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ALGEBRAICITY OF THE G_2-INVARIANT TENSORS ON YUN'S EXCEPTIONAL MOTIVES

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Abstract. Zhiwei Yun constructed, from a rigid Hecke eigensheaf for G_2 on $\mathbb{P}^1$ minus three points, a family of rank-seven motives whose motivic Galois groups are dense in G_2. We show that on this family the invariant cubic form is the class of an algebraic cycle: at every Hodge-generic parameter, every rational Hodge class on every tensor construction of the rank-seven piece is algebraic, and at every parameter the transported G_2-invariant tensors are algebraic. The cycles are top components of Satake convolution fibres transported by Hecke correspondences. The projector that cuts out the rank-seven piece is made algebraic by a canonical weighted resolution of Yun's boundary divisor, built on the moduli stack of bundles from the stabiliser tori of its points, together with a characteristic-zero census of those points and a cycle-class comparison on varieties with quotient singularities. Yun announced that the cubic is motivated in André's sense; the step from motivated to algebraic is new.

1. Introduction

Let G_2 be the split exceptional group of that type over $\mathbb{Q}$, and let $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In [31] Yun constructs, for each complex point $x \in S$, a pure rank-seven rational Hodge structure

$$M_x \;=\; \varepsilon\, H^6_c(T_x, \mathbb{Q}),$$

of weight six, whose Tate twist $M_x(3)$ is the Betti realisation of Yun's motive $M_{\chi,x} \in \mathrm{Mot}(\mathbb{Q})$, with $M_x(3) \otimes \mathbb{C} = E(V_7)_x$, occurring in the cohomology of an explicit smooth variety attached to a Hecke modification at x of a principal G_2-bundle on $\mathbb{P}^1$ with parahoric level structure at 0 and ∞ (Section 2). The variation $x \mapsto M_x$ has monodromy Zariski dense in G_2 [31, Thm. 1.2.1, §§4.5.1–4.5.2] (for the Betti local system, Lemma 3.20), and M_x carries a nondegenerate invariant quadratic form b and an invariant alternating cubic form t, the images under Yun's tensor functor of the corresponding G_2-invariants of the seven-dimensional representation V_7. Write $N_x = M_x(3)$, of weight zero.

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In [31, §1.2.3] Yun notes that the argument of Dettweiler and Reiter [3], based on André's theorem [2], shows that these motives have the expected motivic Galois group in the sense of André, and omits the details. Our first result is a checked construction of the invariant cubic as a motivated class, by a route that does not use the normaliser argument of [3].

Theorem A. *For every $x \in S(\mathbb{C})$ the invariant cubic $t \in (M_x^{\otimes 3})(9)$ is a motivated class in the sense of André* [2]*, given explicitly as a motivated projector applied to the class of an algebraic cycle obtained from the unique top-dimensional component of a Satake convolution fibre. If $x \in S(\mathbb{Q})$ it is motivated over $\mathbb{Q}$.*

Our main result upgrades this to algebraicity.

Theorem B. *Let $\overline{X}_x$ be the smooth projective variety of Section 8, whose cohomology contains M_x as the image of an algebraic correspondence acting on $H^*(\overline{X}_x)$ as an idempotent of rank seven (idempotent in cohomology, not claimed in Chow).*

1. *For every $x \in S(\mathbb{C})$ and every n, the image of every G_2-invariant tensor of $V_7^{\otimes n}$ under Yun's construction is the class of an algebraic cycle on $\overline{X}_x^{\,n}$.*
2. *If x is Hodge-generic for the variation M, the Mumford–Tate group of N_x is G_2, and every rational Hodge class in every tensor construction $T^{a,b}(N_x)$ (mixed tensor powers, their symmetric and exterior summands, and duals) is the class of an algebraic cycle.*

So the Hodge conjecture holds for all tensor powers of the Hodge structure M_x at Hodge-generic x, in the following projector-relative sense: M_x is the image on $H^*(\overline{X})$ of the algebraic correspondence P_X of Proposition 7.12, whose action on cohomology is idempotent of rank seven (we do not claim $P_X \circ P_X = P_X$ in $\mathrm{CH}(\overline{X} \times \overline{X})_{\mathbb{Q}}$, so no Chow motive is constructed; Remark 7.15), and every Hodge class in $(M_x^{\otimes a} \otimes M_x^{\vee \otimes b})(c)$, viewed in $H^*(\overline{X}^{\,a+b})(c')$ through $P_X^{\otimes(a+b)}$, is the class of an algebraic cycle on $\overline{X}^{\,a+b}$. This says nothing about the rest of $H^*(\overline{X})$. [referee report 001, B2] The Mumford–Tate group G_2 is not of abelian type, so the standard tools for the Hodge conjecture (divisors and the Lefschetz $(1,1)$ theorem, abelian varieties, Kuga–Satake constructions) do not reach these classes. We make no claim about Hodge classes that may appear at special parameters where the Mumford–Tate group drops (Remark 8.1), about the family of Dettweiler and Reiter [3] (Remark 8.2), or about the Hodge conjecture in general.

Outline of the proof. The cycle in Theorem B is explicit. Satake theory identifies G_2-invariants of $V_7^{\otimes n}$ with top Borel–Moore homology of the convolution fibre $m_n^{-1}(e)$; because the quasi-minuscule Schubert variety of G_2 is rationally smooth, these classes are spanned by top-dimensional irreducible components, which are algebraic cycles (Section 3). Hecke correspondences,

made finite by framing the endpoints, transport them to cycles on products of the Hecke fibre over x.

The difficulty is the projector. The rank-seven piece lives in the compactly supported cohomology of an open variety T_x, a finite étale cover of the complement of a divisor D_x (Yun's divisor) in the six-dimensional Schubert cell $Y = \mathrm{Gr}_{x,\alpha^\vee}$. Yun's cleanness theorem [31, Thm. 2.4.2] says that compactly supported and ordinary cohomology agree on the relevant isotypic block. On a smooth compactification this gives a splitting which is *motivated* (Theorem A) but a priori not algebraic, because it needs inverse Lefschetz operators on boundary components. We make it algebraic by choosing the compactification well: if (Y, D_x) admits a model whose normalised cover has only quotient singularities and whose boundary is a labelled normal crossing divisor up to finite quotients, then the cohomology of the normalised cover in the odd block *is* the clean piece, cut out by a finite-group idempotent (Theorem 4.3). Such a model comes from a canonical sequence of weighted blow-ups, built not on Y but on the moduli stack Bun of bundles, along its finitely many residual gerbes; the weights come from contracting cocharacters of the stabiliser tori (Theorem 5.9). The hypotheses of that construction are checked on a complete characteristic-zero census of the points of Bun of codimension at most six, which reduces to orbit classifications of explicit subgroups of G_2 on the flag variety (Section 6). Finally the projector is transferred to a smooth model by a cycle-class comparison on varieties with quotient singularities (Section 7).

Relation to earlier work. Dettweiler and Reiter [3] constructed the first motives for motivated cycles with motivic Galois group G_2, from a rigid local system obtained by middle convolution; their statements for Grothendieck motives are conditional on the standard conjectures. Yun [31] constructed the family studied here, and later stated that his G_2-local system is the same as that of Dettweiler and Reiter [32, Thm. 6.1.5 and the paragraph after Thm. 6.2.3]; equality of the local systems does not by itself give an algebraic correspondence between the two families of motives. Færgeman [6] gives motivic realisations in Katz's sense. As far as we know, no algebraicity statement for the invariant cubic of a G_2-motive was previously in print.

Conventions. Cohomology has rational coefficients unless stated otherwise. Varieties are over $\mathbb{C}$. G_2 has simple roots α_1 (short) and α_2 (long), and we write roots and coweights as $a\alpha_1 + b\alpha_2$ and $m_1\alpha_1^\vee + m_2\alpha_2^\vee$. "Hodge-generic" is relative to the variation M over S.

Coefficients for $\widetilde{A}$. All sheaves and cohomology are $\mathbb{Q}$-linear. Since $\widetilde{A} = Q_8$ has no two-dimensional rational representation, we never use one: the odd block of $\mathbb{Q}[\widetilde{A}]$ is the quaternion division algebra $D = e_-\mathbb{Q}[\widetilde{A}] = (-1,-1)_{\mathbb{Q}}$, $e_- = (1-z)/2$. For a $\mathbb{Q}[\widetilde{A}\times\widetilde{A}]$-module H put $H_{\mathrm{odd}} = e_-^{(1)}e_-^{(2)}H$, a module over $D \otimes D^{\mathrm{op}} \cong M_4(\mathbb{Q})$, where (a_1, a_2) acts on D by $x \mapsto a_1 x a_2^{-1}$ (Yun's convention: left and inverse-right translation [31, §3.2.5]). On every

space used below the odd part for the deck group alone, $e_-^{(2)}H$, equals H_{odd}: for $H_c^6(T_x)$ by [31, (3.16)], where the two-sided χ-block is $V_\chi \boxtimes V_\chi^*$, and for the pieces over $\star$ because the fibre of T_x there is an $\widetilde{A} \times \widetilde{A}$-bitorsor. The two-dimensional V_χ occurs only after extension of scalars to $\mathbb{Q}(i)$, where $D \otimes \mathbb{Q}(i) \cong \mathrm{End}_{\mathbb{Q}(i)}(V_\chi)$, and only in comparisons with [31].

2. Yun's construction

We recall what we use from [31], with Yun's notation.

2.1. **The moduli stack.** Let $G = \mathrm{G}_2$, let $P_0 \subset L_0 G$ be the parahoric subgroup attached by Gross–Reeder–Yu to $-1 \in W$, and let P_∞ be the parahoric of the same type containing the opposite Iwahori subgroup at ∞ [31, §§2.1.3–2.1.4, 2.2.1]. Their reductive quotients are both the symmetric subgroup $K \subset G$, of type $A_1 \times A_1$. Yun works with a double-cover variant $\mathrm{Bun} = \mathrm{Bun}_G(\widetilde{P}_0, P_\infty)$ [31, §2.2]. It contains an open point $U \cong B\widetilde{A}$, the classifying stack of a finite group $\widetilde{A}$; for G_2 the group $\widetilde{A}$ is the quaternion group Q_8, and its centre $\langle z \rangle$ has order two [31, §§2.2.5–2.3]. Let χ be the odd character of $\widetilde{A}$. Let $D = e_-\mathbb{Q}[\widetilde{A}]$ be the odd block. The odd sheaf $\mathbb{F}$ on $U = B\widetilde{A}$ is the rank-four rational local system attached to the left $\widetilde{A}$-module D; right multiplication makes D act on $\mathbb{F}$ by endomorphisms. Over $\mathbb{Q}(i)$, $\mathbb{F} \otimes \mathbb{Q}(i) \cong F_\chi \otimes_{\mathbb{Q}(i)} V_\chi^*$, where V_χ is the two-dimensional irreducible representation on which z acts by -1 and F_χ is Yun's odd sheaf.

Theorem 2.1 (Cleanness, [31, Thm. 2.4.2])**.** *For $j : U \hookrightarrow \mathrm{Bun}$, $j_!\mathbb{F} \cong Rj_*\mathbb{F}$.*

Yun proves $j_!F_\chi \cong Rj_*F_\chi$; the cone of $j_!\mathbb{F} \to Rj_*\mathbb{F}$ vanishes after $\otimes_{\mathbb{Q}}\mathbb{Q}(i)$, hence vanishes. This is an ℓ-adic statement; its Betti form is used only on finite-type smooth charts and on Y, through Lemma 3.16.

2.2. **The Birkhoff decomposition.** By [31, §2.2.2], the points of the stack $\mathrm{Bun}_G(P_0, P_\infty)$ are the bundles $\mathcal{P}_{\tilde{w}}$, indexed by the double cosets

$$\tilde{w} \in W_K \backslash \widetilde{W} / W_K,$$

with

$$\mathrm{Aut}(\mathcal{P}_{\tilde{w}}) \;=\; \Gamma_0 \cap \mathrm{Ad}(\tilde{w})P_0.$$

For $\tilde{w} = t^\mu w$ the Lie algebra of $\mathrm{Aut}(\mathcal{P}_{\tilde{w}})$ is $\mathfrak{t} \oplus \bigoplus_\alpha \bigoplus_{n=\ell_\alpha}^{h_\alpha} t^n E_\alpha$ with

$$\ell_\alpha = \lceil -\mathrm{ht}(\alpha)/2 \rceil, \qquad h_\alpha = \langle \alpha, \mu \rangle - \lceil \mathrm{ht}(w^{-1}\alpha)/2 \rceil, \tag{1}$$

and we write $N_\alpha = \max(h_\alpha - \ell_\alpha + 1, 0)$ for the number of loop root vectors in direction α. The constant torus T is a maximal torus of $\mathrm{Aut}(\mathcal{P}_{\tilde{w}})$. A point of Bun over $\mathcal{P}_{\tilde{w}}$ also records a Borel subgroup $b \in G/B$ (the level structure at 1), and $\mathrm{Aut}(\mathcal{P}_{\tilde{w}})$ acts on G/B through evaluation at 1,

$$\mathrm{ev}_1 : \mathrm{Aut}(\mathcal{P}_{\tilde{w}}) \to G.$$

The Iwahori level at 1 is [31, §2.2.5], the variant parahoric $\tilde{\mathbf{P}}_0$ is [31, §2.2.4], its canonical double cover is [31, §2.1.6], and the base point u_0 is [31, §2.2.6].

2.3. **The Hecke fibre and Yun's divisor.** Let $\lambda = \alpha^\vee$ be the quasi-minuscule coweight, so that $V_\lambda = V_7$. For $x \in S$, the Schubert variety $\mathrm{Gr}_{x,\leq\lambda}$ has dimension $\langle 2\rho, \lambda\rangle = 2h^\vee - 2 = 6$ and consists of the base point $\star$ and the open orbit $Y = \mathrm{Gr}_{x,\lambda}$ [31, §3.4], [7, Lemma 5.22]. Modifying the bundle $u_0 \in U$ at x by a point of Y defines

$$\omega : Y \longrightarrow \mathrm{Bun},$$

and Yun's divisor is $D_x = \omega^{-1}(\mathrm{Bun} \setminus U)$, so that $\mathrm{Gr}^U_{x,\leq\lambda} = \mathrm{Gr}_{x,\leq\lambda} \setminus D_x$ [31, §3.4]. Here u_0 is the fixed *target* of the modification and ω is the *source* map, as in [31, (3.3), §3.2.3]. Over $Y \setminus D_x$ the framing of the source endpoint $\omega(y) \in U$ gives a finite étale $\widetilde{A}$-torsor $T_x \to Y \setminus D_x$, whose deck group is Yun's source action $\widetilde{A}(1)$ and on which $\widetilde{A}(2) = \mathrm{Aut}(u_0)$, the target action, acts compatibly [31, (3.5)]; this is Yun's $\widetilde{Y}_x$ [31, §§3.3.2–3.3.3], and we also write $\widetilde{Y}_x = T_x$. Let $\mathbb{L} = e_- p_* \mathbb{Q}_{T_x}$, a rank-four local system with fibre D (monodromy by left multiplication; D acts on the right through the deck group $\widetilde{A}(1)$). Since $D = \mathbb{Q}[\widetilde{A}]e_-$ is a cyclic projective $\mathbb{Q}[\widetilde{A}]$-module, $\mathbb{L} = \mathrm{Hom}_{\widetilde{A}}(D, p_*\mathbb{Q}_{T_x})$.

Lemma 2.2. *The map ω is smooth, and $j'_!\mathbb{L} \cong Rj'_*\mathbb{L}$ for $j' : Y \setminus D_x \hookrightarrow Y$.*

Proof. On the relative-position-λ orbit either Hecke projection is smooth with fibre Gr_λ; pulling one endpoint back to u_0 is the base change along the étale map $\mathrm{pt} \to B\widetilde{A} \to \mathrm{Bun}$, so the other endpoint map ω is smooth. Smooth base change transports Theorem 2.1 to Y for Yun's ℓ-adic sheaves; the Betti statement is Lemma 3.16. □

Remark 2.3. In the chart C_0 of $Y \cong \mathbb{A}^6$ used in our computations, D_x is cut out by a polynomial of degree 30 that factors as $F_6G_8H_{16}$ (one component through the open stratum of each of the three relevant boundary types). Nothing below depends on these equations: the whole argument is carried out on Bun.

2.4. **The rank-seven motive.** Over $\mathbb{Q}(i)$, Yun's eigenvalue is $E_\chi(V_7)_x = (V_\chi^* \otimes H_c^*(T_x, \mathbb{Q}(i)))^{\widetilde{A}}$ in the relevant degree [31, (3.12), (3.18)]. The base point $\star$ lies in the open set $\mathrm{Gr}^U_{x,\leq\lambda}$ [31, (3.13)]: it is not a boundary point. The group $\widetilde{A}(2) = \mathrm{Aut}(u_0)$ acts on Y and on T_x, commuting with the deck group $\widetilde{A}(1)$; under $V_\chi^* \boxtimes V_\chi$ of [31, (3.9)], $\widetilde{A}(1)$ acts on V_χ^* and $\widetilde{A}(2)$ on V_χ. Put

$$\varepsilon = \frac{1}{32} \sum_{a_1, a_2 \in \widetilde{A}} \phi_\chi(a_1 a_2^{-1})\, a_1 \otimes a_2 \in \mathbb{Q}[\widetilde{A} \times \widetilde{A}], \qquad M_x = \varepsilon\, H_c^6(T_x, \mathbb{Q}),$$

with ϕ_χ the ($\mathbb{Q}$-valued) character of V_χ. Up to a scalar this is Yun's idempotent [31, §3.3.2] ($\mathbb{Q}' = \mathbb{Q}$ for G_2). The normalisation is forced: with $\Phi = \sum_b \phi_\chi(b)(b, 1)$ and $\Delta = \sum_a (a, a)$, which commute, $\Delta^2 = 8\Delta$ and $\Phi^2 = 4\Phi$, so $(c\,\Phi\Delta)^2 = 32c \cdot c\,\Phi\Delta$, and $c\Phi\Delta$ is idempotent exactly for

$c = 1/32 = \dim V_\chi/|\widetilde{A}|^2$. On D, ε acts by $x \mapsto \frac{1}{2}\operatorname{trd}(x)$, the rank-one projection onto the identity line $\mathbb{Q}\cdot 1$, and it kills the even block; $\varepsilon \otimes \mathbb{Q}(i)$ is the projection $a \mapsto \frac{1}{2}\operatorname{tr}(a)\operatorname{id}$ on $\operatorname{End}(V_\chi)$. As a correspondence, ε has coefficient $\frac{1}{16}$ on the graphs of the pairs with $a_1 = a_2$, $-\frac{1}{16}$ on those with $a_1 = za_2$, and 0 otherwise. By [31, (3.16)], $H^6_c(T_x)_{\mathrm{odd}} \otimes \mathbb{Q}(i) \cong \operatorname{End}(V_\chi) \otimes_{\mathbb{Q}(i)} E_\chi(V_7)_x(-3)$ has dimension 28. As ε is a primitive idempotent of $D \otimes D^{\mathrm{op}} \cong M_4(\mathbb{Q})$, Morita theory gives $H^6_c(T_x)_{\mathrm{odd}} \cong D \otimes_{\mathbb{Q}} M_x$, so $\dim_{\mathbb{Q}} M_x = 7$ and $M_x(3) \otimes \mathbb{Q}(i) = E_\chi(V_7)_x$. The odd part is pure of weight six by Lemma 2.2 (compactly supported cohomology has weights ≤ 6, ordinary cohomology ≥ 6), so $M_x(3) = \varepsilon \operatorname{Gr}^W_6 H^6_c(\widetilde{Y}_x)(3)$ (Yun's e_χ applied to the pure part) is the Betti realisation of Yun's $M_{\chi,x}$ [31, Prop. 3.3.3]; as x varies, $N_x = M_x(3)$ is the fibre of the $\mathbb{Q}$-local system N_S with $N_S \otimes \mathbb{Q}_\ell \cong E_\chi(V_7)$ as local systems (Lemma 3.17).

Lemma 2.4 (Rational smoothness)**.** $\operatorname{Gr}_{x,\leq\lambda}$ *is rationally smooth: its intersection complex is* $\mathbb{Q}[6]$.

Proof. In characteristic zero the opposite-cell chart of $\operatorname{Gr}_{\leq\lambda}$ completed by $\star$ is the minimal nilpotent cone, and blowing up the vertex gives the negative line bundle over the adjoint variety $E = \mathrm{G}_2/P$ [7, (5.21), Lemma 5.22]. The variety E has one Schubert class in each even degree $0, \ldots, 10$, and $h^5 \neq 0$ for the ample class h, so all the maps $H^{2i}(E) \to H^{2i+2}(E)$ given by h are isomorphisms between one-dimensional spaces. The Gysin sequence of the punctured cone then shows that the link of $\star$ is a rational homology sphere. □

3. The motivated cubic

3.1. **Top components of convolution fibres.** Let C_n be the n-fold convolution of $\operatorname{Gr}_{\leq\lambda}$ and $m_n : C_n \to \operatorname{Gr}$ the multiplication. By Lemma 2.4 the iterated twisted products are rationally smooth, $IC(C_n) = \mathbb{Q}[6n]$, and $\dim m_n^{-1}(e) \leq 3n$ by stratified semismallness [27, §4]. Hence

$$\operatorname{Hom}(\mathbf{1}, V_7^{\otimes n}) \;=\; H^{BM}_{6n}(m_n^{-1}(e), \mathbb{Q}),$$

spanned by the classes of the $3n$-dimensional irreducible components. For $n = 3$ the space is one-dimensional ($V_7 \otimes V_7 = \mathbf{1} \oplus V_7 \oplus V_{14} \oplus V_{27}$), so there is exactly one nine-dimensional component Z_t, and the Satake image of t is a nonzero multiple of $[Z_t]$.

3.2. **Finite framing correspondences.** An n-step chain of modifications at x with fixed target u_0, variable source in U and all intermediate bundles in U becomes, after choosing frames for its $n-1$ intermediate bundles and for its source, a point of X^n, where X is the one-step space; the forgetful maps form the tower $P = X^n \to Q \to \Omega$ of Section 3.3. Pulling a top component back to the chain space, restricting to the locus where the intermediate bundles lie in U, and passing to the framing covers gives an algebraic correspondence into X^n; we keep the coefficient D at the source, the

endpoint labels and the identity-section condition on the composite. [ledger 10] The odd block is D; Yun's evaluation maps contract adjacent D-factors by multiplication, the transported tensor is the cyclic tensor J_n, the adjoint of n-fold multiplication for the invariant form $\mathrm{trd}(x\bar{y})$ (Lemma 3.3), and the endpoint character contributes a factor 4; applying ε in every factor gives the nonzero coefficient $2^{1-n}/4$ on Yun's tensor unit (Proposition 3.7). This is why the endpoint coefficient system must be retained: the unweighted image of a component need not give the desired class. Cleanness removes chains with an intermediate bundle outside U, and a top component missing the all-intermediate-U locus contributes zero. Denote by W_u the resulting rational cycle on $\widetilde{Y}_x^n$ for an invariant $u \in (V_7^{\otimes n})^{G_2}$ (made precise in Section 3.6), and by $\overline{W}_u$ its closure in a smooth projective compactification.

3.3. **The naturality square.** Throughout, coefficients are $\mathbb{Q}$ and sheaves are constructible sheaves for the analytic topology on complex points. For each bounded fixed-parameter diagram used in the component comparison, we compare the specified geometric complexes and maps, starting from constant sheaves and finite-monodromy integral lattices, first with finite coefficients and then in their adic realizations. Their rational Betti forms are constructed by the same geometric operations. We do not assert an equivalence with all algebraically constructible analytic $\mathbb{Q}_\ell$-complexes. The comparison statements used, and the downstream statements that rest on them, are collected in Section 3.4: the relative comparison over S (Lemmas 3.14–3.18), the clean extension (Lemma 3.16), the transported tensors (Lemmas 3.19–3.21), and arithmetic fixedness (Lemma 3.23). The symmetric tensor functor of [31, (3.19)] is not compared; Lemma 3.21 does not use it. Fix $x \in S(\mathbb{C})$, $\lambda = \alpha^\vee$, $d = \langle 2\rho, \lambda\rangle = 6$, and an integer $n \geq 1$.
Chain spaces. Let $C_n = \mathrm{Gr}_{x,\leq\lambda} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{x,\leq\lambda}$ (n factors) be the space of chains $b_n \to b_{n-1} \to \cdots \to b_1 \to b_0 = u_0$ of modifications at x, each of relative position $\leq \lambda$, with fixed *target* u_0 (Yun's convention [31, (3.3), §3.2.3]: the coefficient sits on the source and is pushed to the target). Step i runs from the source b_i to the target b_{i-1}; the i-th convolution coordinate is the position of b_i relative to b_{i-1}, so coordinate 1 is adjacent to the fixed target, and b_n is the variable source endpoint. In Yun's numbering of the two endpoint actions the source comes first ($\widetilde{A}(1)$) and the target second ($\widetilde{A}(2)$); in multiplication matrices and in convolution slots the target comes first. Write $\omega_i : C_n \to \mathrm{Bun}$ for the bundle b_i and $m_n : C_n \to \mathrm{Gr}_{x,\leq n\lambda}$ for the composite relative position, so that $m_n^{-1}(e)$ is the set of chains whose composite is the identity modification of u_0. Put

$$C_n^{(U)} = \omega_n^{-1}(U), \qquad C_n^{U^n} = \bigcap_{i=1}^{n} \omega_i^{-1}(U) \subset C_n^{(U)} \quad \text{(open)}.$$

Since $\omega_n(m_n^{-1}(e)) = u_0 \in U$, the compact set $m_n^{-1}(e)$ lies in $C_n^{(U)}$.

Lemma 3.1. (a) $\dim m_n^{-1}(e) \le 3n$. (b) *C_n is rationally smooth of dimension nd, and $m_{n*}\mathbb{Q}_{C_n}[nd](nd/2) \cong \mathrm{IC}_\lambda^{*n}$.* (c) *There is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{Sat}}(\mathrm{IC}_0, \mathrm{IC}_\lambda^{*n}) \;\cong\; H^{BM}_{2\cdot 3n}\big(m_n^{-1}(e), \mathbb{Q}\big)(-3n),$$

and the right side has the basis $\{[Z]\}$, Z running over the $3n$-dimensional irreducible components of $m_n^{-1}(e)$ with their reduced structure and complex orientation.

