with every finite stabiliser preserving each local branch.

For a good model let X be its coarse space, R the normalisation of X in the total ring of fractions of T_x (the product of the function fields of its components; we do not need T_x to be irreducible), $p : R \rightarrow X$ the finite map, $\partial \subset X$ the preimage of D_x , $j : U_X = X \setminus \partial \hookrightarrow X$, and $\mathcal{F} = e_{-p*}\mathbb{Q}_R$. The interior divisor $E_{\star}$ lies in U_X .

Theorem 5.2. *For every good model, R has only quotient singularities, $\mathcal{F} = R^0 j_{*}\mathbb{L}$, and there is a split exact sequence*

$$0 \rightarrow H_c^k(\widehat{\text{Gr}}_x \setminus D_x, \mathbb{L}) \rightarrow H^k(R)_{\text{odd}} \rightarrow H^k(\partial, \mathcal{F}) \rightarrow 0.$$

Here $H_c^*(\widehat{\text{Gr}}_x \setminus D_x, \mathbb{L})$ is $H_c^6(T_x)_{\text{odd}}$ in degree six together with Tate summands in degrees $2i$, $1 \leq i \leq 5$, isomorphic to $D \otimes H^{2i}(E_{\star})$; as subspaces of $H^*(R)$ these are the odd parts of the Gysin images $\iota_* H^{2i-2}(E_{\star})$ of the copies of $E_{\star}$.

We call $H^*(\partial, \mathcal{F})$ the *defect* of the model.

Proof. Quotient singularities. At a point of the good model the local model is a finite quotient of $(\mathbb{C}^n, \text{coordinate hyperplanes})$; the cover is Kummer over the hyperplanes, so its normalisation is $\mathbb{C}^n$ modulo an extension of the stabiliser by a finite abelian group, which acts because the stabiliser preserves each hyperplane.

Stalks. At a point e with local image $I_e \subset \widetilde{A}$ of the orbifold fundamental group of the complement, $\mathcal{F}_e = e_{-}\mathbb{Q}[I_e \setminus \widetilde{A}] = D^{I_e}$. If $I_e \neq 1$ then $z \in I_e$, because every nontrivial subgroup of Q_8 contains z ; then z acts trivially on $I_e \setminus \widetilde{A}$ while e_{-} requires $z = -1$, so $\mathcal{F}_e = 0$. If $I_e = 1$, $\mathcal{F}_e = D$. Let B° be the set of boundary points with $I_e = 1$ (necessarily off the strict transform of D_x , whose meridians are nontrivial by cleanness) and $\overline{\mathbb{L}}$ the extension of $\mathbb{L}$ across it. Then $\mathcal{F} = j_{U_X \cup B^{\circ}, !} \overline{\mathbb{L}}$, and $\mathcal{F} = R^0 j_{*}\mathbb{L}$ because R is normal.

The split sequence. The triangle $j_!\mathbb{L} \rightarrow \mathcal{F} \rightarrow i_*(\mathcal{F}|_{\partial})$ gives a long exact sequence, and $H^*(X, \mathcal{F}) = H^*(R)_{\text{odd}}$ by finiteness of p . Consider

$$H_c^k(U_X, \mathbb{L}) \xrightarrow{a} H^k(X, \mathcal{F}) \xrightarrow{b} H^k(U_X, \mathbb{L}).$$

The composite $b \circ a$ is forget-supports, and it is an isomorphism: on the proper Schubert model with the interior base point restored, smooth base change of Yun's cleanness (Lemma 2.2) gives a clean extension across D_x , so compactly supported and ordinary cohomology of that extension agree; replacing the interior base point by $E_\star$ changes only a neighbourhood contained in the complement of D_x , where the coefficient system extends. So a is injective with retraction $(b \circ a)^{-1} \circ b$, every connecting map into $H_c^k(U_X, \mathbb{L})$ vanishes, and the long exact sequence breaks into split short exact sequences. Here $U_X = \widehat{\text{Gr}}_x \setminus D_x$. The description of $H_c^*(\widehat{\text{Gr}}_x \setminus D_x, \mathbb{L})$ follows from the sequences for $\tilde{A} \cdot \star \subset \text{GR}_{x, \leq \lambda}^U$ and $E_\star \subset \widehat{\text{Gr}}_x \setminus D_x$, near which $\mathbb{L}$ is constant: removing the $\star$ -orbit creates $H_c^1(Y \setminus D_x, \mathbb{L}) \cong D$, which the exceptional divisor cancels. Finally, when R is projective, as it is for the weighted model (Remark 10.2), $H^*(R)_{\text{odd}}$ is pure.

- A quotient singularity is rationally smooth: locally $R = \mathbb{C}^d/G$ with G finite, and $H_{\{0\}}^*(\mathbb{C}^d/G, \mathbb{Q}) = H_{\{0\}}^*(\mathbb{C}^d, \mathbb{Q})^G = \mathbb{Q}[-2d]$, since a complex-linear group preserves the orientation. So $\mathbb{Q}_R[d]$ is the intersection complex of R , and $H^*(R) = IH^*(R)$ [5, Thm. 5.4.5, Thm. 6.2.5].
- R is then a compact projective V-manifold, and the cohomology of such a variety carries a pure Hodge structure [33, §1, Cor. (1.13)].
- The odd block is the image of the idempotent e_- , a rational combination of automorphisms of R . These act by morphisms of Hodge structures, so the odd block is a direct summand of a pure Hodge structure, hence pure.

□

The defect depends on the model. It vanishes on the weighted model of Section 6 (Theorem 7.1), and not on all good models (Example 7.9). The following example shows the phenomenon in two dimensions.

Example 5.3 (Weighted and ordinary resolutions of a tacnode). On $\mathbb{C}^2$ let $C = C_1 \cup C_2$ with $C_1 = \{x = 0\}$ and $C_2 = \{x = y^2\}$. Then $H_1(\mathbb{C}^2 \setminus C) = \mathbb{Z}m_1 \oplus \mathbb{Z}m_2$; let L be the rank-one $\mathbb{Q}(i)$ -local system with $m_1 \mapsto i$ and $m_2 \mapsto i^{-1}$. (Rationally, by restriction of scalars, it has rank two; inside Q_8 take $m_1 \mapsto \mathbf{i}$, $m_2 \mapsto \mathbf{i}^{-1}$, so that $D = (\text{Res}_{\mathbb{Q}(i)/\mathbb{Q}} L)^{\oplus 2}$ and the defects below are doubled.)

L is clean at 0. The germ is quasi-homogeneous for $\text{wt}(y) = 1$, $\text{wt}(x) = 2$; the loop of the $\mathbb{C}^*$ -action is $h = 2m_1 + 2m_2$, so $\rho(h) = 1$, and the exceptional orbit $\{y = 0\}$ (isotropy μ_2 , off C) has class $m_1 + m_2$, with $\rho = 1$. On $B^\circ = \mathbb{P}(1, 2) \setminus \{[C_1], [C_2]\} \cong \mathbb{C}^*$ the local system $R^0 \varpi_* L$ has monodromy $i \neq 1$ around $[C_1]$, so its cohomology vanishes, and the Gysin sequence of the link gives $H^*(M, L) = 0$.

On the weighted blow-up the defect is 0. The $(1, 2)$ -weighted blow-up has exceptional divisor $\mathbb{P}(1, 2)$, meeting C_1, C_2 in two distinct points, and Lemma 7.4 gives $H_c^*(B^\circ, \overline{L}) = 0$.

On the ordinary resolution the defect is not 0. Blowing up 0 and then the triple point of the strict transforms gives E_1, E_2 with $\text{ord}_{E_1}(x) = \text{ord}_{E_1}(x - y^2) = 1$ and $\text{ord}_{E_2}(x) = \text{ord}_{E_2}(x - y^2) = 2$; both have trivial inertia, and they cross. The trivial-inertia locus is $\tilde{E}_1 \cup (E_2 \setminus \{2 \text{ points}\})$. The closed $\tilde{E}_1 \cong \mathbb{P}^1$ carries the trivial local system, and its complement $O = E_2 \setminus \{3 \text{ points}\}$ carries monodromies $i, i^{-1}, 1$, so $H_c^*(O, \bar{L})$ is $\mathbb{Q}(i)$ in degree one and zero otherwise. The sequence $0 = H_c^2(O) \rightarrow H_c^2(B^\circ) \rightarrow H^2(\mathbb{P}^1) \rightarrow H_c^3(O) = 0$ gives $H_c^2(B^\circ, \bar{L}) \cong \mathbb{Q}(i)(-1) \neq 0$. So the same clean local system has zero defect on the weighted model and nonzero defect on an ordinary simple normal crossing resolution.

6. A WEIGHTED RESOLUTION

We construct a good model by blowing up Bun, not Y , and pulling back along the smooth map ω .

6.1. Hypotheses. Let $\mathfrak{B}$ be the open substack of Bun where the stabiliser dimension is at most six; since ω is smooth and $\dim Y = 6$, $\omega(Y) \subset \mathfrak{B}$. A point y of $\mathfrak{B}$ is a residual gerbe $[\text{pt}/\mathcal{G}_y]$; since Bun is smooth of dimension zero, its codimension is $c = \dim \mathcal{G}_y$, and the deformation space T_y^1 has dimension c .

Hypothesis 6.1. (H0) $\mathfrak{B}$ has finitely many points.

- (H1) For every point y of $\mathfrak{B}$ with $c \geq 1$ there are a maximal torus $T_y \subset \mathcal{G}_y^\circ$ and a cocharacter η_y of T_y whose weights on T_y^1 are all positive.
- (H2) The weights of η_y on $\text{Lie}(\mathcal{G}_y)/\text{Lie}(T_y)$ are all ≤ 0 .

Section 8 verifies these hypotheses. We call η_y a *contracting cocharacter*.

6.2. Charts and the weighted filtration. Let Γ_y be the image of $N_{\mathcal{G}_y}(T_y)$ in $\text{Aut } X_*(T_y)$; it is finite, and we put $\bar{\eta}_y = \sum_{\gamma \in \Gamma_y} \gamma \eta_y$. The normaliser permutes the weights on T_y^1 and on $\text{Lie} \mathcal{G}_y/\text{Lie } T_y$, so $\bar{\eta}_y$ still satisfies (H1) and (H2). (In our case $\Gamma_y = 1$; Proposition 8.3.)

By the local structure theorem of Alper–Hall–Rydh [1, Thm. 1.1], applied to the linearly reductive subgroup $T_y \subset \mathcal{G}_y$, there is a smooth morphism $f : ([W/T_y], w) \rightarrow (\text{Bun}, y)$ with W affine and w fixed by T_y , and the theorem gives $f^{-1}(B\mathcal{G}_y) = BT_y$. The dimension of W is forced. Since f is smooth and Bun is smooth, $[W/T_y]$ is smooth at w , hence so is W . The fibre of f over $y : \text{Spec } \mathbb{C} \rightarrow \text{Bun}$ is $f^{-1}(B\mathcal{G}_y) \times_{B\mathcal{G}_y} \text{Spec } \mathbb{C} = BT_y \times_{B\mathcal{G}_y} \text{Spec } \mathbb{C} = \mathcal{G}_y/T_y$, so the relative tangent space of f at w is $\text{Lie } \mathcal{G}_y/\text{Lie } T_y$. Taking cohomology of the triangle of tangent complexes of f at w gives the T_y -equivariant exact sequence

$$0 \rightarrow \text{Lie } T_y \rightarrow \text{Lie } \mathcal{G}_y \rightarrow \text{Lie } \mathcal{G}_y/\text{Lie } T_y \rightarrow T_w W \rightarrow T_y^1 \rightarrow 0,$$

where we used that w is T_y -fixed, so that the tangent space of $[W/T_y]$ at w is $T_w W$. The middle map is the quotient map, so $T_w W \rightarrow T_y^1$ is a T_y -equivariant isomorphism. Hence $\dim_w W = \dim T_y^1 = c$, and the T_y -weights on $T_w W$ are exactly the weights on T_y^1 listed in Proposition 8.7.

Linearisation. The stack $[W/T_y]$ is smooth at the closed point w , whose stabiliser T_y is linearly reductive. Let $N_w = T_w W$, a representation of T_y , isomorphic to T_y^1 by the sequence above, on which $\bar{\eta}_y$ has positive weights by (H1). By [1, Thm. 1.2] there are an affine étale morphism $(U, u) \rightarrow (N_w//T_y, 0)$ and an étale morphism $([W'/T_y], w') \rightarrow ([W/T_y], w)$, inducing an isomorphism of stabilisers at w' , with $W' = N_w \times_{N_w//T_y} U$. Since all $\bar{\eta}_y$ -weights on N_w are positive, the only T_y -invariant functions on N_w are the constants, so $N_w//T_y$ is a point. Replacing U by the point u , we get $W' = N_w \cong \mathbb{A}^c$ with linear action. Replace W by W' and f by the composite, which is again smooth. The preimage of w in W' is a closed T_y -stable subset, finite because the map is étale; a nonzero point of N_w has an infinite $\bar{\eta}_y$ -orbit, so this preimage is $\{0\}$. Hence still $f^{-1}(B\mathcal{G}_y) = BT_y$. Let $a_1, \dots, a_c > 0$ be the $\bar{\eta}_y$ -weights of the coordinates $y_1, \dots, y_c$, and let $F_k \subset \mathcal{O}_W$ be the ideal of functions of $\bar{\eta}_y$ -weight $\geq k$, generated by monomials and of finite colength. This is the *weighted filtration* at y .

Let $\mathfrak{R} = W \times_{\text{Bun}} W$ with source and target $s, t : \mathfrak{R} \rightarrow W$, both smooth. Pullbacks of ideals below are image ideals in the target structure sheaf. We show that the weighted filtration descends to Bun.

Lemma 6.2 (The closed fibre of the relation). *On all of $\mathfrak{R}$, $Z = s^{-1}(w) = t^{-1}(w) \simeq \mathcal{G}_y$ and $s^* \mathfrak{m}_w \cdot \mathcal{O}_{\mathfrak{R}} = t^* \mathfrak{m}_w \cdot \mathcal{O}_{\mathfrak{R}} = I_Z$. For $\Delta_N = \text{Spec}(\mathcal{O}_{W,w}/\mathfrak{m}_w^N)$, $\mathfrak{R} \times_{s,W} \Delta_N = V(I_Z^N) = \mathfrak{R} \times_{t,W} \Delta_N$.*

Proof. The condition $f^{-1}(B\mathcal{G}_y) = BT_y$ pulls back to $W \times_{\text{Bun}} B\mathcal{G}_y = \{w\}$. So an arrow with source w has target w , and conversely by inversion; the two closed fibres have the same support. Both are smooth, since s and t are, hence reduced, so they are the same reduced closed subscheme, the isomorphism scheme $\mathcal{G}_y$ of w . Equality of the ideals is global on $\mathfrak{R}$; take powers. $\square$

Lemma 6.3 (Section lifting). *Let A be a finite-dimensional local $\mathbb{C}$ -algebra with residue field $\mathbb{C}$, $\Delta = \text{Spec } A$, and $E = \text{Spec } P \rightarrow \Delta$ affine, smooth and of finite presentation. Let $\mathcal{S} = \text{Res}_{\Delta/\mathbb{C}}(E)$ be the scheme of sections. Then $\mathcal{S}$ is affine of finite presentation, and evaluation $\text{ev} : \mathcal{S} \times \Delta \rightarrow E$ is smooth and surjective, hence faithfully flat.*

Proof. Choosing a basis of A and a presentation of P over A and expanding the relations gives $\mathcal{S}$ as an affine scheme of finite presentation. For lifting, let $C' \rightarrow C$ be a surjection of $\mathbb{C}$ -algebras with square-zero kernel J , $p' \in \Delta(C')$, $e' \in E(C')$ over p' , and $\sigma \in \mathcal{S}(C)$ with $\sigma(p' \bmod J) = e' \bmod J$. Put $Q = A \otimes C'$, $L = A \otimes J$, $H = \ker(Q \rightarrow C')$ (evaluation at p'). Then $Q/(L \cap H) \simeq (A \otimes C) \times_C C'$, so σ and e' give an A -algebra map $P \rightarrow Q/(L \cap H)$; since $(L \cap H)^2 = 0$, smoothness of P/A lifts it to $P \rightarrow Q$,

which is a section lifting σ with value e' at p' . This is formal smoothness of ev , which is locally of finite presentation, hence smooth [31, Tag 02H6]. A geometric point of E lies over the closed point of Δ , and lifts through powers of the maximal ideal to a section by smoothness, so ev is surjective. $\square$

Lemma 6.4 (Truncated bisections). *Let $\text{Bis}_N = \text{Res}_{\Delta_N/\mathbb{C}}(\mathfrak{R} \times_{s,W} \Delta_N)$, the scheme of sections σ of s over Δ_N . Then:*

- (1) Bis_N is a smooth affine group scheme of finite type, with product $(\sigma\sigma')(p) = \sigma(t\sigma'(p)) \circ \sigma'(p)$; every section takes the closed point into $\mathcal{G}_y$, and $t \circ \sigma \in \text{Aut}(\Delta_N)$;
- (2) evaluation at the closed point $\text{ev}_N : \text{Bis}_N \rightarrow \mathcal{G}_y$ is a smooth surjective homomorphism with connected unipotent kernel;
- (3) $T_y \hookrightarrow \text{Bis}_N$, $g \mapsto \sigma_g$, is a maximal torus of Bis_N° , and ev_N restricts to the inclusion $T_y \subset \mathcal{G}_y$;
- (4) $\text{Lie } \text{Bis}_N = \Gamma(\Delta_N, \mathcal{A}|_{\Delta_N})$ for the Lie algebroid $\mathcal{A} = \ker(ds)|_{\text{identity}}$, acting on $\mathcal{O}_{\Delta_N}$ through the anchor.

Proof. (1) Bun has affine diagonal, so $\mathfrak{R}$ is affine, and the Weil restriction of the affine finite-type $\mathfrak{R} \times_W \Delta_N$ along the finite flat $\Delta_N \rightarrow \text{Spec } \mathbb{C}$ is affine of finite type [6, §7.6]. By Lemma 6.2, $s^{-1}(w) = \mathcal{G}_y$, and $t \circ \sigma$ fixes the closed point. For $N \geq 2$ smoothness of s, t gives conormal isomorphisms $s_Z^*(\mathfrak{m}_w/\mathfrak{m}_w^2) \cong I_Z/I_Z^2 \cong t_Z^*(\mathfrak{m}_w/\mathfrak{m}_w^2)$, so $(t\sigma)^*$ is invertible on $\mathfrak{m}_w/\mathfrak{m}_w^2$ and hence on every graded piece of the $\mathfrak{m}_w$ -adic filtration, over every test ring: every section is a bisection. The product is algebraic, and a group scheme of finite type in characteristic zero is smooth. (2) The lifts along $\text{Res}_{\Delta_{j+1}} \rightarrow \text{Res}_{\Delta_j}$ form a torsor under a vector bundle, by smoothness of s , which on the kernel over the identity is the vector group of $\text{Lie}(\mathcal{G}_y) \otimes \mathfrak{m}^j/\mathfrak{m}^{j+1}$. So ev_N is smooth and surjective with affine-space fibres. The congruence subgroups are normal with vector-group quotients, because the anchor vanishes at w , so the kernel is connected unipotent. (3) The T_y -action on W gives $W \times T_y \rightarrow \mathfrak{R}$, whose sections σ_g preserve Δ_N . Since $\text{ev}_N(\text{Bis}_N^\circ) = \mathcal{G}_y^\circ$ with unipotent kernel, every torus of Bis_N° maps injectively onto a torus of $\mathcal{G}_y^\circ$, so T_y is maximal. (4) The action of σ on $\mathcal{O}_{\Delta_N}$ is pullback by $t \circ \sigma$, whose derivative is the anchor. Every element of $\text{Lie } \text{Bis}_N$ is $\xi = \sum_j f_j e_j$ with $f_j \in \mathcal{O}_{\Delta_N}$ arbitrary, for a T_y -homogeneous frame (e_j) of $\mathcal{A}$; such a frame exists because graded projective modules over the positively graded $\mathcal{O}_W$ are free with homogeneous basis. $\square$

Lemma 6.5. Bis_N preserves $F_k \bmod \mathfrak{m}_w^N$ for every k .