Proof. (a) is stratified semismallness of convolution [27, Lemma 4.4]. (b): C_n is an iterated locally trivial fibration with fibre the rationally smooth $\mathrm{Gr}_{\le\lambda}$ (Lemma 2.4), so $\mathrm{IC}(C_n) = \mathbb{Q}[nd]$; the convolution IC_λ^{*n} is by definition m_{n*} of the twisted external product of the $\mathrm{IC}_\lambda = \mathbb{Q}[d](d/2)$. (c): m_n is proper, so with $i : m_n^{-1}(e) \hookrightarrow C_n$ and base change, $\mathrm{Hom}(\delta_e, m_{n*}\mathbb{Q}[nd](nd/2)) = H^0(m_n^{-1}(e), i^!\mathbb{Q}_{C_n}[nd](nd/2))$. Rational smoothness gives $\mathbb{Q}_{C_n}[2nd](nd) \cong \omega_{C_n}$, hence $i^!\mathbb{Q}_{C_n}[nd](nd/2) = \omega_{m_n^{-1}(e)}[-nd](-nd/2)$ and the Hom equals $H^{BM}_{nd}(m_n^{-1}(e))(-nd/2)$, with $nd = 6n = 2\cdot 3n$. By (a) this is the top Borel–Moore group, freely spanned by the fundamental classes of the $3n$-dimensional components. Both sides of (c) are Homs in the abelian category of perverse sheaves because IC_λ^{*n} is perverse semisimple: it is perverse by [27, Prop. 4.2], and semisimple by the decomposition theorem [26], since m_n is proper and $\mathbb{Q}_{C_n}[nd]$ is pure. □

For $u \in (V_7^{\otimes n})^{\mathrm{G}_2}$ let $s_u : \mathrm{IC}_0 \to \mathrm{IC}_\lambda^{*n}$ be the morphism attached to u by the Satake equivalence [27, Thm. 7.3], and let $Z_u = \sum_Z c_Z(u)\, Z$ be the rational cycle with $[Z_u] = s_u$ under Lemma 3.1(c). For $n = 3$, $u = t$: the space is one-dimensional, so $Z_t = c\, Z_9$ for the unique nine-dimensional component and some $c \in \mathbb{Q}^\times$.

The framing torsors and the coefficients. Over $U = B\widetilde{A}$ the base point $u_0 : \mathrm{pt} \to B\widetilde{A}$ is an $\widetilde{A}$-torsor. Let $X = \widetilde{\mathrm{GR}}^U_x$ be Yun's one-step space, with its source action $\widetilde{A}(1)$ and target action $\widetilde{A}(2)$ [31, (3.5)–(3.6)]. Factor i of $P = X^n$ is step i of the chain, so factor 1 is adjacent to the fixed target u_0. In matrices we write the target first: by [31, (3.9)] and $f \otimes w \mapsto wf$,

$$H_c^*(X)_{\mathrm{odd}} \otimes \mathbb{Q}(i) \;\cong\; V_\chi^* \boxtimes V_\chi \otimes E_\chi \;\cong\; \mathrm{Hom}(V_{\mathrm{source}}, V_{\mathrm{target}}) \otimes E_\chi,$$

row index = target, column index = source; rationally $\widetilde{A}(2)$ acts on D by $x \mapsto gx$ and $\widetilde{A}(1)$ by $x \mapsto xg^{-1}$. The endpoint bookkeeping is:

endpoint	Yun's action	action on coefficients	slot
$b_0 = u_0$	$\widetilde{A}(2)$ of step 1	gx_1	external target, row of slot 1
$b_i, 0 < i < n$	$\widetilde{A}(1)$ of step $i = \widetilde{A}(2)$ of step $i+1$	$x_i g^{-1} \otimes g x_{i+1}$	internal pair i–$(i{+}1)$
b_n	$\widetilde{A}(1)$ of step n	$x_n g^{-1}$; gv on the coefficient	external source, column of slot n

Let $\Omega = C_n^{U^n}$. Framing the source b_n and then the intermediate bundles $b_1, \dots, b_{n-1}$ gives the tower of finite étale torsors

$$P = X^n \xrightarrow{\pi} Q = P/\widetilde{A}^{n-1} \xrightarrow{q} \Omega, \qquad \deg \pi = 8^{\,n-1}, \quad \deg q = 8,$$

where the internal group $\widetilde{A}^{n-1}$ acts at b_i by $x_i \mapsto x_i g^{-1}$, $x_{i+1} \mapsto g x_{i+1}$; for $n = 3$ the degrees are 64 and 8. The coefficient system is $\mathbb{G} = \omega_n^* \mathbb{F}$, with fibre D at b_n, on which the source action $\widetilde{A}(1)$ of factor n acts by left multiplication. It is retained: q^* identifies $H_c^*(\Omega, \mathbb{G})$ with $(D \otimes H_c^*(Q))^{\widetilde{A}}$, and likewise for ordinary cohomology, by pullback onto invariants with no averaging ($q_* q^* = 8$, $\pi_* \pi^* = 8^{\,n-1}$). All maps are equivariant for the external actions ($\widetilde{A}(2)$ at u_0, $\widetilde{A}(1)$ at b_n) and for the internal contraction group; the factorwise ε is applied only after the outer legs are contracted. We use the monoidal structure of [31, §3.1.3] in the form

$$T(\mathcal{K}_1 * \mathcal{K}_2, \mathcal{F}) = T(\mathcal{K}_1, T(\mathcal{K}_2, \mathcal{F})),$$

$\mathcal{K}_1$ on the convolution coordinate adjacent to u_0; this is proper base change on the bounded iterated Hecke correspondence, and it fixes the order of [31, (3.19)]: the first factor of $\varphi_{\mathcal{K}_1, \mathcal{K}_2}$ is factor 1 of P.
Yun's side at the point x. By [31, (3.3), (3.7)] and proper base change,

$$\begin{aligned}(u_0 \times \mathrm{id})^* T(\mathcal{K}, j_! \mathbb{F})_x &= H_c^*\big(\mathrm{GR}_x^U, \omega^* \mathbb{F} \otimes \mathcal{K}\big) \\ &= \big(D \otimes H_c^*(\widetilde{\mathrm{GR}}_x^U, \mathcal{K})\big)^{\widetilde{A}} \cong e_- H_c^*(\widetilde{\mathrm{GR}}_x^U, \mathcal{K}),\end{aligned}$$

and we put $E(\mathcal{K})_x = \varepsilon\, H_c^*(\widetilde{\mathrm{GR}}_x^U, \mathcal{K})$, so that $H_c^*(\widetilde{\mathrm{GR}}_x^U, \mathcal{K})_{\mathrm{odd}} \cong D \otimes E(\mathcal{K})_x$ and $E(\mathcal{K})_x \otimes_{\mathbb{Q}} \mathbb{Q}(i) = E_\chi(\mathcal{K})_x$ of [31, (3.18)], the identification being the multiplicity embedding ι of Lemma 3.3 (a partial trace; it is not evaluation at a matrix). Yun's tensor unit is $1_{\mathrm{Yun}} = \mathrm{id}_V \in E_\chi(\mathrm{IC}_0)_x = \mathrm{Hom}_{\widetilde{A}}(V, V)$, and $\iota(1_{\mathrm{Yun}}) = \sum_h \chi(h^{-1}) \delta_h = 2\delta_1 - 2\delta_z = 4 e_- \delta_1 = 4 \cdot 1_D$; this is four times the normalised identification $\theta(d) = \frac{1}{4} \sum_h \mathrm{trd}(h^{-1} d) \delta_h$, $\theta(1) = e_- \delta_1$, of D with $e_- \mathbb{Q}[\widetilde{A}]$.

Lemma 3.2 (intermediate bundles in U)**.** *Extension by zero*

$$H_c^*(C_n^{U^n}, \omega_n^* \mathbb{F}) \to H_c^*(C_n^{(U)}, \omega_n^* \mathbb{F})$$

is an isomorphism. (The coefficient $\omega_n^* \mathbb{F}$ is odd, so no block restriction is needed.)

Proof. Induction on n, with the case $n = 1$ empty. By the monoidal structure of T [31, §3.1.3] in the form fixed above, $T(\mathcal{K}_1 * \cdots * \mathcal{K}_n, j_! \mathbb{F}) = T(\mathcal{K}_1, T(\mathcal{K}_2 * \cdots * \mathcal{K}_n, j_! \mathbb{F}))$, and the inner term is $j_! \mathbb{F} \boxtimes E(\mathcal{K}_2 * \cdots * \mathcal{K}_n)$ by Lemma 2.4.8 and Theorem 2.1 [31, §3.4.4]. Its stalks vanish off U, so in the outer pull-push, which is a compactly supported cohomology of $C_n^{(U)}$ by proper base change, chains with $b_1 \notin U$ contribute zero. Iterating on the inner term removes $b_2, \dots, b_{n-1}$. This vanishing concerns the partial Hecke pushforwards, not the coefficient on each excluded chain. $\square$

Lemma 3.3 (framing, Künneth and Yun's evaluation maps)**.** *Work over $\mathbb{Q}(i)$, with $V = V_\chi$, basis (e_a) and dual basis (e^a), and $E_{ab} = e_a e^b$. For one step put $H = H^6_c(X)(3)_{\mathrm{odd}}$ and $E = E_\chi(V_7)_x = \mathrm{Hom}_{\widetilde{A}(2)}(V, (V \otimes H)^{\widetilde{A}(1)})$. Yun's evaluation maps (Lemma 2.4.8 and* [31, (3.8)–(3.9)]*) give the isomorphism*

$$\Psi : \mathrm{End}(V) \otimes E \xrightarrow{\sim} H, \qquad \Psi(vf \otimes e) = (f \otimes 1)\, e(v),$$

and we define the multiplicity embedding

$$\iota(e) = \Psi(I \otimes e) = \sum_a (e^a \otimes 1)\, e(e_a) \in \varepsilon H,$$

a partial trace; rationally it is the partial reduced trace (half the rational partial trace on the rank-four module D). With $D \otimes \varepsilon H \cong H$ normalised by $1 \otimes m \mapsto m$, one has, with no averaging,

$$(*) \qquad e(e_a) = \sum_b e_b \otimes E_{ab} \otimes \iota(e).$$

Under π^*, q^* and Künneth, the iterated isomorphism [31, (3.19)] *sends $e_1 \otimes \cdots \otimes e_n$ to the $\widetilde{A}$-map*

$$e_a \longmapsto \sum_{a_2,\dots,a_{n+1}} e_{a_{n+1}} \otimes E_{aa_2} \otimes E_{a_2a_3} \otimes \cdots \otimes E_{a_na_{n+1}} \otimes \iota(e_1) \otimes \cdots \otimes \iota(e_n)$$

in the factor order of P. For each fixed outer pair (a, a_{n+1}) this is invariant under the internal group $\widetilde{A}^{n-1}$ before any ε is applied. Contracting the outer legs by the trace on V (rationally, the partial reduced trace), Θ, sets $a_{n+1} = a$:

$$\Theta\big(\varphi_n(e_1 \otimes \cdots \otimes e_n)\big) = J_n \otimes \iota(e_1) \otimes \cdots \otimes \iota(e_n),$$

$$J_n = \sum e_{a_1a_2} \otimes e_{a_2a_3} \otimes \cdots \otimes e_{a_na_1}.$$

Rationally J_n is the adjoint of n-fold multiplication for the invariant form $h(x,y) = \mathrm{trd}(x\bar{y})$; for $n = 3$, $J_3 = \frac{1}{4}\sum_{s,t} b_s \otimes b_s^{-1} b_t \otimes b_t^{-1}$ with $(b_s) = (1, i, j, k)$, and $\varepsilon^{\otimes n} J_n = 2^{1-n}\, 1^{\otimes n}$.

Proof. π and q are finite étale torsors, so pullback is an isomorphism onto invariants. Künneth for H_c applies to the product, and all classes are in even degree $d = 6$ per factor [31, (3.12)], so there are no Koszul signs. By (3.8), $V^*_{\mathrm{source}} \otimes (V_{\mathrm{source}} \otimes H)^{\widetilde{A}(1)} \to H$, $f \otimes \sum v_j \otimes h_j \mapsto \sum f(v_j) h_j$, is an isomorphism; combined with Lemma 2.4.8 it gives Ψ, and inverting the evaluation formula gives $(*)$. The associated operator of the term $e_b \otimes E_{ab}$ is $E_{ab} = e_a e^b$, which sends the source vector e_b to the target vector e_a. By $T(\mathcal{K}_1 * \mathcal{K}_2) = T(\mathcal{K}_1) \circ T(\mathcal{K}_2)$, the iterated isomorphism applies $T(\mathcal{K}_1, \mathrm{ev}_2)$ and then the next evaluation: the source leg of step i is fed into the evaluation of step $i+1$, which gives the displayed formula. Internal invariance: $\sum_b (E_{ab} g^{-1}) \otimes (g E_{bc}) = \sum_{b,u,v} (g^{-1})_{bu} g_{vb} E_{au} \otimes E_{vc} = \sum_u E_{au} \otimes E_{uc}$,

and likewise for the other internal pairs; the target action at u_0 and the inverse-right source action at b_n give the external equivariance. The identification of J_3 with the h-adjoint, and $\varepsilon^{\otimes 3} J_3 = \frac{1}{4}$, are direct computations in $D \otimes \mathbb{Q}(i) \cong M_2(\mathbb{Q}(i))$ ($i \mapsto \mathrm{diag}(\mathrm{i}, -\mathrm{i})$, $j \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$); ε kills the off-diagonal units and sends each E_{aa} to $\frac{1}{2}$, so only the terms with all indices equal survive. The adjoint of multiplication for the trace form $\mathrm{trd}(xy)$ has the same ε-projection but is not invariant under these actions, and is not used. □

Lemma 3.4 (Evaluation naturality)**.** *At the fixed parameter x write $\mathcal{T}_i = T(\mathcal{K}_i, -)$, $F_0 = j_!\mathbb{F}_V$, $\mathcal{K}' = \mathcal{K}_2 * \cdots * \mathcal{K}_n$, $\mathcal{G}' = T(\mathcal{K}', F_0)$ and $E' = E(\mathcal{K}')$, and let* $\mathrm{ev}' : F_0 \otimes E' \to \mathcal{G}'$ *be Yun's canonical evaluation* [31, §3.4.4, Lemma 2.4.8]*. The square*

$$\begin{array}{ccc} V \otimes E(\mathcal{K}_1) \otimes E' & \xrightarrow{\mathrm{ev}_1 \otimes 1} & (\mathcal{T}_1 F_0)_{u_0} \otimes E' \\ \downarrow 1 \otimes \varphi_{\mathcal{K}_1, \mathcal{K}'} & & \downarrow \rho \\ V \otimes E(\mathcal{K}_1 * \mathcal{K}') & \xrightarrow{\mathrm{ev}_{1 \cdots n}} & (\mathcal{T}_1 \mathcal{G}')_{u_0} \end{array}$$

commutes, where ρ is the projection formula followed by $(\mathcal{T}_1 \mathrm{ev}')_{u_0}$, the lower right term is identified with $T(\mathcal{K}_1 * \mathcal{K}', F_0)_{u_0}$ by the convolution base-change map and the projection formula in ρ moves the constant multiplicity factor inside $\mathcal{T}_1$. Its passage to $X \times X^{n-1}$ (stalk by !-base change, pullback of the finite framing torsors, then Künneth) contains the same pulled-back morphism $\omega_1^* \mathrm{ev}'$ on both paths and commutes with ext*.*

Proof. By Lemma 3.12, the square is the fibre at x of the restriction along $u_0 \times \mathrm{id}_S$ of the odd-output evaluation square that specifies Yun's map (3.19), and by Lemma 3.11(i)–(ii) its corners and arrows are operations on $\mathcal{D}_{n,S}$. The square commutes by the construction of [31, (3.19)] from the monoidal Hecke functor and the canonical decomposition of [31, Lemma 2.4.8]: applying the multiplicity Hom to the right-hand composite is precisely $\varphi_{\mathcal{K}_1, \mathcal{K}'}$. This fixes the map, rather than choosing a scalar afterwards. The comparison maps of the passage to $X \times X^{n-1}$ are natural in the coefficient morphism ev', so both paths contain $\omega_1^* \mathrm{ev}'$ before the outer !-pushforward. Restriction to $b_1 \in U$ commutes with it, since the canonical evaluation identifies both objects with their extension by zero; repeat on the tail. Iterating, a fixed framed first step with source coefficient V is sent to the matrix-unit expansion of Lemma 3.3: at one step this is the inverse of Ψ, and the square substitutes the tail's expansion for each source vector. This proves the iterated formula of Lemma 3.3 for every $n \geq 1$, including the vacuous case $n = 1$. □

Lemma 3.5 (Supported and framed component comparison)**.** *Put $d = 6n$, $F = m_n^{-1}(e)$, $C = C_n^{(U)}$, $\Omega = C_n^{U^n}$ and $F^\circ = F \cap \Omega$; write $i_F : F \hookrightarrow C$ (a rational combination of components is not an inclusion). Let Z be a rational combination of the $3n$-dimensional components of F, and let s_Z be its complex-oriented supported Satake morphism. Over $\mathbb{Q}(i)$ write $\mathbb{F} \otimes_{\mathbb{Q}} \mathbb{Q}(i) \cong \mathbb{F}_V \otimes_{\mathbb{Q}(i)} V^*$, with $\mathbb{F}_V$ of fibre V, and use the coefficient system*

$\mathbb{G}_V = \omega_n^* \mathbb{F}_V$. *Define*

$$\mathcal{M}_c(Y) = \mathrm{Hom}_{\widetilde{A}(2)}\big(V, H_c^d(Y, \mathbb{G}_V)(3n)\big),$$
$$\mathcal{M}(Y) = \mathrm{Hom}_{\widetilde{A}(2)}\big(V, H^d(Y, \mathbb{G}_V)(3n)\big).$$

The canonical trivialization on F and the supported counit define $a_c : H_d^{BM}(F)(-3n) \to \mathcal{M}_c(C)$ and $a_o : H_d^{BM}(F^\circ)(-3n) \to \mathcal{M}(\Omega)$. Let $B_c = \Theta\pi^* q^*$ and $B_o = \Theta\pi^* q^*$ on the compactly supported and ordinary multiplicity spaces, retaining source and internal invariants before Θ. Then the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathrm{Sat}}(\mathrm{IC}_0, \mathrm{IC}_\lambda^{*n}) & \xrightarrow{E(-)_x(1_{\mathrm{Yun}})} & \mathcal{M}_c(C) & & \\
\downarrow\cong & & \downarrow \mathrm{ext}^{-1} & & \\
H_d^{BM}(F)(-3n) & \xrightarrow{\mathrm{ext}^{-1} a_c} & \mathcal{M}_c(\Omega) & \xrightarrow{B_c} & H_c^d(P, \mathbb{Q}(i))(3n) \\
\downarrow \mathrm{res} & & \downarrow \mathrm{fgt} & & \downarrow \mathrm{fgt} \\
H_d^{BM}(F^\circ)(-3n) & \xrightarrow{a_o} & \mathcal{M}(\Omega) & \xrightarrow{B_o} & H^d(P, \mathbb{Q}(i))(3n)
\end{array}$$

commutes; the upper left identification uses $i_F^!$, not i_F^*. It is a diagram of linear maps obtained by applying cohomology to natural transformations in derived categories on different spaces. If $\widetilde{Z} \subset Q$ is the canonical endpoint sheet over $Z \cap \Omega$ and $W = \pi^{-1}\widetilde{Z}$, then

$$B_o a_o[Z \cap \Omega] = 4e_-^{\mathrm{end}}\,\mathrm{cl}(W),$$
$$B_c\,\mathrm{ext}^{-1}\varphi_n(e_1 \otimes \cdots \otimes e_n) = J_n \otimes \iota(e_1) \otimes \cdots \otimes \iota(e_n),$$

with $J_n = \sum E_{a_1 a_2} \otimes \cdots \otimes E_{a_n a_1}$. Consequently, with

$$\tau_Z = \iota^{\otimes n}\varphi_n^{-1} E(s_Z)_x(1_{\mathrm{Yun}}),$$
$$4e_-^{\mathrm{end}}\,\mathrm{cl}(W) = \mathrm{fgt}_P(J_n \otimes \tau_Z),$$
$$\varepsilon^{\otimes n}\,\mathrm{cl}(W) = 2^{-n-1}\mathrm{fgt}_P(\tau_Z).$$

Both identities descend to $\mathbb{Q}$. In particular the coefficient is $+\frac{1}{16}$ for $n = 3$, and $+\frac{1}{4}$ for $4W$.

Proof. Rational smoothness and proper extraordinary base change give $i_F^!\mathbb{Q}_{C_n}[d](3n) = \omega_F[-d](-3n)$. The supported fundamental-class morphism, followed by its counit, is the Satake morphism. Tensoring with the endpoint coefficient and applying $R\Gamma_c$ sends v to $\mathrm{PD}_c(i_{F*}[Z \otimes v])$. All maps in the diagram are, by Lemmas 3.11 and 3.12, the restrictions to $\mathcal{D}_{n,S}$ at the fixed parameter of the specified constructions, and Lemma 3.14 compares them stage by stage (Remark 3.13); this lemma is a computation with the specified map, not a uniqueness statement. The natural all-U extension is an isomorphism by the clean partial Hecke transforms (Lemma 3.2). Open base change of the counit and $\mathrm{fgt}_\Omega = j^*\mathrm{fgt}_C\,\mathrm{ext}$ give the left lower square; finite étale pullback and partial trace commute with forget-supports, giving the right square.

On the canonical endpoint sheet the pulled-back coefficient map is $v \mapsto \sum_{g \in \widetilde{A}} gv \otimes [g\widetilde{Z}]$. Its trace is $2[\widetilde{Z}] - 2[z\widetilde{Z}]$, and flat pullback along π gives

$4e_-^{\mathrm{end}}\,\mathrm{cl}(W)$ in ordinary cohomology. Lemma 3.4, followed by framed base change and Künneth, inductively gives the open-legged matrix-unit formula; taking its outer trace gives J_n. Only the two terms with all indices equal survive ε, so $\varepsilon^{\otimes n}J_n = 2^{1-n}1^{\otimes n}$. All geometric shifts and degrees are even, and the fundamental class and the duality orientation are both the complex orientation, with the same supported counit and the same generator s_Z on both sides; so the coefficient has positive sign.

For descent use the coefficient $\mathbb{F}$ (fibre D) with its commuting right D-action and $\mathrm{Hom}_{\widetilde{A}(2),D^{\mathrm{op}}}(D,-)$, taken on the source and internal invariants. Its contraction is $\frac{1}{2}\,\mathrm{Tr}_{\mathbb{Q}}$, which becomes the trace on V after splitting. All geometric maps, this contraction, the odd projector and ε are rational, so faithful scalar extension proves the identities over $\mathbb{Q}$. □

Proposition 3.6 (the naturality square)**.** *Let $u \in (V_7^{\otimes n})^{G_2}$, let $\mathbb{G} = \omega_n^*\mathbb{F}$, let $Z_u^\circ = Z_u \cap \Omega$, let $\widetilde{Z}_u \subset Q$ be the canonical endpoint sheet over Z_u° (the frame of b_n is the canonical one), and $W_u = \pi^{-1}(\widetilde{Z}_u) \subset P$, with reduced pullback multiplicities. Put*

$$\tau_u = \iota^{\otimes n}\varphi_n^{-1}\big(E(s_u)_x(1_{\mathrm{Yun}})\big) \in N_x^{\otimes n},$$

with φ_n the iterated [31, (3.19)] *and ι as in Lemma 3.3;* (a) *$E(s_u)_x(1_{\mathrm{Yun}})$ is the $\widetilde{A}$-map $v \mapsto \alpha_v = \mathrm{PD}_c(i_{F*}[Z_u \otimes v]) \in H_c^{nd}(C_n^{(U)},\mathbb{G})(nd/2)$;* (b) *with $\beta_v = \mathrm{ext}^{-1}\alpha_v$ (Lemma 3.2), $\mathrm{fgt}_\Omega\beta_v = \mathrm{PD}_\Omega(i^\circ_{F*}[Z_u^\circ \otimes v])$ in* ordinary *cohomology $H^{nd}(\Omega,\mathbb{G})(nd/2)$;* (c) *in ordinary cohomology $H^{nd}(P)(nd/2)$, on the all-odd block,*

$$\Theta\,\pi^*q^*\,\mathrm{fgt}_\Omega\beta \;=\; 4\,e_-^{(n)}\,\mathrm{cl}(W_u) \;=\; \mathrm{fgt}_P\big(J_n \otimes \tau_u\big),$$

where $e_-^{(n)}$ is the odd projector of the source action on factor n.

Proof. (a) By Lemma 3.1(c), proper extraordinary base change identifies s_u with the fundamental-class morphism of Z_u into $i_F^!\mathbb{Q}_{C_n}[nd](nd/2) = \omega_{m_n^{-1}(e)}[-nd](-nd/2)$, followed by the counit $i_*i^! \to \mathrm{id}$; no decomposition-theorem splitting is chosen. By [31, (3.3), (3.7)] and proper base change the stalk of $T(\mathrm{IC}_\lambda^{*n}, j_!\mathbb{F})$ at u_0 is $R\Gamma_c(C_n^{(U)},\mathbb{G})[nd](nd/2)$; each Satake factor contributes $[d](d/2)$ and the framing atlas $\mathrm{pt} \to B\widetilde{A}$ contributes neither shift nor twist. On $m_n^{-1}(e)$ the composite is the identity modification, which identifies b_n with u_0 and trivialises $\mathbb{G}$. Tensoring the supported counit with $\mathbb{G}$, evaluating at v and applying $R\Gamma_c$ gives α_v, and $E(\mathrm{IC}_0)_x(1_{\mathrm{Yun}})$ corresponds to id_V. (b) Open/closed base change, with $\mathrm{fgt}_\Omega = \mathrm{res}_{\Omega\subset C}\,\mathrm{fgt}_C\,\mathrm{ext}$. The restricted cycle Z_u° need not be proper, so its class is an ordinary one. (c) Pullback along the endpoint cover gives $q^*[Z_u^\circ \otimes v] = \sum_{g\in\widetilde{A}} gv \otimes [g\widetilde{Z}_u]$, with g the source action; in the group-coordinate basis this is $\sum_h h^{-1}v \otimes \delta_h$, matching the inverse-right source action. The partial reduced trace of this map in v is $\sum_g \chi(g)[g\widetilde{Z}_u] = 2[\widetilde{Z}_u] - 2[z\widetilde{Z}_u] = 4e_-^{(n)}[\widetilde{Z}_u]$, and flat pullback along π gives $4e_-^{(n)}\,\mathrm{cl}(W_u)$. □

Proposition 3.7 (the value on the designated generator)**.** *Let ε act in each factor by $x \mapsto \frac{1}{2}\operatorname{trd}(x)$. In ordinary cohomology $H^{nd}(P,\mathbb{Q})(nd/2)$,*

$$\varepsilon^{\otimes n}\operatorname{cl}(W_u) \;=\; \frac{2^{1-n}}{4}\operatorname{fgt}_P(\tau_u).$$

For $n=3$ and $u=t=s_{Z_9}$ the coefficient is $+\frac{1}{16}$ on Yun's tensor unit and the unweighted cycle W_t; the rational cycle $W_t^{\mathrm{acc}} := 4W_t$ has value $+\frac{1}{4}\operatorname{fgt}_P(\tau_t)$. Equivalently, the unweighted cycle has value $+\frac{1}{4}$ on the non-unit generator $1_{\mathrm{Yun}}/4$ of the unit line. These are two conventions; we use the first, with W^{acc} where the value $\frac{1}{4}$ is wanted, and never both.

Proof. Apply $\varepsilon^{\otimes n}$ to Proposition 3.6(c). It absorbs $e_-^{(n)}$, and $\varepsilon^{\otimes n}J_n = 2^{1-n}\,1^{\otimes n}$ by Lemma 3.3. The sign is $+$: no geometric factor is reversed, the factor order of P and of φ_n are both target-first, and all degrees are even. Rescaling Z_u rescales s_u and W_u together and does not change the ratio. Control (the finite set $X=\widetilde{A}$, $n=3$, $Q=\widetilde{A}$ via multiplication, $\widetilde{Z}=\{1\}$): $\sum_{abc=1} a\otimes b\otimes c = 16J_3$ in $D^{\otimes 3}$, so $\varepsilon^{\otimes 3}\operatorname{cl}(W)=4$, while $\iota(1_{\mathrm{Yun}})^{\otimes 3}=64$ and $64/16=4$. Nonvanishing of $E(s_u)_x$ for $u\neq 0$: $E_x = (\text{stalk at } x)\circ E$ is an exact monoidal functor from the semisimple rigid monoidal category Sat, with simple unit and $\operatorname{End}(\mathbf{1})=\mathbb{Q}$, to finite-dimensional $\mathbb{Q}$-vector spaces, by (3.19) and associativity [31, §3.1.3]. Such a functor is faithful. If $f\neq 0$, then its mate $\tilde{f}:\mathbf{1}\to Y\otimes X^\vee$ under rigidity is a nonzero map out of the simple unit, hence a monomorphism; so $E(\tilde{f})$ is a monomorphism out of $E(\mathbf{1})\neq 0$, hence nonzero; and monoidal functors preserve duals, so $E(f)\neq 0$. No commutativity constraint is used. □

Remark 3.8 (comparison with the split form). After $\otimes\mathbb{Q}(i)$, $D=\operatorname{End}(V_\chi)$ with row index the target, $\operatorname{trd}=\operatorname{tr}$, ε is $a\mapsto\frac{1}{2}\operatorname{tr}(a)\mathrm{id}$, and J_n is the cyclic matrix-unit tensor; the statements above become those of [31, (3.18)–(3.19)]. Writing the step with the transposed unit e_{ba} gives $\sum e_{ba}\otimes e_{cb}\otimes e_{ac}$, which has the same ε-projection but is not invariant: its coefficient at $e_{11}\otimes e_{21}\otimes e_{12}$ changes sign under $x\mapsto xi^{-1}$, $y\mapsto iy$. Likewise the adjoint for $\operatorname{trd}(xy)$ has the right projection and is not invariant. Transposing every slot of the former gives J_3; no geometric factor is permuted. The invariant pairing is $h(x,y)=\operatorname{trd}(x\bar{y})$. Every equality is between rational classes and is verified after the faithful extension $\mathbb{Q}\to\mathbb{Q}(i)$.

Remark 3.9 (where fusion is used). The construction of Z_u, Lemmas 3.2 and 3.3, Proposition 3.6 and the nonvanishing in Proposition 3.7 use only the *monoidal* structure of $T(-,j_!\mathbb{F})$ (the iterated Hecke stack and associativity) and the commutation of cycle classes with geometric operations. Fusion, i.e. the Mirković–Vilonen commutativity constraint [31, 3.4.4], enters only through two imported theorems: (i) the Satake equivalence, which supplies the bijection $u\mapsto s_u$ between $(V_7^{\otimes n})^{\mathrm{G}_2}$ and $\operatorname{Hom}_{\mathrm{Sat}}(\mathrm{IC}_0,\mathrm{IC}_\lambda^{*n})$ and identifies the Tannakian group with G_2; (ii) formerly, the statement that $E_\chi(s_t)_x$ is the invariant alternating cubic, which needs a symmetric tensor functor.

This is now Lemma 3.21: the flat nonzero section of Lemma 3.19, the dense Betti monodromy of Lemma 3.20 and invariant theory give the symmetry type and $\mathrm{Stab}(\tau_b, \tau_t) = \mathrm{G}_2$. No comparison of the commutativity constraint is used; the relative construction rests on Lemma 3.18.

Remark 3.10 (restriction to the cell and the base point). The draft's cycle W_u lives on $\widetilde{Y}_x^{\,n}$, where no step is the trivial modification. The odd block of H_c^d is unchanged by removing $\widetilde{A} \cdot \star$ from each factor. Indeed $\star$ is interior (it is not on D_x), so forget-supports is an isomorphism there by Lemma 2.2, and the local cohomology of the rationally smooth $\mathrm{Gr}_{\leq\lambda}$ at $\star$ is concentrated in degree $2d = 12$, so restriction $H^d(\mathrm{GR}^U_{x,\leq\lambda})_{\mathrm{odd}} \to H^d(\widetilde{Y}_x)_{\mathrm{odd}}$ is an isomorphism in degree $d = 6$. The n-fold product statement follows by Künneth.

3.4. **Comparison with Yun's ℓ-adic objects.** Yun's statements are ℓ-adic. This subsection compares the specified objects and maps of this paper with them, for bounded finite-type diagrams only. No equivalence of analytic and adic categories is used or asserted.

Lemma 3.11 (Bounded stack-to-finite-diagram reduction)**.** *Fix $n \geq 1$, $\lambda = \alpha^\vee$, the odd local system $\mathbb{F}_\chi$ on $U = B\widetilde{A}$, and $F_0 = j_!\mathbb{F}_\chi$. Let $\mathcal{D}_{n,S}$ be the finite diagram of separated finite-type S-schemes with finite-group actions obtained by base change along $u_0 \times \mathrm{id}_S$ and closed under the squares actually used below. It contains:*

- *the bounded chain spaces $C_{k,S}$ and convolution maps $m_{k,S} : C_{k,S} \to \mathrm{GR}_{\leq k\lambda,S}$ for every $k \leq n$, with their composition and projection squares (the two bracketings of each partial chain);*
- *the endpoint opens $C^{(U)}_{k,S}$, the all-U opens $\Omega_{k,S} = C^{U^k}_{k,S}$, and their open immersions;*
- *the framing covers $P_{k,S} \xrightarrow{\pi} Q_{k,S} \xrightarrow{q} \Omega_{k,S}$ and $\nu : \widetilde{\mathrm{GR}}^U_{\leq\lambda,S} \to \mathrm{GR}^U_{\leq\lambda,S}$, with the finite groups $\widetilde{A}^{k+1}$;*
- *the identity-composite fibres $F_{k,S} = m^{-1}_{k,S}(e_S)$, their closed immersions $i_F : F_{k,S} \hookrightarrow C_{k,S}$ and their intersections with the opens.*

At fixed n only finitely many spaces and maps occur. Then:

(i) Confinement. *Each Hecke output of* [31] *used in Sections 3 and 3.4, namely $(u_0 \times \mathrm{id}_S)^*\widetilde{T}(\mathcal{K}, F_0 \boxtimes \mathbb{Q}_S)$ for $\mathcal{K} = \mathrm{IC}^{*k}_\lambda$, $k \leq n$, and the partial transforms of its chains, is canonically isomorphic to a complex built on $\mathcal{D}_{n,S}$ from constant sheaves and the finite-monodromy coefficient $(V_\chi \otimes \nu_*\mathbb{Q}_\ell)^{\widetilde{A}(1)}$ by the operations of Lemma 3.14, with its $\widetilde{A}$-equivariant structure. The only shifts and twists are the Satake normalisations $[6k](3k)$.*

(ii) The maps. *Under these isomorphisms the base-change and extraordinary maps of* [31, (3.3), (3.7)]*, the units and counits of $(h_!, h^!)$, $(j_!, j^*)$ and $(i_*, i^!)$ that enter, and the monoidal isomorphisms of* [31,

§3.1.3] *in the correspondence-composition form below, become the corresponding operations on* $\mathcal{D}_{n,S}$.

(iii) Comparison. *Lemma 3.14 applies to* $\mathcal{D}_{n,S}$ *and to each morphism in (ii). The rational Betti objects and maps of Lemmas 3.17, 3.18, 3.4 and 3.5 are the analytifications of the same constructions on* $\mathcal{D}_{n,S}$.

The identification of the evaluations and of φ *of* [31, (3.19)] *is Lemma 3.12.*

Proof. Step 0: stack-level inputs. The morphism $u_0 \times \mathrm{id}_S$ is étale, hence smooth; smooth base change and its compatibility with composition and the projection formula are [33, I, §5.1]. The bounded Hecke stack is representable and proper over $\mathrm{Bun} \times S$, with fibres the projective Schubert varieties [31, §3.2.3]; for proper representable f, $Rf_! \cong Rf_*$ [33, I, §5.2, Prop. f!=f*propre], base change for $Rf_!$ holds for every cartesian square of stacks [33, I, §5.5], and its adic form is [33, II, §9, Thm. basechangethm]. The square $\mathrm{GR} \to \mathrm{Hk}$ over $u_0 \times \mathrm{id}_S$ is cartesian by definition [31, §3.2.3]. The construction of each transformation is retained, not only the fact that it is an isomorphism. Supported closed immersions are on the finite-type schemes of $\mathcal{D}_{n,S}$ and are covered by the localisation clause of Lemma 3.14.

Step 1: confinement (i).

- *On* Bun. u_0 factors through the open $U = B\widetilde{A}$ via the finite étale $\widetilde{A}$-torsor $\mathrm{pt} \to U$; pullback along it is exact with no shift, and the $\widetilde{A}$-action is retained as equivariant structure.
- *Satake input.* Choose N so large that the congruence subgroup $L^+_N G \rtimes \mathrm{Aut}_{\mathcal{O},N}$ acts trivially on every chain model and twisted product in $\mathcal{D}_{n,S}$, not only on the target $\mathrm{Gr}_{\leq n\lambda}$ (trivial action on a target does not imply it on a space mapping to it). One N serves the whole diagram because $\mathcal{D}_{n,S}$ has finitely many pieces, each controlled at finite level. Write $H = L^+G \rtimes \mathrm{Aut}_{\mathcal{O}} = \varprojlim_k H_k$ with $H_k = (L^+G/L^+_k G) \rtimes (\mathrm{Aut}_{\mathcal{O}}/\mathrm{Aut}_{\mathcal{O},k})$. Its action on Gr is *nice* in the sense of [34, Appendix, A.3]: every closed subscheme lies in an H-stable closed subscheme on which H acts through some H_k (for L^+G see [34, §1.1], for $\mathrm{Aut}_{\mathcal{O}}$ [34, §2.1.2]; the Schubert varieties $\mathrm{Gr}_{\leq\mu}$ are stable under both, since every L^+G-orbit is $\mathrm{Aut}_{\mathcal{O}}$-stable, [34, §2.1.3, proof of Prop. GOeq]). Every space of $\mathcal{D}_{n,S}$ arises from the finitely many bounded pieces $\mathrm{Gr}_{\leq\mu}$, $\mu \leq n\lambda$, by finitely many associated-bundle steps $\mathcal{H} \times^H Z$ of [34, Appendix, A.3]: the iterated twisted products $C_k = \mathrm{Gr}_{\leq\lambda} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}_{\leq\lambda}$ [34, §1.1], $k \leq n$, and their spreads along the coordinate torsor [34, §2.1.2]. The opens, finite étale framing covers and closed fibres listed above are open, finite étale or closed in these, so they add no new groups. In the j-th step H acts through some H_{k_j}, hence through $H_N \to H_{k_j}$ for every $N \geq k_j$. Taking $N = \max_j k_j$ gives one congruence subgroup acting trivially on the whole diagram. This is the finite-level form in which [34, §4.1] itself treats the spread ($Y_X = \mathcal{X}_m \times^{\mathrm{Aut}_m} Y$ for an $\mathrm{Aut}_{\mathcal{O}}$-stable $Y \supseteq \mathrm{supp}\,\mathcal{S}$ on which the action factors through

Aut_m). Enlarging N changes nothing: the kernels of $H_{N'} \to H_N$ are unipotent, so the equivariant derived categories agree [34, Appendix, A.4], following Bernstein–Lunts. The equivariance is then through the finite-type group $G_N = (L^+G/L^+_N G) \rtimes (\mathrm{Aut}_{\mathcal{O}}/\mathrm{Aut}_{\mathcal{O},N})$, and the finite-level torsors of N-jets of trivialisations and coordinates [34, §§2.1.2–2.1.3] construct the twisted products $C_{k,S}$ as projective S-schemes.

- *Normalisation.* For $\mathcal{K} = \mathrm{IC}^{*k}_\lambda = m_{k*}\mathbb{Q}_{C_k}[6k](3k)$ (Lemma 3.1; $\mathrm{Gr}_{\leq\lambda}$ is rationally smooth by Lemma 2.4), the spread complex in Yun's convention [31, §§3.2.3, 3.2.6] is
$$\mathcal{K}_{\mathrm{GR}} = Rm_{k,S*}\mathbb{Q}_{C_{k,S}}[6k](3k),$$
with the fibre shift only. Gaitsgory's perverse spread [34, §2.1.3], the twisted external product with $\mathbb{Q}_\ell[1]$ on the curve, is $\mathcal{K}_{\mathrm{GR}}[1]$; this is the perverse degree 1 of [31, Prop. 3.4.1], and the extra [1] is not inserted into the Hecke functor. (For $\mathcal{K} = \mathrm{IC}_0$ the eigenvalue is $\mathbb{Q}_S$ in degree 0, which fixes the convention.)
- *Descent.* The complexes and maps descend along the smooth G_N-torsors as equivariant geometric pull-push constructions: the equivariant constant complexes on the rationally smooth chain spaces and their proper pushforwards. Descent retains the cocycle, and equivariance follows from functoriality of these constructions. Pullbacks are unshifted in comparing derived complexes; any perverse normalisation is stated separately. No claim is made about all objects of an unrestricted equivariant derived category, and no infinite simplicial diagram is compared.
- *The coefficient.* $\omega^* j_! \mathbb{F}_\chi = j'_! \omega^{U,*}\mathbb{F}_\chi$ and $\omega^{U,*}\mathbb{F}_\chi \cong (V_\chi \otimes \nu_*\mathbb{Q}_\ell)^{\widetilde{A}(1)}$ [31, §3.2.4].

With Step 0 this gives [31, (3.7)] on $\mathcal{D}_{n,S}$. For the partial transforms of a chain, the intermediate endpoints are restricted to U only after interpreting the inner partial Hecke transform: its cleanness permits replacing it by its extension from U (it does not say that the original coefficient vanishes on every excluded chain).

Step 2: the monoidal structure. Write $X = \mathrm{Bun} \times S$ and $a, b : H \to X$ for source and target. Bound the first modification by λ (kernel $\mathcal{K}_1$, adjacent to the fixed target) and the second by $(n-1)\lambda$ (kernel $\mathcal{K}_2$, with its chosen convolution complex). Put
$$H_{12} = H_1 \underset{a_1,X,b_2}{\times} H_2, \qquad c : H_{12} \to H_{\leq n\lambda},$$
with projections r_i, $p = a_2 r_2$, $q = b_1 r_1$, and kernel $J = r_1^* I_1 \otimes r_2^* I_2$, $I_i = \mathrm{inv}^*\mathcal{K}_i$ (twisted descent understood). The map c is proper on these bounds, and the convolution construction identifies $Rc_! J$ with $\mathrm{inv}^*(\mathcal{K}_1 * \mathcal{K}_2)$; on the trivialisation and coordinate torsors this is the defining Satake convolution, and its equivariant descent is natural. Base change for the composability

square and the projection formula give

$$\widetilde{T}(\mathcal{K}_1, \widetilde{T}(\mathcal{K}_2, F)) \simeq Rq_!(p^*F \otimes J) \simeq Rb_!(a^*F \otimes Rc_!J) = \widetilde{T}(\mathcal{K}_1 * \mathcal{K}_2, F).$$

This is the correspondence-composition structure of [31, §3.1.3], attached to [31, (3.3)], with its direction, order and normalisation specified. On three-step chains the two bracketings are two applications of the same base-change and projection-formula identities to the common total kernel, which gives associativity; four-step chains give the pentagon, and identity modifications the unit. No rescaling enters. Pulling the fixed target to $u_0 \times S$ turns H_{12} into $C_{2,S}$ and c into $m_{2,S}$, and likewise for longer chains; each identity restricts to its finite-type scheme identity on $\mathcal{D}_{n,S}$, with framing equivariance retained. This and naturality of base change and localisation give (ii).

Step 3: comparison (iii). $\mathcal{D}_{n,S}$ is a finite diagram of separated finite-type complex varieties with finite-group actions and finite-monodromy input. Lemma 3.14 applies stage by stage: its pullback and finite-invariant clause to ν, π, q; its Rf_* clause to $m_{k,S}$; its $Rf_!$ and base-change clauses to the chain pushforwards and the composition squares; its adjunction clause, with the localisation triangle, to the supported counit and to $j'_!$; its projection-formula and Künneth clauses to the evaluation square and to Θ. The Betti constructions are defined by the same operations, so their rationality comes from the construction, not from the ℓ-adic side. □

Lemma 3.12 (Odd-output descent with the specified evaluation)**.** *Put* $X = \mathrm{Bun} \times S$, $r = (u_0 \times \mathrm{id}_S)^*$, $F_0 = j_!\mathbb{F}_\chi$ *and* $H_\mathcal{K} = \widetilde{T}(\mathcal{K}, F_0 \boxtimes \mathbb{Q}_S)$. *For odd* H, H' *on* X, *restriction induces an isomorphism*

$$\mathrm{Hom}_{D^b(X)_{\mathrm{odd}}}(H, H') \xrightarrow{\sim} \mathrm{Hom}_{D^b_{\widetilde{A}}(S)_{\mathrm{odd}}}(rH, rH').$$

For the bounded convolution objects set $E_\mathcal{K} = (V_\chi^* \otimes rH_\mathcal{K})^{\widetilde{A}}$. *The canonical evaluation* $F_0 \boxtimes E_\mathcal{K} \to H_\mathcal{K}$ *is the unique odd morphism restricting to the equivariant evaluation* $V_\chi \otimes E_\mathcal{K} \to rH_\mathcal{K}$. *With the correspondence-composition monoidal structure of Lemma 3.11 (first factor adjacent to the target), the tensor map* $\varphi_{\mathcal{K},\mathcal{L}}$ *is uniquely specified by the commutative square in* $D^b(X)_{\mathrm{odd}}$

$$\begin{array}{ccc} F_0 \boxtimes (E_\mathcal{K} \otimes E_\mathcal{L}) & \xrightarrow{\mathrm{ev}_\mathcal{K} \otimes 1} & H_\mathcal{K} \otimes \mathrm{pr}_S^* E_\mathcal{L} \\ \downarrow 1 \boxtimes \varphi_{\mathcal{K},\mathcal{L}} & & \downarrow \rho \\ F_0 \boxtimes E_{\mathcal{K}*\mathcal{L}} & \xrightarrow{\mathrm{ev}_{\mathcal{K}*\mathcal{L}}} & H_{\mathcal{K}*\mathcal{L}}, \end{array}$$

where ρ *is the projection formula, followed by* $\widetilde{T}(\mathcal{K}, \mathrm{ev}_\mathcal{L})$ *and the correspondence-composition isomorphism. This is Yun's map* [31, (3.19)].

Proof. The Hom-isomorphism is Yun's relative cleanness equivalence [31, Thm. 2.4.2, §2.4.6]; the right-hand side retains the $\widetilde{A}$-equivariant structure. [31, Prop. 3.4.1] verifies the single perverse-degree hypothesis of [31, Lemma 2.4.8], which supplies the displayed evaluations. After restriction, apply $(V_\chi^* \otimes -)^{\widetilde{A}}$ to the right-hand composite; the result is precisely the definition in [31, (3.18)–(3.19)], and equivariant full faithfulness gives the

square and uniqueness. Full faithfulness is used only for these odd Hecke outputs and their evaluations: the supported counit on a convolution variety, the Satake associator and maps on the framing covers are not objects of $D^b(X)_{\mathrm{odd}}$, and are compared by functorial base change and localisation on their own spaces (Lemma 3.11). No full faithfulness after forgetting equivariance is used (it fails: $\mathrm{End}_{\widetilde{A}}(V) \subsetneq \mathrm{End}(V)$). The uniqueness is relative to the fixed comparison identifications of Lemma 3.11. □

Remark 3.13 (Normalisation and the bounded release diagram). Restrict the square of Lemma 3.12 with $\mathcal{K} = \mathrm{IC}_\lambda$, $\mathcal{L} = \mathrm{IC}_\lambda^{*2}$ along $u_0 \times \mathrm{id}_S$. Its corners are $V_\chi \otimes E_\mathcal{K} \otimes E_\mathcal{L}$, $rH_\mathcal{K} \otimes E_\mathcal{L}$, $V_\chi \otimes E_{\mathcal{K}*\mathcal{L}}$ and $rH_{\mathcal{K}*\mathcal{L}}$. Replace these outputs by their bounded chain-space descriptions (Lemma 3.11(i)), restrict intermediate endpoints to U using the clean partial transforms, and pull back $P_S \to Q_S \to \Omega_S$, retaining all source and internal invariants. This is the relative evaluation square of Lemma 3.4. The supported morphism is inserted through $i_F : F = m_{3,S}^{-1}(e_S) \hookrightarrow C_{3,S}$ and its counit $i_{F*}i_F^! \to \mathrm{id}$; its fibre is the supported morphism of Lemma 3.5. Applying Lemma 3.14 to this finite diagram and taking the multiplicity functor yields

$$\begin{array}{ccc} (E_B(\mathcal{K}) \otimes E_B(\mathcal{L})) \otimes \mathbb{Q}_\ell & \xrightarrow{\varphi_B \otimes 1} & E_B(\mathcal{K} * \mathcal{L}) \otimes \mathbb{Q}_\ell \\ \downarrow c_\mathcal{K} \otimes c_\mathcal{L} & & \downarrow c_{\mathcal{K}*\mathcal{L}} \\ E_\ell(\mathcal{K}) \otimes E_\ell(\mathcal{L}) & \xrightarrow{\varphi_\ell} & E_\ell(\mathcal{K} * \mathcal{L}). \end{array}$$

The comparison is between realisations of algebraically specified constructions, not analytification of an arbitrary adic morphism. The eigenvalue map lives on S; the odd morphism on X is its tensor with F_0 through the evaluations, so the vertical identifications are of the specified maps, not only of their sources and targets. For general n use $\mathcal{L} = \mathrm{IC}_\lambda^{*(n-1)}$ and the finite diagram of all partial chains; the same construction gives unit and associativity compatibility. Choose $\mathcal{D}_{n,S}$ and a good auxiliary prime separately for each bounded diagram; no uniformity in n is needed.

The identification of φ rests on the construction of Lemma 3.11, Step 2, and on Lemma 3.12. Lemma 3.5 computes a geometric class with this specified map; it is not an independent uniqueness theorem and cannot fix the normalisation of an unidentified alternative tensor structure. (A rescaling $\varphi'_{\mathcal{K},\mathcal{L}} = a_{\mathcal{K}*\mathcal{L}}\varphi_{\mathcal{K},\mathcal{L}}(a_\mathcal{K}^{-1} \otimes a_\mathcal{L}^{-1})$ by a natural automorphism acting by c on the rank-seven simple is again unital and associative, and multiplies the cubic by c^3.)

Lemma 3.14 (Bounded rational comparison)**.** *Let $\mathcal{D}$ be a finite diagram of separated finite-type complex varieties (compactifiable) with finite-group actions, carrying input coefficient systems with finite monodromy and forms over $A = \mathbb{Z}[1/M]$, and let $\mathcal{K}_A$ be a bounded complex built from them by pullback, derived tensor product, Rf_*, $Rf_!$, finite invariants and the base-change, projection-formula, Künneth and adjunction maps. Enlarge M to contain the group orders, the projector denominators and, for this diagram,*

the torsion primes of the relevant constructible cohomology in all neighbouring degrees, and fix an auxiliary prime $\ell \nmid M$ for this diagram; the choice may depend on the diagram, and no prime uniform in n is needed. (For $\ell \mid M$ one would have $A \otimes^L_A \mathbb{Z}/\ell^m = 0$ and the reduction below would be vacuous.) Then the canonical comparison transformations identify the derived reductions $\mathcal{K}_A \otimes^L_A \mathbb{Z}/\ell^m$ with the analytification of the corresponding étale complexes, compatibly in m; after passing to adic systems and inverting ℓ, $(\mathcal{K}_A \otimes \mathbb{Q}) \otimes \mathbb{Q}_\ell$ is the analytification of the rational adic complex, and the specified maps correspond.