Proof. Grade $\mathcal{O}_W$ by $\deg y_i = a_i > 0$. A vector field of $\bar{\eta}_y$ -weight ν has degree $-\nu$. By T_y -equivariance of the anchor, $a(e_j)$ has the weight ν_j of $e_j(w) \in \text{Lie } \mathcal{G}_y$, and $\nu_j \leq 0$ by (H2), with $\nu = 0$ on $\text{Lie } T_y$. So $a(e_j)$ maps $F_k = \bigoplus_{b \geq k} \mathcal{O}_b$ into itself, and so does $f_j a(e_j)$, since F_k is an ideal. So $\text{Lie } \text{Bis}_N$ preserves $F_k \bmod \mathfrak{m}^N$, and in characteristic zero so does the connected group Bis_N° . For an arbitrary $b \in \text{Bis}_N$, the tori $bT_y b^{-1}$ and T_y are maximal in Bis_N° , so there is $h \in \text{Bis}_N^\circ$ with $m = hb$ normalising T_y . Since ev_N is

injective on T_y , the action of m on T_y is that of an element of Γ_y ; so m fixes $\bar{\eta}_y$, preserves the grading and hence each F_k , and so does $b = h^{-1}m$. $\square$

Proposition 6.6 (Descent). *For every k , $s^*F_k = t^*F_k$ as ideals of $\mathcal{O}_{\mathfrak{R}}$. Hence the ideals F_k descend to coherent ideals on the open image $f(W) \subset \text{Bun}$, and the Rees algebra $\bigoplus_k F_k u^k$ descends to a finitely generated graded algebra there.*

Proof. Fix k and choose N with $\mathfrak{m}_w^N \subset F_k$. By Lemma 6.3, applied to $\mathfrak{R} \times_{s,W} \Delta_N \rightarrow \Delta_N$, the evaluation map $\Phi : \text{Bis}_N \times \Delta_N \rightarrow V(I_Z^N)$ is faithfully flat, with $s\Phi = \text{pr}_2$ and $t\Phi$ the action a_N of the bisection group. Put $J = F_k/\mathfrak{m}_w^N \subset \mathcal{O}(\Delta_N)$. By Lemma 6.5, J is stable under all complex points of the group Bis_N . This group is reduced, since every group scheme of finite type over a field of characteristic zero is (Cartier's theorem), so its complex points are dense and J is a sub-comodule, so $a_N^*J = \text{pr}_2^*J$. Hence $\Phi^*(s^*J) = \Phi^*(t^*J)$, and faithful flatness gives $s^*J = t^*J$ on $V(I_Z^N)$. Both s^*F_k and t^*F_k contain I_Z^N on all of $\mathfrak{R}$ (Lemma 6.2), so they are equal. This is a descent datum along the smooth surjection $W \rightarrow f(W)$ (its cocycle condition is an equality of submodules), and effective flat descent gives a coherent ideal. The descents respect multiplication; on W the Rees algebra is generated by $y_i u^j$, $1 \leq j \leq a_i$, and finite type is smooth-local [14, IV, 2.7.1]. $\square$

6.3. One centre. The stack-theoretic weighted blow-up of $W = \mathbb{A}^c$ at the origin [25, Def. 3.2.1] has, since $y_1, \dots, y_c$ is a regular sequence, the chart of [25, Cor. 5.2.5–5.2.6]: $V' = (W \setminus 0) \times \mathbb{A}_s^1$ modulo $\mathbb{G}_m$ acting by $(y, s) \mapsto (\bar{\eta}_y(\tau)y, \tau^{-1}s)$, with blow-down $\beta(y, s) = (s^{a_i}y_i)$ and exceptional divisor $\{s = 0\}$. Let $\gamma(y, s) = y$ be the radial projection.

Lemma 6.7 (Radial lemma). *Let I be an ideal of $\mathcal{O}_{\text{Bun}}$ near y and $J = f^*I$. Then $(\beta^*J : s^\infty) = \gamma^*J$, degree by degree for filtrations, and $(f\gamma, s) : V' \rightarrow \text{Bun} \times \mathbb{A}^1$ is smooth with image avoiding y .*

Proof. J is T_y -stable, hence $\bar{\eta}_y$ -stable, so on $s \neq 0$ the action identifies β^*J with γ^*J . On an affine open A of $W \setminus 0$, $\mathcal{O}/\gamma^*J = (A/J)[s]$ has no s -torsion, so γ^*J is s -saturated. Smoothness holds because $(f\gamma, s)$ is $f|_{W \setminus 0} \times \text{id}$ after the product identification, and $f^{-1}(B\mathcal{G}_y)$ is the origin. $\square$

Let $D_W = f^{-1}(\mathfrak{B} \setminus U)$, $W^\circ = W \setminus D_W$, and $\mathcal{L}_W = f^*\mathbb{F}|_{[W^\circ/T_y]}$, a T_y -equivariant local system on W° with structure maps $\psi_g : \mathcal{L}_{W,v} \rightarrow \mathcal{L}_{W,gv}$, continuous in $g \in T_y$, with $\psi_{gh} = \psi_g\psi_h$. Let $\mu_y = \ker(T_y \rightarrow \text{Aut } W)$, a finite group; for $g \in \mu_y$, ψ_g is left multiplication on the fibre D by $\kappa(g)$, where $\kappa : \mu_y \rightarrow \tilde{A}$ is the stabiliser homomorphism of the $\tilde{A}$ -torsor.

Lemma 6.8 (The equivariant diagram at one centre). *Let $\mathbb{H} = \mathbb{G}_m \times T_y$ act on V' by $(\tau, g)(v, s) = (\bar{\eta}_y(\tau)gv, \tau^{-1}s)$, and put $\delta_0(\tau, g) = g$, $\delta_1(\tau, g) = \bar{\eta}_y(\tau)g$. Then β is δ_0 -equivariant, γ is δ_1 -equivariant, and on $W^\circ \times \mathbb{C}_s^*$*

$$\Psi : \gamma^*\mathcal{L}_W \xrightarrow{\sim} \beta^*\mathcal{L}_W, \quad \Psi_{(v,s)} = \psi_{\bar{\eta}_y(s)},$$

is an isomorphism of $\mathbb{H}$ -equivariant local systems, $\gamma^*\mathcal{L}_W$ carrying the δ_1 -structure and $\beta^*\mathcal{L}_W$ the δ_0 -structure. Consequently $\beta^*\mathcal{L}_W|_{s \neq 0}$ extends across $s = 0$ to the $\mathbb{H}$ -equivariant local system $\gamma^*\mathcal{L}_W$ on $W^\circ \times \mathbb{A}_s^1$, and at a point q of the blow-up over $(v, 0)$:

- (1) the stabiliser of q contains $\mu_{e(v)} = \{\tau : \bar{\eta}_y(\tau)v = v\}$, acting on s by τ^{-1} and on the stalk $\mathcal{L}_{W,v}$ by $\psi_{\bar{\eta}_y(\tau)}$;
- (2) the meridian of the exceptional divisor at q is the orbifold loop formed by the path $s \mapsto e^{2\pi i \theta/e(v)}s$ and the element $\zeta_{e(v)}$, with monodromy $\psi_{\bar{\eta}_y(\zeta_{e(v)})}$; at a generic point $e(v) = e_0$, and the monodromy is $\kappa(\bar{\eta}_y(\zeta_{e_0}))$.

Proof. (τ, g) sends (v, s) to $(\bar{\eta}_y(\tau)gv, \tau^{-1}s)$, and $\psi_{\bar{\eta}_y(\tau^{-1}s)} \circ \psi_{\delta_1(\tau, g)} = \psi_{\bar{\eta}_y(s)g} = \psi_{\delta_0(\tau, g)} \circ \psi_{\bar{\eta}_y(s)}$ by the cocycle rule and commutativity of T_y ; so Ψ is equivariant. That β is the weighted blow-up is the chart description [25, Prop. 5.2.2, Cor. 5.2.5–5.2.6], compatible with flat base change [25, Cor. 3.2.14(iii)], applied on an atlas. After base change to Y , on which T_y acts freely through the torsor $Y \times_{\mathfrak{B}} W$, the stabiliser of a point over $(v, 0)$ has trivial T_y -component, so it lies in $\mu_{e(v)}$, and (1) holds by the δ_1 -structure. Off the exceptional divisor the $\mathbb{H}$ -action is free; the displayed path, closed by $(\zeta_{e(v)}, 1)$, is the meridian; transport of $\gamma^*\mathcal{L}_W$ along it is trivial, because $\gamma^*\mathcal{L}_W$ is constant in s , and the group element acts by $\psi_{\bar{\eta}_y(\zeta)}$. $\square$

Lemma 6.9 (Descent of extensions). *Let $\mathfrak{M}$ be a smooth complex algebraic stack, $\mathfrak{D} \subset \mathfrak{M}$ an open substack whose complement has positive codimension on every smooth atlas, L a local system on $\mathfrak{D}$, and $p : V \rightarrow \mathfrak{M}$ a smooth surjective scheme atlas. Suppose a local system L_V on V is given with an isomorphism $\alpha : L_V|_{V \times_{\mathfrak{M}} \mathfrak{D}} \xrightarrow{\sim} p^*L$. Then there is a unique descent datum on L_V extending that of L ; it is effective, and yields a local system on $\mathfrak{M}$ extending L , unique up to unique isomorphism.*

Proof. For a smooth complex space Z and a closed analytic subset A of positive codimension, restriction of local systems from Z to $Z \setminus A$ is fully faithful: on a small ball both local systems are trivial and the complement of A in the ball is connected. Put $V_2 = V \times_{\mathfrak{M}} V$. On $V_2 \times_{\mathfrak{M}} \mathfrak{D}$ the comparison $(p_2^*\alpha)^{-1}(p_1^*\alpha)$ extends uniquely to V_2 , since V_2 is smooth, and so does its inverse; the cocycle identity holds on the dense open of V_3 , hence everywhere. Descent of local systems along a smooth atlas is effective. The whole Čech groupoid, including self-overlaps, is used, so isotropy is recorded. $\square$

6.4. The construction. By (H0) and Proposition 8.6, ordering the points of $\mathfrak{B}$ with $c \geq 2$ by decreasing c (ties arbitrary) as $x_1, x_2, \dots, x_N$ makes each x_m closed in $O_{m-1} = \mathfrak{B} \setminus \{x_1, \dots, x_{m-1}\}$: a point of the closure of x_m other than x_m has larger stabiliser dimension. For each m let $F_{m,\bullet}^{(0)}$ be the contraction to $\mathfrak{B}$ of the descended weighted filtration of x_m , extended by

the unit ideal on $O_{m-1} \setminus \{x_m\}$:

$$F_{m,k}^{(0)} = \ker(\mathcal{O}_{\mathfrak{B}} \rightarrow j_{m*}(\mathcal{O}_{O_{m-1}}/F_{m,k})), \quad j_m : O_{m-1} \hookrightarrow \mathfrak{B}.$$

These are coherent ideals (j_m is quasi-compact and quasi-separated), compatible with flat base change, supported on the closure of x_m , and equal to the weighted ideals near x_m . The *tracked ideals* are the reduced boundary ideal of $\mathfrak{B} \setminus U$, the reduced ideals of the closures of the points of $\mathfrak{B}$, and the filtrations $F_{m,\bullet}^{(0)}$.

Definition 6.10 (Intrinsic transforms). Put $\mathfrak{B}_0 = \mathfrak{B}$, and suppose the stack-theoretic weighted blow-ups $b_m : \mathfrak{B}_m \rightarrow \mathfrak{B}_{m-1}$, $m \leq n$, have been constructed, with exceptional Cartier divisors E_m [25, Def. 3.2.2]. For a tracked ideal J (degree by degree for a filtration) its *intrinsic transform* is $J^{(0)} = J$ and

$$J^{(n)} = (b_n^{-1}J^{(n-1)} \cdot \mathcal{O}_{\mathfrak{B}_n} : \mathcal{I}_{E_n}^\infty),$$

an ideal on the smooth site of $\mathfrak{B}_n$. The older divisors are the total transforms $E_i^{(n)} = b_n^{-1}E_i^{(n-1)}$, $i < n$. The $(n+1)$ -st centre is $I_\bullet^{(n)} = (F_{n+1,\bullet}^{(0)})^{(n)}$, and $\mathfrak{B}_n^\natural$ is the complement of the intrinsic transform of $\mathfrak{B} \setminus U$.

The invariant (I_n) . The stack $\mathfrak{B}_n$ has a smooth atlas by schemes V , each with a smooth morphism $(g_V, s) : V \rightarrow O_n \times \mathbb{A}^r$, such that:

- (a) $J^{(n)}|_V = g_V^{-1}J \cdot \mathcal{O}_V$ for every tracked ideal J , degree by degree;
- (b) $E_i^{(n)}|_V = V(s_i)$ for the r exceptional divisors through V , and $E_i^{(n)} \cap V = \emptyset$ for the others;
- (c) there are a local system $\mathbb{F}_n$ on $\mathfrak{B}_n^\natural$ extending $\mathbb{F}$ and isomorphisms $\alpha_V : \mathbb{F}_n|_{V^\natural} \xrightarrow{\sim} g_V^*\mathbb{F}$ on $V^\natural = g_V^{-1}(U)$.

For $n = 0$ take the charts $f : W \rightarrow \mathfrak{B}$, $g_W = f$, $r = 0$, $\mathbb{F}_0 = \mathbb{F}$ and $\alpha_W = \text{id}$. Agreement of (a) on overlaps holds because both sides are restrictions of the one ideal $J^{(n)}$ on the stack.

Lemma 6.11 (Intrinsic transforms and the next centre). *Let $\mathfrak{X}$ be a locally noetherian algebraic stack, $E \subset \mathfrak{X}$ an effective Cartier divisor and J a coherent ideal. Then $(J : \mathcal{I}_E^\infty)$ is a coherent ideal on the smooth site, its formation commutes with flat pullback, and degreewise saturation preserves multiplicative decreasing filtrations with degree-zero part $\mathcal{O}$. If $(I_n)(a)$ holds, then $I_\bullet^{(n)}$ is a finitely generated Rees algebra defining a regular weighted centre.*

Proof. On a smooth affine chart with $E = (t)$, saturation is $\ker(A \rightarrow (A/J)[t^{-1}])$, finite since A is noetherian; flat tensor product preserves the kernel and the localisation, so these ideals agree on overlaps and descend. If $t^a u \in J_i$ and $t^b v \in J_j$ then $t^{a+b} uv \in J_{i+j}$, which gives multiplicativity. By $(I_n)(a)$ the next filtration on each chart is $g_V^{-1}F_{n+1,\bullet} \cdot \mathcal{O}_V$, pulled back from O_n , where x_{n+1} is closed. Near x_{n+1} , on the chart W , the algebra is generated by $y_i u^j$, $1 \leq j \leq a_i$, with y_i regular parameters of positive weights a_i , and the same holds after smooth pullback; away from it the filtration is the

unit filtration. Finite generation descends and regularity of a weighted centre is smooth-local [25, Def. 5.2.7, Def. 5.3.1], so the stack-theoretic blow-up is defined [25, Def. 3.2.1] and regular [25, Cor. 5.3.2]. $\square$

Lemma 6.12 (Chart square). *Assume (I_n) , and let $F_\bullet$ be the descended weighted filtration of x_{n+1} on O_n (unit off x_{n+1}). For every chart V there are canonical isomorphisms of stacks over V*

$$V \times_{\mathfrak{B}_n} \mathfrak{B}_{n+1} \simeq \mathrm{Bl}_{g_V^{-1}F_\bullet \cdot \mathcal{O}_V} V \simeq V \times_{g_V, O_n} \mathrm{Bl}_{F_\bullet} O_n,$$

under which $\mathcal{I}_{E_{n+1}}$ is the pullback of the exceptional ideal of $\mathrm{Bl}_{F_\bullet} O_n$.

Proof. Relative Spec of a graded algebra, its irrelevant-open locus and the quotient by $\mathbb{G}_m$ commute with base change, and $\mathrm{Proj}^{\mathrm{st}}$ is defined over a stack [25, §1.2]. Flatness identifies the pullback of the filtration with its image filtration, as in the proof of [25, Cor. 3.2.14(iii)]. Apply this to the smooth $V \rightarrow \mathfrak{B}_n$, using $(I_n)(a)$, and then to the smooth g_V . The exceptional ideal is $\mathcal{O}(1) = \mathcal{O}(-E)$ [25, Def. 3.2.2, Rem. 3.2.3], compatible with both base changes. $\square$

Lemma 6.13 ((I_n) implies (I_{n+1})). *Let V' be the radial chart of Section 6.3 at x_{n+1} , built inside O_n , with blow-down $\beta : V' \rightarrow O_n$, radial map $\gamma : V' \rightarrow O_{n+1}$ and exceptional coordinate t , and for a chart V of (I_n) put $P_V = V \times_{g_V, O_n, \beta} V'$, $g_{P_V} = \gamma \circ \mathrm{pr}_2$ and $s_{P_V} = (s \circ \mathrm{pr}_1, t \circ \mathrm{pr}_2)$. The charts P_V , together with the unchanged charts $V \setminus g_V^{-1}(x_{n+1})$, form a smooth atlas of $\mathfrak{B}_{n+1}$ satisfying (I_{n+1}) .*

Proof. Smoothness. The morphism $(g_{P_V}, s \circ \mathrm{pr}_1, t \circ \mathrm{pr}_2)$ factors as

$$P_V \xrightarrow{(\mathrm{pr}_2, s \circ \mathrm{pr}_1)} V' \times \mathbb{A}^r \xrightarrow{(\gamma, \mathrm{id}, t)} O_{n+1} \times \mathbb{A}^{r+1},$$

the first arrow is the base change of the smooth (g_V, s) along (β, id) , and the second is smooth by Lemma 6.7. By Lemma 6.12 these charts cover the preimage of the chart images.