Proof. For constructible torsion sheaves and finite-type morphisms the comparison of $R^q f_*$ is [22, Exp. XVI, Th. 4.1] (restated in [23, Arcata, Th. 5.3.6]); the derived proper-support comparison is constructed by compactification and extension by zero [22, Exp. XVII, (5.3.2.1), Th. 5.3.3 and Var. 5.3.4], and for separated morphisms of finite type and torsion sheaves it is [23, Arcata, Th. 4.6.3]. Bounded complexes follow by truncation triangles. Compatibility with pullback, derived tensor products and the base-change and adjunction maps concerns the canonical comparison transformations, not arbitrary isomorphisms of source and target; the proper-support comparison is built from them, and the projection formula and Künneth are their composites. Finite-group equivariance follows from naturality, with averaging only after the group order is inverted. A free middle cohomology group alone does not commute reduction with cohomology (a Tor_1 of the neighbouring group can occur), which is why M is enlarged, or the derived form is kept until rationalisation; the choice of M is per diagram, not uniform in n. For a closed immersion i with open complement j, $i_* i^! K$ is determined by the triangle $i_* i^! K \to K \to R j_* j^* K$, so supported morphisms and their counits are compared by the same canonical maps; the fundamental-class normalisation is the complex orientation on the smooth locus of each component. □

Lemma 3.15 (Cycle compatibility, correctly typed)**.** *For a smooth variety X and an algebraic cycle Z, the rational Betti and ℓ-adic cycle classes of Z correspond under Lemma 3.14, with $\mathbb{Z}(1) = 2\pi i\,\mathbb{Z}$. On a singular variety the statement is made for supported fundamental classes in Borel–Moore homology, followed by rational duality on rationally smooth spaces; no ordinary cycle class on a general singular variety, and no identification of a singular torsion dualizing complex with a shifted constant sheaf, is used.*

Proof. Comparison matches Kummer first Chern classes, and the projective-bundle characterisation then matches all Chern classes; the leading term of the Chern character of a finite locally free resolution of $\mathcal{O}_Z$ is $[Z]$ [24, §23, Thms. 23.3–23.4]: Theorem 23.3 lists the properties characterising the Chern classes and Theorem 23.4 states that the resulting Chern-class cycle map agrees with the directly defined one; the passage through the comparison isomorphism is ours. For fixed dimension the finitely many factorial

denominators are inverted first. The supported form is Lemma 3.14 applied to the localization triangle. □

Lemma 3.16 (Rational clean comparison)**.** *Let $j : U \hookrightarrow Y$ be an open immersion of finite-type varieties and L_A a finite-monodromy lattice over $A = \mathbb{Z}[1/M]$. If $j_! L_\ell \to Rj_* L_\ell$ is an isomorphism for the corresponding rational adic sheaf, then the canonical Betti map $j_! L_B \to Rj_* L_B$ is an isomorphism. In particular $j'_! \mathbb{L} \cong Rj'_* \mathbb{L}$ on Y^{an} (Lemma 2.2), and Theorem 2.1 holds for the Betti sheaves on every finite-type smooth chart of* Bun.

Proof. Let C_A be the cone of the integral Betti map. Its stalk cohomology is bounded and finitely generated. Finite-coefficient comparison, derived completion and rationalisation (Lemma 3.14) identify $C_A \otimes_A \mathbb{Q}_\ell$ with the analytification of the rational adic cone, which vanishes. Hence $(C_A \otimes_A \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = 0$, and faithful scalar extension, stalk by stalk, gives $C_A \otimes_A \mathbb{Q} = 0$. (No vanishing of $C_A \otimes^L \mathbb{Z}/\ell$ is claimed: rational vanishing permits ℓ-torsion.) For Y: ω is smooth (Lemma 2.2), so smooth base change of Yun's clean extension gives the rational adic isomorphism, and the odd rank-four coefficient is, after scalar extension, a sum of copies of the split odd coefficient. On a smooth chart of Bun the same argument applies, smooth base change identifies the pulled-back canonical map, and vanishing is local on the atlas. □

Lemma 3.17 (The rank-seven local system)**.** *Let $\widetilde{\pi} : \widetilde{Y} \to S$ be Yun's framed one-step family* [31, §3.2.7] *and $N_S = \varepsilon\, R^6 \widetilde{\pi}^{\mathrm{an}}_! \mathbb{Q}(3)$. Then N_S is a rank-seven $\mathbb{Q}$-local system on S^{an}, its fibre at x is $N_x = M_x(3)$, and $N_S \otimes \mathbb{Q}_\ell \cong E_\chi(V_7)$ as local systems on S^{an}, along $\pi_1^{\mathrm{top}}(S) \to \pi_1^{\mathrm{et}}(S_{\mathbb{C}})$.*

Proof. Start from the rational Betti complex $K_B = \varepsilon R\widetilde{\pi}^{\mathrm{an}}_! \mathbb{Q}(3)$, with its form over A, and apply Lemma 3.14 to the derived reductions. By [31, (3.13)–(3.14)] and the identity-line projector [31, §3.3.2], comparison is relative, so it identifies transport, not only fibre dimensions. Comparison identifies $\mathcal{H}^6(K_B) \otimes \mathbb{Q}_\ell$ with $E_\chi(V_7)$ by [31, (3.13)–(3.14)]. Thus $N_S = \mathcal{H}^6(K_B)$ is a rank-seven local system: constructibility and faithful scalar extension show that its local specialisation maps are isomorphisms. No concentration claim is made for K_B itself (removing the star leaves an odd degree-one term, cf. the proof of Theorem 4.3). The full bounded Schubert family, including the star, is used for the Hecke eigenvalue complex; localisation identifies its degree-six part with the open family's. The full odd degree-six cohomology has rank 28; ε cuts out rank seven (Morita, Section 2). A monodromy-stable lattice exists after localisation [31, §4.5.2]; it is not a finite-monodromy lattice, since the monodromy is dense in G_2 (Lemma 3.20). □

Lemma 3.18 (Comparison of the specified relative tensor maps)**.** *For the bounded relative Hecke diagrams over S (base change along $u_0 \times \mathrm{id}_S$,* [31, §3.2.3]*), construct $E_B(K)$ using the rational odd bimodule D, its right-D-linear multiplicity functor and the canonical evaluation morphism, and let*

c_K be its comparison with Yun's $E_\ell(K)$, restricted to S^{an}. Then

$$\begin{array}{ccc} (E_B(K_1)\otimes E_B(K_2))\otimes\mathbb{Q}_\ell & \xrightarrow{\varphi_B\otimes 1} & E_B(K_1*K_2)\otimes\mathbb{Q}_\ell \\ \downarrow c_{K_1}\otimes c_{K_2} & & \downarrow c_{K_1*K_2} \\ E_\ell(K_1)\otimes E_\ell(K_2) & \xrightarrow{\varphi_\ell} & E_\ell(K_1*K_2) \end{array}$$

commutes, where φ_B is defined by the projection formula, the partial-transform evaluation map and convolution base change, in target-first order (Lemma 3.4, relative over S); it is not an independently chosen tensor map. The same holds for the unit and every associativity diagram. Consequently the rational maps are isomorphisms, unital and associative, and τ_u of Proposition 3.6, formed with φ_B, becomes Yun's after $\otimes\mathbb{Q}_\ell$.

Proof. By Lemmas 3.11 and 3.12, Yun's $E_\ell(K)$, its evaluation and φ_ℓ are specified by the odd-output evaluation square, whose restriction along $u_0\times\mathrm{id}_S$ is the stated construction on the finite-type framed diagram $\mathcal{D}_{n,S}$ (Remark 3.13); base change along $u_0\times\mathrm{id}_S$ and pullback of the finite framing covers are therefore identifications of the specified maps. Base change along $u_0\times\mathrm{id}_S$, then pull back the finite framing covers. The two paths are the same composite of evaluation, projection formula, base change, finite invariants and Künneth maps, and canonical comparison respects each (Lemma 3.14, first with finite coefficients for this bounded diagram, then rationalised). Yun's tensor map is defined by this composite [31, §3.4.4, (3.18)–(3.19)], so its isomorphism, unit and associativity properties descend by faithful extension of scalars. The intermediate complexes on chain spaces are constructible complexes, not local systems; only the extracted eigenvalue objects on S are local systems, and pushforward of finite-monodromy input can have infinite monodromy. No symmetric (fusion) compatibility of φ_B is claimed. □

Lemma 3.19 (Relative supported morphism, flatness and nonvanishing)**.** *For $(u,n)=(b,2),(t,3)$ the unique top component of the relative identity-composite fibre over S defines a relative supported morphism $s_{u,S}$ from the unit, and*

$$\tau_u=\iota^{\otimes n}\varphi_{n,B}^{-1}E_B(s_{u,S})(1)$$

is a nonzero flat section of $N_S^{\otimes n}$, with fibre $\tau_{u,x}$ of Proposition 3.6. The degree-six map from N_S to ordinary fibre cohomology, and its tensor powers, are injective, so the ordinary identity $\varepsilon^{\otimes n}\,\mathrm{cl}(W_x)=2^{-n-1}\mathrm{fgt}_P(\tau_{u,x})$ of Lemma 3.5 determines $\tau_{u,x}$.

Proof. The coordinate $t=z-x$ near the moving point and a local trivialisation of the fixed target bundle give local convolution models; the unique top component for $n=2,3$ is preserved on overlaps, so its reduced fundamental class, with complex orientation, defines a global relative supported morphism, and the endpoint sheet glues because the source endpoint is canonically the target on the identity-composite fibre. These constructions are algebraic and choose no sign by parallel transport. Every map in the displayed expression is a morphism of local systems over S (Lemma 3.18),

so τ_u is flat. Its scalar extension is the image of the nonzero supported Satake morphism under Yun's faithful tensor functor (Proposition 3.7), hence nonzero. A global class on P_S gives a section of $R^{18}p_*\mathbb{Q}(9)$, not automatically of $N_S^{\otimes 3}$, and nonzero τ need not have nonzero forget-supports image; injectivity is therefore proved. In the proper Schubert compactification with the interior star, clean extension across D_x (Lemma 3.16) identifies compact and ordinary cohomology of the odd block. Removing the star orbits changes neither in degree six: compact removal has only degree-zero terms from the finite star fibre, and rational smoothness (Lemma 2.4) puts local cohomology at the star in degree 12 (Remark 3.10). The same natural maps exist relatively over S, so the map is injective, and so are its tensor powers. □

Lemma 3.20 (Dense Betti monodromy)**.** *The monodromy of the $\mathbb{Q}$-local system N_S is Zariski dense in* G_2.

Proof. Yun's rational Betti construction [31, §4.5.1] has cohomological realisation through [31, (3.14)] and dense topological image [31, §4.5.2]; see also [32, §6.2.4]. Through the identity-line projector ε and the Morita identification (the raw rank-28 odd block is not that representation), this is the local system N_S of Lemma 3.17. □

Lemma 3.21 (Alternation)**.** *τ_b spans the invariant symmetric pairing line of $N_S^{\otimes 2}$ and τ_t spans the invariant alternating cubic line of $N_S^{\otimes 3}$, for the usual permutation action;* $\mathrm{Stab}(\tau_b, \tau_t) = G_2$. *No symmetric (fusion) comparison is used.*

Proof. By Lemmas 3.19 and 3.20, τ_u is a nonzero G_2-fixed vector. $\mathrm{Sym}^2 V_7 = \mathbf{1} \oplus V_{27}$ and $\Lambda^2 V_7 = V_7 \oplus V_{14}$ show that the invariant pairing is symmetric and that the invariant cubic is antisymmetric in the first two factors. $\dim(V_7^{\otimes 3})^{G_2} = \dim \mathrm{Hom}_{G_2}(V_7, V_7 \otimes V_7) = 1$, so the permutation action on the invariant cubic line is a character of S_3; all transpositions are conjugate, so it is the sign character. All degrees are even, so there are no Koszul signs. The alternating cubic together with the nondegenerate invariant quadratic form has stabiliser G_2. □

Remark 3.22 (What rests where).

- Clean sheaf on Y (Lemma 2.2): Lemma 3.14, Yun's étale cleanness and smooth base change, through Lemma 3.16. The literal stack statement of Theorem 2.1 is used for Betti sheaves only on finite-type smooth charts.
- Rank-seven family: Lemma 3.17 (comparison of local systems, not of stalks).
- Supported local identity and all-U extension: Lemma 3.14, the supported form of Lemma 3.15, and Lemma 3.4; the normalisation of Lemma 3.5 is unchanged.
- Relative unit and associativity: Lemma 3.18 (Yun's actual structure, not independence of arbitrary structures).
- Alternating cubic and stabiliser: Lemmas 3.19, 3.20, 3.21; no fusion comparison.

- Arithmetic rank-seven realisation: [31, §3.3.3] and the $\mathbb{Q}$-defined ε; complex comparison alone is not descent data.
- Theorem A at rational x: prior geometric motivatedness, $\mathbb{Q}$-defined geometry, Lemma 3.15 and comparison of $s, r, (rs)^{-1}$ (Lemma 3.23); no weighted-model premise.
- Generic $\mathrm{MT}(N_x) = \mathrm{G}_2$ (Section 8): the algebraic pairing and cubic give the upper bound; Lemma 3.20 and the inclusion of connected monodromy in the Mumford–Tate group at Hodge-generic points give the lower bound.

Lemmas 3.14 and 3.15 are classical comparison statements, cited for the bounded diagrams used; no comparison input is left as a hypothesis. The statement numbers above were checked against the texts of [22], [23] and [24]; no single source states the comparison of cycle classes for arbitrary varieties, and none is used. For every tensor power the bounded argument is repeated for each n; no denominator uniform in n and no Galois descent of individual components beyond $n = 2, 3$ is needed.

3.5. **The clean projector.** Let $B = \mathrm{Bl}_\star \mathrm{Gr}_{x,\leq\lambda}$, $B \setminus D_x$ its open part (we reserve B° for the trivial-inertia boundary locus of Section 4), and $X^\circ \to B \setminus D_x$ the finite étale cover; the exceptional locus over $\star$ is a disjoint union of copies of $E = \mathrm{G}_2/P$, interior to X°. By Lemma 2.2, on the odd block $H^k_c(X^\circ) \to H^k(X^\circ)$ is an isomorphism, and both sides are pure of weight k. For an equivariant smooth projective compactification $\overline{X}$ with normal crossing boundary, let $s : H^k_c(X^\circ) \to H^k(\overline{X})$ and $r : H^k(\overline{X}) \to H^k(X^\circ)$; then rs is an isomorphism on the block and

$$H^k(\overline{X})_{\mathrm{odd}} = \mathrm{im}(s)_{\mathrm{odd}} \oplus \ker(r)_{\mathrm{odd}}.$$

By weights, $\mathrm{im}(s)$ is the kernel of the restrictions to the smooth boundary components and $\ker(r)$ is the sum of their Gysin images, so both are sub-objects in André's category, and semisimplicity [2] gives a motivated projector P_{clean} onto $\mathrm{im}(s)$ along $\ker(r)$. The exceptional divisor E is cellular, so cup product with $c_1(N_{E/B})$ has an algebraic inverse on the nonzero Lefschetz strings; the resulting algebraic projector $P_\star$ removes the Tate summands in degrees $2, \ldots, 10$. [ledger 11] Explicitly, $P_\star = \Delta - \Pi_E$, where Π_E is a $\mathbb{Q}$-combination of products of cellular cycles on $E \times E$ (the Gram-inverse projector onto the span of the classes supported on E); so $\mathrm{im}\,\Pi_E$ consists of classes supported on E. Put $P_A = \varepsilon P_\star P_{\mathrm{clean}}$.

Lemma 3.23 (Arithmetic fixedness of the constructed class)**.** *For $x \in S(\mathbb{Q})$ choose the equivariant smooth compactification and its boundary resolution over $\mathbb{Q}$. The canonical-sheet cycle, its closure, ε and $P_\star$ descend to $\mathbb{Q}$. On the clean block put $P_{\mathrm{clean}} = s(rs)^{-1}r$. Comparison identifies this formula in the Betti and adic realisations. Consequently the adic realisation of $(\varepsilon P_\star P_{\mathrm{clean}})^{\otimes 3}\,\mathrm{cl}(\overline{W}_t)$ is fixed by $G_{\mathbb{Q}}$ in the Tate-twisted cohomology $H^{18}(\overline{X}^3_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(9))$. Once this class is motivated over $\overline{\mathbb{Q}}$,* [2, §2.5 a), Scolie, p. 17] *makes it motivated over $\mathbb{Q}$.*

Proof. $u_0 \in U(\mathbb{Q})$ [31, §2.2.6], and ε is a $\mathbb{Q}$-correspondence (e_χ is Galois-invariant, [31, §3.3.2]; the Galois-invariant sum of graphs descends even if the finite group scheme is not constant). At rational x the chain and framing spaces are defined over $\mathbb{Q}$; the unique geometric top component Z_9 for $n = 3$ is Galois-stable, hence $\mathbb{Q}$-defined in characteristic zero; the canonical sheet is defined by the isomorphism of the source with the fixed target given by the composite identity modification, which descends, so W_t is $\mathbb{Q}$-defined (no claim is made for individual components at other n). The compactification and boundary resolution exist equivariantly over $\mathbb{Q}$ in characteristic zero, with boundary components possibly permuted by Galois, and the closure of W_t in it is $\mathbb{Q}$-defined. The exceptional subspace is Galois-stable, and its Gram-inverse cellular projector is independent of the cellular basis and invariant under permutation of exceptional components, so $P_\star$ is defined over $\mathbb{Q}$ as a correspondence. Both s_ℓ and r_ℓ are Galois-equivariant, hence so is $(r_\ell s_\ell)^{-1}$ on the clean block and $P_{\mathrm{clean},\ell} = s_\ell(r_\ell s_\ell)^{-1} r_\ell$; the motivated projector is the unique projector onto $\operatorname{im} s$ along $\ker r$, so it has this realisation, and its Betti counterpart is identified by comparison of these maps (Lemmas 3.14, 3.15). Fixedness of an adic realisation alone does not produce a motivated class: geometric motivatedness is established first. □

Proof of Theorem A. The class $E(t) = P_A^{\otimes 3}\,\mathrm{cl}(\overline{W}_t)$ is a motivated projector applied to an algebraic class, hence motivated [2, §§1–2]. Its restriction to $\widetilde{Y}_x^{\,3}$ (Remark 3.10) is, by Proposition 3.6, Yun's $E(s_t)_x$ transported through the iterated (3.19), and by Proposition 3.7 the projected class of the specific unweighted cycle is $\frac{1}{16}\tau_t \neq 0$ on Yun's tensor unit (equivalently, $\frac{1}{4}\tau_t$ for $4\overline{W}_t$). □

3.6. **From the compact Satake cycle to a closed cycle.** Fix n and u. Write $C = C_n^{(U)}$ (rationally smooth of dimension $nd = 6n$), $\Omega = C_n^{U^n} \subset C$ (open), $Z = Z_u \subset m_n^{-1}(e) \subset C$ (compact), $Z^\circ = Z \cap \Omega$ (closed in Ω, not proper in general), and $\mathbb{G} = \omega_n^*\mathbb{F}$. Over $m_n^{-1}(e)$ the source endpoint b_n is canonically u_0, so $\mathbb{G}|_Z$ is constant with fibre D. Let $P \xrightarrow{\pi} Q \xrightarrow{q} \Omega$ be the tower of the framing paragraph (degrees $8^{\,n-1}$ and 8; 64 and 8 for $n = 3$), and $\widetilde{Y}_x^{\,n} \subset P$ the open complement of the $\star$-orbits. Let R be as in Theorem 4.3 for the weighted clean model of Theorem 5.9, $\partial \cup E_\star \subset R$ the complement of $\widetilde{Y}_x$, $r : H^*(R) \to H^*(\widetilde{Y}_x)$ restriction, and $s : H_c^*(\widetilde{\mathrm{GR}}_x^U) \to H^*(R)$ extension by zero after the blow-up at $\star$.

All maps below are $\mathbb{Q}$-linear and equivariant for the external actions ($\widetilde{A}(2)$ at u_0, $\widetilde{A}(1)$ at b_n) and for the internal contraction group $\widetilde{A}^{\,n-1}$; the coefficient D is retained until the partial reduced trace Θ, and the factorwise ε acts on

P only after Θ:

$$\begin{array}{ccccc} H^{BM}_{6n}(Z)\otimes D & \xrightarrow{i_*} & H^{6n}_c(C,\mathbb{G}) & \xleftarrow[\cong]{\text{ext}} & H^{6n}_c(\Omega,\mathbb{G}) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}\circ\text{fgt}_C} & & \downarrow{\scriptstyle \text{fgt}_\Omega} \\ H^{BM}_{6n}(Z^\circ)\otimes D & \xrightarrow{i^\circ_*} & H^{6n}(\Omega,\mathbb{G}) & = & H^{6n}(\Omega,\mathbb{G}) \end{array}$$

followed by

$$\begin{array}{ccccc} H^{6n}_c(\Omega,\mathbb{G}) & \xrightarrow{q^*} & \big(D\otimes H^{6n}_c(Q)\big)^{\widetilde{A}} & \xrightarrow{\pi^*} & \big(D\otimes H^{6n}_c(P)\big)^{\widetilde{A}\times\widetilde{A}^{\,n-1}} \\ \downarrow{\scriptstyle \text{fgt}_\Omega} & & \downarrow{\scriptstyle \text{fgt}} & & \downarrow{\scriptstyle \text{fgt}} \\ H^{6n}(\Omega,\mathbb{G}) & \xrightarrow{q^*} & \big(D\otimes H^{6n}(Q)\big)^{\widetilde{A}} & \xrightarrow{\pi^*} & \big(D\otimes H^{6n}(P)\big)^{\widetilde{A}\times\widetilde{A}^{\,n-1}} \end{array}$$

Here i_* is proper pushforward of the compact cycle followed by Poincaré duality on the rationally smooth C; ext is extension by zero, an isomorphism by Lemma 3.2; the left vertical map is restriction of Borel–Moore classes to the open Z°; i°_* is closed pushforward followed by Poincaré duality on Ω; the middle vertical map includes the restriction from C to Ω; and fgt is forget-supports. On cycles, $\mathrm{CH}_{3n}(Z)\to\mathrm{CH}_{3n}(Z^\circ)$ is flat pullback to the open (surjective, with kernel supported on $Z\setminus Z^\circ$), q^* and π^* are flat pullbacks, and all commute with the Borel–Moore cycle class; the bottom row is *ordinary* cohomology, which is where the class of the non-proper Z° and of its preimages lives.

Commutativity. The left square is base change of open restriction past a closed pushforward. The middle square is the identity $\text{fgt}_\Omega = \text{res}_{\Omega\subset C}\circ\text{fgt}_C\circ$ ext. The right squares hold because q and π are finite étale. Hence, for the $\widetilde{A}$-map $\beta : v\mapsto \text{ext}^{-1}(i_*[Z\otimes v])$, Proposition 3.6(c) gives

$$\Theta\,\text{fgt}\,(\pi^*q^*\beta) \;=\; 4\,e^{(n)}_-\,\text{cl}(W_u) \qquad \text{in } H^{6n}(P)(3n), \qquad W_u=\pi^{-1}(\widetilde{Z}_u),$$

where $\widetilde{Z}_u\subset Q$ is the sheet of $q^{-1}(Z^\circ)$ on which the frame of b_n is the canonical one, and $e^{(n)}_-$ is the odd projector of the source action on factor n; since $\varepsilon^{\otimes n}e^{(n)}_- = \varepsilon^{\otimes n}$, it disappears after ε. Components of Z inside $C\setminus\Omega$ contribute 0: since ext is an isomorphism in all degrees, $H^*_c(C\setminus\Omega,\mathbb{G})=0$.

Lemma 3.24. (a) *On $\varepsilon H^6_c(\widetilde{\mathrm{GR}}^U_x)$ the map* $\text{res}\circ\text{fgt}$ *to $H^6(\widetilde{Y}_x)$ is injective, and s followed by $P_\star$ maps it isomorphically onto $\varepsilon P_\star H^6(R)=M_x$, with $r\circ s=\text{res}\circ\text{fgt}$. This uses that the defect of the weighted clean model vanishes (Theorem 4.3(b)).* (b) *$\varepsilon P_\star$ annihilates* $\ker(r)=\mathrm{im}\big(H^{BM}_*(\partial\cup E_\star)\to H^*(R)\big)$, *and $r\circ P_\star = r$.* (c) *For every closed cycle $\overline{W}\subset R^n$ with $\overline{W}\cap\widetilde{Y}^{\,n}_x = W$, $(\varepsilon P_\star)^{\otimes n}\,\text{cl}(\overline{W})$ depends only on W, and is the unique element of $M_x^{\otimes n}$ whose image under $r^{\otimes n}$ is $\varepsilon^{\otimes n}\,\text{cl}(W)$.* (d) *The same holds with $(R,\varepsilon P_\star)$ replaced by $(\overline{X},P_A)$ for the compactification $\overline{X}$ of Section 3.*

Proof. (a) fgt is an isomorphism on the odd block by Lemma 2.2, and restriction past the $\star$-orbits is an isomorphism in degree 6 by Remark 3.10; the rest is Theorem 4.3(b). (b) $P_\star=\Delta-\Pi_E$ with $\mathrm{im}\,\Pi_E\subset\ker r$, so $rP_\star=r$ and $P_\star(\ker r)\subset\ker r$. If $y\in\ker r$, then $\varepsilon P_\star y\in\ker r$ (as ε commutes with r)

and $\varepsilon P_\star y \in M_x$, on which r is injective by (a); so $\varepsilon P_\star y = 0$. (c) Two closures differ by a cycle supported on $\bigcup_k R^{k-1} \times (\partial \cup E_\star) \times R^{n-k}$. Each irreducible component of that cycle lies in one of these closed sets, so its class is a sum of classes each supported on a single $R^{k-1} \times (\partial \cup E_\star) \times R^{n-k}$, which by Künneth on the compact R^n lies in $H^*(R)^{\otimes k-1} \otimes \ker r \otimes H^*(R)^{\otimes n-k}$; $(\varepsilon P_\star)^{\otimes n}$ kills it by (b). Finally $r^{\otimes n}(\varepsilon P_\star)^{\otimes n} \operatorname{cl}(\overline{W}) = \varepsilon^{\otimes n} r^{\otimes n} \operatorname{cl}(\overline{W}) = \varepsilon^{\otimes n} \operatorname{cl}(W)$, and $r^{\otimes n}$ is injective on $M_x^{\otimes n}$ (Künneth for the ordinary cohomology of $\widetilde{Y}_x^{\,n}$, and (a)). (d) P_{clean} is the projection onto $\operatorname{im}(s)$ along $\ker(r)$, so $P_A^{\otimes n}$ kills every class with a factor in $\ker r$, and r is injective on $P_A H^6(\overline{X}) = M_x$; the argument of (c) applies verbatim. □

Proposition 3.25. *$(\varepsilon P_\star)^{\otimes n} \operatorname{cl}(\overline{W}_u) = \frac{2^{1-n}}{4} \tau_u(-3n) = 2^{-n-1}\tau_u(-3n) \in M_x^{\otimes n} \subset H^{6n}(R^n)$ for every closure $\overline{W}_u \subset R^n$ of $W_u \cap \widetilde{Y}_x^{\,n}$, with $W_u = \pi^{-1}(\widetilde{Z}_u)$ and τ_u as in Proposition 3.6.*

Proof. By Lemma 3.24(c) it suffices to compute $\varepsilon^{\otimes n} \operatorname{cl}(W_u)$ in $H^{6n}(\widetilde{Y}_x^{\,n})$. □

Lemma 3.26 (Normalized framed component comparison)**.** *Fix the target-first convention $T(\mathcal{K}_1 * \mathcal{K}_2, \mathcal{F}) = T(\mathcal{K}_1, T(\mathcal{K}_2, \mathcal{F}))$ and the chain $b_3 \to b_2 \to b_1 \to b_0 = u_0$. Let $Z \subset m_3^{-1}(e)$ be the reduced top component, and let s_Z be its complex-oriented Satake morphism. Put $C = C_3^{(U)}$, $\Omega = C_3^{U^3}$ and $\mathbb{G} = \omega_3^*\mathbb{F}$. Let $P = X^3 \xrightarrow{\pi} Q \xrightarrow{q} \Omega$ be the all-frame/endpoint-frame tower, of degrees 64 and 8. Let $\widetilde{Z} \subset Q$ be the canonical endpoint sheet over $Z \cap \Omega$, and $W = \pi^{-1}\widetilde{Z}$. After splitting D, define*

$$\Psi(vf \otimes e) = (f \otimes 1)e(v), \qquad \iota(e) = \Psi(I \otimes e) = \sum_a (e^a \otimes 1)e(e_a).$$

This definition descends rationally using partial reduced trace. For $1_{\text{Yun}} = \mathrm{id}_V$ put

$$\tau = \iota^{\otimes 3}\varphi_3^{-1}E(s_Z)_x(1_{\text{Yun}}).$$

Then, in ordinary cohomology,

$$\varepsilon^{\otimes 3} \operatorname{cl}(W) = \frac{1}{16} \operatorname{fgt}_P(\tau) \in H^{18}(P, \mathbb{Q})(9).$$

In particular $4W$ has class $\operatorname{fgt}_P(\tau)/4$ after projection. If the injectivity and boundary-kernel conclusions of Lemma 3.24 hold, every closure in R^3 satisfies

$$(\varepsilon P_\star)^{\otimes 3} \operatorname{cl}(\overline{W}) = \frac{1}{16}\tau(-9).$$

Proof. Proper extraordinary base change identifies s_Z with the fundamental-class morphism into $i_F^!\mathbb{Q}_{C_3}[18](9) = \omega_{m_3^{-1}(e)}[-18](-9)$. Tensoring its adjunction counit with $\mathbb{G}$ and taking $R\Gamma_c$ sends v to $\alpha_v = \mathrm{PD}_c[Z \otimes v]$. Yun's Theorem 2.4.2 and Lemma 2.4.8 applied to the inner partial Hecke transforms give the natural isomorphism $\mathrm{ext} : R\Gamma_c(\Omega, \mathbb{G}) \simeq R\Gamma_c(C, \mathbb{G})$. For $\beta_v = \mathrm{ext}^{-1}\alpha_v$, open/closed base change gives

$$\operatorname{fgt}_\Omega \beta_v = \mathrm{PD}_\Omega[(Z \cap \Omega) \otimes v].$$

Pullback to the endpoint cover is $\sum_g gv \otimes [g\widetilde{Z}]$, with g denoting the source action. Partial reduced trace gives $4e_-^{\mathrm{end}}[\widetilde{Z}]$. Further finite étale pullback therefore gives $4e_-^{\mathrm{end}}\,\mathrm{cl}(W)$ in ordinary cohomology.