(a) *Ideals.* The map $P_V \rightarrow \mathfrak{B}_{n+1} \rightarrow \mathfrak{B}_n$ is 2-isomorphic to $P_V \rightarrow V \rightarrow \mathfrak{B}_n$, so $b_{n+1}^{-1}J^{(n)} \cdot \mathcal{O}|_{P_V} = \mathrm{pr}_1^{-1}g_V^{-1}J \cdot \mathcal{O} = \mathrm{pr}_2^{-1}(\beta^{-1}J \cdot \mathcal{O}_{V'}) \cdot \mathcal{O}_{P_V}$, and the exceptional ideal is $(t \circ \mathrm{pr}_2)$. Saturation commutes with the flat pr_2 , and Lemma 6.7 gives $(\beta^{-1}J : t^\infty) = \gamma^{-1}J$; so $J^{(n+1)}|_{P_V} = g_{P_V}^{-1}J \cdot \mathcal{O}$, degree by degree. On the unchanged charts b_{n+1} is an isomorphism.

(b) *Divisors.* By the factorisation above, the old s_i are independent parameters relative to V' , so $\mathcal{O}_{P_V}/g_{P_V}^{-1}J$ is locally smooth over $(\mathcal{O}_{V'}/\gamma^{-1}J)[s_1, \dots, s_r]$ and has no s_i -torsion; total and strict transforms of the old divisors agree, and they remain smooth and transverse to $E_{n+1} = V(t \circ \mathrm{pr}_2)$.

(c) *Coefficients.* On the complement of E_{n+1} in $\mathfrak{B}_{n+1}^\natural$ there is the one local system $b_{n+1}^*\mathbb{F}_n$. On the punctured radial chart the composite

$$g_{P_V}^*\mathbb{F} = \mathrm{pr}_2^*\gamma^*\mathbb{F} \xrightarrow{\mathrm{pr}_2^*\Psi} \mathrm{pr}_2^*\beta^*\mathbb{F} \xrightarrow{\phi} \mathrm{pr}_1^*g_V^*\mathbb{F} \xrightarrow{\mathrm{pr}_1^*\alpha_V^{-1}} b_{n+1}^*\mathbb{F}_n|_{P_V},$$

with Ψ of Lemma 6.8 and ϕ the 2-isomorphism of the fibre product, identifies $g_{P_V}^* \mathbb{F}$ with $b_{n+1}^* \mathbb{F}_n$. The former is defined on all of $g_{P_V}^{-1}(U)$, including where old s_i vanish. Lemma 6.9 constructs $\mathbb{F}_{n+1}$ with α_{P_V} . Monodromy stays finite: extending across a divisor adds relations to the monodromy of the dense open, not new image generators. $\square$

Theorem 6.14. *Assume Hypothesis 6.1. Blowing up $x_1, \dots, x_N$ in order, the n -th time along the regular weighted centre $I_\bullet^{(n-1)}$ (Lemma 6.11), gives a smooth stack $\mathfrak{B}_N \rightarrow \mathfrak{B}$ with finite relative inertia such that the total transform of $\mathfrak{B} \setminus U$ is a divisor with smooth, individually labelled components meeting as coordinate hyperplanes in smooth charts, and (I_N) holds. Consequently the stack $\mathcal{X}_x$ obtained by gluing $Y \times_{\mathfrak{B}} \mathfrak{B}_N$ to a neighbourhood of $E_\star$ in $\widehat{\text{Gr}}_x$ along $Y \setminus D_x$ is a good model of (Y, D_x) , for every $x \in S$.*

Proof. The logical order at each stage is: the initial coherent ideals; their intrinsic transforms along the blow-ups already constructed; their local form from (I_n) ; finite generation and regularity of the next centre (Lemma 6.11); the next blow-up [25, §1.2, Def. 3.2.1], which is regular [25, Cor. 5.3.2]; then (I_{n+1}) (Lemma 6.13).

Disjointness. Two distinct points of equal codimension have closures meeting only in deeper points (Proposition 8.6). On a chart their transforms have supports g_V^{-1} of their closures, and g_V avoids the processed points, so these supports are disjoint once the deeper points have been processed.

Termination. After the points with $c \geq 2$ are processed, g_V lands in U or in a point with $c = 1$, where the minimal transverse chart is a smooth curve and the boundary is its origin. So the strict boundary is locally a smooth Cartier divisor or empty, and by smoothness of (g_V, s) it is transverse to the exceptional coordinates.

Finite inertia and branches. Each weighted blow-up has finite relative inertia, and so do composites. The stack $\mathfrak{B}_N$ need not be Deligne–Mumford, but $Y \times_{\mathfrak{B}} \mathfrak{B}_N$ is: Y is a scheme, so its stabilisers are the kernels of the maps to stabilisers in $\mathfrak{B}_N$, which are finite; and it is smooth because ω and $\mathfrak{B}_N$ are. Étale locally at a geometric point it is a finite quotient of a smooth chart in which the boundary is a union of coordinate hyperplanes; each E_i is a globally labelled smooth Cartier divisor with one local branch, and the strict boundary is smooth with at most one branch, so the stabilisers preserve each branch. The modification is an isomorphism over $Y \setminus D_x$, and D_x is closed and disjoint from $\star$, so the gluing is defined. $\square$

We call the good model $\mathcal{X}_x$ of Theorem 6.14 the *weighted model*. It depends on the chosen contracting cocharacters η_y and on the order of points of equal codimension; the results below hold for every such choice. The gluing over $\widehat{\text{Gr}}_x$ is written out at the start of Section 10. There Lemma 10.1 and Remark 10.2 show that the coarse space X and the normalisation R are projective, using only the construction of this section.

7. THE DEFECT OF THE WEIGHTED MODEL VANISHES

Let $\mathcal{X} = \mathcal{X}_x$ be the weighted model, X its coarse space, $\pi_{\mathcal{X}} : X \rightarrow \widehat{\text{Gr}}_x$ the modification, $\partial = \pi_{\mathcal{X}}^{-1}(D_x)$, $U_X = X \setminus \partial$ and $j : U_X \hookrightarrow X$. Let $E_1, \dots, E_N \subset X$ be the pulled-back exceptional divisors and $\tilde{D}$ the strict transform of D_x , so $\partial = \tilde{D} \cup E_1 \cup \dots \cup E_N$. Write $\mathfrak{X}_n = Y \times_{\mathfrak{B}} \mathfrak{B}_n$ and $\mathfrak{t}_n \subset \mathfrak{X}_{n-1}$ for the centre of the n -th blow-up, with coarse spaces X_n, Σ_n . For $q \in X$ let Γ_q be the orbifold fundamental group of $N_q \cap U_X$, N_q a small conical neighbourhood, and $I_q \subset \tilde{A}$ the image of the local monodromy of $\mathbb{L}$. With $p : R \rightarrow X$ the normalisation in the function field of T_x ,

$$\mathcal{F} = R^0 j_* \mathbb{L} = e_{-p*} \mathbb{Q}_R, \quad \mathcal{F}_q = D^{I_q},$$

because R is normal: sections of $p_* \mathbb{Q}_R$ over N_q are the Γ_q -invariant sections over $N_q \cap U_X$.

Theorem 7.1 (Defect vanishing). $H^*(\partial, \mathcal{F}|_{\partial}) = 0$. Consequently $H_c^*(U_X, \mathbb{L}) \rightarrow H^*(R)_{\text{odd}}$ is an isomorphism.

The rest of this section proves Theorem 7.1.

7.1. Coefficients across the exceptional divisors.

Lemma 7.2. *For every n the local system $\mathbb{F}_n$ of (I_n) on $\mathfrak{B}_n^{\natural}$ pulls back to a local system L^{st} on the Deligne–Mumford stack $\mathfrak{X}_n^{\natural} = Y \times_{\mathfrak{B}} \mathfrak{B}_n^{\natural}$ extending $\mathbb{L}$, and on its coarse space*

$$R^0 j_* \mathbb{L}|_{X_n^{\natural}} = \pi_*^{\text{inv}} L^{\text{st}},$$

where $\pi : \mathfrak{X}_n^{\natural} \rightarrow X_n^{\natural}$ is the coarse map and π_*^{inv} takes invariants of the finite stabilisers. In particular $\mathcal{F}_q = (L_q^{\text{st}})^{H_q} = D^{I_q}$, where H_q is the stabiliser of q , acting on $L_q^{\text{st}} = D$ by left multiplication through $H_q \rightarrow \text{Stab}_{\mathfrak{B}}(g_V(q))$, and I_q is the image.

Proof. The extension is $(I_n)(c)$. Near q , $\mathfrak{X}_n$ is $[Z/H_q]$ with Z smooth and contractible and the boundary a union of coordinate hyperplanes; $\pi_1(N_q \cap U_X) \rightarrow \pi_1^{\text{orb}}([Z/H_q] \setminus \text{boundary}) \rightarrow H_q \rightarrow 1$ is exact, $\mathbb{L}$ is the restriction of L^{st} , whose monodromy on Z is trivial, so the invariants of $\pi_1(N_q \cap U_X)$ on $\mathbb{L}$ are the H_q -invariants of L_q^{st} . This holds at multiple crossings of exceptional divisors too, since each branch is preserved and the atlas meridians act trivially on L^{st} . $\square$

Trivial meridians on smooth atlases of the stack do not mean trivial meridians on its coarse space: on $[\Delta/\mu_e]$ with $s \mapsto \zeta s$, a constant local system upstairs with stabiliser representation χ extends across $s = 0$, while a once-around coarse loop has monodromy $\chi(\zeta)^{\pm 1}$. Coarse stalks are computed by invariants, as in Lemma 7.2.

Corollary 7.3 (Stalks). *At every $q \in \partial$ off $\tilde{D}$, $\mathcal{F}_q = D^{I_q}$, which is D if $I_q = 1$ and 0 otherwise, since every nontrivial subgroup of Q_8 contains*

z . On $\tilde{D}$ the stalk is 0: a point of $\tilde{D}$ maps to a codimension-one point of $\mathfrak{B} \setminus U$, its local image contains the original meridian, which acts without fixed vectors by cleanness.

7.2. The link at a centre.

Lemma 7.4 (Duality on the link). *Let x_m be a centre with chart W , contracting cocharacter $\bar{\eta}$ of weights $a_1, \dots, a_c > 0$ and $\bar{\eta}$ -stable boundary D_W . Let $M = S_\epsilon \setminus D_W$ for a $\bar{\eta}$ -adapted sphere S_ϵ about 0, $B_m^\circ = (W \setminus (0 \cup D_W)) / \mathbb{C}^*$, an orbifold open in $\mathbb{P}(a)$, and $\varpi : M \rightarrow B_m^\circ$ the Seifert map, with regular fibre the orbit loop h and exceptional fibres ℓ with $\ell^k = h$. Let $\mathcal{M}$ be a local system of $\mathbb{Q}$ -vector spaces on M with finite monodromy such that $H^*(M, \mathcal{M}) = 0$, and $\mathcal{Q} = R^0 \varpi_* \mathcal{M}$, with stalks $\mathcal{M}^{(h)}$ generically and $\mathcal{M}^{(\ell)}$ at orbifold points. Then $H^*(B_m^\circ, \mathcal{Q}) = 0$ and $H_c^*(B_m^\circ, \mathcal{Q}) = 0$.*

Proof. Since h is central in the local fundamental group, $\mathcal{M} = \mathcal{M}^{\bar{h}} \oplus \mathcal{M}'$ as local systems, where h acts trivially on $\mathcal{M}^{\bar{h}}$. Every Leray sheaf of $\mathcal{M}'$ vanishes, $\mathcal{M}^{\bar{h}}$ descends to the orbifold base, and $H^*(M, \mathcal{M}^{\bar{h}})$ is a summand of $H^*(M, \mathcal{M}) = 0$; so we may assume $\mathcal{M} = \mathcal{M}^{\bar{h}}$. Rationally the Leray sheaves of ϖ are then $R^0 = \mathcal{Q}$ and $R^1 \cong \mathcal{Q}$: invariants equal coinvariants for a finite-order operator, and the orientation of the $\mathbb{C}^*$ -orbits trivialises $R^1 \varpi_* \mathbb{Q}$ rationally, including at exceptional fibres, where $H^1(\ell) \rightarrow H^1(h)$ is multiplication by k . The Leray spectral sequence has two rows, and d_2 is cup product with the rational Euler class e , giving the Gysin sequence

$$\cdots \rightarrow H^k(B_m^\circ, \mathcal{Q}) \rightarrow H^k(M, \mathcal{M}) \rightarrow H^{k-1}(B_m^\circ, \mathcal{Q}) \xrightarrow{\cup e} H^{k+1}(B_m^\circ, \mathcal{Q}) \rightarrow \cdots$$

Since $H^*(M, \mathcal{M}) = 0$, $\cup e$ is an isomorphism, and induction from negative degrees gives $H^*(B_m^\circ, \mathcal{Q}) = 0$.

Compact supports. Finite monodromy gives an invariant positive definite symmetric form on $\mathcal{M}$, for which the splitting above is orthogonal; so $\mathcal{M} \cong \mathcal{M}^\vee$, and $\mathcal{M}^\vee$ satisfies the hypothesis. Locally on the coarse space $B_m^\circ = \mathbb{C}^d / H$ and $\mathcal{Q} = (q_* \mathcal{E})^H$ for a local system $\mathcal{E}$ and a finite map q ; put $\mathcal{Q}^\vee := (q_* \mathcal{E}^\vee)^H$. The form restricts to a nondegenerate form on the invariants of every finite group, so $R^0 \varpi_* (\mathcal{M}^\vee) \cong \mathcal{Q}^\vee$. Since $\mathbb{D} \circ q_* = q_* \circ \mathbb{D}$ and H -invariants commute with $\mathbb{D}$, $\mathbb{D} \mathcal{Q} \cong \mathcal{Q}^\vee[2d]$, and Verdier duality gives $H_c^i(B_m^\circ, \mathcal{Q})^\vee \cong H^{2d-i}(B_m^\circ, \mathcal{Q}^\vee) = 0$ by the first half applied to $\mathcal{M}^\vee$. $\square$

Lemma 7.5 (The exceptional divisor on a smooth refinement). *Let $\mathfrak{M} = \mathfrak{X}_{k-1}$ with next centre supported on $\mathfrak{S} = \mathfrak{t}_k$. Let $V \rightarrow \mathfrak{M}$ be an atlas obtained from (I_{k-1}) , with $(g_V, s_1, \dots, s_r)$ smooth, let $W = \mathbb{A}^c$ be the chart at x_k , with weights $a_1, \dots, a_c > 0$, and put $C = V \times_{O_{k-1}} W$ and $C_0 = C \times_W \{0\}$. Then $C \rightarrow V$ is smooth and covers a neighbourhood of the centre, and the exceptional divisor $\mathfrak{E}_k$ satisfies*

$$\mathfrak{E}_k \times_{\mathfrak{S}} C_0 \simeq C_0 \times \mathbb{P}^{\text{st}}(a),$$

in which the complement of the transformed boundary is $C_0 \times \mathcal{B}_k$, $\mathcal{B}_k = [W^\circ/\mathbb{G}_m]$ for the weighted action. The old exceptional equations remain equations on C_0 , and the formula holds where any subset of them vanishes.

Proof. Both projections of C are smooth, as base changes of smooth maps, and its centre is C_0 . Flatness identifies the pulled-back Rees algebra and its associated graded with those on W . In the radial presentation $y_i = t^{a_i}v_i$ with $v \neq 0$; setting $t = 0$ gives the weighted quotient of v , hence $C_0 \times \mathbb{P}^{\text{st}}(a)$. The boundary ideal, saturated by t , is the pullback of the ideal evaluated at v , whose complement is W° . Smoothness of (g_V, s) makes the s_i independent parameters relative to W . $\square$

Lemma 7.6 (Coarse fibres). *Let $f : \mathfrak{A} \rightarrow \mathfrak{S}$ be a morphism of separated finite-type Deligne–Mumford stacks over $\mathbb{C}$, L a rational local system on $\mathfrak{A}$, and $\bar{f} : A \rightarrow S'$ the induced map of coarse spaces. For $\sigma \in S'(\mathbb{C})$ choose an object s above it with stabiliser H , put $\mathfrak{F} = \mathfrak{A} \times_{\mathfrak{S}} \text{Spec } \mathbb{C}$ (along s), and let Z be the coarse space of $\mathfrak{F}$ and $\mathcal{R} = \pi_{\mathfrak{F}*}(L|_{\mathfrak{F}})$. Then*

$$R\Gamma_c(\bar{f}^{-1}(\sigma), (\pi_{\mathfrak{A}*}L)|_{\bar{f}^{-1}(\sigma)}) \simeq (R\Gamma_c(Z, \mathcal{R}))^H.$$

Proof. $\mathfrak{F} \rightarrow \mathfrak{A} \times_{\mathfrak{S}} BH$ is a finite étale H -torsor, whose deck transformations act on $(Z, \mathcal{R})$. The coarse space of $\mathfrak{A} \times_{\mathfrak{S}} BH$ is Z/H , with coarse pushforward $(\rho_*\mathcal{R})^H$ for the finite quotient $\rho : Z \rightarrow Z/H$. The reduced fibre of $\mathfrak{S}$ over σ is BH , so tame coarse base change [4, Thm. 3.2(d), Cor. 3.3(a)] identifies the coarse space with $\bar{f}^{-1}(\sigma)$; ρ is finite, so its pushforward commutes with compact supports, and the averaging idempotent of H commutes with $R\Gamma_c$. $\square$

Lemma 7.7 (Local vanishing at a centre). *Let x_k be a centre, $f : [W/T_{x_k}] \rightarrow \text{Bun}$ its chart with $W = \mathbb{A}^c$ and fixed point $w = 0$, $D_W = f^{-1}(\mathfrak{B} \setminus U)$, which contains 0, $W^\circ = W \setminus D_W$, and $\mathcal{L}_W$ the local system on W° of Subsection 6.3. Then $R\Gamma(W^\circ, \mathcal{L}_W) = 0$.*

Proof. Let $f_W : W \rightarrow \text{Bun}$ be the composite of f with $W \rightarrow [W/T_{x_k}]$, a smooth morphism, and $j_W : W^\circ \hookrightarrow W$ the base change of $j : U \hookrightarrow \text{Bun}$, so that $\mathcal{L}_W = j_W^* f_W^* \mathbb{F}$. By Theorem 2.1 and Lemma A.3, $j_! \mathbb{F} \rightarrow Rj_* \mathbb{F}$ is an isomorphism of Betti sheaves on smooth charts. Writing f_W° for the restriction of f_W to W° , there are canonical isomorphisms $f_W^* j_! \simeq j_W^* f_W^{\circ*}$ and, by smooth base change for constructible complexes, $f_W^* Rj_* \simeq Rj_W^* f_W^{\circ*}$. So $j_W^* \mathcal{L}_W \rightarrow Rj_W^* \mathcal{L}_W$ is an isomorphism on W . Its stalk at $0 \in D_W$ is therefore $(j_W^* \mathcal{L}_W)_0 = 0$, and

$$(Rj_W^* \mathcal{L}_W)_0 = \varinjlim_{B \ni 0} R\Gamma(B \cap W^\circ, \mathcal{L}_W),$$

the limit over open neighbourhoods B of 0. The cocharacter $\bar{\eta}$ acts on $W = \mathbb{A}^c$ linearly with weights $a_1, \dots, a_c > 0$ and preserves D_W , hence W° . For an $\bar{\eta}$ -adapted ball B_ϵ (the set where $\sum_i |y_i|^{2/a_i} < \epsilon$), the inclusion $B_\epsilon \cap W^\circ \subseteq W^\circ$ is a homotopy equivalence, by radial rescaling $u \mapsto \bar{\eta}(e^{-s})u$ with a point-dependent time $s \geq 0$. So the restriction $R\Gamma(W^\circ, \mathcal{L}_W) \rightarrow R\Gamma(B_\epsilon \cap W^\circ, \mathcal{L}_W)$

is an isomorphism, as is each transition map between such balls, and these balls are cofinal. Hence $R\Gamma(W^\circ, \mathcal{L}_W) = (Rj_{W*}\mathcal{L}_W)_0 = 0$. $\square$

7.3. Proof of Theorem 7.1. For $1 \leq k \leq N$ let $Z_k = \bigcup_{i \geq k} E_i$, closed, with $Z_{N+1} = \emptyset$, and $A_k^\flat = Z_k \setminus Z_{k+1}$. The triangles for $\partial \setminus Z_1 \subset \partial$ and $A_k^\flat \subset Z_k$ give, all Z_k being compact,

$$\begin{aligned} H_c^*(\partial \setminus Z_1, \mathcal{F}) &\rightarrow H^*(\partial, \mathcal{F}) \rightarrow H^*(Z_1, \mathcal{F}) \rightarrow, \\ H_c^*(A_k^\flat, \mathcal{F}) &\rightarrow H^*(Z_k, \mathcal{F}) \rightarrow H^*(Z_{k+1}, \mathcal{F}) \rightarrow. \end{aligned}$$

(a) $\partial \setminus Z_1 \subset \tilde{D}$, where $\mathcal{F} = 0$ (Corollary 7.3). By downward induction on k it suffices to show $H_c^*(A_k^\flat, \mathcal{F}) = 0$ for every k .

(b) $A_k^\flat = A_k \sqcup (A_k^\flat \cap \tilde{D})$, with A_k the open set of points whose radial image lies in U : if the image is a codimension-one point of $\mathfrak{B} \setminus U$ the point lies on $\tilde{D}$ by (I_N) , and if it is x_m with $m > k$ the point lies on a later centre, hence its preimage in Z_{k+1} . Since $\mathcal{F}$ vanishes on $\tilde{D}$, $H_c^*(A_k^\flat, \mathcal{F}) = H_c^*(A_k, \mathcal{F})$.

(c) The later blow-ups are isomorphisms off their centres, and on A_k all later centre ideals are units; so A_k is the coarse space of the open $\mathfrak{A}_k \subset \mathfrak{E}_k$ of Lemma 7.5, and $\mathcal{F}$ near it is computed at stage k , since R^0j_* is local. Let $f_k : \mathfrak{A}_k \rightarrow \mathfrak{t}_k$ be the restriction of the blow-down.

(d) *Coefficients on the fibres.* Let $\mathcal{L}_k$ be the local system on $\mathcal{B}_k = [W^\circ/\mathbb{G}_m]$ obtained by restricting the T_{x_k} -equivariance of $\mathcal{L}_W$ along $\bar{\eta}$. On the radial chart of Lemma 6.13 at $t = 0$, which is $C_0 \times (W \setminus 0)$, the coefficient on $C_0 \times W^\circ$ is $\mathrm{pr}_2^*\mathcal{L}_W$; comparing the equivariance maps where both sheaves exist (Lemma 6.8 at $t \neq 0$, then full faithfulness) gives

$$L^{\mathrm{st}}|_{C_0 \times \mathcal{B}_k} \simeq \mathrm{pr}_2^*\mathcal{L}_k,$$

relative to the chart, and compatibly with the groupoid transitions; the right action is δ_1 . Setting old s_i to zero preserves it.

(e) *The fibre vanishes on the stack.* By Lemma 7.7, $R\Gamma(W^\circ, \mathcal{L}_W) = 0$. The map $W^\circ \rightarrow \mathcal{B}_k$ is a principal $\mathbb{G}_m$ -bundle over the stack, and $\mathcal{L}_W$ is the pullback of $\mathcal{L}_k$, so the Gysin sequence gives $H^*(\mathcal{B}_k, \mathcal{L}_k) = 0$ by induction from negative degrees. An invariant positive definite form gives the same vanishing for the dual, and duality on the smooth Deligne–Mumford stack $\mathcal{B}_k$ gives $R\Gamma_c(\mathcal{B}_k, \mathcal{L}_k) = 0$. Passing to the coarse weighted projective open takes finite isotropy invariants, with coefficient sheaf $R^0\varpi_*\mathcal{L}_W|_M$; so this is the situation of Lemma 7.4.

(f) *Coarse vanishing.* The atlases C_0 of Lemma 7.5 cover $\mathfrak{t}_k$, and the fibre of f_k over an object s of $\mathfrak{t}_k$ is $\mathcal{B}_k$ with the coefficient of (d). The coarse spaces are separated of finite type, hence locally compact, and the stalk of $R\bar{f}_{k!}$ is the compactly supported cohomology of the fibre [28, Thm. 4.5.3, (4.5.4)]. By Lemma 7.2, $\mathcal{F}|_{A_k} = \pi_*^{\mathrm{inv}} L^{\mathrm{st}}$, and Lemmas 7.6 and 7.4 give

$$(R\bar{f}_{k!}(\mathcal{F}|_{A_k}))_\sigma \simeq (R\Gamma_c(B_k^\circ, R^0\varpi_*\mathcal{L}_W|_M))^{H_\sigma} = 0.$$

Hence $R\bar{f}_{k!}(\mathcal{F}|_{A_k}) = 0$ and $R\Gamma_c(A_k, \mathcal{F}) = 0$. With (a)–(c) this gives $H^*(\partial, \mathcal{F}) = 0$, and the second statement follows from Theorem 5.2. $\square$

The proof uses the weighted structure at each centre (the Seifert link, and the fibration of A_k over Σ_k with fibre B_k°), the extension of the coefficients across the exceptional loci (Lemmas 6.8, 6.13 and 7.2), and cleanness at every point of $\mathfrak{B}$. The group theory of Q_8 enters only in Corollary 7.3.

7.4. The projector on R . The interior divisors $E_\star \subset U_X$ lift isomorphically to R (the cover is étale there), so Definition 4.1 defines an algebraic correspondence $P_\star = [\Delta_R] - \Pi_E$ on R , with Π_E supported on $\bigsqcup E_\star \times E_\star$, in the smooth locus of R . Lemma 4.2 holds on R with the same proof, since R is a rational homology manifold (Theorem 5.2) and the $E_\star$ lie in its smooth locus.

Corollary 7.8. *On the weighted model, $\varepsilon P_\star$ acts on $H^*(R)$ as an idempotent of rank seven with image $M_x \subset H^6(R)$, and restriction to T_x is injective on M_x .*

Proof. By Theorems 5.2 and 7.1, $H^*(R)_{\text{odd}} = D \otimes M_x \oplus \bigoplus_{i=1}^5 D \otimes H^{2i}(E_\star)$. The correspondence $P_\star$ removes the second summand, and ε kills the even block and has rank one on D . Injectivity of restriction is Lemma 2.2 and Remark 3.9. $\square$

Example 7.9 (A good model of (Y, D_x) with nonzero defect). *Step 1: a divisor with trivial inertia.* Fix a centre x_m and let $\Sigma = \tilde{D} \cap E_m$ on the weighted model; it is nonempty, being the weighted projectivisation of the cone D_W . At a generic point of Σ the local group is $\mathbb{Z}^2$, generated by the meridians of $\tilde{D}$ and E_m , with commuting images $a \neq 1$ and b . Abelian subgroups of Q_8 are cyclic, so $a = g^\alpha$, $b = g^\beta$ for some g of order $n \mid 4$. Iterated blow-ups of codimension-two crossing strata over Σ produce, for every primitive $(p, q) \in \mathbb{Z}_{>0}^2$, a divisor whose meridian has image $a^p b^q$, and the congruence $\alpha p + \beta q \equiv 0 \pmod{n}$ has a solution with $p, q \geq 1$ coprime: if $\gcd(\beta, n) \mid \alpha$ take $p = 1$; if $\gcd(\alpha, n) \mid \beta$ take $q = 1$; otherwise $n = 4$, α and β are both 2 modulo 4, and $(p, q) = (1, 1)$ works. This gives a divisor E' with trivial inertia at its generic points, and each step keeps a good model.

Step 2: a point blow-up. Choose $q_0 \in E'$ generic, off the other components, and blow it up (or its residual gerbe). The result is a good model, the new divisor being one labelled branch. Since $I_{q_0} = 1$, the cover is trivial near q_0 , $p^{-1}(q_0)$ is a free $\tilde{A}$ -orbit, and the new normalisation is $R' = \text{Bl}_{p^{-1}(q_0)} R$, with

$$H^*(R')_{\text{odd}} = H^*(R)_{\text{odd}} \oplus D \otimes \bigoplus_{j=1}^5 \mathbb{Q}(-j)[-2j],$$

while $H_c^*(\widehat{\text{Gr}}_x \setminus D_x, \mathbb{L})$ is unchanged. So the defect of this good model is nonzero. Merely creating trivial-inertia points does not create defect: the nonzero defect comes from the closed $\mathbb{P}^5$ on which the extended local system is trivial, as the closed $\tilde{E}_1 \cong \mathbb{P}^1$ does in Example 5.3.

8. THE POINTS OF $\mathfrak{B}$

This section verifies Hypothesis 6.1. A point of $\mathfrak{B}$ is a pair $(\mathcal{P}_{\tilde{w}}, b)$, $b \in G/B$, up to $H_{\tilde{w}} = \text{Aut}(\mathcal{P}_{\tilde{w}})$. Let $j(\tilde{w}) = \dim H_{\tilde{w}} - \dim K$ be the codimension of the bundle-type locus. The stabiliser of b is $S_b = \text{ev}_1^{-1}(B_b)$, of dimension $c = \dim H_{\tilde{w}} - \dim(H_{\tilde{w}} \cdot b)$, so $c \geq j$.

8.1. Bundle types.

Lemma 8.1. *The bundle types with $j \leq 6$ are exactly seven: one for each $j \in \{0, 1, 3, 4, 5\}$ and two with $j = 6$, with $\dim H_{\tilde{w}} = 6, 7, 9, 10, 11, 12, 12$. Representatives are $\star = t^{(-2, -2)} s_1 s_2 s_1$, $B_1 = t^{(-2, -2)} s_1 s_2 s_1 s_2$, $B_3 = t^{(-3, -3)} s_1 s_2 s_1 s_2$, $B_4 = t^{(-2, -3)} s_1 s_2 s_1$, $B_5 = t^{(-3, -3)} s_1 s_2 s_1$, $B_6 = t^{(-2, -3)} s_1 s_2$ and $C_6 = t^{(-2, -2)} s_1$, with $\mu = (m_1, m_2)$ the coordinates of t^μ in the basis $\alpha_1^\vee, \alpha_2^\vee$.*

Proof. With $v = \rho^\vee/2$ and $\nu = \mu + v - wv$, (1) gives $N_\alpha = \max(\langle \alpha, \nu \rangle + 1 - (\pi(\text{ht } \alpha) + \pi(\text{ht } w^{-1}\alpha))/2, 0)$, where $\pi(m) \in \{0, 1\}$ is the parity of m . So $N_\alpha + N_{-\alpha} \geq |\langle \alpha, \nu \rangle|$, and $j = \sum_\alpha N_\alpha - 4 \leq 6$ forces $\sum_{\alpha > 0} |\langle \alpha, \nu \rangle| \leq 10$. Writing $\nu = (x, y)$, the six positive pairings are $2x - y$, $-3x + 2y$, $-x + y$, x , $3x - y$, y ; the sum of the first, third, fourth, fifth and sixth is $5x$, and the sum of the last five is $3y$, so $5|x| \leq 10$ and $3|y| \leq 10$, and since $\nu \in \frac{1}{2}\mathbb{Z}^2$, $|x| \leq 2$ and $|y| \leq 3$. As $wv - v$ varies by at most 3 in the first coordinate and 5 in the second, $|m_1| \leq 5$ and $|m_2| \leq 8$. An exact enumeration over this box and all $w \in W$, with both parahoric Weyl groups the stabilisers of $-\rho^\vee/2$ (the only choice for which $\dim H_{\tilde{w}}$ is constant on double cosets), gives the list, with B_6 and C_6 in distinct double cosets (Appendix B). If affine roots are written $f_{\alpha, k}(x) = \langle \alpha, x \rangle + k$ and counted by $f_{\alpha, k}(p) \geq 0$, $f_{\alpha, k}(gp) \leq 0$ with $p = -\rho^\vee/2$ and $g = t^\mu w$, that count is $N_{-\alpha}$ in our convention; summing over all roots gives the same dimension and the same census. $\square$

Proposition 8.2. *The image $\text{ev}_1(H_{\tilde{w}})$ is the connected subgroup with Lie algebra $\mathfrak{t} \oplus \bigoplus_{N_\alpha \geq 1} \mathfrak{g}_\alpha$:*

- for $\star$, the symmetric subgroup K ;
- for B_6 , a Borel subgroup B' ;
- for C_6 , a Borel subgroup B'' (the roots with $N_\alpha \geq 1$ are α_2 , $-\alpha_1$, $-(\alpha_1 + \alpha_2)$, $-(2\alpha_1 + \alpha_2)$, $-(3\alpha_1 + \alpha_2)$, $-(3\alpha_1 + 2\alpha_2)$, a positive system);
- for B_5 , a parabolic subgroup P' with Levi of type GL_2 (roots $\pm(\alpha_1 + \alpha_2)$);
- for B_1, B_3, B_4 , the subgroup $T \cdot U_0$ where U_0 is the product of the positive root groups of a Borel subgroup B' other than one simple root γ (respectively $\gamma = -\alpha_1$, $-(3\alpha_1 + 2\alpha_2)$, $-(2\alpha_1 + \alpha_2)$).

The kernel of ev_1 on $H_{\tilde{w}}$ is unipotent and meets every torus trivially.

Proof. The root sets are read off from (1). For the solvable types, if $\text{ev}_1(tu) = 1$ with t semisimple and u unipotent then $\text{ev}_1(t) = \text{ev}_1(u)^{-1}$ is semisimple

and unipotent, hence $t = 1$; for B_5 evaluation identifies the Levi with the root Levi of P' and maps the unipotent part into its unipotent radical; for $\star$ evaluation is the inclusion of K . $\square$

Proposition 8.3. *For every point b of every type and every maximal torus $T_S \subset S_b^\circ$, the normaliser $N_{S_b}(T_S)$ centralises T_S . In particular $\Gamma_y = 1$ and $\bar{\eta}_y = \eta_y$.*

Proof. In a connected solvable group the projection to the quotient by the unipotent radical is injective on every torus and has commutative target, so if n normalises a torus T_0 then ntn^{-1} and t have the same image, hence are equal. Apply this inside the Borel subgroup B_b to $\text{ev}_1(T_S)$, using Proposition 8.2. $\square$

Proposition 8.4 (Covered stabilisers). *The stabilisers of Yun's stack involve the simply connected cover $\tilde{K}$ of K [40, §2.1.6], while $H_{\tilde{w}}$ and S_b above are computed in G_2 . Let $\tilde{S} \rightarrow S_b$ be the induced central double cover and $\tilde{T} \rightarrow T_S$ the isogeny of maximal tori. For every cocharacter η of T_S satisfying (H1) and (H2), 2η lifts to $\tilde{T}$ and satisfies (H1) and (H2) there, with weights twice those of η ; and the normaliser of $\tilde{T}$ in $\tilde{S}$ centralises it.*

Proof. The central μ_2 changes neither orbits nor Lie algebras. The cokernel of $X_*(\tilde{T}) \rightarrow X_*(T_S)$ is killed by 2, so 2η lifts, and the deformation and adjoint representations factor through S_b , so the weights of the lift are twice those of η . For the normaliser: an element normalising $\tilde{T}$ induces a commutator map from $\tilde{T}$ to the finite central kernel, a morphism from a connected torus to a finite group, hence trivial; so it centralises $\tilde{T}$. Doubling the weights can add finite inertia to the weighted blow-up; the weighted model allows finite stabilisers and takes their invariants, so this changes nothing in Section 7. $\square$

8.2. Orbits.

Proposition 8.5. *The $H_{\tilde{w}}$ -orbits on G/B are finite in number for each type: 12 Bruhat cells for each of B_6 and C_6 ; 6 orbits for B_5 ; 18 orbits for each of B_1, B_3, B_4 , namely one orbit vB for each $v \in W$ with $\gamma \in v\Phi^+$ and two orbits $vB, u_\gamma(1)vB$ for each v with $\gamma \notin v\Phi^+$; and 10 orbits for $\star$. Every orbit with $c \geq 1$ has a representative fixed by T or by $(\ker \gamma)^\circ$ for a root γ ; in particular no orbit of positive codimension has a stabiliser without a torus.*

Proof. For B_6, C_6 and B_5 this is the Bruhat decomposition. For B_1, B_3, B_4 , U_0 is normal in B' and $B'/U_0 = T \ltimes U_\gamma$, with image of $T \cdot U_0$ equal to T . With $Q_v = B' \cap vBv^{-1}$, the image of Q_v in $T \ltimes U_\gamma$ is T if $\gamma \notin v\Phi^+$ and all of $T \ltimes U_\gamma$ otherwise; in the first case $T \backslash (T \ltimes \mathbb{G}_a)/T$ has the two double cosets 0 and 1, since T scales the coordinate through the nontrivial character γ .

For $\star$, K is connected of type $A_1 \times A_1$ with Weyl group W_K of order four. By Matsuki [22] and Richardson–Springer [26], $K \backslash G/B$ is in bijection with $\bigsqcup_H W(G, H)/W_K(H)$ over the K -classes of θ -stable Cartan subgroups

H . By Sugiura [34, Thm. 6, Def. 9, Thm. 7] (Theorem 7 as corrected in [35]), the conjugacy classes of Cartan subalgebras of the split real form correspond to the classes of strongly orthogonal sets of noncompact roots, realised by iterated Cayley transforms from the compact Cartan subalgebra; for split G_2 there are four classes [34, p. 429]. Cayley transforms from T give them: compact, one mixed class with a real short root, one with a real long root, and split. Their contributions are $12/4 = 3$ (as $|W_K| = 4$), $12/4 = 3$, $12/4 = 3$ and $12/12 = 1$: for a mixed Cartan H with real root α , $W_K(H) = \{1, s_\alpha, s_\beta, -1\}$ with $\beta \perp \alpha$ noncompact imaginary and s_β realised in K , the full centraliser of s_α in W . The total is 10. The exact computation of Appendix B realises each of these Weyl group elements by an explicit element of K over $\mathbb{Q}$ or $\mathbb{Q}(i)$, so the counts 3, 3, 3, 1 hold.