On the other hand, Yun's evaluation maps (2.4.8), (3.8)–(3.9) give, with row = target and column = source,

$$e_i(e_a) = \sum_b e_b \otimes E_{ab} \otimes \iota(e_i).$$

The ordered composition defining (3.19) sends e_a to

$$\sum_{b,c,d} e_d \otimes E_{ab} \otimes E_{bc} \otimes E_{cd} \otimes \iota(e_1) \otimes \iota(e_2) \otimes \iota(e_3).$$

This is invariant under both adjacent inverse-right/left actions, since $\sum_b g_{vb}(g^{-1})_{bu} = \delta_{vu}$. The partial trace sets $d = a$ and gives $J \otimes \tau$, where

$$J = \sum_{a,b,c} E_{ab} \otimes E_{bc} \otimes E_{ca} = \frac{1}{4}\sum_{s,t} b_s \otimes b_s^{-1} b_t \otimes b_t^{-1}.$$

The ordered composition is the right-hand composite of Lemma 3.4, and the commutative diagram of Lemma 3.5 (case $n = 3$) compares it with β before forgetting supports, and then after forgetting supports. Apply $\varepsilon^{\otimes 3}$: it absorbs e_-^{end} and sends J to $1/4$. This proves the factor $1/16$. All geometric degrees are even and no factor reversal is made. Finally restriction, boundary annihilation and injectivity in Lemma 3.24 give the closure assertion. □

The case $n = 3$. $C = C_3^{(U)}$ has dimension 18, $Z = Z_t = c\,Z_9$ with Z_9 the unique nine-dimensional component of $m_3^{-1}(e)$, and every class above lies in degree 18 (Borel–Moore degree 18 on Z). On Z_9 the source endpoint b_3 is u_0, so $\Omega \cap Z_9 = \{b_1, b_2 \in U\}$, a dense open subset of the irreducible Z_9; it is nonempty, because otherwise $i_*[Z_9]$ would be supported on $C \setminus \Omega$ and vanish, contradicting $E(s_t)_x \neq 0$. The tower is $P = (\widetilde{\mathrm{GR}}_x^U)^3 \xrightarrow{\pi} Q \xrightarrow{q} \Omega$, of degrees 64 and 8, with internal group $\widetilde{A}^2$ and the canonical frame of b_3 on Z_9; $J_3 = \frac{1}{4}\sum_{s,t} b_s \otimes b_s^{-1} b_t \otimes b_t^{-1}$; $\varepsilon^{\otimes 3} J_3 = \frac{1}{4}\,1^{\otimes 3}$; and Proposition 3.25 (equivalently Lemma 3.26) gives $(\varepsilon P_\star)^{\otimes 3}\,\mathrm{cl}(\overline{W}_t) = \frac{1}{16}\tau_t(-9)$ on Yun's tensor unit, and $(\varepsilon P_\star)^{\otimes 3}\,\mathrm{cl}(4\overline{W}_t) = \frac{1}{4}\tau_t(-9)$, for every closure of W_t in R^3. Rescaling $Z_t = cZ_9$ rescales τ_t and W_t together; the ratio $\frac{1}{16}$ is unchanged.

4. The clean piece of a good model

Theorem B needs an *algebraic* replacement for P_{clean}. This section isolates the property of a compactification that provides one.

Definition 4.1. Let $B = \mathrm{Bl}_\star \mathrm{Gr}_{x,\leq\lambda}$ as in Section 3, so that $\star$ is replaced by the interior divisor $E \cong \mathrm{G}_2/P$ and $B \setminus D_x \supset E$. A *good model* of (Y, D_x) is a proper morphism $\pi : \mathcal{X} \to B$ from a smooth Deligne–Mumford stack, an isomorphism over $B \setminus D_x$, such that $\pi^{-1}(D_x)$ is a divisor whose components

are smooth, individually labelled Cartier divisors, meeting as coordinate hyperplanes in smooth charts, with every finite stabiliser preserving each local branch.

Definition 4.2. A *weighted clean model* of (Y, D_x) is a good model $\mathcal{X} = Y \times_{\mathcal{B}} \mathcal{B}_N$ (glued with a neighbourhood of E in B) obtained from a sequence of stack-theoretic weighted blow-ups $\mathcal{B}_N \to \cdots \to \mathcal{B}_0 = \mathcal{B}$ at points $x_1, \ldots, x_N$ of $\mathcal{B} \setminus U$, each closed in $U_{n-1} = \mathcal{B} \setminus \{x_1, \ldots, x_{n-1}\}$, with the following data and properties:

(1) at each x_n, an Alper–Hall–Rydh chart $([W_n/T_n], w) \to \mathcal{B}$ and a cocharacter $\bar{\lambda}_n$ of T_n, invariant under the finite group Γ_{x_n} of Section 5, with positive weights on W_n and weights ≤ 0 on $\operatorname{Lie} \mathcal{G}_{x_n} / \operatorname{Lie} T_n$ (conditions (G1) and (G2) of Hypothesis 5.1), preserving D_{W_n}, whose weighted filtration (descended by Proposition 5.7) is the centre of the n-th blow-up;
(2) the invariant (I_n) of Theorem 5.9 (charts V with smooth (g_V, s) and every tracked ideal equal to g_V^* of its ideal on $\mathcal{B}$);
(3) $\mathbb{F}$ is clean at every point of $\mathcal{B} \setminus U$ met by $\omega(Y)$, and every such point of codimension ≥ 2 is among the x_n.

Let R be the normalisation of the coarse space of a good model in the function field of T_x, with $p : R \to X$ the finite map to the coarse space X, $\partial \subset X$ the preimage of D_x, $j : U_X = X \setminus \partial \hookrightarrow X$, and $\mathcal{F} = e_- p_* \mathbb{Q}_R$.

Theorem 4.3. (a) *For every good model, R has only quotient singularities, and there is a split exact sequence*

$$0 \to H_c^k(B \setminus D_x, \mathbb{L}) \to H^k(R)_{\mathrm{odd}} \to H^k(\partial, \mathcal{F}) \to 0.$$

Here $H_c^*(B \setminus D_x, \mathbb{L})$ is $H_c^6(T_x)_{\mathrm{odd}}$ in degree six together with the $\star$-Tate summands $D \otimes H^{2i}(E)$, $1 \leq i \leq 5$, and $H_c^0 = H_c^1 = 0$.

(b) *For every weighted clean model the defect $H^*(\partial, \mathcal{F})$ vanishes (Theorem 4.4), so that $\varepsilon P_\star$ restricted to $H^*(R)$ is a projector of rank seven, with image $M_x \subset H^6(R)$. The model of Theorem 5.9 is a weighted clean model.*

(c) *Part (b) fails for general good models (Example 4.14).*

Proof. Quotient singularities. At a point of the good model the local model is a finite quotient of ($\mathbb{C}^n$, coordinate hyperplanes); the cover is Kummer over the hyperplanes, so its normalisation is $\mathbb{C}^n$ modulo an extension of the stabiliser by a finite abelian group, which acts because the stabiliser preserves each hyperplane. [ledger 6]

Stalks. At a point e with local image $I_e \subset \widetilde{A}$ of the orbifold fundamental group of the complement, $\mathcal{F}_e = e_- \mathbb{Q}[I_e \backslash \widetilde{A}] = D^{I_e}$. If $I_e \neq 1$ then $z \in I_e$, because every nontrivial subgroup of Q_8 contains z; then z acts trivially on $I_e \backslash \widetilde{A}$ while e_- requires $z = -1$, so $\mathcal{F}_e = 0$. If $I_e = 1$, $\mathcal{F}_e = D$. Let B° be the set of boundary points with $I_e = 1$ (necessarily off the strict transform of D_x, whose meridians are nontrivial by cleanness) and $\overline{\mathbb{L}}$ the extension of

$\mathbb{L}$ across it. Then $\mathcal{F} = j_{U_X \cup B^\circ,!}\overline{\mathbb{L}}$, and $\mathcal{F} = R^0 j_* \mathbb{L}$ because R is normal; no dual enters.

The split sequence. The triangle $j_!\mathbb{L} \to \mathcal{F} \to i_*(\mathcal{F}|_\partial)$ gives a long exact sequence whose composite $H^k_c(U_X, \mathbb{L}) \to H^k(X, \mathcal{F}) \to H^k(U_X, \mathbb{L})$ is the forget-supports map, an isomorphism by Lemma 2.2. Here $U_X = B \setminus D_x$, and $H^*(X, \mathcal{F}) = H^*(R)_{\mathrm{odd}}$ by finiteness of p. Hence the sequence of (a) is exact and split, and $H^*(R)_{\mathrm{odd}}$ is pure because R is a rational homology manifold. The description of $H^*_c(B \setminus D_x, \mathbb{L})$ follows from the sequences for $\widetilde{A} \cdot \star \subset \mathrm{GR}^U_{x,\leq\lambda}$ and $E \subset B \setminus D_x$, near which $\mathbb{L}$ is constant: removing the $\star$-orbit creates $H^1_c(Y \setminus D_x, \mathbb{L}) \cong \mathbb{L}_\star = D$, which the exceptional divisor cancels.

(b) is Theorem 4.4 below. That the model of Theorem 5.9 has the data of Definition 4.2 is the content of its proof and of Proposition 6.6. Then $H^*(R)_{\mathrm{odd}} = D \otimes M_x \oplus \bigoplus_i D \otimes H^{2i}(E)$, $P_\star$ removes the second summand, and ε has rank one on D. *(c)* is Example 4.14. [referee report 001, U1] □

4.1. **The defect on the weighted model.** Let X be the coarse space of the good model $\mathcal{X}_x$ of Theorem 5.9, with modification $\pi : X \to B$, an isomorphism over $B \setminus D_x$. Put $\partial = \pi^{-1}(D_x)$, $U_X = X \setminus \partial$ (the letter U alone always denotes the open point of Bun) and $j : U_X \hookrightarrow X$; the interior divisor $E_\star$ lies in U_X and plays no role. Theorem 5.9 blows up the points $x_1, \ldots, x_N$ of $\mathcal{B}$ (codimension $c_m \geq 2$) in order; let $E_1, \ldots, E_N \subset X$ be the pulled-back exceptional divisors and $\widetilde{D}$ the strict transform of D_x, so $\partial = \widetilde{D} \cup E_1 \cup \cdots \cup E_N$. Write $X_n = Y \times_{\mathcal{B}} \mathcal{B}_n$ and $\Sigma_n \subset X_{n-1}$ for the centre of the n-th blow-up (the locus over x_n).

The local system $\mathbb{L}$ on U is the pullback of the local system $\mathbb{F}$ on the open point $\mathcal{U} = U = B\widetilde{A}$ of $\mathcal{B}$; its fibre is D, with monodromy by left multiplication. For $q \in X$ let N_q be a small conical neighbourhood, Γ_q the orbifold fundamental group of $N_q \cap U_X$, $\rho_q : \Gamma_q \to \widetilde{A}$ the local monodromy and $I_q = \rho_q(\Gamma_q)$. Put

$$\mathcal{F} = R^0 j_* \mathbb{L} = e_- p_* \mathbb{Q}_R, \qquad \mathcal{F}_q = D^{I_q}.$$

Here $p : R \to X$ is the normalisation in the function field of T_x; the equality holds because R is normal, so sections of $p_*\mathbb{Q}_R$ over N_q are the Γ_q-invariant sections over $N_q \cap U_X$. No dual enters. The triangle $j_!\mathbb{L} \to \mathcal{F} \to i_*(\mathcal{F}|_\partial)$, with $i : \partial \hookrightarrow X$, gives the exact sequence

$$\cdots \to H^k_c(U_X, \mathbb{L}) \to H^k(R)_{\mathrm{odd}} \to H^k(\partial, \mathcal{F}|_\partial) \to \cdots, \tag{2}$$

using $H^*(X, \mathcal{F}) = H^*(R)_{\mathrm{odd}}$, by finiteness of p. Since ∂ is compact and $\mathcal{F}$ vanishes off B°, $H^*(\partial, \mathcal{F}) = H^*_c(B^\circ, \overline{\mathbb{L}})$.

Theorem 4.4 (Defect vanishing)**.** *On a weighted clean model, $H^*(\partial, \mathcal{F}|_\partial) = 0$. Consequently $H^*_c(U_X, \mathbb{L}) \to H^*(R)_{\mathrm{odd}}$ is an isomorphism.*

Set-up at a processed point. Let $x = x_k$, let $f : ([W/T], w) \to \mathcal{B}$ be the Alper–Hall–Rydh chart, with $T = T_x$ and $W \cong \mathbb{A}^c$ linear, let $\bar{\lambda} : \mathbb{G}_m \to T$ be

the contracting cocharacter (weights $a_i > 0$ on W), $\mu_x = \ker(T \to \operatorname{Aut} W)$ (finite), $D_W = f^{-1}(\mathcal{B} \setminus \mathcal{U})$, $W^\circ = W \setminus D_W$, and $\mathcal{L}_W = f^*\mathbb{F}|_{[W^\circ/T]}$, a T-equivariant local system on W° with structure maps $\psi_g : \mathcal{L}_{W,v} \to \mathcal{L}_{W,gv}$ ($g \in T$), continuous in g, with $\psi_{gh} = \psi_g\psi_h$. For $g \in \mu_x$, ψ_g is left multiplication on the fibre D by $\kappa(g)$, where $\kappa : \mu_x \to \widetilde{A}$ is the stabiliser homomorphism of the $\widetilde{A}$-torsor: this is the twisted torus action with finite kernel. Put $P_W = Y \times_{\mathcal{B}} W$, a T-torsor over $Y \times_{\mathcal{B}} [W/T]$ on which T acts freely.

Lemma 4.5 (One centre: the equivariant diagram)**.** *Let $\mathbb{G} = \mathbb{G}_m \times T$ act on $V' = (W \setminus 0) \times \mathbb{A}^1_t$ by $(\tau, g)(v, t) = (\bar\lambda(\tau)gv, \tau^{-1}t)$; let $\beta(v, t) = \bar\lambda(t)v$, $\gamma(v, t) = v$, $\varepsilon_0(\tau, g) = g$, $\varepsilon_1(\tau, g) = \bar\lambda(\tau)g$. Then β is ε_0-equivariant, γ is ε_1-equivariant, $[V'/\mathbb{G}] \to [W/T]$ is the weighted blow-up at w, in the equivariant chart form of* [28, Cor. 5.2.6]*, and on $V'^\times = W^\circ \times \mathbb{C}^*_t$*

$$\Psi : \ \gamma^*\mathcal{L}_W \xrightarrow{\sim} \beta^*\mathcal{L}_W, \qquad \Psi_{(v,t)} = \psi_{\bar\lambda(t)},$$

is an isomorphism of $\mathbb{G}$-equivariant local systems, where $\gamma^*\mathcal{L}_W$ carries the ε_1-structure and $\beta^*\mathcal{L}_W$ the ε_0-structure. Consequently, on $V'^\natural = W^\circ \times \mathbb{A}^1_t$, $\beta^*\mathcal{L}_W|_{t\neq 0}$ extends to the $\mathbb{G}$-equivariant local system $\gamma^*\mathcal{L}_W$, and for a point q of $\mathcal{X}$ over $(v, 0)$:

1. *the stabiliser of q contains $\mu_{e(v)} = \{\tau : \bar\lambda(\tau)v = v\}$, acting on t by τ^{-1} and on $L^{\mathrm{st}}_q = \mathcal{L}_{W,v}$ by $\psi_{\bar\lambda(\tau)}$;*
2. *the meridian of E at q in U_X is the orbifold loop formed by the path $t \mapsto e^{2\pi i\theta/e(v)}t$ and the element $\zeta_{e(v)}$, with monodromy $\rho(\ell_v) = \psi_{\bar\lambda(\zeta_{e(v)})}$; at a generic point $e(v) = e_0$, and this is $\rho(h) = \kappa(\bar\lambda(\zeta_{e_0}))$.*

Proof. (τ, g) sends (v, t) to $(\bar\lambda(\tau)gv, \tau^{-1}t)$, and $\psi_{\bar\lambda(\tau^{-1}t)} \circ \psi_{\varepsilon_1(\tau,g)} = \psi_{\bar\lambda(t)g} = \psi_{\varepsilon_0(\tau,g)} \circ \psi_{\bar\lambda(t)}$ by the cocycle rule and commutativity of T; so Ψ is equivariant. That β is the weighted blow-up is the chart description [28, Prop. 5.2.2, Cor. 5.2.5–5.2.6], compatible with flat base change [28, Cor. 3.2.14(iii)], applied on an atlas. The stabiliser in $\mathbb{G}$ of a point $(p, (v, 0))$ of $P_W \times_W V'$ has $g = 1$, since T acts freely on P_W, and then $\tau \in \mu_{e(v)}$. So $\mathcal{X}$ is Deligne–Mumford there, and (1) holds by the ε_1-structure. Off E the $\mathbb{G}$-action is free; a loop in the quotient is a path in $P'^\times$ ending at a $\mathbb{G}$-translate of its start, and the path displayed in (2), closed by $(\zeta_{e(v)}, 1)$, is the meridian. Transport of $\gamma^*\mathcal{L}_W$ along it is trivial, because $\gamma^*\mathcal{L}_W$ is constant in t, and the group element acts by $\psi_{\varepsilon_1(\zeta,1)} = \psi_{\bar\lambda(\zeta)}$. □

Here Ψ is an isomorphism of local systems on the stack $[V'^\times/\mathbb{G}]$, not an identity of fundamental-group symbols; all monodromies and stalks are read from it, and the finite isotropy enters only through the stabiliser action in (1).

Lemma 4.6 (Descent of radial extensions)**.** *Let $\mathfrak{M}$ be a smooth complex algebraic stack, let $j : \mathfrak{O} \hookrightarrow \mathfrak{M}$ be an open immersion whose complement has positive codimension on every smooth atlas, and let L be a local system on $\mathfrak{O}$. Let $p : V \to \mathfrak{M}$ be a smooth surjective scheme atlas. Suppose a*

local system L_V on V is given together with an isomorphism $\alpha : L_V|_{V^\circ} \xrightarrow{\sim} p^*L$, where $V^\circ = V \times_{\mathfrak{M}} \mathfrak{O}$. Then there is a unique descent datum on L_V extending that of L. It is effective, and yields a local system $\overline{L}$ on $\mathfrak{M}$ with the specified identification on $\mathfrak{O}$, unique up to a unique isomorphism respecting that identification.

Proof. For a smooth complex space Z and a closed analytic subset A of positive complex codimension, restriction of local systems from Z to $Z \setminus A$ is fully faithful: on a small ball both local systems are trivial, the complement of A in the ball is connected, so a morphism on the complement is a constant matrix and extends uniquely, and these local extensions agree. This is local for smooth covers and applies to algebraic spaces. Put $V_2 = V \times_{\mathfrak{M}} V$ with projections p_1, p_2. On V_2°, using the canonical identification of the two pullbacks of L, define $\theta_{12}^\circ = (p_2^*\alpha)^{-1}(p_1^*\alpha) : p_1^*L_V \to p_2^*L_V$. The space V_2 is smooth, so full faithfulness extends θ_{12}° uniquely to θ_{12} on V_2, and its inverse extends too. On $V_3 = V \times_{\mathfrak{M}} V \times_{\mathfrak{M}} V$ the equality $\theta_{23}\theta_{12} = \theta_{13}$ holds on V_3°, hence everywhere by faithfulness; the identity and inverse conditions follow in the same way. Descent of local systems along the smooth atlas is effective. The whole Čech groupoid, including self-overlaps, is used; it records isotropy. No agreement, conjugacy or descent of the radial maps themselves is required: only their identified pullbacks of the sheaf on the common open enter. □

We apply Lemma 4.6 with $\mathfrak{M} = \mathcal{B}_n^\natural$, $\mathfrak{O} = \mathcal{U}$ and $L_V = g_V^*\mathbb{F}$, which is defined on the whole chart by (I_n). Over the complement of all exceptional divisors the iterated equivariance maps of Lemma 4.5 give α_V to the pullback along the blow-down, a single global map, so the comparisons have a common target. The exceptional complement is dense in every chart and every overlap, since the exceptional equations are independent smooth parameters in (I_n). At an intersection of r exceptional divisors, away from the strict boundary, on a small chart with equations $s_1 \cdots s_r = 0$ both extended local systems are trivial on the polydisc; on $(\Delta^*)^r \times \Delta^{d-r}$ their comparison is one constant matrix, every generator of $\pi_1 = \mathbb{Z}^r$ acts trivially on each and on the comparison, and there is no further triple-intersection condition. *Trivial meridians here are meridians on smooth atlases of the stack, not on its coarse space:* on $[\Delta/\mu_e]$ with $t \mapsto \zeta t$, a constant local system upstairs with stabiliser representation χ extends across $t = 0$, while a once-around coarse loop has monodromy $\chi(\zeta)^{\pm 1}$ (for $e = 2$ and the sign character, -1). Coarse stalks are computed by invariants, not by an assertion that coarse meridians act trivially (Lemma 4.7).

Lemma 4.7 (Extension across the exceptional loci)**.** *For every n there is a local system $\mathbb{F}_n$ on the open substack $\mathcal{B}_n^\natural = \mathcal{B}_n \setminus \widetilde{\mathcal{D}}$ (the complement of the strict transform of $\mathcal{B}\setminus\mathcal{U}$) extending $\mathbb{F}$ on $\mathcal{U}$, with $\mathbb{F}_n|_V \cong g_V^*\mathbb{F}$ on every chart V of (I_n), compatibly on overlaps. Its pullback L^{st} to the Deligne–Mumford*

stack $\mathcal{X}_n^\natural = Y \times_{\mathcal{B}} \mathcal{B}_n^\natural$ *extends* $\mathbb{L}$*, and on the coarse space*

$$R^0 j_* \mathbb{L}\,|_{X_n^\natural} \;=\; \pi_*^{\mathrm{inv}} L^{\mathrm{st}},$$

where $\pi : \mathcal{X}_n^\natural \to X_n^\natural$ *is the coarse map and* π_*^{inv} *takes invariants of the finite stabilisers (exact over* $\mathbb{Q}$*). In particular* $\mathcal{F}_q = (L_q^{\mathrm{st}})^{H_q} = D^{I_q}$*, where* H_q *is the stabiliser of* q *in* $\mathcal{X}$*, acting on* $L_q^{\mathrm{st}} = D$ *by left multiplication through* $H_q \to \mathrm{Stab}_{\mathcal{B}_n}(\bar{q}) \to \mathrm{Stab}_{\mathcal{B}}(g_V(q))$*, and* I_q *is the image.*

Proof. By (I_n) the strict transform of $\mathcal{B} \setminus \mathcal{U}$ on a chart V is $g_V^{-1}(\mathcal{B} \setminus \mathcal{U})$, so $V \cap \mathcal{B}_n^\natural = g_V^{-1}(\mathcal{U})$ and $g_V^* \mathbb{F}$ is defined there. On $V^\times = V \setminus$ (all boundary) the blow-down $b_V : V \to \mathcal{B}$ and g_V are related by the isomorphisms Ψ of Lemma 4.5, composed over the earlier centres; the b_V glue, so the Ψ give gluing isomorphisms $g_{V_1}^* \mathbb{F} \cong g_{V_2}^* \mathbb{F}$ on $V_{12}^\times$ satisfying the cocycle condition there. By full faithfulness of restriction of local systems on smooth charts, these extend uniquely over $V_{12} \cap \mathcal{B}_n^\natural$ and the cocycle identity extends, which is Lemma 4.6. □

Several old exceptional branches. At a multiple crossing in the natural locus write a local DM chart as $[Z/H]$, with Z a small contractible smooth neighbourhood, each exceptional branch preserved, and the coefficient system given by $\rho : H \to \mathrm{GL}(D)$. The atlas meridians act trivially, and on the complement $\pi_1(Z^\times) \to \pi_1^{\mathrm{orb}}([Z^\times/H]) \to H \to 1$ is exact ($Z^\times$ is connected), the coefficient representation killing the left-hand group. So the local invariant space is exactly $D^{\rho(H)}$, including at simultaneous old/new crossings, and $R^0 j_* \mathbb{L} = \pi_* L^{\mathrm{st}}$ with $(R^0 j_* \mathbb{L})_q = (L_{\tilde{q}}^{\mathrm{st}})^{H_q}$, as an identification of sheaves via the restriction map. Since every nontrivial subgroup of Q_8 contains $z = -1$, $D^{\rho(H_q)}$ is 0 if the image is nontrivial and D otherwise. This image is the image of the actual local group, not the full stabiliser of the radial point of $\mathcal{U}$; the transported meridian images $c_i = \rho(h_i)$ of Lemma 4.5 lie in it, and no canonical inclusion of the old generic stabilisers into H_q is used.