The ten orbits are as follows. The twelve T -fixed flags form three closed orbits K/B_K : a closed orbit $K \cdot wB$ meets $(G/B)^T$ exactly in $\{vB : v \in W_K w\}$, and $W_K \backslash W$ has three elements. The six flags $u_\gamma(1)vB$ of Table 1 have stabiliser dimensions $c = 1, 2, 3$, maximal stabiliser torus $(\ker \gamma)^\circ$ of rank one, and unipotent radical of dimension $c - 1$. The flag in general position has $c = 0$ and lies in the open orbit. Orbits of different dimension are distinct, and the two orbits of Table 1 of each codimension are distinct by the following invariant. In characteristic zero stabilisers are smooth, so the stabiliser Lie algebra is $\mathfrak{s}_x = \mathfrak{k} \cap \mathfrak{b}_x$, and $y = k \cdot x$ implies $\mathfrak{s}_y = \text{Ad}(k)\mathfrak{s}_x$. For each of the six representatives, $\mathfrak{s}_x = \mathbb{C}h \ltimes \mathfrak{n}$ with $h \in \mathfrak{t}$, $\mathfrak{n}$ abelian and spanned by $\text{ad } h$ -eigenvectors (each a root vector or, at the short-root orbit with $c = 2$, the combination $E_{\alpha_1 + \alpha_2} + 3E_{-(3\alpha_1 + \alpha_2)}$), and $\text{ad } h$ invertible on $\mathfrak{n}$. Hence the nonzero semisimple elements of $\mathfrak{s}_x$ are the $\text{Ad}(K_x^\circ)$ -conjugates of nonzero multiples of h , and

$$\varrho(x) := \max |\text{spec ad } h| / \min |\text{spec ad } h \setminus 0|$$

on $\mathfrak{g}_2$ is an invariant of the orbit. If h is proportional to the coroot of a long (respectively short) root, the values $\gamma(h)$, $\gamma \in \Phi$, are proportional to $\{0, \pm 1, \pm 2\}$ (respectively $\{0, \pm 1, \pm 2, \pm 3\}$), so $\varrho = 2$ (respectively 3). The stabiliser torus is $(\ker \gamma)^\circ$, and in G_2 the line orthogonal to a short root is spanned by a long coroot and vice versa; so in Table 1 the short-root column has $\varrho = 2$ and the long-root column $\varrho = 3$, and the two orbits of each codimension are distinct, and they belong to the two mixed Cartan classes. $\square$

TABLE 1. Representatives $u_\gamma(1)vB$ of the K -orbits of codimension 1, 2, 3.

c	short root γ	long root γ
1	$-(2\alpha_1 + \alpha_2), v = e$	$-(3\alpha_1 + 2\alpha_2), v = e$
2	$-(2\alpha_1 + \alpha_2), v = s_1$	$-\alpha_2, v = s_1$
3	$-\alpha_1, v = e$	$-\alpha_2, v = e$

The orbit counts over finite fields agree with this list: with $|K(\mathbb{F}_q)| = q^2(q^2 - 1)^2$, Lang's theorem gives each orbit q -count $|K(\mathbb{F}_q)|/((q - 1)^r q^u)$ for a stabiliser with torus quotient of rank r and unipotent radical of dimension u , and the ten counts sum to $(q + 1)^2(q^4 + q^2 + 1) = |G/B(\mathbb{F}_q)|$.

Proposition 8.6 (Frontier order). *$\mathfrak{B}$ has finitely many points, and a point in the closure of a distinct point y has strictly larger stabiliser dimension than y .*

Proof. Finiteness follows from Lemma 8.1 and Proposition 8.5. On a smooth atlas of pure relative dimension d over the smooth stack of dimension zero, the inverse image of a residual gerbe with stabiliser dimension c is smooth of pure dimension $d - c$ and open in its closure, so a distinct stratum in its boundary has smaller dimension, that is, larger c . $\square$

8.3. Contracting cocharacters. At a T -fixed point $(\mathcal{P}_{\tilde{w}}, vB)$ the stabiliser dimension and the weights of T on T^1 and on $\mathrm{Lie} \mathcal{G}/\mathrm{Lie} T$ are given by the affine root counts:

$$c = 2 + \sum_{\alpha \in v\Phi^+} N_\alpha + \sum_{\alpha \in v\Phi^-} (N_\alpha - 1)_+;$$

the T^1 -weights are the roots $\alpha \in v\Phi^-$ with $N_\alpha = 0$, together with each α repeated $(-d_\alpha - 1)_+$ times, where $d_\alpha = h_\alpha - \ell_\alpha$; and the stabiliser weights are the roots $\alpha \in v\Phi^+$ with multiplicity N_α and $\alpha \in v\Phi^-$ with multiplicity $N_\alpha - 1$.

Derivation. Let $\mathrm{ad}_{\tilde{w}}$ be the sheaf on $\mathbb{P}^1$ of infinitesimal automorphisms of $\mathcal{P}_{\tilde{w}}$ preserving the level structures at 0 and ∞ . Its global sections are $\mathrm{Lie} H_{\tilde{w}} = \mathfrak{t} \oplus \bigoplus_\alpha \mathrm{span}\{t^n E_\alpha : \ell_\alpha \leq n \leq h_\alpha\}$, and it splits T -equivariantly as $\mathfrak{t} \otimes \mathcal{O} \oplus \bigoplus_\alpha L_\alpha$, where L_α is the line bundle whose sections near 0 and ∞ are $t^n E_\alpha$ with $n \geq \ell_\alpha$, resp. $n \leq h_\alpha$; so $L_\alpha \cong \mathcal{O}(d_\alpha)$. Deformations of the point $(\mathcal{P}_{\tilde{w}}, vB)$ are governed by the two-term complex $\mathrm{ad}_{\tilde{w}} \rightarrow (\mathfrak{g}/v\mathfrak{b})_1$ (evaluation at 1 into a skyscraper), whose hypercohomology gives the exact sequence

$$0 \rightarrow \mathrm{Lie} \mathcal{G}_y \rightarrow \mathrm{Lie} H_{\tilde{w}} \xrightarrow{\mathrm{ev}_1} \mathfrak{g}/v\mathfrak{b} \rightarrow T_y^1 \rightarrow H^1(\mathbb{P}^1, \mathrm{ad}_{\tilde{w}}) \rightarrow 0.$$

So, noncanonically as a representation of T (only the multiset of weights is canonical), $T_y^1 \cong \mathrm{coker}(\mathrm{ev}_1) \oplus H^1$ with $H^1 = H^1(\mathbb{P}^1, \mathrm{ad}_{\tilde{w}}) = \bigoplus_\alpha H^1(\mathbb{P}^1, \mathcal{O}(d_\alpha))$, of dimension $(-d_\alpha - 1)_+$ in weight α by Riemann–Roch. The weights of $\mathfrak{g}/v\mathfrak{b}$ are $v\Phi^-$, and E_α is in the image of ev_1 exactly when $N_\alpha \geq 1$, which gives the cokernel. Counting dimensions in the sequence, and using that $\dim T_y^1 = c$ because Bun has dimension zero, gives the formula for c and the stabiliser weights. At a point $u_\gamma(1)vB$ the stabiliser torus is $(\ker \gamma)^\circ$, and its weights on the same two terms are computed from a semisimple element of the stabiliser.

An example: the row $(\star, T, e)$ of Table 2. Here $\tilde{w} = t^{(-2, -2)} s_1 s_2 s_1$, and $\mathrm{Lie} H_{\tilde{w}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_K} \mathbb{C} E_\alpha$ with $\Phi_K = \{\pm(\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2)\}$, the roots of even height, each with $N_\alpha = 1$; all other N_α vanish. With $v = e$:

$c = 2 + N_{\alpha_1 + \alpha_2} + N_{3\alpha_1 + \alpha_2} = 4$. The roots of Φ^- with $N_\alpha = 0$ are $-\alpha_1$, $-\alpha_2$, $-(2\alpha_1 + \alpha_2)$, $-(3\alpha_1 + 2\alpha_2)$; these four already account for $\dim T^1 = c = 4$, so $H^1 = 0$. For $\eta = -3\alpha_1^\vee - 5\alpha_2^\vee = -\rho^\vee$, using $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and $\langle \alpha_2, \alpha_1^\vee \rangle = -3$, their pairings with η are 1, 1, 3, 5, all positive, which is (H1). The stabiliser weights are those of $\alpha_1 + \alpha_2$ and $3\alpha_1 + \alpha_2$, namely -2 and -4 , which is (H2).

Proposition 8.7. *Hypothesis 6.1 holds, for the stabilisers computed in G_2 and, by Proposition 8.4, for those of Yun's stack.*

Proof. (H0) is Proposition 8.6. For (H1) and (H2), Table 2 of Appendix B lists 57 representative points, covering all 46 orbits with $1 \leq c \leq 6$ over the seven types, each with an integral cocharacter η , all T^1 -weights positive, $\dim T^1 = c$, and all stabiliser weights ≤ 0 . By Proposition 8.3 the averaging of Section 6 is trivial. $\square$

9. CLOSED CYCLES ON R

Let R be the normalisation for the weighted model, and $\partial_R \cup E_\star \subset R$ the complement of T_x . Let $r : H^*(R) \rightarrow H^*(T_x)$ be restriction.

Lemma 9.1. (a) $\varepsilon P_\star H^6(R) = M_x$, and r is injective on it. (b) $\varepsilon P_\star$ annihilates $\ker(r) = \text{im}(H_\star^{BM}(\partial_R \cup E_\star) \rightarrow H^*(R))$, and $r \circ P_\star = r$. (c) For every closed cycle $\overline{W} \subset R^n$ with $\overline{W} \cap T_x^n = W$, the class $(\varepsilon P_\star)^{\otimes n} \text{cl}(\overline{W})$ depends only on W , and is the unique element of $M_x^{\otimes n}$ whose image under $r^{\otimes n}$ is $\varepsilon^{\otimes n} \text{cl}(W)$.

Proof. (a) is Corollary 7.8. (b) $P_\star = \Delta - \Pi_E$ with $\text{im } \Pi_E \subset \ker r$, so $r P_\star = r$ and $P_\star(\ker r) \subset \ker r$. If $y \in \ker r$, then $\varepsilon P_\star y \in \ker r$, as ε commutes with r , and $\varepsilon P_\star y \in M_x$, on which r is injective; so $\varepsilon P_\star y = 0$. (c) Two closures differ by a cycle supported on $\bigcup_k R^{k-1} \times (\partial_R \cup E_\star) \times R^{n-k}$. Each irreducible component lies in one of these closed sets, so by Künneth on the compact R^n its class lies in $H^*(R)^{\otimes k-1} \otimes \ker r \otimes H^*(R)^{\otimes n-k}$, which $(\varepsilon P_\star)^{\otimes n}$ kills by (b). Finally $r^{\otimes n}(\varepsilon P_\star)^{\otimes n} \text{cl}(\overline{W}) = \varepsilon^{\otimes n} \text{cl}(W)$, and $r^{\otimes n}$ is injective on $M_x^{\otimes n}$ by Künneth for T_x^n and (a). $\square$

Proposition 9.2. *For every $u \in (V_7^{\otimes n})^{G_2}$ and every closure $\overline{W}_u \subset R^n$ of $W_u \cap T_x^n$,*

$$(\varepsilon P_\star)^{\otimes n} \text{cl}(\overline{W}_u) = 2^{-n-1} \tau_u(-3n) \in M_x^{\otimes n} \subset H^{6n}(R^n).$$

For the cubic this is $\frac{1}{16} \tau_t(-9)$.

Proof. By Lemma 9.1(c) it suffices to compute $\varepsilon^{\otimes n} \text{cl}(W_u)$ in $H^{6n}(T_x^n)$, which is $2^{-n-1} \tau_u$ by Proposition 3.7, τ_u being placed in compactly supported cohomology through the ε -line of each factor and Remark 3.9. Removing the twists places the class in $H^{6n}(R^n)$. $\square$

10. ALGEBRAIC PROJECTORS THROUGH QUOTIENT SINGULARITIES

Throughout this section a *scheme* is a separated scheme of finite type over $\mathbb{C}$, $\mathrm{CH}_*(-)_{\mathbb{Q}}$ is the rational Chow group [17, Ch. 1], and for a morphism $f : T \rightarrow Y$, $\mathrm{CH}^*(T \xrightarrow{f} Y)_{\mathbb{Q}}$ is Fulton's bivariant group with rational coefficients [17, Ch. 17], with cap product, pullback, proper pushforward and product [17, §17.1]. H_*^{BM} is rational Borel–Moore homology, and $\mathrm{cl} : \mathrm{CH}_k(T)_{\mathbb{Q}} \rightarrow H_{2k}^{BM}(T)$ is the cycle map, compatible with proper pushforward, exterior product and refined Gysin maps of regular embeddings [17, §§19.1–19.2]. When T is a compact oriented rational homology manifold of real dimension $2m$ we write $\mathrm{cl}_H = \mathrm{PD}_T^{-1} \circ \mathrm{cl} : \mathrm{CH}_k(T)_{\mathbb{Q}} \rightarrow H^{2m-2k}(T; \mathbb{Q})$.

10.1. Setting.

- (T1) R is the normalisation of Section 9, for the weighted model: a normal projective scheme of pure dimension d with quotient singularities (projectivity: Lemma 10.1 and Remark 10.2).
- (T2) $q : \overline{X} \rightarrow R$ is a projective birational morphism from a smooth projective variety, an isomorphism over the smooth locus R^{sm} ; $\Gamma_q \subset \overline{X} \times R$ is its graph and ${}^t\Gamma_q$ the transpose.
- (T3) A finite group A acts on R and on $\overline{X}$, and q is A -equivariant.
- (T4) $p \in \mathrm{CH}_d(R \times R)_{\mathbb{Q}}$ is an algebraic correspondence. The results below hold for every such p ; in this paper $p = \varepsilon P_*$.

A *test space* is a product $R^a \times \overline{X}^b$, $a, b \geq 0$. A product of quotient singularities is a quotient singularity, so every test space has quotient singularities and is equidimensional.

Gluing the stages over $\widehat{\mathrm{Gr}}_x$. The cell Y is not proper; the projective base is $\widehat{\mathrm{Gr}}_x$. Put $\mathcal{X}_0 = \widehat{\mathrm{Gr}}_x$, $V_0 = \widehat{\mathrm{Gr}}_x \setminus D_x$, and glue $\mathcal{X}_i|_{\widehat{\mathrm{Gr}}_x \setminus E_*} = Y \times_{\mathfrak{B}} \mathfrak{B}_i$ with $\mathcal{X}_i|_{V_0} = V_0$. On the overlap $Y \setminus D_x$ the weighted filtration is the unit filtration, whose Rees algebra is canonically $\mathcal{O}[t]$; gluing the pulled-back Rees algebra to $\mathcal{O}_{V_0}[t]$ degree by degree gives a finitely generated graded algebra $\mathcal{A}_i$ on $\mathcal{X}_{i-1}$ with $(\mathcal{A}_i)_0 = \mathcal{O}$, and $b_i : \mathcal{X}_i = \mathrm{Proj}_{\mathcal{X}_{i-1}}^{\mathrm{st}}(\mathcal{A}_i) \rightarrow \mathcal{X}_{i-1}$ with $L_i = \mathcal{O}_{\mathcal{X}_i}(1)$. On the Y -part flat base change applies [25, Cor. 3.2.14(iii)]. The centre is disjoint from the unchanged locus, where every graded ideal is $\mathcal{O}$, and Lemma 6.12 identifies $\mathrm{Proj}^{\mathrm{st}}(\mathcal{A}_i)$ with the prescribed stage. Each $\mathcal{X}_i$ is of finite type, proper over $\widehat{\mathrm{Gr}}_x$, with finite diagonal, hence a tame Deligne–Mumford stack with a separated proper coarse space C_i .

Lemma 10.1 (Projectivity). *Let $\mathcal{X}_0$ be a projective scheme, and for $1 \leq i \leq N$ let $b_i : \mathcal{X}_i = \mathrm{Proj}_{\mathcal{X}_{i-1}}^{\mathrm{st}}(\mathcal{A}_i) \rightarrow \mathcal{X}_{i-1}$ with $\mathcal{A}_i$ finitely generated, non-negatively graded and quasi-coherent, $(\mathcal{A}_i)_0 = \mathcal{O}$, and $L_i = \mathcal{O}_{\mathcal{X}_i}(1)$. Then the coarse spaces $c_i : \mathcal{X}_i \rightarrow C_i$ exist, each C_i is projective, and $f_i : C_i \rightarrow C_{i-1}$ is projective, with an f_i -ample line bundle M_i such that $c_i^* M_i \simeq L_i^{\otimes n_i}$ for some $n_i > 0$. Consequently the normalisation of C_N in a finite extension of its function field is projective.*

Proof. By [25, Prop. 1.6.1(i)–(ii)] the $\mathcal{X}_i$ are proper of finite type with finite diagonal, hence tame Deligne–Mumford with proper coarse spaces. Choose d_i with the Veronese $\mathcal{A}_i^{(d_i)}$ generated in degree one [25, Prop. 1.6.3], and by [25, Setup 1.7.2] choose n_i , divisible by d_i , such that $L_i^{\otimes n_i}$ descends to M_i on C_i . To see that M_i is f_i -ample, choose étale locally on C_{i-1} an affine V with $\mathcal{X}_{i-1} \times_{C_{i-1}} V = [U'/G']$, U' affine and finite over V and G' finite [4, Thm. 3.2, Cor. 3.3]. Put $P' = \text{Proj}_{U'}((\mathcal{A}_i|_{U'})^{(d_i)})$ with $H = \mathcal{O}_{P'}(1)$; then $C_i \times_{C_{i-1}} V = P'/G'$, the finite surjection $\rho : P' \rightarrow P'/G'$ has $\rho^* M_i \simeq H^{\otimes n_i/d_i}$, which is ample, and finite-surjective descent of ampleness [31, Lemma 30.17.2] with fpqc-locality [31, Lemma 74.13.1] makes M_i f_i -ample. If H_{i-1} is ample on C_{i-1} , then $M_i^{\otimes a} \otimes f_i^* H_{i-1}^{\otimes b}$ is ample for suitable $a, b > 0$. Finally, finite-type $\mathbb{C}$ -schemes are excellent, so normalisation in a finite extension is finite, and the pullback of an ample bundle is ample. $\square$

Remark 10.2 (Projectivity of R , and the choice of q). For the weighted model, the coarse space $C_N = X$ is projective by Lemma 10.1 applied to the stages glued over $\widehat{\text{Gr}}_x$, so R is projective. A functorial resolution [7, Thm. 1.1] is a finite sequence of blow-ups with smooth centres, an isomorphism over the regular locus and functorial for automorphisms; it therefore gives q satisfying (T2) and (T3), with $\overline{X}$ projective, and the group $A = \tilde{A}(1) \times \tilde{A}(2)$, which acts on R by functoriality of normalisation, lifts to $\overline{X}$. The neighbourhood of $E_\star$ is smooth, so it is untouched. Below, properness of R and degree one of q are what the transfer uses; projectivity makes $\overline{X}$ projective.