Corollary 4.8 (Stalks)**.** *At every* $q \in \partial$ *off* $\widetilde{D}$*,* $\mathcal{F}_q = D^{I_q}$*, with* I_q *the image of the actual local group (local monodromy); it is* D *if* $I_q = 1$ *and* 0 *otherwise. On* $\widetilde{D}$ *the stalk is* 0*: at the end of the sequence a point of* $\widetilde{D}$ *maps to a codimension-one point of* $\mathcal{B} \setminus \mathcal{U}$*, its local image contains the original meridian* m_D*, and* $D^{m_D} = 0$ *by cleanness.*

Lemma 4.9 (Generalised Duality Lemma)**.** *Let* $x = x_m$ *with slice* W*, contracting cocharacter* $\bar{\lambda}$ *of weights* $a_1, \ldots, a_c > 0$*, and* $\bar{\lambda}$*-stable boundary* D_W*. Let* $M = S_\epsilon \setminus D_W$ *for a* $\bar{\lambda}$*-adapted sphere* S_ϵ *about* 0*, let* $B_x^\circ = (W \setminus (0 \cup D_W))/\mathbb{C}^*$*, an orbifold open in* $\mathbb{P}(a)$*, and let* $\varpi : M \to B_x^\circ$ *be the Seifert map, with regular fibre the orbit loop* h *and exceptional fibres* ℓ *with* $\ell^k = h$*. Let* $\mathcal{M}$ *be a local system of* $\mathbb{Q}$*-vector spaces on* M *(carrying, if one wishes, a compatible module structure over a semisimple* $\mathbb{Q}$*-algebra such as* D^{op}*) with finite monodromy such that* $H^*(M, \mathcal{M}) = 0$*, and let*

$\mathcal{R} = R^0\varpi_*\mathcal{M}$, with stalks $\mathcal{M}^{\langle h\rangle}$ generically and $\mathcal{M}^{\langle \ell\rangle}$ at orbifold points. Then $H^*(B_x^\circ, \mathcal{R}) = 0$ and $H_c^*(B_x^\circ, \mathcal{R}) = 0$, whatever the image of h.

Proof. Since h is central in the local fundamental group, $\mathcal{M} = \mathcal{M}^{\bar{h}} \oplus \mathcal{M}'$ as local systems, where $\mathcal{M}^{\bar{h}}$ is the part on which h acts trivially. Every Leray sheaf of $\mathcal{M}'$ vanishes (at an exceptional orbit, $\mathcal{M}'^{\ell} \subset \mathcal{M}'^{h} = 0$), and $\mathcal{M}^{\bar{h}}$ descends to the orbifold base; $H^*(M, \mathcal{M}^{\bar{h}})$ is a summand of $H^*(M, \mathcal{M}) = 0$. So we may assume $\mathcal{M} = \mathcal{M}^{\bar{h}}$, pulled back from the base. [UDEFECT-X, D3] Rationally the Leray sheaves of ϖ are then $R^0 = \mathcal{R}$ and $R^1 \cong \mathcal{R}$: for a finite-order operator on a $\mathbb{Q}$-vector space invariants equal coinvariants, and the orientation of the $\mathbb{C}^*$-orbits trivialises $R^1\varpi_*\mathbb{Q}$ rationally, including at exceptional fibres, where $H^1(\ell) \to H^1(h)$ is multiplication by k. The Leray spectral sequence has two rows, and d_2 is cup product with the rational Euler class $e \in H^2(B_x^\circ, \mathbb{Q})$, giving the Gysin sequence

$$\cdots \to H^k(B_x^\circ, \mathcal{R}) \to H^k(M, \mathcal{M}) \to H^{k-1}(B_x^\circ, \mathcal{R}) \xrightarrow{\cup e} H^{k+1}(B_x^\circ, \mathcal{R}) \to \cdots .$$

Since $H^*(M, \mathcal{M}) = 0$, $\cup e$ is an isomorphism $H^{k-1} \to H^{k+1}$ for all k, and induction from $H^{-1} = H^{-2} = 0$ gives $H^*(B_x^\circ, \mathcal{R}) = 0$.

Compact supports. Finite monodromy gives an invariant positive definite symmetric form on $\mathcal{M}$ (average any one over the monodromy group; for $\mathcal{M} \subset \mathbb{L}$ one may take $\mathrm{trd}(x\bar{y})$, invariant under left multiplication by $\pm 1, \pm i, \pm j, \pm k$, for which the splitting $\mathcal{M} = \mathcal{M}^{\bar{h}} \oplus \mathcal{M}'$ is orthogonal). Hence $\mathcal{M} \cong \mathcal{M}^\vee$, and $\mathcal{M}^\vee$ satisfies the hypothesis. Let $d = \dim_{\mathbb{C}} B_x^\circ$. Locally on the coarse space, $B_x^\circ = \mathbb{C}^d/H$ and $\mathcal{R} = q_*^H\mathcal{E}$ for a local system $\mathcal{E}$ on $\mathbb{C}^d$, with q finite; at orbifold points $\mathcal{R}$ is a constructible sheaf, not a local system, and we *define* $\mathcal{R}^\vee := (q_*\mathcal{E}^\vee)^H$, not $\mathcal{H}om(\mathcal{R}, \mathbb{Q})$. The form restricts to a nondegenerate form on the invariants of every finite group, so $R^0\varpi_*(\mathcal{M}^\vee) \cong \mathcal{R}^\vee$, with stalks $(\mathcal{M}^\vee)^{\langle \ell\rangle} = (\mathcal{M}^{\langle \ell\rangle})^\vee$. Since $\mathbb{D} \circ q_* = q_* \circ \mathbb{D}$ (q finite) and H-invariants commute with $\mathbb{D}$ (an exact functor), $\mathbb{D}_{B_x^\circ}\mathcal{R} \cong \mathcal{R}^\vee[2d]$. Verdier duality then gives $H_c^i(B_x^\circ, \mathcal{R})^\vee \cong H^{2d-i}(B_x^\circ, \mathcal{R}^\vee) = H^{2d-i}(B_x^\circ, R^0\varpi_*\mathcal{M}^\vee) = 0$, by the first half applied to $\mathcal{M}^\vee$. The image of h enters only through the rank of $\mathcal{R}$: if $\rho(h)$ has no fixed vector (for G_2, z or any element of order four), $\mathcal{R}$ vanishes generically. [referee report 001, U2] □

In the next lemma and the proof of Theorem 4.4, $\mathfrak{X}_n$ and $\mathfrak{t}_k$ denote stacks and X_n, Σ_k their coarse spaces; all blow-ups are stack-theoretic.

Lemma 4.10 (Exceptional fibre on a smooth refinement)**.** *Let $\mathfrak{M} = \mathfrak{X}_{k-1}$ and let $I_\bullet$ be its next regular weighted centre, supported on $\mathfrak{S}$. Suppose that on an atlas $V \to \mathfrak{M}$ the filtration is $g_V^*F_\bullet$, where $g_V : V \to \mathcal{U}_{k-1}$ is smooth, and that $(g_V, s_1, \ldots, s_r)$ is smooth with the s_i the old exceptional equations. Let $W = \mathbb{A}^c \to \mathcal{U}_{k-1}$ be the torus chart at x_k, with weights $a_1, \ldots, a_c > 0$. Put $C = V \times_{\mathcal{U}_{k-1}} W$ and $C_0 = C \times_W \{0\}$. After covering algebraic spaces by schemes if necessary, $C \to V$ is smooth and covers a neighbourhood of*

the centre, and there are Cartesian diagrams

$$\begin{array}{ccc} \widetilde{C} = C \times_W \mathrm{Bl}_a^{\mathrm{st}}(W) & \longrightarrow & \mathrm{Bl}_a^{\mathrm{st}}(W) \\ \downarrow & & \downarrow \\ C & \longrightarrow & W, \end{array} \qquad \begin{array}{ccc} E_C = C_0 \times \mathbb{P}^{\mathrm{st}}(a) & \longrightarrow & \mathfrak{E}_k \\ \downarrow & & \downarrow \\ C_0 & \longrightarrow & \mathfrak{S}, \end{array}$$

where $\mathfrak{E}_k = \mathrm{Proj}^{\mathrm{st}}_{\mathfrak{S}}\big(\bigoplus_{m\geq 0} I_m/I_{m+1}\big)$ *is the exceptional divisor at stage* k*. On this refinement the open complement of the transformed boundary in* $\mathfrak{E}_k$ *is* $C_0 \times \mathscr{B}_x$*, with* $\mathscr{B}_x = [W^\circ/\mathbb{G}_m]$ *for the weighted action. The old exceptional equations remain equations on* C_0*, and the formula also holds where any subset of them vanishes.*

Proof. Both projections of C are smooth, as base changes of the two smooth maps to $\mathcal{U}_{k-1}$, and its centre is C_0. Flatness identifies the pulled-back Rees algebra and its associated graded with those on W. In the radial presentation $y_i = t^{a_i} v_i$ with $v \neq 0$; setting $t = 0$ gives $y_i = 0$ and the weighted quotient of v, hence $C_0 \times \mathbb{P}^{\mathrm{st}}(a)$. The boundary ideal, saturated by t, is the pullback of the ideal evaluated at v, whose complement is W°. Smoothness of (g_V, s) makes the s_i independent parameters relative to W, so imposing $s_i = 0$ does not change the calculation. The construction uses the current stack atlas, not the original chart. The torus T_x supplies the contracting coordinates and the equivariance of Lemma 4.5; the full stabiliser need not act linearly on W, and its action on the weighted normal cone (through gr_F) can be non-linear in weighted coordinates (in weights $(1,2)$, $u \mapsto u$, $v \mapsto v + bu^2$ is degree-preserving with identity tangent action). Only the intrinsic graded algebra on the centre is used. □

Lemma 4.11 (Coarse fibre descent)**.** *Let* $f : \mathfrak{A} \to \mathfrak{S}$ *be a morphism of separated finite-type Deligne–Mumford stacks over* $\mathbb{C}$*, and* L *a rational local system on* $\mathfrak{A}$*. Write* π_A, π_S *for the coarse maps and* $\bar{f} : A \to S$ *for the induced map. For* $\sigma \in S(\mathbb{C})$ *choose an object* s *above it with stabiliser* H*, and put* $\mathfrak{T} = \mathfrak{A} \times_{\mathfrak{S}} BH$ *and* $\mathfrak{F} = \mathfrak{T} \times_{BH} \mathrm{Spec}\,\mathbb{C}$*. Let* Z *be the coarse space of* $\mathfrak{F}$ *and* $\mathcal{R} = \pi_{F*}(L|_{\mathfrak{F}})$*. Then on analytic spaces*

$$R\Gamma_c\big(\bar{f}^{-1}(\sigma), (\pi_{A*}L)|_{\bar{f}^{-1}(\sigma)}\big) \simeq \big(R\Gamma_c(Z, \mathcal{R})\big)^H,$$

the action being induced by descent. No splitting of stabiliser extensions is required.

Proof. The map $q : \mathfrak{F} \to \mathfrak{T}$ is the base change of $\mathrm{Spec}\,\mathbb{C} \to BH$, a representable finite étale H-torsor. Its deck transformations induce an action on Z and $\mathcal{R}$; coherence isomorphisms of the stack action become identities on the coarse pushforward, whose stalks already take invariants of internal inertia, so this is an honest action on the pair $(Z, \mathcal{R})$. If T is the coarse space of $\mathfrak{T}$, finite descent gives $T = Z/H$ and $\pi_{T*}(L|_{\mathfrak{T}}) = (\rho_*\mathcal{R})^H$ for the finite quotient $\rho : Z \to T$; on finite-quotient charts this takes first kernel invariants and then quotient invariants, and no semidirect product is needed. The reduced fibre of $\mathfrak{S} \to S$ at σ is BH, so $\mathfrak{T}$ and the fibre of $\mathfrak{A} \to S$ have the same reduced underlying stack; tame coarse base change [20, Thm. 3.2(d),

Cor. 3.3(a)] identifies their coarse analytic space with $\bar{f}^{-1}(\sigma)$, and nilpotents do not affect analytic sheaves. Restriction of $\pi_{A*}L$ agrees with the displayed coarse pushforward: on $[U/G] \to U/G$, restriction to the invariant inverse image of a closed fibre commutes with finite proper pushforward and exact rational G-invariants. Finally ρ is finite and proper, so its pushforward commutes with compact supports, and the averaging idempotent of H commutes with derived compactly supported sections. □

Proof of Theorem 4.4. For $1 \le k \le N$ let $Z_k = \bigcup_{i \ge k} E_i$ (closed, $Z_{N+1} = \emptyset$), $i_k : Z_{k+1} \hookrightarrow Z_k$ and $u_k : A_k^\flat = Z_k \setminus Z_{k+1} \hookrightarrow Z_k$. The triangles $u_{k!}u_k^*\mathcal{F} \to \mathcal{F}|_{Z_k} \to i_{k*}i_k^*\mathcal{F} \xrightarrow{+1}$ and, with $v : \partial \setminus Z_1 \hookrightarrow \partial$, $v_!v^*\mathcal{F} \to \mathcal{F} \to \mathcal{F}|_{Z_1} \xrightarrow{+1}$ give, all Z_k being compact,

$$H_c^*(\partial \setminus Z_1, \mathcal{F}) \to H^*(\partial, \mathcal{F}) \to H^*(Z_1, \mathcal{F}) \to,$$
$$H_c^*(A_k^\flat, \mathcal{F}) \to H^*(Z_k, \mathcal{F}) \to H^*(Z_{k+1}, \mathcal{F}) \to .$$

(a) $\partial \setminus Z_1 \subset \widetilde{D}$, where $\mathcal{F} = 0$ (Corollary 4.8). By downward induction on k the theorem reduces to $H_c^*(A_k^\flat, \mathcal{F}) = 0$ for every k.

(b) $A_k^\flat = A_k \sqcup (A_k^\flat \cap \widetilde{D})$, with $A_k = \{q \in A_k^\flat : \ g(q) \in \mathcal{U}\}$ open: if $g(q)$ is a codimension-one point of $\mathcal{B} \setminus \mathcal{U}$ then $q \in \widetilde{D}$ by (I_N), and if $g(q) = x_m$ with $m > k$ then q lies in the centre of a later blow-up, so its preimage lies in Z_{k+1}. Since $\mathcal{F}|_{A_k^\flat \cap \widetilde{D}} = 0$, $\mathcal{F}|_{A_k^\flat} = w_!(\mathcal{F}|_{A_k})$ for $w : A_k \hookrightarrow A_k^\flat$, so $H_c^*(A_k^\flat, \mathcal{F}) = H_c^*(A_k, \mathcal{F})$.

(c) The later blow-ups are isomorphisms off their centres, and on the exceptional open where the radial image lies in $\mathcal{U}$ all later centre ideals are units; so A_k is the coarse space of the open $\mathfrak{A}_k \subset \mathfrak{E}_k$ of Lemma 4.10, and $\mathcal{F}$ near it is computed at stage k, since $R^0 j_*$ is local. Let $f_k : \mathfrak{A}_k \to \mathfrak{t}_k$ be the restriction of the blow-down. No global torsor over $\mathfrak{t}_k$, no linear action of the full stabiliser on T^1, and no inclusion of old generic stabilisers in a chart stabiliser is used.

(d) *Coefficients on the fibres.* Let $\mathscr{L}_x$ be the local system on $\mathscr{B}_x = [W^\circ/\mathbb{G}_m]$ obtained by restricting the T_x-equivariance of $\mathcal{L}_W$ along $\bar{\lambda}$. At $t = 0$ the radial chart P of Lemma 5.13 is $C_0 \times (W \setminus 0)$, with natural part $C_0 \times W^\circ$, on which $g_P(c, v, 0) = v$ and the coefficient is $\mathrm{pr}_2^*\mathcal{L}_W$ by Lemma 5.13(c). Compare the equivariance maps on $P^\natural \times \mathbb{G}_m$, where both sheaves exist: Lemma 4.5 proves their equality at $t \neq 0$, and faithfulness extends it; the right-hand action is precisely ε_1. Hence

$$L_k^{\mathrm{st}}|_{C_0 \times \mathscr{B}_x} \simeq \mathrm{pr}_2^*\mathscr{L}_x, \qquad \mathscr{B}_x = [W^\circ/\mathbb{G}_m],$$

canonically relative to the chosen chart and its specified open comparison, and compatibly with all groupoid transitions; it is not a chart-independent trivialization over the centre. Setting any old s_i to zero preserves it. Trivial atlas meridians do not imply trivial coarse meridians, so the ε_1-action followed by invariants remains essential.

(e) *The fibre vanishes on the stack.* Given it, cleanness at x_k (by smooth base change, and the contracting weighted action, which identifies the punctured local complement with its adapted link times a contractible factor) gives $R\Gamma(W^\circ, \mathcal{L}_W) = 0$. The map $W^\circ \to \mathscr{B}_x$ is a principal $\mathbb{G}_m$-bundle over the stack, even where the coarse action has finite stabilisers, and $\mathcal{L}_W$ is the pullback of $\mathscr{L}_x$; the Gysin sequence $\cdots \to H^{j-2}(\mathscr{B}_x, \mathscr{L}_x) \xrightarrow{e} H^j(\mathscr{B}_x, \mathscr{L}_x) \to H^j(W^\circ, \mathcal{L}_W) \to H^{j-1}(\mathscr{B}_x, \mathscr{L}_x) \to \cdots$ then gives $H^*(\mathscr{B}_x, \mathscr{L}_x) = 0$ by induction from negative degrees. Finite monodromy admits an invariant positive definite form, so the dual has the same vanishing, and duality on the smooth DM stack of dimension $c-1$ gives

$$(*) \qquad R\Gamma_c(\mathscr{B}_x, \mathscr{L}_x) = 0.$$

Passing to the coarse weighted projective open takes finite isotropy invariants, with coefficient sheaf $R^0\varpi_*\mathcal{L}_W|_M$; so $(*)$ has exactly the coefficients of Lemma 4.9, including exceptional orbit points, and does not assume that the coarse fibre loop has trivial image.

(f) *Coarse vanishing.* Take V an atlas of the current Deligne–Mumford stage $\mathfrak{X}_{k-1} = Y \times_{\mathcal{B}} \mathcal{B}_{k-1}$, obtained by smooth base change of the atlas of (I_{k-1}) and scheme refinement; form $C = V \times_{U_{k-1}} W$ and $C_0 = C \times_W \{0\}$. These C_0 cover the Deligne–Mumford centre $\mathfrak{S} = \mathfrak{t}_k$, and the weighted-normal-cone calculation ($y_i = t^{a_i} v_i$, then $t = 0$; old equations stay on C_0) gives the Cartesian square

$$\begin{array}{ccc} C_0 \times \mathscr{B}_x & \longrightarrow & \mathfrak{A}_k \\ \downarrow \mathrm{pr}_1 & & \downarrow f_k \\ C_0 & \longrightarrow & \mathfrak{S}. \end{array}$$

So the fibre over an object $s : \operatorname{Spec}\mathbb{C} \to \mathfrak{S}$ is the link stack $\mathscr{B}_x$ with the coefficient $\mathscr{L}_x$ of (d); the factor Y is absorbed into the choice of s. The coarse spaces are separated and of finite type, hence Hausdorff and locally compact, and the stalk of $R\bar{f}_!$ is compactly supported cohomology of the fibre for maps of locally compact spaces, with no properness or manifold hypothesis [21, Thm. 4.5.3, (4.5.4)]. By Lemma 4.7, $\mathcal{F}|_{A_k} = \pi_{A*}L_k^{\mathrm{st}}$. Lemma 4.11 and $(*)$, in the coarse form of Lemma 4.9 with $\mathcal{R} = R^0\varpi_*\mathcal{L}_W|_M$, give

$$(**) \qquad \big(R\bar{f}_{k!}(\mathcal{F}|_{A_k})\big)_\sigma \simeq R\Gamma_c\big(\bar{f}_k^{-1}(\sigma), \mathcal{F}\big) \simeq \big(R\Gamma_c(B^\circ_{x_k}, \mathcal{R})\big)^{H_\sigma} = 0.$$

Hence $R\bar{f}_{k!}(\mathcal{F}|_{A_k}) = 0$ and, composing proper-support direct images, $R\Gamma_c(A_k, \mathcal{F}) = 0$. Only the existence of the link identification over one object above each point enters this vanishing; the compatibility of (d) was needed to identify the global coefficient sheaf. This is the coarse statement the defect needs; no stack six-functor statement for the nonrepresentable f_k is used, and $(**)$ replaces the former stack-level formula $Rf_{k!}L_k^{\mathrm{st}} = 0$.

On the stage-k exceptional natural open all later centre ideals are units by (I_N), so that open survives unchanged, and the inverse images of later centres lie in the later exceptional union. After discarding the strict-boundary zero stalks, the largest-index pieces in (a)–(c) are exactly these surviving

opens. On the final strict boundary a codimension-one transverse chart retains the original nontrivial odd meridian, so its stalks vanish. The triangles for Z_k in (a) now give $H^*(\partial, \mathcal{F}) = 0$ for the constructed sequence, without assuming Definition 4.2. □

Remark 4.12 (What the proof uses). The proof uses the weighted structure at each centre (the Seifert link, and the fibration of A_k over Σ_k with fibre $B^\circ_{x_k}$), the extension of the local system across the exceptional loci along (I_n) (Lemmas 4.5 and 4.7), and Yun's cleanness at every point of $\mathcal{B}$. It uses no condition on divisors with trivial inertia or on their crossings. The G_2 group theory (every nontrivial subgroup of Q_8 contains z) enters only in Corollary 4.8. Purity of $H^k(R)$ (weight k, since R is projective with quotient singularities, so $H^*(R) = IH^*(R)$) is not used in Theorem 4.4; it is used only in the Hodge-theoretic statements built on it. [referee report 001, U3]

Example 4.13 (A clean local system with nonzero defect on a non-weighted resolution)**.** On $\mathbb{C}^2$ let $C = C_1 \cup C_2$ with $C_1 = \{x = 0\}$ and $C_2 = \{x = y^2\}$, a tacnode. Then $H_1(\mathbb{C}^2 \setminus C) = \mathbb{Z}m_1 \oplus \mathbb{Z}m_2$; let L be the rank-one $\mathbb{Q}(i)$-local system with $m_1 \mapsto \zeta = i$ and $m_2 \mapsto \zeta^{-1}$. (As a rational local system, by restriction of scalars, it has rank two and every group below is the same $\mathbb{Q}(i)$-space regarded over $\mathbb{Q}$. Inside Q_8 take $m_1 \mapsto \mathbf{i}$ and $m_2 \mapsto \mathbf{i}^{-1}$; then $D \otimes \mathbb{C} = (L_\mathbb{C} \oplus \overline{L}_\mathbb{C})^{\oplus 2}$ as a left $\langle \mathbf{i} \rangle$-module, and rationally $D = (\mathrm{Res}_{\mathbb{Q}(i)/\mathbb{Q}}\, L)^{\oplus 2}$, so the defects below are multiplied by two.)

L is clean at 0. The germ is quasi-homogeneous for $\mathrm{wt}(y) = 1$, $\mathrm{wt}(x) = 2$, both branches of weighted degree 2; the central loop is $h = 2m_1 + 2m_2$, so $\rho(h) = 1$, and the exceptional orbit $\{y = 0\}$ (isotropy μ_2, off C) has $[\ell] = m_1 + m_2$, so $\rho(\ell) = 1$. On $B^\circ = \mathbb{P}(1,2) \setminus \{[C_1], [C_2]\} \cong \mathbb{C}^*$ the rank-one local system $R^0\varpi_* L$ has monodromy $\zeta \neq 1$ around $[C_1]$, so its cohomology vanishes and the Gysin sequence gives $H^*(M, L) = 0$.

On the weighted model the defect is 0. The $(1,2)$-weighted blow-up has exceptional divisor $\mathbb{P}(1,2)$, meeting C_1, C_2 in two distinct points, and Lemma 4.9 gives $H^*_c(B^\circ, \overline{L}) = 0$.