10.2. Intersection theory on test spaces.

Proposition 10.3 (Alexander property). *Let Y be a test space, $n = \dim Y$, and $f : T \rightarrow Y$ any morphism of schemes. Then $\text{ev}_Y : \text{CH}^r(T \xrightarrow{f} Y)_{\mathbb{Q}} \rightarrow \text{CH}_{n-r}(T)_{\mathbb{Q}}$, $\kappa \mapsto \kappa \cap [Y]$, is an isomorphism. For Y smooth its inverse sends w to the class κ_w with $\kappa_w \cap z = \gamma^!(z \times w)$, γ the graph of f .*

Proof. In characteristic zero, schemes with quotient singularities are Alexander schemes [36, Thm. 5.1], [37, Cor. 6.4]; by [36, Def. 2.1] this means that ev_Y is an isomorphism for every $T \rightarrow Y$. For Y smooth this is [17, Props. 17.3.1, 17.4.2], and the inverse is the formula in the proof of [13, Prop. 4]. $\square$

Definition 10.4 (Composition through R). For a test space Y , morphisms $T \rightarrow Y$ and $Y' \rightarrow Y$, $x \in \text{CH}_*(T)_{\mathbb{Q}}$ and $y \in \text{CH}_*(Y')_{\mathbb{Q}}$, put $x \cdot_Y y = \text{ev}_Y^{-1}(x) \cap y \in \text{CH}_*(T \times_Y Y')_{\mathbb{Q}}$ [36, Note 2.4]. For test spaces Q_1, Q_2 and $x \in \text{CH}_a(Q_1 \times R)_{\mathbb{Q}}$, $y \in \text{CH}_b(R \times Q_2)_{\mathbb{Q}}$, define

$$y \circ_R x = p_{13*}(x \cdot_R y) \in \text{CH}_{a+b-d}(Q_1 \times Q_2)_{\mathbb{Q}},$$

where p_{13} is proper because R is. Replacing R by the smooth $\overline{X}$ gives Fulton’s composition of correspondences [17, Def. 16.1], by Proposition 10.3; replacing R by R^m gives $\circ_{R^m}$.

The product $\cdot_Y$ has the formal properties of [17, Prop. 8.1.1], and the projection formula holds [36, Note 2.4]. On a test space $Y = Z/G$ with Z smooth, the resulting ring is the equivariant one [13, Thm. 4, Prop. 15]. *Presentations.* By [37, Prop. 2.8], R is the moduli space of a smooth Deligne–Mumford stack $\mathcal{R}$ with generically trivial stabiliser, and by [15, Thm. 2.18, Cor. 2.20], $\mathcal{R} \cong [Z/G]$ with Z a smooth algebraic space and G a linear algebraic group of dimension g , so $R = Z/G$ and the generic stabiliser is trivial. The action is proper: the diagonal of $\mathcal{R}$ is finite since R is separated, and the action map is its base change. Fix a representation V of dimension l and an open $U_V \subset V$ on which G acts freely, with complement of codimension at least c , and put $Z_U = (Z \times U_V)/G$, a smooth algebraic space of dimension $d+l$ [13, §2.2]. For a scheme S with trivial G -action (S will be a test space, the *inert factor*), $S \times Z \rightarrow S \times R$ is a geometric quotient, since geometric quotients are stable under the flat base change $S \times R \rightarrow R$ [19, Ch. 0, §2]. Write $\rho : S \times Z_U \rightarrow S \times R$ and

$$\hat{\pi} = \hat{\pi}_S : \mathrm{CH}_k(S \times R)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}_{k+l}(S \times Z_U)_{\mathbb{Q}},$$

the isomorphism of [13, Thm. 3(a)], valid for $k > \dim(S \times R) - c$. On cycles it is $[W] \mapsto e_W [(\pi^{-1}W)_{\mathrm{red}}]$ [13, Remark (6) after Thm. 4], where e_W is the order of the generic stabiliser in G of $\pi^{-1}W$. The map ρ is smooth of relative dimension l onto the stack followed by a bijection on points, so ρ^{-1} of a pure k -dimensional closed set is pure of dimension $k+l$. The Chow operations on the algebraic spaces Z_U are those of [13, §6.1].

Topology. A complex variety with quotient singularities is an oriented rational homology manifold: at the image of a fixed point of a finite linear group $H \subset GL_n(\mathbb{C})$, transfer [8, Ch. III, §7] gives $H^*(\mathbb{C}^n/H, \mathbb{C}^n/H \setminus 0; \mathbb{Q}) = H^*(\mathbb{C}^n, \mathbb{C}^n \setminus 0; \mathbb{Q})^H = \mathbb{Q}[-2n]$, because H preserves the complex orientation. So test spaces, and the spaces $S \times Z_U$, satisfy Poincaré and Alexander duality over $\mathbb{Q}$ [9, Ch. V]. For compact oriented rational homology manifolds Q_1, Q_2 and $\gamma \in H_*(Q_1 \times Q_2)$ of even degree put $\gamma_*\xi = \mathrm{PD}_{Q_2}^{-1} \mathrm{pr}_{2*}(\mathrm{pr}_1^*\xi \cap \gamma)$, and for an algebraic class x write $x_* = \mathrm{cl}(x)_*$.

10.3. The cycle class on quotient-singular test spaces.

Lemma 10.5 (Supported quotient multiplicities). *Let $a : M' \rightarrow M = M'/L$ be a generically free finite quotient of complex oriented rational homology manifolds of the same dimension, with L acting effectively. For a closed $W \subset M$ define $a_W^! : H_j^{BM}(W) \rightarrow H_j^{BM}(a^{-1}W)$ by duality with supports and pullback in cohomology with supports. Then $(a|_{a^{-1}W})_* a_W^! = |L|$. If W is integral and the generic stabiliser above W has order e , then $a_W^![W] = e[(a^{-1}W)_{\mathrm{red}}]$, the bracket summing the reduced components dominating W .*

Proof. For $\alpha \in H_W^*(M)$ the supported projection formula gives $a_*(a^*\alpha \cap [M']) = |L|\alpha \cap [M]$, in the homology of W . The class $a_W^![W]$ lies in top degree on the reduced inverse image and is L -invariant; L permutes the

components over W transitively, so their coefficients agree, and their total generic pushforward degree is $|L|/e$, which forces the coefficient e . $\square$

Lemma 10.6. *Let S be a test space, $Q = S \times R$, $n = \dim Q$, $Q_U = S \times Z_U$, and $c_{\text{top}} = \text{PD}_{Q_U} \circ \rho^* \circ \text{PD}_Q^{-1} : H_m^{BM}(Q) \rightarrow H_{m+2l}^{BM}(Q_U)$, with its supported version $c_{\text{top}}^W : H_m^{BM}(W) \rightarrow H_{m+2l}^{BM}(\rho^{-1}W)$ for closed $W \subset Q$.*

- (a) c_{top} is an isomorphism for $m > 2(n - c) + 1$.
- (b) For $\beta \in \text{CH}_k(Q)_{\mathbb{Q}}$, $\text{cl}(\hat{\pi}\beta) = c_{\text{top}}(\text{cl}\beta)$, and likewise with supports.

Proof. (a) Put $E_Q = S \times (Z \times V)/G \xrightarrow{p} Q$ and $j : Q_U \hookrightarrow E_Q$, so $\rho = p \circ j$. Scalar multiplication on V retracts E_Q onto the zero section, so p is a homotopy equivalence, and since E_Q and Q have quotient singularities, $\text{PD}_{E_Q} \circ p^* \circ \text{PD}_Q^{-1}$ is an isomorphism. The complement $E_Q \setminus Q_U$ has complex dimension at most $n + l - c$, so by the localisation sequence j^* is an isomorphism on H_{m+2l}^{BM} for $m + 2l > 2(n + l - c) + 1$.

(b) Take $\beta = [W]$, W closed integral of dimension k . Both $\text{cl}(\hat{\pi}[W]) = e_W[W_U]$, $W_U = \rho^{-1}W$, and $c_{\text{top}}^W(\text{cl}[W])$ lie in the top Borel–Moore group of W_U , which is the vector space on its irreducible components and injects into the same group of any dense open subset [17, §19.1]; so it suffices to compare them near a general point of each component. There, analytically locally, $S = S'/K'$ with K' finite acting effectively and generically freely on a smooth S' , and $\mathcal{R} = [\Omega/H]$ with H finite and Ω smooth [37, proof of Prop. 2.8]. Put $\Omega_U = \Omega \times_{\mathcal{R}} Z_U$, a smooth U_V -fibration over Ω on which H acts freely, $A' = S' \times \Omega$, $A'_U = S' \times \Omega_U$, $\varphi : A' \rightarrow Q$ and $\varphi_U : A'_U \rightarrow Q_U$ the quotients by $K' \times H$, and $\text{pr} : A'_U \rightarrow A'$, smooth of relative dimension l . Let the generic stabiliser orders of the inverse image of W be s in K' and r in H ; then $e_W = r$, the generic stabiliser of $\varphi^{-1}(W)_{\text{red}}$ has order sr , and that of $\varphi_U^{-1}(W_U)_{\text{red}}$ has order s , since H acts freely there. Lemma 10.5 gives $\varphi^![W] = sr[A'_W]$, $\varphi_U^![W_U] = s[A'_{U,W}]$, and $\text{pr}^![A'_W] = [A'_{U,W}]$ since pr is smooth. From $\rho\varphi_U = \varphi\text{pr}$, $\varphi_U^!(c_{\text{top}}^W \text{cl}[W]) = \text{pr}^!\varphi^![W] = sr[A'_{U,W}] = \varphi_U^!(r[W_U])$, and $\varphi_U^!$ is injective over $\mathbb{Q}$. Hence $c_{\text{top}}^W \text{cl}[W] = e_W[W_U] = \text{cl}(\hat{\pi}[W])$. $\square$

Lemma 10.7 (Topological composition). *Let Q_1, Q_2, Q_3 be compact oriented rational homology manifolds, $N = Q_1 \times Q_2 \times Q_3$, and $\gamma \in H_*(Q_1 \times Q_2)$, $\delta \in H_*(Q_2 \times Q_3)$ of even degrees. Put $\gamma \cdot_{Q_2}^{\text{top}} \delta = \text{PD}_N(\text{pr}_{12}^* \text{PD}^{-1} \gamma \cup \text{pr}_{23}^* \text{PD}^{-1} \delta)$ and $\delta \otimes \gamma = p_{13*}(\gamma \cdot_{Q_2}^{\text{top}} \delta)$. Then $(\delta \otimes \gamma)_* = \delta_* \circ \gamma_*$. For a morphism $f : Q_1 \rightarrow Q_2$ with graph class Γ_f , $(\Gamma_f)_* = f_* := \text{PD}_{Q_2}^{-1} f_* \text{PD}_{Q_1}$ and $({}^t\Gamma_f)_* = f^*$, and $({}^t\gamma)_*$ is the adjoint of γ_* for the Poincaré pairings.*

Proof. For $\xi \in H^*(Q_1)$, using $\text{pr}_{12}^* \theta \cap [N] = (\theta \cap [Q_1 \times Q_2]) \times [Q_3]$, the projection formula for pr_{23} and even degrees, $\text{pr}_{23*}(\text{pr}_1^* \xi \cap (\gamma \cdot_{Q_2}^{\text{top}} \delta)) = \text{pr}_2^*(\gamma_* \xi) \cap \delta$; applying pr_{3*} and PD^{-1} gives $\delta_* \gamma_* \xi$. For the graph, $\text{pr}_1^* \xi \cap (1, f)_*[Q_1] = (1, f)_*(\xi \cap [Q_1])$ and $\text{pr}_2 \circ (1, f) = f$; the transpose is similar, and adjointness is the symmetry of $\int \text{pr}_1^* \xi \cup \text{pr}_2^* \eta \cap \gamma$. Only cup and cap products,

Poincaré duality and the projection formula over $\mathbb{Q}$ are used, all valid on compact oriented rational homology manifolds [9, Ch. V]. $\square$

Lemma 10.8 (Bivariant classes and the coarse quotient). *Let A, B be schemes, $\kappa \in \mathrm{CH}^t(A \times R \rightarrow R)_{\mathbb{Q}}$ and $y \in \mathrm{CH}_b(R \times B)_{\mathbb{Q}}$. In the stable range,*

$$\hat{\pi}_{A \times B}(\kappa \cap y) = \rho^* \kappa \cap \hat{\pi}_B(y) \quad \text{in } \mathrm{CH}_{b-t+l}(A \times Z_U \times B)_{\mathbb{Q}}.$$

Proof. Choose the finite surjective diagram of [13, Prop. 10]: $g : Z' \rightarrow Z$ over $f : R' \rightarrow R$, with $Z' \rightarrow R'$ a principal G -bundle; its product with an inert factor is still a diagram of geometric quotients. Write $g_U : Z'_U \rightarrow Z_U$ and $\rho' : Z'_U \rightarrow R'$, so that $\rho g_U = f \rho'$. Choose y' with $(f \times 1_B)_* y' = y$ and put $\kappa' = f^* \kappa$. For the principal bundle the comparison is flat pullback along $\rho' \times 1_B$, so $\hat{\pi}'_{A \times B}(\kappa' \cap y') = \rho'^* \kappa' \cap \hat{\pi}'_B(y')$. Every square used for the bivariant proper or flat axiom is the Cartesian square of $1_A \times h$ over h , with $h = f \times 1_B$, $g_U \times 1_B$ or $\rho' \times 1_B$. With [13, Prop. 11(a)] ($g_* \hat{\pi}' = \hat{\pi}_* f_*$) and the proper axiom,

$$\begin{aligned} \hat{\pi}_{A \times B}(\kappa \cap y) &= (1_A \times g_U \times 1_B)_* \hat{\pi}'_{A \times B}(\kappa' \cap y') \\ &= (1_A \times g_U \times 1_B)_*(\rho'^* \kappa' \cap \hat{\pi}'_B(y')) = \rho^* \kappa \cap \hat{\pi}_B y. \end{aligned}$$

Choose the representation with complement of codimension large enough for the finitely many degrees occurring. $\square$

Proposition 10.9 (Cycle classes and $\circ_R$). *Let Q_1, Q_2 be test spaces, $x \in \mathrm{CH}_a(Q_1 \times R)_{\mathbb{Q}}$ and $y \in \mathrm{CH}_b(R \times Q_2)_{\mathbb{Q}}$. Then $\mathrm{cl}(x \cdot_R y) = \mathrm{cl} x \cdot_R^{\mathrm{top}} \mathrm{cl} y$ in $H_*(Q_1 \times R \times Q_2)$; hence $\mathrm{cl}(y \circ_R x) = \mathrm{cl} y \otimes \mathrm{cl} x$ and $(y \circ_R x)_* = y_* \circ x_*$. The same holds with R replaced by R^m , using the presentation Z^m/G^m .*

Proof. Let $N = Q_1 \times R \times Q_2$, $N_U = Q_1 \times Z_U \times Q_2$ and $\kappa = \mathrm{ev}_R^{-1}(x)$, so that $x \cdot_R y = \kappa \cap y$; take the codimension c large. *Step 1.* By Lemma 10.8, $\hat{\pi}_{Q_1 \times Q_2}(\kappa \cap y) = \rho^* \kappa \cap \hat{\pi}_{Q_2}(y)$, and for $y = [R]$, $\rho^* \kappa \cap [Z_U] = \hat{\pi}_{Q_1}(x)$. *Step 2.* Since Z_U is smooth, Proposition 10.3 gives $\rho^* \kappa \cap z = \Delta_{Z_U}^! (\hat{\pi}_{Q_1}(x) \times z)$, so $\hat{\pi}(x \cdot_R y) = \Delta_{Z_U}^! (\hat{\pi}_{Q_1} x \times \hat{\pi}_{Q_2} y)$. *Step 3.* The cycle class commutes with exterior products and with $\Delta_{Z_U}^!$ [17, §§19.1–19.2], [13, §§2.8, 6.1]. The topological $\Delta^!$ is cap product with the orientation class of the diagonal of the smooth Z_U^2 , pulled back to $Q_1 \times Z_U \times Z_U \times Q_2$, a rational homology manifold; with ι the diagonal inclusion, $u \cap \mathrm{PD}(\theta) = \mathrm{PD}(\iota^* \theta)$ and $\iota^*(\theta_1 \times \theta_2) = \mathrm{pr}_{12}^* \theta_1 \cup \mathrm{pr}_{23}^* \theta_2$, so $\mathrm{cl}(\Delta_{Z_U}^! (\hat{\pi} x \times \hat{\pi} y)) = \mathrm{PD}_{N_U}(\mathrm{pr}_{12}^* \mathrm{PD}^{-1} \mathrm{cl}(\hat{\pi} x) \cup \mathrm{pr}_{23}^* \mathrm{PD}^{-1} \mathrm{cl}(\hat{\pi} y))$. *Step 4.* Lemma 10.6(b) on $Q_1 \times R$, $R \times Q_2$ and N gives $\mathrm{cl}(\hat{\pi}(-)) = c_{\mathrm{top}} \mathrm{cl}(-)$, and naturality of the cup product gives $c_{\mathrm{top}}(\mathrm{cl} x \cdot_R^{\mathrm{top}} \mathrm{cl} y) = \mathrm{PD}_{N_U}(\mathrm{pr}_{12}^* \mathrm{PD}^{-1} c_{\mathrm{top}} \mathrm{cl} x \cup \mathrm{pr}_{23}^* \mathrm{PD}^{-1} c_{\mathrm{top}} \mathrm{cl} y)$. So $c_{\mathrm{top}} \mathrm{cl}(x \cdot_R y) = c_{\mathrm{top}}(\mathrm{cl} x \cdot_R^{\mathrm{top}} \mathrm{cl} y)$, and c_{top} is injective in this degree by Lemma 10.6(a). Finally cl commutes with p_{13*} , and Lemma 10.7 gives the action. For R^m use Z^m , G^m , V^m and U_V^m . $\square$

10.4. The transferred projector.

Lemma 10.10 (Graphs). *The classes $[\Gamma_q]$, $[{}^t\Gamma_q]$ and $[\Gamma_g]$, $g \in A$, act on cohomology by $(\Gamma_q)_* = q_*$, $({}^t\Gamma_q)_* = q^*$ and $(\Gamma_g)_* = g_* = (g^{-1})^*$, all of degree 0, and $q_*q^* = \text{id}_{H^*(R)}$. A class $(i \times i)_*\zeta$ with ζ on $E_* \times E_*$, $E_* \subset R^{\text{sm}}$, acts by the smooth-variety formula on a neighbourhood of E_* .*

Proof. The formulas are Lemma 10.7. For q_*q^* : $q_*(q^*\eta \cap [\overline{X}]) = \eta \cap q_*[\overline{X}] = \eta \cap [R]$, as q has degree one. For the last claim, $\text{PD}^{-1} \text{cl}((i \times i)_*\zeta)$ is supported on $E_* \times E_*$, so only the smooth neighbourhood R^{sm} of E_* enters. $\square$

Proposition 10.11. *Let $p \in \text{CH}_d(R \times R)_{\mathbb{Q}}$.*

- (i) $P_X = {}^t\Gamma_q \circ_R (p \circ_R \Gamma_q)$ is a well-defined class in $\text{CH}_d(\overline{X} \times \overline{X})_{\mathbb{Q}}$, independent of any presentation of R .
- (ii) $(P_X)_* = q^*p_*q_*$ on $H^*(\overline{X})$.
- (iii) If p_* is idempotent on $H^*(R)$, then $(P_X)_*$ is idempotent, with image $q^*\text{im}(p_*) \cong \text{im}(p_*)$.
- (iv) For $n \geq 1$, the same construction for $(R^n, \overline{X}^n, q^n, p^{\times n})$ gives $P_X^{(n)} \in \text{CH}_{nd}(\overline{X}^n \times \overline{X}^n)_{\mathbb{Q}}$ with $(P_X^{(n)})_* = (q^*p_*q_*)^{\otimes n}$. For $W \in \text{CH}_k(R^n)_{\mathbb{Q}}$ put $(q^n)^!W = {}^t\Gamma_{q^n} \circ_{R^n} W \in \text{CH}_k(\overline{X}^n)_{\mathbb{Q}}$. Then

$$\text{cl}_H((q^n)^!W) = (q^n)^*\text{cl}_H(W), \quad \text{cl}_H((P_X^{(n)})_*((q^n)^!W)) = (q^n)^*p_*^{\otimes n}\text{cl}_H(W).$$

Proof. (i) $p \circ_R \Gamma_q \in \text{CH}_d(\overline{X} \times R)_{\mathbb{Q}}$ and then $P_X \in \text{CH}_d(\overline{X} \times \overline{X})_{\mathbb{Q}}$; ev_R and $\cap$ are intrinsic to R , and presentations enter only the proof of Proposition 10.9. (ii) Proposition 10.9 twice and Lemma 10.10. (iii) $(q^*p_*q_*)^2 = q^*p_*(q_*q^*)p_*q_* = q^*p_*q_*$; the image is $q^*\text{im}(p_*)$ because q_* is onto, and q^* is injective. (iv) R^n has quotient singularities with smooth locus $(R^{\text{sm}})^n$, and q^n is a projective resolution that is an isomorphism over it. Γ_{q^n} is the reordered exterior product of copies of Γ_q , and cross products are compatible with cl , $\cap$, PD and pushforward, so under Künneth $(q^n)_* = q_*^{\otimes n}$, $(q^n)^* = (q^*)^{\otimes n}$ and $\text{cl}(p^{\times n})_* = p_*^{\otimes n}$. Proposition 10.9 with $Q_1 = \text{pt}$ gives $\text{cl}_H((q^n)^!W) = (q^n)^*\text{cl}_H(W)$, and the last identity follows from (ii) and $(q^n)_*(q^n)^* = 1$. $\square$

In this paper $p = \varepsilon P_*$. As a correspondence, ε is a $\mathbb{Q}$ -combination (coefficients $\pm \frac{1}{16}$ and 0) of graphs of the automorphisms of R given by the elements of $\tilde{A}(1) \times \tilde{A}(2)$, and $P_* = [\Delta_R] - \Pi_E$ with Π_E built from classes on $E_* \times E_*$. By Corollary 7.8, $(\varepsilon P_*)_*$ is idempotent on $H^*(R)$ with image M_x , so Proposition 10.11(iii) gives a correspondence P_X on $\overline{X}$ acting as an idempotent of rank seven with image $q^*M_x \cong M_x$.