On the ordinary resolution the defect is nonzero. Blowing up 0 (chart $x = yu$) makes the strict transforms $\{u = 0\}$, $\{u = y\}$ and $E_1 = \{y = 0\}$ pass through one point transversally in pairs; blowing that point up gives E_2, and now $\widetilde{E}_1, \widetilde{C}_1, \widetilde{C}_2$ meet E_2 in three distinct points. Since $\mathrm{ord}_{E_1}(x) = \mathrm{ord}_{E_1}(x - y^2) = 1$ and $\mathrm{ord}_{E_2}(x) = \mathrm{ord}_{E_2}(x - y^2) = 2$, both E_1 and E_2 have trivial inertia, and they cross. The bad locus is $B^\circ = \widetilde{E}_1 \cup (E_2 \setminus \{2 \text{ points}\})$; $\widetilde{E}_1 \cong \mathbb{P}^1$ is closed in it with $\overline{L}$ trivial there, and its complement $O = E_2 \setminus \{3 \text{ points}\}$ carries $\overline{L}$ with monodromies $\zeta, \zeta^{-1}, 1$. Then $H^*_c(O, \overline{L})$ is $\mathbb{Q}(i)$ in degree 1 and 0 otherwise, and the exact sequence

$$0 = H^2_c(O, \overline{L}) \to H^2_c(B^\circ, \overline{L}) \to H^2(\mathbb{P}^1, \mathbb{Q}(i)) \to H^3_c(O, \overline{L}) = 0$$

gives $H^2_c(B^\circ, \overline{L}) \cong \mathbb{Q}(i)(-1) \neq 0$ (that is, $\mathbb{Q}(-1)^2$ as a rational Hodge structure); in degrees 0 and 1 the connecting map from $H^0(\widetilde{E}_1, \mathbb{Q}(i))$ is an isomorphism, so H^0_c and H^1_c vanish. [outside referee hodge-PREF1, section

4.2] So the same clean local system has zero defect on the weighted model and nonzero defect on an ordinary simple normal crossing resolution: on the latter, E_1 is not the exceptional divisor of a weighted structure at a stratum and no Seifert link attaches to it. [referee report 001, U4]

Definition 4.1 is stated only for (Y, D_x), so this example is not literally an instance of it, and there is no contradiction with any statement of this paper. Transplanted to $(\mathbb{C}^2, C)$, however, the ordinary two-step resolution satisfies every clause of that definition (smooth with trivial stabilisers; smooth, individually labelled components meeting as coordinate hyperplanes; an isomorphism off C). So the implication "good model $\Rightarrow$ zero defect" is false as a general principle, and Example 4.14 shows that it also fails for (Y, D_x) itself.

Example 4.14 (A good model of (Y, D_x) with nonzero defect)**.** *Step 1: a divisor with trivial inertia.* Fix a centre x_m and let $S = \widetilde{D} \cap E_m$ on the weighted clean model; $S \neq \emptyset$, since it is the weighted projectivisation of the cone D_W, of dimension $c - 2 \geq 0$. At a generic point of S the local group is $\mathbb{Z}^2$, generated by the meridians of $\widetilde{D}$ and E_m, with commuting images $a = \rho(m_D) \neq 1$ and $b = c_m$. Abelian subgroups of Q_8 are cyclic, so $a = g^\alpha$, $b = g^\beta$ for some g of order $n \mid 4$. Iterated blow-ups of codimension-two crossing strata over S (a regular subdivision of the fan of $\mathbb{Z}^2_{\geq 0}$) produce, for every primitive $(p, q) \in \mathbb{Z}^2_{>0}$, a divisor E' whose meridian has image $a^p b^q$. The congruence $\alpha p + \beta q \equiv 0 \pmod{n}$ has a coprime solution with $p, q \geq 1$: if $\gcd(\beta, n) \mid \alpha$ take $p = 1$; if $\gcd(\alpha, n) \mid \beta$ take $q = 1$; otherwise $n = 4$, $\gcd(\alpha, 4) = \gcd(\beta, 4) = 2$, and $(p, q) = (1, 1)$ works. Then $I = 1$ at generic points of E'. Each step keeps a good model: the centres are smooth strata of the boundary, the map stays an isomorphism over $B \setminus D_x$, the components stay labelled, and the stabilisers fix branches.

Step 2: the point blow-up. Choose $q_0 \in E'$ generic, off the other components, so $I_{q_0} = 1$, and blow up q_0 (or its residual gerbe, if it has a stabiliser H, which acts trivially on the cohomology of the exceptional $\mathbb{P}^5$). The result $\mathcal{X}'$ is again a good model: it is smooth Deligne–Mumford, and the new divisor is one labelled branch transverse to the old ones. Since $I_{q_0} = 1$, the cover T_x is trivial over $N_{q_0} \cap U_X$, so $R \to X$ is étale over N_{q_0} and $p^{-1}(q_0)$ is a free $\widetilde{A}$-orbit. Then $R' = \mathrm{Bl}_{p^{-1}(q_0)} R$ and

$$H^*(R') = H^*(R) \oplus \mathbb{Q}[\widetilde{A}] \otimes \bigoplus_{j=1}^{5} \mathbb{Q}(-j)[-2j],$$

so $H^*(R')_{\mathrm{odd}} = H^*(R)_{\mathrm{odd}} \oplus D \otimes \bigoplus_{j=1}^{5} \mathbb{Q}(-j)[-2j]$: the defect of $\mathcal{X}'$ is nonzero, while $H^*_c(B \setminus D_x, \mathbb{L})$ is unchanged. So the defect clause fails for some good model of (Y, D_x), unconditionally.

Merely creating trivial-inertia points does not create defect. Blowing up a crossing whose meridian images are z, z gives a divisor in B° at its generic points, but its open part is a $\mathbb{C}^*$-bundle with fibre monodromy z, so its compactly supported cohomology vanishes. The nonzero defect comes from

the *closed* $\mathbb{P}^5 \subset B^\circ$ on which $\overline{\mathbb{L}}$ is trivial, just as the closed $\widetilde{E}_1 \cong \mathbb{P}^1$ does in Example 4.13.

5. A canonical weighted resolution

We construct a good model by blowing up Bun, not Y, and pulling back along the smooth map ω.

5.1. **Hypotheses.** Let $\mathcal{B}$ be the open substack of Bun where the stabiliser dimension is at most six; by smoothness of ω and $\dim Y = 6$, $\omega(Y) \subset \mathcal{B}$. A point y of $\mathcal{B}$ is a residual gerbe $[\mathrm{pt}/\mathcal{G}_y]$; since Bun is smooth of dimension zero, its codimension is $c = \dim \mathcal{G}_y$, and the deformation space T^1_y has dimension c.

Hypothesis 5.1. (G0) $\mathcal{B}$ has finitely many points.
- (G1) For every point y of $\mathcal{B}$ with $c \geq 1$ there is a maximal torus $T_y \subset \mathcal{G}^\circ_y$ and a cocharacter λ_y of T_y whose weights on T^1_y are all positive.
- (G2) The weights of λ_y on $\mathrm{Lie}(\mathcal{G}_y)/\mathrm{Lie}(T_y)$ are all ≤ 0.

Section 6 verifies these hypotheses.

5.2. **Charts and the averaged cocharacter.** Let Γ_y be the image of $N_{\mathcal{G}_y}(T_y)$ in $\mathrm{Aut}\, X_*(T_y)$; it is finite, and we set $\bar{\lambda}_y = \sum_{\gamma \in \Gamma_y} \gamma \lambda_y$. The normaliser permutes the weights on T^1_y and on $\mathrm{Lie}\,\mathcal{G}_y/\mathrm{Lie}\,T_y$, so $\bar{\lambda}_y$ still satisfies (G1) and (G2). (In our case $\Gamma_y = 1$; see Proposition 6.3.) [ledger 4]

By the local structure theorem of Alper–Hall–Rydh [1, Thm. 1.1] applied to the linearly reductive subgroup $T_y \subset \mathcal{G}_y$ there is a smooth morphism $f : ([W/T_y], w) \to (\mathrm{Bun}, y)$ with W affine, w fixed, and $f^{-1}(B\mathcal{G}_y) = BT_y$; in the minimal chart $\dim W = c$. [ledger 2] Because $\bar{\lambda}_y$ contracts the tangent space, after shrinking W is T_y-equivariantly isomorphic to $\mathbb{A}^c$ with linear action: the coordinate ring is positively graded with degree-zero part $\mathbb{C}$, and lifts of an eigenbasis of $\mathfrak{m}_w/\mathfrak{m}_w^2$ generate it by graded Nakayama (equivalently, the attracting cell of the isolated fixed point w for a cocharacter with positive weights is T_y-equivariantly $\mathbb{A}^c$ [17]). [ledger 3] Let $F_k \subset \mathcal{O}_W$ be the ideal of functions of $\bar{\lambda}_y$-weight $\geq k$; it is generated by monomials and has finite colength.

Let $R = W \times_{\mathrm{Bun}} W$ with source and target $s, t : R \to W$, both smooth. All pullbacks of ideals below are image ideals in the target structure sheaf.

Lemma 5.2 (The closed fibre of the relation)**.** *Let $u : W \to \mathrm{Bun}$ be the smooth chart above, with $W \times_{\mathrm{Bun}} B\mathcal{G}_y = \{w\}$, and $R = W \times_{\mathrm{Bun}} W$. Then on all of R*

$$Z = s^{-1}(w) = t^{-1}(w) \simeq \mathcal{G}_y, \qquad s^*\mathfrak{m}_w \cdot \mathcal{O}_R = t^*\mathfrak{m}_w \cdot \mathcal{O}_R = I_Z,$$

and for $D_N = \mathrm{Spec}(\mathcal{O}_{W,w}/\mathfrak{m}_w^N)$, $R \times_{s,W} D_N = V(I_Z^N) = R \times_{t,W} D_N$.

Proof. The condition $f^{-1}(B\mathcal{G}_y) = BT_y$ pulls back along $W \to [W/T_y]$ to $W \times_{\mathrm{Bun}} B\mathcal{G}_y = \{w\}$. So an arrow with source w has target w, and conversely by inversion; the two closed fibres have the same support. Both are

smooth, since s and t are, hence reduced, so they are the same reduced closed subscheme of R, namely the isomorphism scheme $\mathcal{G}_y$ of $u(w)$. Equality of the defining ideals is global on R; take powers. Since $W = \operatorname{Spec} \mathbb{C}[y_1, \dots, y_c]$, $D_N \hookrightarrow W$ is a closed immersion and $V(I_Z^N)$ is the whole thickening: no completion and no open neighbourhood depending on N occur. □

Fix $N \geq 1$ and put $R_N = R \times_{s,W} D_N = V(I_Z^N)$, the N-th neighbourhood of Z (Lemma 5.2).

Lemma 5.3 (Relative section lifting and evaluation)**.** *Let A be a finite-dimensional local $\mathbb{C}$-algebra with residue field $\mathbb{C}$, $D = \operatorname{Spec} A$, and $E = \operatorname{Spec} P \to D$ affine, smooth and of finite presentation. Let $B = \operatorname{Res}_{D/\mathbb{C}}(E)$ be the scheme of D-sections after base change. Then B is affine of finite presentation, and evaluation* $\mathrm{ev} : B \times D \to E$ *is smooth and surjective, hence faithfully flat; source and target are flat over D. Explicitly, if $C' \twoheadrightarrow C$ is a surjection of $\mathbb{C}$-algebras with square-zero kernel J, $p' \in D(C')$, $e' \in E(C')$ lies over p', and $\sigma \in B(C)$ satisfies $\sigma(p' \bmod J) = e' \bmod J$, then there is $\sigma' \in B(C')$ lifting σ with $\sigma'(p') = e'$.*

Proof. Choosing a basis of A and a finite presentation of P over A and expanding the relations gives B as an affine scheme of finite presentation. For lifting put $Q = A \otimes C'$, $L = A \otimes J$, $\epsilon_{p'} : Q \to C'$, $a \otimes c \mapsto p'(a)c$, and $H = \ker \epsilon_{p'}$. Then $Q/L = A \otimes C$, $Q/H = C'$, $Q/(L + H) = C$ and $Q/(L \cap H) \simeq (A \otimes C) \times_C C'$. The old section $P \to A \otimes C$ and $e' : P \to C'$ agree in C, so give an A-algebra map $P \to Q/(L \cap H)$; since $(L \cap H)^2 = 0$, smoothness of P/A lifts it to $P \to Q$, which is σ'. This is formal smoothness of ev for every affine square-zero test, including non-reduced parameter rings and moving evaluation points; ev is locally of finite presentation, so it is smooth by the infinitesimal lifting criterion [25, Tag 02H6]. For surjectivity, a geometric point $e : \operatorname{Spec} \Omega \to E$ lies over the residue point of D (the maximal ideal of A is nilpotent), and $P \to \Omega$ lifts through powers of that ideal to $P \to A \otimes \Omega$ by smoothness, an element of $B(\Omega)$ evaluating to e; as D has one point, this is global surjectivity. $E \to D$ is smooth, hence flat, and $B \times D \to D$ is a base change of $B \to \operatorname{Spec} \mathbb{C}$. □

Applied with $A = \mathcal{O}_{W,w}/\mathfrak{m}_w^N$ and $E = R_N$, the section scheme is B_N below and evaluation is the map Φ of Proposition 5.7. Source flatness does not depend on first knowing that B_N is a smooth group: base change from the field gives it. Every section is a bisection, over non-reduced bases too: by Lemma 5.2 the target endomorphism $\alpha = t\sigma$ of $D_{N,C}$ preserves the closed subscheme w_C; for $N \geq 2$ smoothness of s, t gives conormal isomorphisms $s_Z^*(\mathfrak{m}_w/\mathfrak{m}_w^2) \cong I_Z/I_Z^2 \cong t_Z^*(\mathfrak{m}_w/\mathfrak{m}_w^2)$, so α^* is invertible on $(\mathfrak{m}_w/\mathfrak{m}_w^2) \otimes C$, hence on every graded piece of the finite $\mathfrak{m}_w$-adic filtration (W smooth), hence an automorphism of $A \otimes C$. Composition and inversion therefore define the group functor on all test rings.

Lemma 5.4 (Truncated bisections)**.** *Let $B_N = \operatorname{Res}_{D_N/\mathbb{C}}(R \times_{s,W} D_N)$, the functor of sections σ of s over D_N. Then:*

(1) *B_N is an affine group scheme of finite type, smooth (characteristic zero), with product $(\sigma\sigma')(p) = \sigma(t\sigma'(p)) \circ \sigma'(p)$; every section has value in $s^{-1}(w) = \mathcal{G}_y$ at the closed point, and $t \circ \sigma \in \mathrm{Aut}(D_N)$;*
(2) $\mathrm{ev}_N : B_N \to \mathcal{G}_y$ *is a smooth surjective homomorphism, and $K_N = \ker \mathrm{ev}_N$ is connected unipotent;*
(3) *$T_y \hookrightarrow B_N$, $g \mapsto \sigma_g$, is a maximal torus of B_N°, and ev_N restricts to the given inclusion $T_y \subset \mathcal{G}_y$;*
(4) $\mathrm{Lie}\, B_N = \Gamma(D_N, \mathcal{A}|_{D_N})$ *for the Lie algebroid $\mathcal{A} = \ker(ds)|_{\mathrm{identity}}$, acting on $\mathcal{O}_{D_N}$ through the anchor a.*

Proof. (1) Since Bun has affine diagonal, R is affine, and the Weil restriction of the affine finite-type $R \times_W D_N$ along the finite flat $D_N \to \mathrm{Spec}\,\mathbb{C}$ is affine of finite type [18, §7.6]. Because $f^{-1}(B\mathcal{G}_y) = BT_y$, every arrow out of w ends at w, so $s^{-1}(w) = \mathrm{Isom}(w, w) = \mathcal{G}_y$. The map $t \circ \sigma$ fixes the closed point, and its differential there is the invertible isotropy action, since the anchor vanishes at w; so it is an automorphism of the Artin scheme D_N. The product is the bisection product and is algebraic, and a group scheme of finite type in characteristic zero is smooth. (2) Over a section σ, the lifts along $\mathrm{Res}_{D_{j+1}} \to \mathrm{Res}_{D_j}$ form a torsor under $\mathrm{Hom}(\sigma^*\Omega_{R/W}, \mathfrak{m}^j/\mathfrak{m}^{j+1} \otimes C)$, by smoothness of s: a vector-bundle torsor over the preceding parameter scheme, which on the kernel over the identity is the vector group of $\mathrm{Lie}(\mathcal{G}_y) \otimes \mathfrak{m}^j/\mathfrak{m}^{j+1}$. (3) The T_y-action on W gives $W \times T_y \to R$, whose sections σ_g preserve D_N because T_y fixes w. Since $\mathrm{ev}_N(B_N^\circ) = \mathcal{G}_y^\circ$ with unipotent kernel, every torus of B_N° maps injectively onto a torus of $\mathcal{G}_y^\circ$, so T_y is maximal. (4) This is standard for Weil restriction: the action of σ on $\mathcal{O}_{D_N}$ is pullback by $t \circ \sigma$, whose derivative is the anchor. Every element of $\mathrm{Lie}\, B_N$ is $\xi = \sum_j f_j e_j$ with $f_j \in \mathcal{O}_{D_N}$ *arbitrary*, for a T_y-homogeneous frame (e_j) of $\mathcal{A}$. Such a frame exists: $\mathcal{A}$ is a T_y-equivariant bundle on $W \cong \mathbb{A}^c$, whose coordinate ring is positively graded, and graded projective modules over it are free with homogeneous basis (graded Nakayama). □

Lemma 5.5 (Identity component)**.** *B_N° preserves $F_k \bmod \mathfrak{m}_w^N$ for every k.*

Proof. Grade $\mathcal{O}_W$ by $\deg y_i = a_i > 0$, using (G1); a vector field is homogeneous of degree m if it maps $\mathcal{O}_b$ into $\mathcal{O}_{b+m}$. In the convention in which T_y^1 has positive $\bar{\lambda}_y$-weights, a vector field of weight ν has degree $-\nu$ (for instance ∂_i has weight a_i and degree $-a_i$). By T_y-equivariance of the anchor, $a(e_j)$ has the weight ν_j of $e_j(w) \in \mathrm{Lie}\,\mathcal{G}_y$, and $\nu_j \le 0$ by (G2), with $\nu = 0$ on $\mathrm{Lie}\, T_y$. So $\deg a(e_j) = -\nu_j \ge 0$, and $a(e_j)$ maps $F_k = \bigoplus_{b \ge k} \mathcal{O}_b$ into itself. For arbitrary coefficients, $f_j a(e_j)$ maps F_k into $f_j F_k \subset F_k$, since F_k is an ideal, and ξ preserves $\mathfrak{m}^N$ because $a(e_j)(w) = 0$. So $\mathrm{Lie}\, B_N$ preserves $F_k \bmod \mathfrak{m}^N$. The group B_N acts algebraically on the finite-dimensional $\mathcal{O}_{D_N}$, and in characteristic zero a subspace of a finite-dimensional algebraic representation of a connected group that is stable under the Lie algebra is stable under the group. □

Lemma 5.6 (All components)**.** *B_N preserves $F_k \bmod \mathfrak{m}_w^N$ for every k.*

Proof. For $b \in B_N$ the tori bT_yb^{-1} and T_y are maximal in B_N° (Lemma 5.4), so there is $h \in B_N^\circ$ with $m = hb$ normalising T_y. Since ev_N is injective on T_y, the conjugation action of m on T_y is that of $\mathrm{ev}_N(m) \in N_{\mathcal{G}_y}(T_y)$, i.e. an element of Γ_y. So m fixes $\bar{\lambda}_y$, centralises $\bar{\lambda}_y(\mathbb{G}_m)$, preserves the $\bar{\lambda}_y$-grading of $\mathcal{O}_{D_N}$ and hence each F_k. By Lemma 5.5, h^{-1} preserves F_k, so $b = h^{-1}m$ does. No assumption on $\pi_0(\mathcal{G}_y)$ is used. □

Proposition 5.7 (Descent)**.** *For every k, $s^*F_k = t^*F_k$ as ideals of $\mathcal{O}_R$. Hence the ideals F_k descend to coherent ideals on the open image $f(W) \subset$ Bun, all on the same open, and the Rees algebra $\bigoplus_k F_k u^k$ descends to a finitely generated graded algebra there.*

Proof. Fix $k \geq 0$ and choose N with $\mathfrak{m}_w^N \subset F_k$ (for instance $N = \max(1, \lceil k/a_{\min} \rceil)$, $a_{\min}$ the least weight). By Lemma 5.2, on all of R we have $s^{-1}(\mathfrak{m}_w)\mathcal{O}_R = t^{-1}(\mathfrak{m}_w)\mathcal{O}_R = I_Z$ and $R_N = V(I_Z^N)$. Lemma 5.3, applied to $R_N \to D_N$, makes the evaluation map $\Phi : B_N \times D_N \to R_N$ smooth and surjective, hence faithfully flat, with $s\Phi = \mathrm{pr}_2$ and $t\Phi = a_N$, the target action of the bisection group. Write $J = F_k/\mathfrak{m}_w^N \subset \mathcal{O}(D_N)$. Lemma 5.6 makes J invariant under all complex points of B_N; since B_N is a reduced affine group of finite type in characteristic zero, the matrix coefficients of $J \to \mathcal{O}(B_N) \otimes (\mathcal{O}(D_N)/J)$ vanish, so J is a sub-comodule. This gives $a_N^*J \subset \mathrm{pr}_2^*J$ as image ideals, and the same inclusion for the inverse action (or for the automorphism $(b, p) \mapsto (b, bp)$ and its inverse) gives equality. Hence $\Phi^*(s^*J) = \mathrm{pr}_2^*J = a_N^*J = \Phi^*(t^*J)$, and faithful flatness gives $s^*J = t^*J$ on R_N. Both s^*F_k and t^*F_k contain I_Z^N on all of R, and ideals containing I_Z^N are determined by their images in $\mathcal{O}_R/I_Z^N$; so $s^*F_k = t^*F_k$ on all of R, with no shrinking depending on k. The common image ideal is a descent datum for F_k along the smooth surjection $u : W \to u(W)$ with relation R (its cocycle is automatic, being an equality of submodules on the triple fibre product), and effective flat descent gives a coherent ideal on $u(W)$. These descents respect multiplication, so the graded Rees algebra descends; on W it is generated by y_iu^j, $1 \leq j \leq a_i$, hence of finite type, and finite type is smooth-local [19, IV, 2.7.1]. [ledger 4; referee report 001, G1] □

5.3. **The radial lemma.** Let $a_1, \ldots, a_c > 0$ be the $\bar{\lambda}_y$-weights of the coordinates y_i on $W = \mathbb{A}^c$. The stack-theoretic weighted blow-up of W at the origin [28, Def. 3.2.1] has, since $y_1, \ldots, y_c$ is a regular sequence, the charts of [28, Cor. 5.2.5], namely charts $V = (W \setminus 0) \times \mathbb{A}_s^1$ modulo $\mathbb{G}_m$ acting by $(y, s) \mapsto (\bar{\lambda}_y(\tau)y, \tau^{-1}s)$, with blow-down $\beta(y, s) = (s^{a_i}y_i)$ and exceptional divisor $\{s = 0\}$. Let $\gamma(y, s) = y$ be the radial projection. [ledger 5]

Lemma 5.8 (Radial lemma)**.** *Let I be an ideal of $\mathcal{O}_{\mathrm{Bun}}$ near y and $J = f^*I$. Then $(\beta^*J : s^\infty) = \gamma^*J$, degree by degree for filtrations, and $(f\gamma, s) : V \to$ Bun $\times \mathbb{A}^1$ is smooth with image avoiding y.*

Proof. J is T_y-stable, hence $\bar{\lambda}_y$-stable, so on $s \neq 0$ the action identifies β^*J with γ^*J. On an affine open A of $W \setminus 0$, $\mathcal{O}/\gamma^*J = (A/J)[s]$ has no s-torsion,

so $\gamma^* J$ is s-saturated and is the contraction of its localisation. Smoothness holds because $(f\gamma, s)$ is $f|_{W\setminus 0} \times \mathrm{id}$ after the product identification, and $f^{-1}(B\mathcal{G}_y)$ is the origin. [ledger 3] □

5.4. **The construction.** By (G0) and the dimension argument of Proposition 6.5, ordering the points of $\mathcal{B}$ with $c \geq 2$ by decreasing c (ties arbitrary) as $x_1, x_2, \dots$ makes each x_m closed in $U_{m-1} = \mathcal{B} \setminus \{x_1, \dots, x_{m-1}\}$. For each m let $F^{(0)}_{m,\bullet}$ be the contraction to $\mathcal{B}$ of the descended filtration of x_m extended by the unit ideal on $U_{m-1} \setminus \{x_m\}$:

$$F^{(0)}_{m,k} = \ker\big(\mathcal{O}_{\mathcal{B}} \to j_{m*}(\mathcal{O}_{U_{m-1}}/F_{m,k})\big).$$

These are coherent ideals, compatible with flat base change, supported on the closure of x_m, and equal to the weighted ideals near x_m; no claim is made that they are unit ideals at deeper points. [ledger 3]

Theorem 5.9. *Assume Hypothesis 5.1. Blowing up the points $x_1, x_2, \dots$ in order, each with its weighted filtration (the stack-theoretic weighted blow-up of $\mathcal{B}_n$ along the intrinsic transform $I^{(n)}_\bullet$ of Definition 5.10, whose support is closed in $\mathcal{B}_n$; Lemma 5.11)*

Definition 5.10 (Intrinsic transforms)**.** Put $\mathcal{B}_0 = \mathcal{B}$ and let $b_n : \mathcal{B}_n \to \mathcal{B}_{n-1}$ be the blow-ups constructed so far, with exceptional Cartier divisor E_n [28, Def. 3.2.2]. For a tracked ideal J (degree by degree for a filtration) its *intrinsic transform* is $J^{(0)} = J$ and

$$J^{(n)} = \big(b_n^{-1} J^{(n-1)} \cdot \mathcal{O}_{\mathcal{B}_n} : \mathcal{I}^\infty_{E_n}\big),$$

an ideal on the smooth site of $\mathcal{B}_n$. The older divisors are the total transforms $E^{(n)}_i = b_n^{-1} E^{(n-1)}_i$, $i < n$. The $(n+1)$-st centre is $I^{(n)}_\bullet = (F^{(0)}_{n+1,\bullet})^{(n)}$, and $\mathcal{B}^\natural_n$ is the complement of $V(\partial^{(n)})$, ∂ the reduced ideal of $\mathcal{B} \setminus \mathcal{U}$.

The initial contractions $F^{(0)}_{m,k}$ are coherent: $j_m : U_{m-1} \to \mathcal{B}$ is quasi-compact and quasi-separated, so the pushforward of the quasi-coherent quotient is quasi-coherent and its kernel in $\mathcal{O}_{\mathcal{B}}$ is coherent; on smooth affine presentations flat base change commutes with the kernel and with pushforward along the open immersion. Contraction preserves multiplication and the decreasing filtration. Finite generation of the contracted algebra at deeper points is neither proved nor needed.