Example 10.12 (The graph convention on $\mathbb{P}(1, 1, 2)$). Let $\overline{R} = \mathbb{P}(1, 1, 2)$, with one A_1 singularity, resolved by the Hirzebruch surface $\overline{X}_0 = \mathbb{F}_2 \xrightarrow{q} \overline{R}$, which contracts the (-2) -section C and sends a fibre f to a line ℓ through the vertex, with $\ell \cdot \ell = \frac{1}{2}$. In the basis (f, C) , with $f^2 = 0$, $f \cdot C = 1$ and $C^2 = -2$, the operational pullback is $q^![\ell] = f + \frac{1}{2}C$, and the cohomological

pullback is the class Θ with $\Theta \cdot f = \langle \ell, q_* f \rangle = \frac{1}{2}$ and $\Theta \cdot C = 0$, also $f + \frac{1}{2}C$. The composite ${}^t\Gamma_q \circ_{\overline{R}} \Gamma_q$ acts by $f \mapsto f + \frac{1}{2}C$ and $C \mapsto 0$ on H_2 , and as the identity on H_0 and H_4 ; since $\mathbb{F}_2 \times \mathbb{F}_2$ has a cellular Chow basis, this gives

$${}^t\Gamma_q \circ_{\overline{R}} \Gamma_q = [\Delta_{\overline{X}_0}] + \frac{1}{2}[C \times C] \quad \text{in } \text{CH}_2(\overline{X}_0 \times \overline{X}_0)_{\mathbb{Q}}.$$

With $E_0 = [C \times C]$, $E_0 \circ E_0 = (C \cdot C)E_0 = -2E_0$, so $(\Delta + \mu E_0)^{\circ 2} = \Delta + (2\mu - 2\mu^2)E_0$, a Chow idempotent at $\mu = \frac{1}{2}$. The naive fundamental cycle $\Delta + E_0$ of the reduced fibre product $\overline{X}_0 \times_{\overline{R}} \overline{X}_0 = \Delta \cup (C \times C)$ acts by $C \mapsto -C$ and is not idempotent; the coefficient $\frac{1}{2}$ is the rational multiplicity supplied by the bivariant composite.

Remark 10.13 (Idempotence in the Chow group). Proposition 10.11(iii) is a statement in cohomology, and Theorem B uses nothing more. Idempotence $P_X \circ P_X = P_X$ in $\text{CH}_d(\overline{X} \times \overline{X})_{\mathbb{Q}}$ would follow from associativity of mixed compositions through $\overline{X}$ and R (the argument of [17, Prop. 16.1.1(a)] uses only the formal properties that $\cdot_R$ has, but we have not written it out), from $\Gamma_q \circ_{\overline{X}} {}^t\Gamma_q = [\Delta_R]$ (which holds: the intersection is transverse and q has degree one), and from $p \circ_R p = p$ in the Chow group of $R \times R$ (which holds for the group-algebra part, and for $P_{\star}$ if Π_E is equivariant and of Gram form on the cellular $E_{\star} \times E_{\star}$). We do not claim it.

11. PROOF OF THEOREM B

Let $\overline{X}_x = \overline{X}$ and $q : \overline{X} \rightarrow R$ be as in Remark 10.2, for the weighted model of Theorem 6.14, which exists for every $x \in S$ by Proposition 8.7; let $p = \varepsilon P_{\star}$ and let P_X be the correspondence of Proposition 10.11, which acts on $H^*(\overline{X})$ as an idempotent of rank seven with image $q^* M_x$.

Proof of Theorem B. (1) For $u \in (V_7^{\otimes n})^{G_2}$ let W_u be the cycle of Proposition 3.5 and $\overline{W}_u$ its closure in R^n . By Proposition 9.2, $(\varepsilon P_{\star})^{\otimes n} \text{cl}_H(\overline{W}_u) = 2^{-n-1} \tau_u(-3n)$. Consider the cycle

$$\mathcal{Z}_u = (P_X^{(n)})_*((q^n)^![\overline{W}_u]) \in \text{CH}_*(\overline{X}^n)_{\mathbb{Q}}.$$

By Proposition 10.11(iv), $\text{cl}_H(\mathcal{Z}_u) = (q^n)^* p_{\star}^{\otimes n} \text{cl}_H(\overline{W}_u) = 2^{-n-1} (q^n)^* \tau_u(-3n)$, and $(q^n)^*$ is injective. So $2^{n+1} \mathcal{Z}_u$ is an algebraic cycle on $\overline{X}^n$ whose class is the transported tensor $(q^n)^* \tau_u(-3n)$; for the cubic the multiplier is 16. By Lemma A.8 and Proposition 3.7, τ_u is Yun's transported tensor, and it is nonzero for $u \neq 0$.

(2) The pairing $b = \tau_b$ and the cubic $t = \tau_t$ are algebraic Hodge classes by (1), so $\text{MT}(N_x) \subseteq \text{Aut}(b, t)$, which is a $\mathbb{Q}$ -form of G_2 (Lemma A.10). The local system N_S underlies a polarisable variation of Hodge structure of weight 0 with fibres the Hodge structures N_x . Indeed, it is the ε -part of $R^6 \tilde{\pi}_! \mathbb{Q}(3)$ for the algebraic family $\tilde{\pi}$ over the smooth curve S , and this sheaf is locally constant (Lemma A.6). The locally constant cohomology sheaves of a geometric direct image with compact supports over a smooth curve carry a canonical admissible graded-polarisable variation of mixed Hodge structure

[32, §4]. The idempotent ε is induced by automorphisms of the family, so admissibility and graded polarisability pass to its image. The fibres N_x are pure of weight 0 by Lemma 2.2, so the weight filtration has only Gr_0^W , and the variation is polarisable. At a Hodge-generic x , the identity component of the algebraic monodromy group is a normal subgroup of the derived Mumford–Tate group [2, §5, Thm. 1, p. 10], which applies here because the variation is pure and polarisable, the case in which its proof rests on the theorem of the fixed part of Griffiths and Schmid; in particular this group is contained in $\mathrm{MT}(N_x)$. By Lemma A.9 it is $\mathrm{Aut}(b, t)$, which is connected; so $\mathrm{MT}(N_x) = \mathrm{Aut}(b, t)$, and the Hodge classes in $T^{a,b}(N_x)$ are exactly the G_2 -invariants. Every rational Hodge class in $T^{a,b}(N_x)$ is obtained from these invariants by operations that preserve algebraic classes:

- duals, through the pairing and its coevaluation, which are invariants with $n = 2$;
- contractions, which are compositions with algebraic correspondences;
- Schur summands (symmetric, exterior and the others), cut out by rational permutation idempotents, that is, rational combinations of graphs of permutations of factors;
- external products and rational linear combinations.

The twists are those forced by weight. So all of them are algebraic by (1). $\square$

Remark 11.1 (Special parameters). At special x the Mumford–Tate group may be a proper subgroup of G_2 , with more invariant tensors; such points form a countable set. The construction gives algebraicity of the transported G_2 -invariants there, but not of the additional invariants. We do not know whether such points occur.

Remark 11.2 (The family of Dettweiler and Reiter). Yun’s local system is the same as that of Dettweiler and Reiter [41, Thm. 6.1.5 and the paragraph after Thm. 6.2.3]. The corresponding statement for their family therefore reduces to the algebraicity of the Hodge class in $\mathrm{Hom}(M_x, M_x^{\mathrm{DR}})$ identifying the two motives, which we do not address.

Remark 11.3 (Other rigid data). The method uses rigidity of the Hecke eigensheaf: the odd sheaf is a representation of the finite group $\tilde{A}$ on the open point of a zero-dimensional moduli stack, defined without reference to the local system, and the Hecke operators produce M_x from it. Yun’s other rigid data (types A_1 , D_{2n} , E_7 , E_8) have the same shape. For D_4 , E_7 and E_8 the analogue of Corollary 7.3 needs a meridian-by-meridian analysis, since not every nontrivial subgroup of $\tilde{A}$ contains the central element.

APPENDIX A. COMPARISON WITH YUN’S ℓ -ADIC OBJECTS

Yun’s statements are ℓ -adic. This appendix compares the objects and maps of Section 3 with his, for bounded finite-type diagrams only; no equivalence of analytic and adic categories is used. The outcome is: cleanness

holds for the rational Betti sheaves (Lemma A.3); N_x is the fibre of a rational local system N_S with $N_S \otimes \mathbb{Q}_\ell \cong E_\chi(V_7)$ (Lemma A.6); the rational tensor maps are Yun's (Lemma A.7); and τ_t is the invariant alternating cubic, with stabiliser G_2 (Lemma A.10).

Lemma A.1 (Bounded rational comparison). *Let $\mathcal{D}$ be a finite diagram of separated finite-type complex varieties with finite-group actions, carrying input coefficient systems with finite monodromy and forms over $A = \mathbb{Z}[1/M]$, and let K_A be a bounded complex built from them by pullback, derived tensor product, Rf_* , $Rf_!$, finite invariants and the base-change, projection-formula, Künneth and adjunction maps. Enlarge M to contain the group orders, the projector denominators and the torsion primes of the relevant constructible cohomology in all neighbouring degrees, and fix a prime $\ell \nmid M$ for this diagram. Then the canonical comparison transformations identify the derived reductions $K_A \otimes^L \mathbb{Z}/\ell^m$ with the analytifications of the corresponding étale complexes, compatibly in m ; after passing to adic systems and inverting ℓ , $(K_A \otimes \mathbb{Q}) \otimes \mathbb{Q}_\ell$ is the analytification of the rational adic complex, and the specified maps correspond.*

Proof. For constructible torsion sheaves and finite-type morphisms the comparison of $R^q f_*$ is [29, Exp. XVI, Th. 4.1] (see [30, Arcata, Th. 5.3.6]); the proper-support comparison is constructed by compactification and extension by zero [29, Exp. XVII, 5.3], [30, Arcata, Th. 4.6.3]. Bounded complexes follow by truncation triangles. The compatibilities concern the canonical comparison transformations, of which the projection formula and Künneth maps are composites; finite-group equivariance follows from naturality, with averaging after the group order is inverted. For a closed immersion i with open complement j , $i_* i^! K$ is determined by the triangle $i_* i^! K \rightarrow K \rightarrow Rj_* j^* K$, so supported morphisms and their counits are compared by the same maps. $\square$

Lemma A.2 (Cycle classes). *For a smooth variety and an algebraic cycle, the rational Betti and ℓ -adic cycle classes correspond under Lemma A.1, with $\mathbb{Z}(1) = 2\pi i\mathbb{Z}$. On a singular variety the statement is made for supported fundamental classes in Borel–Moore homology.*

Proof. The comparison matches Kummer first Chern classes, and the projective bundle formula then matches all Chern classes; the leading term of the Chern character of a locally free resolution of $\mathcal{O}_Z$ is $[Z]$ [23, §23, Thms. 23.3–23.4]. The supported form is Lemma A.1 applied to the localisation triangle. $\square$

Lemma A.3 (Rational cleanness). *Let $j : U' \hookrightarrow Y'$ be an open immersion of finite-type varieties and L_A a finite-monodromy lattice over A . If $j_! L_\ell \rightarrow Rj_* L_\ell$ is an isomorphism for the rational adic sheaf, then the Betti map $j_! L_B \rightarrow Rj_* L_B$ is an isomorphism. In particular $j_! \mathbb{L} \cong Rj_* \mathbb{L}$ on Y , and Theorem 2.1 holds for the Betti sheaves on every finite-type smooth chart of Bun .*

Proof. Let C_A be the cone of the integral Betti map; its stalk cohomology is bounded and finitely generated. By Lemma A.1, $C_A \otimes \mathbb{Q}_\ell$ is the analytification of the rational adic cone, which vanishes; so $C_A \otimes \mathbb{Q} = 0$ by faithful scalar extension, stalk by stalk. For Y : ω is smooth, so smooth base change of Yun's clean extension gives the adic isomorphism, and the odd coefficient is, after scalar extension, a sum of copies of the split odd coefficient. On a smooth chart of Bun the same argument applies. $\square$

Lemma A.4 (Bounded diagrams). *Fix $n \geq 1$ and let $\mathcal{D}_{n,S}$ be the finite diagram of separated finite-type S -schemes with finite-group actions obtained by base change along $u_0 \times \text{id}_S$: the chain spaces $C_{k,S}$ and convolution maps $m_{k,S}$ for $k \leq n$ with their composition squares, the opens $C_{k,S}^{(U)}$ and $C_{k,S}^{U^k}$, the framing covers, and the identity-composite fibres. Then:*

- (i) *each Hecke output of [40] used in Section 3, $(u_0 \times \text{id}_S)^*T(\text{IC}_\lambda^{*k}, F_0 \boxtimes \mathbb{Q}_S)$ for $k \leq n$, and the partial transforms of its chains, is canonically isomorphic to a complex built on $\mathcal{D}_{n,S}$ from constant sheaves and the finite-monodromy coefficient $(V_\chi \otimes \nu_* \mathbb{Q}_\ell)^{\tilde{A}(1)}$, ν the framing cover, by the operations of Lemma A.1, the only shifts and twists being $[6k](3k)$;*
- (ii) *under these isomorphisms the base-change and extraordinary maps of [40, (3.3), (3.7)], the units and counits that enter, and the monoidal isomorphisms of [40, §3.1.3] become the corresponding operations on $\mathcal{D}_{n,S}$;*
- (iii) *Lemma A.1 applies to $\mathcal{D}_{n,S}$ and to each of these maps.*

Proof. Stack-level inputs. $u_0 \times \text{id}_S$ is étale, and smooth base change is [21, I, §5.1]. The bounded Hecke stack is representable and proper over $\text{Bun} \times S$, with fibres the projective Schubert varieties [40, §3.2.3]; for proper representable morphisms $Rf_! \cong Rf_*$, and base change for $Rf_!$ holds for every Cartesian square of stacks [21, I, §§5.2, 5.5], with its adic form [21, II, §9].

Satake input. The group $H_\infty = L^+G \rtimes \text{Aut}_\mathcal{O} = \varprojlim H_k$ acts nicely on Gr in the sense of [18, Appendix, A.3]: every closed subscheme lies in an H_∞ -stable closed subscheme on which H_∞ acts through some H_k [18, §§1.1, 2.1.2–2.1.3]. Every space of $\mathcal{D}_{n,S}$ arises from the finitely many pieces $\text{Gr}_{\leq \mu}$, $\mu \leq n\lambda$, by finitely many associated-bundle steps, and the opens, framing covers and fibres add no new groups; so one congruence level N serves the whole diagram, and enlarging N changes nothing, the kernels being unipotent [18, Appendix, A.4]. The twisted products $C_{k,S}$ are then projective S -schemes built from finite-level torsors [18, §§2.1.2–2.1.3, 4.1]. For $\mathcal{K} = \text{IC}_\lambda^{*k} = m_{k*} \mathbb{Q}_{C_k}[6k](3k)$ the spread complex in Yun's convention [40, §§3.2.3, 3.2.6] is $Rm_{k,S*} \mathbb{Q}_{C_{k,S}}[6k](3k)$; Gaitsgory's perverse spread is this shifted by [1], the perverse degree one of [40, Prop. 3.4.1], and the extra shift is not inserted into the Hecke functor. The complexes and maps descend along the smooth finite-level torsors as equivariant geometric pull-push constructions. The coefficient is $\omega^* j_! \mathbb{F}_\chi = j'_!(V_\chi \otimes \nu_* \mathbb{Q}_\ell)^{\tilde{A}(1)}$ [40, §3.2.4].

The monoidal structure. Write $\mathcal{Y} = \text{Bun} \times S$ and a, b for source and target. For kernels $\mathcal{K}_1$ (adjacent to the target) and $\mathcal{K}_2$, put $H_{12} = H_1 \times_{a_1, \mathcal{Y}, b_2} H_2$ with the proper map $c : H_{12} \rightarrow H_{\leq n\lambda}$, projections $r_i, p = a_2 r_2, q = b_1 r_1$ and kernel $J = r_1^* I_1 \otimes r_2^* I_2$. The convolution construction identifies $Rc_! J$ with the kernel of $\mathcal{K}_1 * \mathcal{K}_2$, and base change and the projection formula give

$$T(\mathcal{K}_1, T(\mathcal{K}_2, F)) \simeq Rq_!(p^* F \otimes J) \simeq Rb_!(a^* F \otimes Rc_! J) = T(\mathcal{K}_1 * \mathcal{K}_2, F).$$

This is the monoidal structure of [40, §3.1.3], with its order and normalisation specified; associativity and the unit follow from the same identities on longer chains. Pulling the target back to $u_0 \times S$ turns H_{12} into $C_{2,S}$ and c into $m_{2,S}$, and likewise for longer chains. With naturality of base change and localisation this gives (ii), and (iii) is Lemma A.1 applied stage by stage. $\square$

Lemma A.5 (The evaluation maps). *Put $\mathcal{Y} = \text{Bun} \times S$, $r = (u_0 \times \text{id}_S)^*$, $F_0 = j_! \mathbb{F}_\chi$ and $H_\mathcal{K} = T(\mathcal{K}, F_0 \boxtimes \mathbb{Q}_S)$. For odd H, H' on $\mathcal{Y}$, restriction is an isomorphism $\text{Hom}_{D^b(\mathcal{Y})_{\text{odd}}}(H, H') \rightarrow \text{Hom}_{D_{\tilde{A}}^b(S)_{\text{odd}}}(rH, rH')$. With $E_\mathcal{K} = (V_\chi^* \otimes rH_\mathcal{K})^{\tilde{A}}$, the canonical evaluation $F_0 \boxtimes E_\mathcal{K} \rightarrow H_\mathcal{K}$ is the unique odd morphism restricting to the equivariant evaluation, and Yun's tensor map $\varphi_{\mathcal{K}, \mathcal{L}}$ [40, (3.19)] is the unique map making the square*

$$\begin{array}{ccc} F_0 \boxtimes (E_\mathcal{K} \otimes E_\mathcal{L}) & \xrightarrow{\text{ev}_\mathcal{K} \otimes 1} & H_\mathcal{K} \otimes \text{pr}_S^* E_\mathcal{L} \\ \downarrow 1 \boxtimes \varphi_{\mathcal{K}, \mathcal{L}} & & \downarrow \rho \\ F_0 \boxtimes E_{\mathcal{K} * \mathcal{L}} & \xrightarrow{\text{ev}_{\mathcal{K} * \mathcal{L}}} & H_{\mathcal{K} * \mathcal{L}} \end{array}$$

commute, where ρ is the projection formula followed by $T(\mathcal{K}, \text{ev}_\mathcal{L})$ and the monoidal isomorphism of Lemma A.4.

Proof. The isomorphism of Homs is Yun's relative cleanness equivalence [40, Thm. 2.4.2, §2.4.6], the right side retaining the $\tilde{A}$ -equivariant structure. [40, Prop. 3.4.1] verifies the perverse-degree hypothesis of [40, Lemma 2.4.8], which supplies the evaluations. After restriction, applying $(V_\chi^* \otimes -)^{\tilde{A}}$ to the right-hand composite is the definition in [40, (3.18)–(3.19)], and equivariant full faithfulness gives the square and its uniqueness. Full faithfulness is used only for these odd Hecke outputs; the supported counit, the Satake associator and the maps on the framing covers are compared by base change and localisation on their own spaces (Lemma A.4). $\square$

A rescaling $\varphi'_{\mathcal{K}, \mathcal{L}} = a_{\mathcal{K} * \mathcal{L}} \varphi_{\mathcal{K}, \mathcal{L}} (a_\mathcal{K}^{-1} \otimes a_\mathcal{L}^{-1})$ by a natural automorphism acting by c on the rank-seven simple would multiply the cubic by c^3 .

Lemma A.6 (The rank-seven local system). *Let $\tilde{\pi} : \tilde{Y} \rightarrow S$ be Yun's framed one-step family [40, §3.2.7] and $N_S = \varepsilon R^6 \tilde{\pi}_! \mathbb{Q}(3)$. Then N_S is a rank-seven $\mathbb{Q}$ -local system on S , its fibre at x is N_x , and $N_S \otimes \mathbb{Q}_\ell \cong E_\chi(V_7)$ as local systems on S^{an} , along $\pi_1^{\text{top}}(S) \rightarrow \pi_1^{\text{et}}(S_\mathbb{C})$.*

Proof. Apply Lemma A.1 to $\varepsilon R \tilde{\pi}_! \mathbb{Q}(3)$, with its form over A . By [40, (3.13)–(3.14)] and the identity-line projector [40, §3.3.2], the comparison identifies

$\mathcal{H}^6 \otimes \mathbb{Q}_\ell$ with $E_\chi(V_7)$, relatively over S ; so $N_S = \mathcal{H}^6$ is a rank-seven local system, its specialisation maps being isomorphisms by constructibility and faithful scalar extension. The full odd degree-six cohomology has rank 28, and ε cuts out rank seven. $\square$

Lemma A.7 (The tensor maps). *For the bounded relative Hecke diagrams over S , let $E_B(\mathcal{K})$ be constructed with the rational odd bimodule D , its multiplicity functor and the canonical evaluation, and let $c_{\mathcal{K}}$ be its comparison with Yun's $E_\ell(\mathcal{K})$. Then $c_{\mathcal{K}_1 * \mathcal{K}_2} \circ (\varphi_B \otimes 1) = \varphi_\ell \circ (c_{\mathcal{K}_1} \otimes c_{\mathcal{K}_2})$, where φ_B is defined by the projection formula, the evaluation and convolution base change, in target-first order; likewise for the unit and associativity. So the rational maps are unital and associative isomorphisms, and τ_u formed with φ_B becomes Yun's after $\otimes \mathbb{Q}_\ell$.*

Proof. By Lemmas A.4 and A.5, Yun's $E_\ell(\mathcal{K})$, its evaluation and φ_ℓ are specified by the evaluation square, whose restriction along $u_0 \times \text{id}_S$ is the stated construction on $\mathcal{D}_{n,S}$. The two paths are the same composite of evaluation, projection formula, base change, finite invariants and Künneth maps, and the comparison respects each (Lemma A.1). $\square$

Lemma A.8 (Flatness and nonvanishing). *For $(u, n) = (b, 2), (t, 3)$ the unique top component of the relative identity-composite fibre over S defines a relative supported morphism $s_{u,S}$ from the unit, and $\tau_u = \iota^{\otimes n} \varphi_{n,B}^{-1} E_B(s_{u,S})(1)$ is a nonzero flat section of $N_S^{\otimes n}$, with fibre the τ_u of Proposition 3.5. The degree-six map from N_S to ordinary fibre cohomology, and its tensor powers, are injective.*

Proof. A coordinate near the moving point and a local trivialisation of the fixed target bundle give local convolution models; the unique top component for $n = 2, 3$ is preserved on overlaps, so its reduced fundamental class defines a global relative supported morphism, and the endpoint sheet glues, the source being canonically the target on the identity-composite fibre. Every map in the expression for τ_u is a morphism of local systems over S (Lemma A.7), so τ_u is flat, and it is nonzero by Proposition 3.7. Injectivity: in the proper Schubert compactification with the interior base point, clean extension across D_x identifies compact and ordinary cohomology of the odd block, and removing the base-point orbits changes neither in degree six (Remark 3.9); the same maps exist relatively over S . $\square$

Lemma A.9 (Dense monodromy). *The monodromy of the $\mathbb{Q}$ -local system N_S is Zariski dense in G_2 .*

Proof. Yun's rational Betti construction [40, §4.5.1] has cohomological realisation through [40, (3.14)] and dense topological monodromy [40, §4.5.2]; see also [41, §6.2.4]. Through ε and the Morita identification this is the local system N_S of Lemma A.6. $\square$

Lemma A.10 (Alternation). τ_b spans the invariant symmetric pairing line of $N_S^{\otimes 2}$ and τ_t spans the invariant alternating cubic line of $N_S^{\otimes 3}$; $\text{Stab}(\tau_b, \tau_t)$ is a $\mathbb{Q}$ -form of G_2 .

Proof. By Lemmas A.8 and A.9, τ_u is a nonzero G_2 -fixed vector. $\text{Sym}^2 V_7 = \mathbf{1} \oplus V_{27}$ and $\Lambda^2 V_7 = V_7 \oplus V_{14}$ show that the invariant pairing is symmetric and that the invariant cubic is antisymmetric in the first two factors. Since $\dim(V_7^{\otimes 3})^{G_2} = 1$, the permutation action on the cubic line is a character of S_3 , and all transpositions are conjugate, so it is the sign character. All degrees are even, so there are no Koszul signs. The alternating cubic together with the nondegenerate invariant quadratic form has stabiliser G_2 . $\square$

APPENDIX B. THE FINITE COMPUTATIONS

All computations are exact over $\mathbb{Q}$ (and $\mathbb{Q}(i)$ where stated) in SageMath 10.7 [27], from the formulas of Section 8.

Bundle types (Lemma 8.1). For each $\mu = (m_1, m_2)$ with $|m_1| \leq 5$, $|m_2| \leq 8$ and each $w \in W$ (2,244 elements in all), compute N_α from (1) and $\dim H_{\bar{w}} = 2 + \sum_\alpha N_\alpha$; keep those with $j \leq 6$ (92 elements), and group them into double cosets for the stabilisers of $-\rho^\vee/2$ on both sides, using a canonical representative. There are exactly seven classes, with the representatives and dimensions of Lemma 8.1, and B_6 and C_6 lie in distinct classes.

Orbits and stabilisers (Propositions 8.2, 8.5). In a Chevalley basis of $\mathfrak{g}_2$, the Lie algebra of $\text{ev}_1(H_{\bar{w}})$ is $\mathfrak{t} \oplus \bigoplus_{N_\alpha \geq 1} \mathfrak{g}_\alpha$; the stabiliser of a flag gB is the intersection with $\text{Ad}(g)\mathfrak{b}$, computed exactly. For the base-point type, the four classes of θ -stable Cartan subalgebras are constructed by Cayley transforms ($\theta = \text{Ad}(s)$, $s \in T$), and for each the group $W_K(H)$ is realised element by element by explicit $k = C_S n_w D(y) C_S^{-1} \in K$ over $\mathbb{Q}$ or $\mathbb{Q}(i)$, checked by $\theta(k) = k$, $\text{Ad}(k)\mathfrak{h}_S = \mathfrak{h}_S$ and the permutation of the twelve root lines; since $K = G_2^\theta$ is connected, this places k in K . This gives $|W_K(H)| = 4, 4, 4, 12$ and the counts 3, 3, 3, 1. The same code gives the known counts 3 for SL_2/SO_2 and 6 for $SL_3/S(GL_2 \times GL_1)$.

Weights (Proposition 8.7). For each of the 108 candidate representatives (T -fixed flags vB , and flags $u_\gamma(1)vB$ with $\gamma \notin v\Phi^+$ for the types $\star, B_1, B_3, B_4$), compute c , the torus of the stabiliser, the weights on T^1 and on the stabiliser modulo its torus, and search for an integral cocharacter η satisfying (H1) and (H2). The 57 representatives with $1 \leq c \leq 6$ are listed in Table 2; they cover the 46 orbits with $1 \leq c \leq 6$ (several T -fixed flags represent one orbit, for instance twelve flags over $\star$ represent its three closed orbits). On every row $\dim T^1 = c$, and 2η lifts to the torus of the double cover with both inequalities preserved (Proposition 8.4). For the rank-one rows the stabiliser torus is $(\ker \gamma)^\circ$, and η lies in its cocharacter lattice.

Table 2: Contracting cocharacters $\eta = m_1\alpha_1^\vee + m_2\alpha_2^\vee$, written (m_1, m_2) .
 “Point”: T for the T -fixed flag vB , or a root $\gamma = (a, b) = a\alpha_1 + b\alpha_2$ for the flag $u_\gamma(1)vB$. The last two columns are the weights of η on T^1 and on the stabiliser Lie algebra modulo its torus.

type	point	v	c	η	T^1 -weights	stabiliser weights
*	T	e	4	$(-3, -5)$	$(1, 1, 3, 5)$	$(-4, -2)$
*	T	s_1	4	$(-1, -3)$	$(1, 1, 3, 3)$	$(-2, 0)$
*	T	s_2	4	$(-1, -1)$	$(1, 1, 1, 1)$	$(-2, 0)$
*	T	$s_1 s_2$	4	$(-1, -3)$	$(1, 1, 3, 3)$	$(-2, 0)$
*	T	$s_2 s_1$	4	$(-1, -1)$	$(1, 1, 1, 1)$	$(-2, 0)$
*	T	$s_1 s_2 s_1$	4	$(1, -1)$	$(1, 1, 3, 5)$	$(-4, -2)$
*	T	$s_2 s_1 s_2$	4	$(-1, 1)$	$(1, 1, 3, 5)$	$(-4, -2)$
*	T	$s_1 s_2 s_1 s_2$	4	$(1, 1)$	$(1, 1, 1, 1)$	$(-2, 0)$
*	T	$s_2 s_1 s_2 s_1$	4	$(1, 3)$	$(1, 1, 3, 3)$	$(-2, 0)$
*	T	$s_1 s_2 s_1 s_2 s_1$	4	$(1, 1)$	$(1, 1, 1, 1)$	$(-2, 0)$
*	T	$s_2 s_1 s_2 s_1 s_2$	4	$(1, 3)$	$(1, 1, 3, 3)$	$(-2, 0)$
*	T	$s_2 s_1 s_2 s_1 s_2 s_1$	4	$(3, 5)$	$(1, 1, 3, 5)$	$(-4, -2)$
*	$(-2, -1)$	e	1	$(0, -1)$	(2)	—
*	$(-3, -2)$	e	1	$(-1, 0)$	(2)	—
*	$(-2, -1)$	s_1	2	$(0, -1)$	$(1, 2)$	(-1)
*	$(0, -1)$	s_1	2	$(-2, -3)$	$(2, 3)$	(-3)
*	$(-1, 0)$	e	3	$(-1, -2)$	$(1, 1, 2)$	$(-1, -1)$
*	$(0, -1)$	e	3	$(-2, -3)$	$(1, 2, 3)$	$(-3, -1)$
B_1	T	s_2	6	$(-2, -3)$	$(1, 1, 2, 3, 3, 3)$	$(-3, -3, -2, -1)$
B_1	T	$s_1 s_2$	6	$(-1, -3)$	$(1, 1, 2, 3, 3, 3)$	$(-3, -3, -2, -1)$
B_1	T	$s_2 s_1$	5	$(-1, -1)$	$(1, 1, 1, 1, 2)$	$(-2, -1, -1)$
B_1	T	$s_1 s_2 s_1$	5	$(0, -1)$	$(1, 1, 1, 1, 2)$	$(-2, -1, -1)$
B_1	T	$s_2 s_1 s_2$	4	$(-1, -1)$	$(1, 1, 1, 2)$	$(-2, -1)$
B_1	T	$s_1 s_2 s_1 s_2$	4	$(0, -1)$	$(1, 1, 1, 2)$	$(-2, -1)$
B_1	T	$s_2 s_1 s_2 s_1$	3	$(-1, -1)$	$(1, 1, 2)$	(-2)
B_1	T	$s_1 s_2 s_1 s_2 s_1$	3	$(0, -1)$	$(1, 1, 2)$	(-2)
B_1	T	$s_2 s_1 s_2 s_1 s_2$	2	$(-1, -1)$	$(1, 1)$	—
B_1	T	$s_2 s_1 s_2 s_1 s_2 s_1$	2	$(0, -1)$	$(1, 1)$	—
B_1	$(-1, 0)$	e	6	$(-1, -2)$	$(1, 1, 1, 1, 2, 2)$	$(-2, -1, -1, -1, -1)$
B_1	$(-1, 0)$	s_2	5	$(-1, -2)$	$(1, 1, 1, 2, 2)$	$(-2, -1, -1, -1)$
B_1	$(-1, 0)$	$s_2 s_1$	4	$(-1, -2)$	$(1, 1, 2, 2)$	$(-2, -1, -1)$
B_1	$(-1, 0)$	$s_2 s_1 s_2$	3	$(-1, -2)$	$(1, 1, 2)$	$(-1, -1)$
B_1	$(-1, 0)$	$s_2 s_1 s_2 s_1$	2	$(-1, -2)$	$(1, 2)$	(-1)
B_1	$(-1, 0)$	$s_2 s_1 s_2 s_1 s_2$	1	$(-1, -2)$	(2)	—
B_3	T	e	6	$(0, -1)$	$(1, 1, 1, 1, 2, 2)$	$(-2, -2, -1, -1)$
B_3	T	s_2	5	$(0, -1)$	$(1, 1, 1, 1, 2)$	$(-2, -1, -1)$
B_3	T	$s_2 s_1$	4	$(0, -1)$	$(1, 1, 1, 2)$	$(-2, -1)$
B_3	T	$s_2 s_1 s_2$	4	$(1, 1)$	$(1, 1, 1, 2)$	$(-2, -1)$
B_3	T	$s_2 s_1 s_2 s_1$	5	$(1, 1)$	$(1, 1, 1, 1, 2)$	$(-2, -1, -1)$
B_3	T	$s_2 s_1 s_2 s_1 s_2$	6	$(1, 1)$	$(1, 1, 1, 1, 2, 2)$	$(-2, -2, -1, -1)$
B_3	$(-3, -2)$	e	5	$(1, 0)$	$(1, 2, 3, 3, 3)$	$(-3, -3, -3, -1)$
B_3	$(-3, -2)$	s_1	6	$(1, 0)$	$(1, 2, 2, 3, 3, 3)$	$(-3, -3, -3, -2, -1)$
B_3	$(-3, -2)$	s_2	4	$(1, 0)$	$(1, 2, 3, 3)$	$(-3, -3, -1)$
B_3	$(-3, -2)$	$s_2 s_1$	3	$(1, 0)$	$(2, 3, 3)$	$(-3, -3)$
B_4	T	s_1	6	$(-1, 1)$	$(1, 1, 3, 4, 5, 5)$	$(-5, -4, -4, -2)$
B_4	T	$s_1 s_2$	5	$(-1, 1)$	$(1, 1, 3, 5, 5)$	$(-5, -4, -2)$

type	point	v	c	η	T^1 -weights	stabiliser weights
B_4	T	$s_1 s_2 s_1$	5	(1, 3)	(1, 1, 3, 3, 3)	(-3, -2, 0)
B_4	T	$s_1 s_2 s_1 s_2$	6	(1, 3)	(1, 1, 3, 3, 3, 3)	(-3, -3, -2, 0)
B_4	(-2, -1)	e	6	(0, 1)	(1, 1, 1, 1, 2, 2)	(-2, -1, -1, -1, -1)
B_4	(-2, -1)	s_1	5	(0, 1)	(1, 1, 1, 2, 2)	(-2, -1, -1, -1)
B_4	(-2, -1)	$s_1 s_2$	4	(0, 1)	(1, 1, 2, 2)	(-2, -1, -1)
B_5	T	e	6	(1, 1)	(1, 1, 1, 1, 1, 2)	(-2, -2, -1, 0)
B_5	T	s_2	5	(1, 1)	(1, 1, 1, 1, 2)	(-2, -2, 0)
B_5	T	$s_2 s_1$	5	(1, 1)	(1, 1, 1, 1, 2)	(-2, -2, 0)
B_5	T	$s_2 s_1 s_2$	6	(1, 1)	(1, 1, 1, 1, 1, 2)	(-2, -2, -1, 0)
B_6	T	s_1	6	(1, 2)	(1, 1, 1, 1, 2, 2)	(-2, -2, -1, -1)
C_6	T	s_2	6	(1, 1)	(1, 1, 1, 1, 2, 2)	(-2, -2, -1, -1)

Files. The scripts, their logs and the data behind this appendix are in the certificate archive [39], [doi:10.5281/zenodo.23549520](https://doi.org/10.5281/zenodo.23549520), whose README lists every file named here with its path in the archive. They consist of: the program computing the census and Table 2 (`u1.sage`, `u2.sage`, with outputs `u1_census.json` and `u2_rows.json`); two further, separately written implementations of the census and the weights, which agree with it on every row; and the Cartan and Weyl-group computation of Proposition 8.5. Each part has a README with the commands and the SHA-256 hashes of the outputs.

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Email address: `benzsevern@gmail.com`