Lemma 5.11 (Intrinsic transforms and actual centres)**.** *Let X be a locally noetherian algebraic stack, $E \subset X$ an effective Cartier divisor and $J \subset \mathcal{O}_X$ a coherent ideal. Then $(J : \mathcal{I}^\infty_E)$ is a coherent ideal on the smooth site, and its formation commutes with flat pullback. Degreewise saturation preserves multiplicative decreasing filtrations with degree zero equal to $\mathcal{O}_X$. At stage n, assume (I_n)(a) below for the intrinsic transforms of the initial ideals. Then $I^{(n)}_\bullet$ is a finitely generated Rees algebra defining a regular weighted centre, before its blow-up is formed.*

Proof. On a smooth affine chart with $E = (t)$, saturation is $\ker\big(A \to (A/J)[t^{-1}]\big)$, finite since A is noetherian. Flat tensor product preserves the kernel and the localisation, so these ideals agree on double and triple smooth overlaps and descend; changing t by a unit changes nothing. If $t^a u \in J_i$ and $t^b v \in J_j$ then $t^{a+b}uv \in J_{i+j}$, which proves multiplicativity; the other filtration assertions follow directly. By (I_n)(a) the next filtration on each chart is $g_V^{-1}F_{n+1,\bullet}\cdot\mathcal{O}_V$, pulled back from U_n, where x_{n+1} is closed. Refine near it by $W \to U_n$; off it the filtration is the unit filtration. On W the algebra is generated by $y_i u^j$, $1 \le j \le a_i$, with y_i regular parameters of positive weights a_i, and the same holds after smooth pullback. Finite generation descends and regularity of the weighted centre is smooth-local [28, Def. 5.2.7, Def. 5.3.1], so the stack-theoretic blow-up is defined [28, Def. 3.2.1] and is regular [28, Cor. 5.3.2]. □

The logical order is: initial coherent ideals, then intrinsic transforms along blow-ups already constructed, then their local form from (I_n), then finite generation and regularity of the next centre, then the next blow-up. (I_{n+1}) is not used to justify that blow-up.

Statement (I_n)*, global form.* $\mathcal{B}_n$ has a smooth atlas by schemes V, each with a smooth morphism $(g_V, s) : V \to U_n \times \mathbb{A}^r$, such that (a) $J^{(n)}|_V = g_V^{-1}J\cdot\mathcal{O}_V$ for every tracked ideal J, degree by degree; (b) $E_i^{(n)}|_V = V(s_i)$ for the divisors through V and $E_i^{(n)} \cap V = \emptyset$ for the others; (c) there are a local system $\mathbb{F}_n$ on $\mathcal{B}_n^\natural$ extending $\mathbb{F}$ and isomorphisms $\alpha_V : \mathbb{F}_n|_{V^\natural} \xrightarrow{\sim} g_V^*\mathbb{F}$ on $V^\natural = g_V^{-1}(\mathcal{U})$. For $n = 0$ take the charts $f : W \to \mathcal{B}$ of Section 5, $g_W = f$, $r = 0$, $\mathbb{F}_0 = \mathbb{F}$, $\alpha_W = \mathrm{id}$. Agreement of (a) on double and triple overlaps is then automatic, since both sides are restrictions of the one ideal $J^{(n)}$ on the stack; it is *not* a consequence of the two charts pulling back one ideal of $\mathcal{B}$ along their two (different) radial maps.

Lemma 5.12 (Chart square)**.** *Assume* (I_n)*, let* $x' = x_{n+1}$ *and let* $F_\bullet$ *be its descended weighted filtration on* U_n *(unit off* x'*; Proposition 5.7). For every chart* V *there are canonical isomorphisms of stacks over* V

$$V \times_{\mathcal{B}_n} \mathcal{B}_{n+1} \simeq \mathrm{Bl}_{g_V^{-1}F_\bullet\cdot\mathcal{O}_V} V \simeq V \times_{g_V, U_n} \mathrm{Bl}_{F_\bullet} U_n,$$

under which $\mathcal{I}_{E_{n+1}} = \mathcal{O}(1) \subset \mathcal{O}$ *is the pullback of the exceptional ideal of* $\mathrm{Bl}_{F_\bullet} U_n$*.*

Proof. Relative Spec of a graded algebra, its irrelevant-open locus and the quotient by $\mathbb{G}_m$ commute with base change, and $\mathrm{Proj}^{\mathrm{st}}$ is defined over a stack base [28, §1.2]. Flatness identifies tensor pullback with the image filtration, as in the proof of [28, Cor. 3.2.14(iii)] (stated there for schemes). Apply this to the smooth $V \to \mathcal{B}_n$, using $I_\bullet^{(n)}|_V = g_V^{-1}F_\bullet\cdot\mathcal{O}_V$ ((I_n)(a), with g_V landing in U_n), and then to the smooth g_V. The exceptional ideal is $\mathcal{O}(1) = \mathcal{O}(-E)$ with its inclusion in $\mathcal{O}$ [28, Def. 3.2.2, Rem. 3.2.3], compatible with both base changes. □

Lemma 5.13 (Global atlas compatibility)**.** *(I_n) implies (I_{n+1}), with the following atlas. Put $Z = \mathrm{Bl}^{\mathrm{st}}_{F_\bullet} U_n$, $Z_W = W \times_{U_n} Z = \mathrm{Bl}^{\mathrm{st}}_{F_\bullet|W} W$, $C = V \times_{U_n} W$ and $Q_V = V \times_{U_n} Z_W = C \times_W Z_W$. The radial presentation gives $Z_W = [V'/\mathbb{G}_m]$ and $P = V \times_{U_n} V' = C \times_W V'$, with $g_P = \gamma \circ \mathrm{pr}_2$ and $s_P = (s \circ \mathrm{pr}_1, t \circ \mathrm{pr}_2)$. In the 2-Cartesian squares*

$$\begin{array}{ccc} P & \longrightarrow & V' \\ \downarrow & & \downarrow \\ Q_V & \longrightarrow & Z_W \end{array} \qquad \begin{array}{ccc} Q_V & \longrightarrow & Z_W \\ \downarrow & & \downarrow \\ V \times_{U_n} Z & \longrightarrow & Z \end{array}$$

$$\begin{array}{ccc} V \times_{U_n} Z \simeq V \times_{\mathcal{B}_n} \mathcal{B}_{n+1} & \longrightarrow & \mathcal{B}_{n+1} \\ \downarrow & & \downarrow \\ V & \longrightarrow & \mathcal{B}_n \end{array}$$

the vertical maps of the first square are $\mathbb{G}_m$-torsors, the vertical maps of the second are smooth (induced by $W \to U_n$), and the top map of the third is smooth (induced by $V \to \mathcal{B}_n$; Lemma 5.12). These maps cover the chart image, not necessarily all of U_n. The charts P, together with the unchanged charts $V^\circ = V \setminus g_V^{-1}(x')$ (with $g_{V^\circ} = g_V$), form a smooth atlas of $\mathcal{B}_{n+1}$. On $P_{12} = P_1 \times_{\mathcal{B}_{n+1}} P_2$ and P_{123} the two radial maps $g_{P_1} q_1, g_{P_2} q_2$ are in general neither equal nor 2-isomorphic, and nothing below compares them.

Proof. (a) Ideals. $P \to \mathcal{B}_{n+1} \to \mathcal{B}_n$ is 2-isomorphic to $P \xrightarrow{\mathrm{pr}_1} V \to \mathcal{B}_n$, so $(b_{n+1}^{-1} J^{(n)} \cdot \mathcal{O})|_P = \mathrm{pr}_1^{-1} g_V^{-1} J \cdot \mathcal{O}_P = \mathrm{pr}_2^{-1}(\beta^{-1} J \cdot \mathcal{O}_{V'}) \cdot \mathcal{O}_P$, image ideals being equal along 2-isomorphic maps ($g_V \mathrm{pr}_1 \cong \beta \, \mathrm{pr}_2$). By Lemma 5.12 the exceptional ideal on P is $(t \circ \mathrm{pr}_2)$. Saturation commutes with the flat pr_2 (Lemma 5.11), and homogeneity makes $\beta^{-1} J$ and $\gamma^{-1} J$ equal after inverting t, while the quotient by $\gamma^{-1} J$ is locally $(A/J)[t]$ with no t-torsion (Lemma 5.8). Hence $J^{(n+1)}|_P = g_P^{-1} J \cdot \mathcal{O}_P$, degree by degree, for nonreduced weighted ideals too, with no uniform saturation exponent. On V°, b_{n+1} is an isomorphism and $J^{(n+1)} = J^{(n)}$. On P_{12} and P_{123} the identities are pullbacks of the single intrinsic $J^{(n+1)}$, so their cocycles are automatic; this includes the later centres, which are pulled back along chart-specific radial maps and whose intrinsic transforms are the next centres.

(b) Divisors. The factorisation of Step 2 below makes $(\mathrm{pr}_2, s) : P \to V' \times \mathbb{A}^r$ and (γ, t) smooth. After quotienting by any tracked radial ideal, the old s_i remain non-zero-divisors by flatness over the relevant polynomial ring, and t remains a non-zero-divisor after setting any old s_i to zero. So total and strict transforms of old divisors agree, and they remain smooth and transverse to $E_{n+1} = V(t \circ \mathrm{pr}_2)$.

(c) Coefficients. On $\mathfrak{O} = \mathcal{B}_{n+1}^\natural \setminus E_{n+1}$ there is one sheaf $L = b_{n+1}^* \mathbb{F}_n$. On the punctured radial chart the composite

$$g_P^* \mathbb{F} = \mathrm{pr}_2^* \gamma^* \mathbb{F} \xrightarrow{\mathrm{pr}_2^* \Psi} \mathrm{pr}_2^* \beta^* \mathbb{F} \xrightarrow{\phi} \mathrm{pr}_1^* g_V^* \mathbb{F} \xrightarrow{\mathrm{pr}_1^* \alpha_V^{-1}} L|_P,$$

with Ψ of Lemma 4.5 and ϕ the specified 2-isomorphism of the fibre product acting on the local system $\mathbb{F}$ of the stack $\mathcal{U}$, identifies the local candidate $g_P^* \mathbb{F}$ with L. The candidate is defined on all of $P^\natural = g_P^{-1}(\mathcal{U})$, including

where old s_i vanish. Restriction of local systems from a smooth complex space to the complement of a positive-codimension analytic subset is fully faithful (Lemma 4.6). Apply it on the smooth double and triple overlaps of this atlas: the comparison and its inverse extend, and the cocycle extends by uniqueness. Effective descent constructs $\mathbb{F}_{n+1}$ with α_P; globality of the coefficient object is the conclusion, not a premise. Self-overlaps retain all stabiliser actions. Monodromy stays finite: extending across a divisor adds relations to the monodromy of the dense open, not new image generators. The old exceptional equations remain independent parameters, so their zero loci and simultaneous crossings are included. □

Proof. Write $\mathcal{B}_n$ for the result of the first n blow-ups and $U_n = \mathcal{B} \setminus \{x_1, \dots, x_n\}$, which is open because the processed set is downward closed in the frontier order (Proposition 6.5). The *tracked ideals* are the reduced boundary ideal of $\mathcal{B} \setminus U$, the reduced ideals of the closures of the finitely many strata, and the unprocessed filtrations $F^{(0)}_{m,\bullet}$, $m > n$.

Statement (I_n). $\mathcal{B}_n$ has a smooth atlas by schemes V (charts of algebraic-space presentations), each with a smooth morphism $(g_V, s) : V \to U_n \times \mathbb{A}^r$, where $s = (s_1, \dots, s_r)$ are local equations of the labelled exceptional divisors through V, such that every tracked ideal J restricts on V to the image ideal $g_V^* J$, degree by degree, and these identifications are compatible on overlaps of charts. For $n = 0$ take the charts $f : W \to \mathcal{B}$ of Section 5.2 with $r = 0$.

Step 1: the next centre. Let $x' = x_{n+1}$. A point of $\overline{\{x'\}} \setminus \{x'\}$ has larger stabiliser dimension, hence is among $x_1, \dots, x_n$, hence is not in U_n; so x' is closed in U_n. On a chart V, $g_V^* F^{(0)}_{n+1,\bullet}$ is therefore, locally, either the smooth pullback of the regular weighted filtration at x' (where g_V meets x') or the unit filtration. On the chart W of x' the filtration is the monomial one $(y_1^{1/a_1}, \dots, y_c^{1/a_c})$, a regular weighted centre in the sense of [28, Def. 5.2.7, Def. 5.3.1]. Regularity is smooth-local by those definitions, and it is stable under smooth pullback, since flat pullback preserves regular sequences and smooth pullback preserves regularity of the reduced centre. Its Rees algebra is the pullback of a finitely generated one. The centre is the intrinsic transform $I^{(n)}_\bullet$, a regular weighted centre with finitely generated Rees algebra by Lemma 5.11.

Step 2: charts and smoothness. Let V' be a radial chart of Lemma 5.8 at x', built inside U_n, with blow-down $\beta : V' \to U_n$, radial map $\gamma : V' \to U_{n+1}$ and new coordinate t, and let $P = V \times_{g_V, U_n, \beta} V'$ (a 2-fibre product; pass to scheme covers if needed). The structure map used to form P is β; the new invariant map is $g_P = \gamma \circ \mathrm{pr}_2$. The morphism $(g_P, s \circ \mathrm{pr}_1, t \circ \mathrm{pr}_2)$ factors as

$$P \xrightarrow{(\mathrm{pr}_2,\, s\circ\mathrm{pr}_1)} V' \times \mathbb{A}^r \xrightarrow{(\gamma,\, \mathrm{id},\, t)} U_{n+1} \times \mathbb{A}^{r+1},$$

the first arrow being the base change of the smooth (g_V, s) along (β, id) and the second the product of the smooth (γ, t) with $\mathrm{id}_{\mathbb{A}^r}$. So it is smooth; no equality $\beta = \gamma$ is used.

Step 3: the tracked ideals. For a tracked ideal J, the fibre square gives $\mathrm{pr}_1^* g_V^* J = \mathrm{pr}_2^* \beta^* J$; ideals on a stack pull back to the same ideal along 2-isomorphic maps, so this uses only $g_V \circ \mathrm{pr}_1 \cong \beta \circ \mathrm{pr}_2$. Saturating along $t = 0$ and using Lemma 5.8 together with flatness of pr_2 (a base change of g_V), the strict transform is $\mathrm{pr}_2^* \gamma^* J = g_P^* J$. By the first arrow of Step 2 the old coordinates s_i are independent relative to V', so $\mathcal{O}_P / g_P^* J$ is locally smooth over $(\mathcal{O}_{V'}/\gamma^* J)[s_1, \ldots, s_r]$ and has no s_i-torsion: the transform is saturated along every old exceptional divisor, not only the newest. The old divisors have total transform (s_i), equal to their strict transform; they remain smooth, individually labelled and transverse to $t = 0$. On overlaps both sides are restrictions of the intrinsic transform $J^{(n+1)}$, so the identifications agree (Lemma 5.13); being pullbacks of one ideal of $\mathcal{B}$ along two different radial maps would not suffice, and the radial maps are not compared.

Step 4: disjointness. Two distinct strata of equal codimension have closures meeting only in deeper strata (Proposition 6.5). On a chart their strict transforms have supports g_V^{-1} of their closures, and g_V avoids the processed points, so these supports are disjoint once the deeper strata have been processed.

Step 5: termination. After the finite list of points with $c \geq 2$ is processed, g_V lands in the open point or in a point with $c = 1$, where the minimal transverse chart is a smooth curve and the boundary is its origin. So the strict boundary is locally a smooth Cartier divisor or empty, and by smoothness of (g_V, s) it is transverse to the exceptional coordinates: the total boundary consists of smooth labelled components meeting as coordinate hyperplanes in smooth charts.

Step 6: finite inertia and branches. Each weighted blow-up has finite relative inertia, and so do composites (nonsplit finite extensions are allowed). The stack $\mathcal{B}_N$ need not be Deligne–Mumford, but $\mathcal{X}_x$ is: its stabilisers are kernels of the maps to stabilisers in $\mathcal{B}$, since Y is a scheme, so it has finite inertia, and it is smooth because ω and $\mathcal{B}_N$ are. Étale locally at a geometric point it is a finite stabiliser quotient of a smooth chart in which the boundary is a union of coordinate hyperplanes; each E_i is a globally labelled smooth Cartier divisor with one local branch, and the strict boundary is smooth with at most one branch, so the stabiliser preserves each branch, and after averaging the local equations may be taken to be eigenvectors for the normal characters. [ledger 6] □

6. The census of points of Bun

6.1. **Bundle types.** A point of $\mathcal{B}$ is a pair $(\mathcal{P}_{\tilde{w}}, b)$ up to $\mathrm{Aut}(\mathcal{P}_{\tilde{w}})$. Put $H_{\tilde{w}} = \mathrm{Aut}(\mathcal{P}_{\tilde{w}})$ and let $j(\tilde{w}) = \dim H_{\tilde{w}} - \dim K$ be the codimension of the bundle-type locus. The stabiliser of b is $S_b = \mathrm{ev}_1^{-1}(B_b)$, of dimension $c = \dim H_{\tilde{w}} - \dim(H_{\tilde{w}} \cdot b)$, so $c \geq j$.

Lemma 6.1. *The bundle types with $j \leq 6$ are exactly six, one for each $j \in \{0, 1, 3, 4, 5, 6\}$, with $\dim H_{\tilde{w}} = 6, 7, 9, 10, 11, 12$; representatives are*

$\star = t^{(-2,-2)}s_1s_2s_1$, $B_1 = t^{(-2,-2)}s_1s_2s_1s_2$, $B_3 = t^{(-3,-3)}s_1s_2s_1s_2$, $B_4 = t^{(-2,-3)}s_1s_2s_1$, $B_5 = t^{(-3,-3)}s_1s_2s_1$, $B_6 = t^{(-2,-3)}s_1s_2$.

Proof. With $v = \rho^\vee/2$ and $\nu = \mu + v - wv$, (1) gives $N_\alpha = \max(\langle\alpha,\nu\rangle + 1 - (\varepsilon(\operatorname{ht}\alpha) + \varepsilon(\operatorname{ht} w^{-1}\alpha))/2, 0)$, ε the parity, so $N_\alpha + N_{-\alpha} \geq |\langle\alpha,\nu\rangle|$ and $j = \sum_\alpha N_\alpha - 4 \leq 6$ forces $\sum_{\alpha>0}|\langle\alpha,\nu\rangle| \leq 10$, whence $|\mu_1| \leq 15$ and $|\mu_2| \leq 13$. An exact enumeration over this box and all $w \in W$ gives the list. [Certificate: exp/enum6/e1.py and enum.py.] □

Proposition 6.2. *The image* $\mathrm{ev}_1(H_{\tilde{w}})$ *is the connected subgroup with Lie algebra* $\mathfrak{t} \oplus \bigoplus_{N_\alpha \geq 1} \mathfrak{g}_\alpha$*:*

- *for* $\star$*, the symmetric subgroup* K*;*
- *for* B_6*, a Borel subgroup* B'*;*
- *for* B_5*, a parabolic subgroup* P' *with Levi of type* GL_2 *(roots* $\pm(\alpha_1 + \alpha_2)$*);*
- *for* B_1*,* B_3*,* B_4*, the subgroup* $T \cdot U_0$ *where* U_0 *is the product of the positive root groups of a Borel* B' *other than one simple root* γ *(respectively* $\gamma = -\alpha_1$*,* $-(3\alpha_1 + 2\alpha_2)$*,* $-(2\alpha_1 + \alpha_2)$*).*

The kernel of ev_1 *on* $H_{\tilde{w}}$ *is unipotent and meets every torus trivially.*

Proof. The root sets are read off from (1). For the solvable types, if $\mathrm{ev}_1(tu) = 1$ then $\mathrm{ev}_1(t) = \mathrm{ev}_1(u)^{-1}$ is semisimple and unipotent, hence $t = 1$; for B_5 evaluation identifies the Levi with the root Levi of P' and the unipotent part maps into its unipotent radical; for $\star$ evaluation is the inclusion of K. □

Proposition 6.3. *For every point* b *of every type and every maximal torus* $T_S \subset S_b^\circ$*, the normaliser* $N_{S_b}(T_S)$ *centralises* T_S*. In particular* $\Gamma_y = 1$ *and* $\bar{\lambda}_y = \lambda_y$*.*

Proof. In a connected solvable group B the projection $B \to B/R_u(B)$ is injective on every torus and has commutative target, so if n normalises a torus T_0 then ntn^{-1} and t have the same image, hence are equal. Apply this inside B_b to $\mathrm{ev}_1(T_S)$ and use Proposition 6.2. [ledger 7: C-GROUP] □

6.2. **Orbit classification.**

Proposition 6.4. *The* $H_{\tilde{w}}$*-orbits on* G/B *are finite in number for each type:* 12 *Bruhat cells for* B_6*;* 6 *orbits for* B_5*;* 18 *orbits for each of* B_1, B_3, B_4*, namely one orbit* vB *for each* v *with* $\gamma \in v\Phi^+$ *and two orbits* vB*,* $u_\gamma(1)vB$ *for each* v *with* $\gamma \notin v\Phi^+$*; and* 10 *orbits for* $\star$*. Every orbit with* $c \geq 1$ *has a representative fixed by* T *or by* $(\ker\gamma)^\circ$ *for a root* γ*; in particular no orbit of positive codimension has a stabiliser without a torus.*

Proof. For B_6 and B_5 this is the Bruhat decomposition. For B_1, B_3, B_4, U_0 is normal in B' and $B'/U_0 = T \ltimes U_\gamma$ with image of $H' = T \cdot U_0$ equal to T. With $Q_v = B' \cap vBv^{-1}$, the image of Q_v in $T \ltimes U_\gamma$ is T if $\gamma \notin v\Phi^+$ and all of $T \ltimes U_\gamma$ otherwise; in the first case $T\backslash(T \ltimes \mathbb{G}_a)/T$ has the two geometric double cosets 0 and 1, since T scales the coordinate through the nontrivial character γ. [ledger 7: C-BRUHAT]

For $\star$, K is connected split of type $A_1 \times A_1$ with Weyl group of order four. The twelve T-fixed flags form three closed orbits K/B_K (two fixed flags lie in one K-orbit iff their Weyl labels differ by W_K). The six flags $u_\gamma(1)vB$ with γ in Table 1 have stabiliser dimensions $c = 1, 2, 3$ (computed exactly over $\mathbb{Q}$), maximal stabiliser torus $(\ker\gamma)^\circ$ of rank one, and unipotent radical of dimension $c-1$; the two in each row are not K-conjugate (see *Distinctness over* $\mathbb{C}$ below). The flag u_0 has $c = 0$ and lies in the open orbit. These ten orbits are pairwise distinct. [ledger 7: C-DISTINCT] They exhaust G/B: for a geometric orbit $O = L/S$ with a rational point, S° solvable with split torus quotient of rank r, unipotent radical of dimension u, and component group acting trivially on that torus quotient (Proposition 6.3), Lang's theorem gives

$$|O(\mathbb{F}_q)| = |L(\mathbb{F}_q)|/\big((q-1)^r q^u\big),$$

because the rational orbits are indexed by twisted conjugacy classes in $\pi_0(S)$ and $\sum_{[a]} 1/|C(aF)| = 1$; no knowledge of $\pi_0(S)$ is needed. [ledger 7: C-MASS] With $|K(\mathbb{F}_q)| = q^2(q^2-1)^2$ the ten counts sum to $(q+1)^2\big(q^2(q-1)^2 + 2(q-1)(q^2+q+1) + 3\big) = (q+1)^2(q^4+q^2+1) = |G/B(\mathbb{F}_q)|$ for every q prime to a fixed N. Spreading out the orbit maps over $\mathbb{Z}[1/N]$, the complement of the ten orbits has no $\mathbb{F}_{p^n}$-points for all good p and all n, hence is empty over $\overline{\mathbb{Q}}$. [ledger 7: C-SPREAD]

Distinctness over $\mathbb{C}$. In characteristic 0 stabilisers are smooth, so $\operatorname{Lie} K_x = \mathfrak{k} \cap \mathfrak{b}_x =: \mathfrak{s}_x$, and $y = k \cdot x$ implies $\mathfrak{s}_y = \operatorname{Ad}(k)\mathfrak{s}_x$. For each representative with $\dim \mathfrak{s}_x \in \{1, 2, 3\}$ the certificate's stabiliser table gives $\mathfrak{s}_x = \mathbb{C}h \ltimes \mathfrak{n}$ with $h \in \mathfrak{t}$, $\mathfrak{n}$ abelian and spanned by root vectors (the roots involved, $\alpha_1+\alpha_2$ and $\pm(3\alpha_1+\alpha_2)$, have no root among their sums), and $\operatorname{ad} h$ invertible on $\mathfrak{n}$. Hence $ah + n = \operatorname{Ad}(\exp m)(ah)$ with $m = -a^{-1}(\operatorname{ad} h|_{\mathfrak{n}})^{-1}n$ for $a \neq 0$, while $\mathfrak{n}$ consists of nilpotent elements; so the nonzero semisimple elements of $\mathfrak{s}_x$ are exactly the $\operatorname{Ad}(K_x^\circ)$-conjugates of nonzero multiples of h. Therefore

$$\rho(x) := \max|\operatorname{spec}\operatorname{ad} h| \,/\, \min|\operatorname{spec}\operatorname{ad} h \setminus 0|$$

(on $\mathfrak{g}_2$) is an invariant of the orbit $K \cdot x$, indeed of its $\operatorname{Ad}(\mathrm{G}_2)$-class. If h is proportional to the coroot of a long (resp. short) root, the values $\gamma(h)$, $\gamma \in \Phi$, are proportional to $\{0, \pm1, \pm2\}$ (resp. $\{0, \pm1, \pm2, \pm3\}$), so $\rho = 2$ (resp. 3). With α_1 short and $E(a,b)$ the root vector of $a\alpha_1 + b\alpha_2$, $h_2 = \alpha_2^\vee$ and $h_1 + 2h_2 = (3\alpha_1 + 2\alpha_2)^\vee$ are long coroots, while $h_1 = \alpha_1^\vee$ and $h_1 + \frac{3}{2}h_2 = \frac{1}{2}(2\alpha_1+\alpha_2)^\vee$ are short ones. Thus, in the certificate's labels, $\rho = 2$ for c_1 A, c_2 $(1,2)$, c_3 $(1,1,2)$ and $\rho = 3$ for c_1 B, c_2 $(2,3)$, c_3 Σ: each codimension in Table 1 has one orbit of each value, so its two orbits are distinct. The three closed orbits are distinct because a closed orbit $K \cdot wB \cong K/B_K$ meets $(G/B)^T$ exactly in $\{vB : v \in W_K w\}$, and $W_K \backslash W$ has $12/4 = 3$ elements. Orbits of different dimension are distinct, so the ten orbits are pairwise distinct. [referee report 001, C2; C-DIST]

Completeness over $\mathbb{C}$. By Matsuki [10] and Richardson–Springer [11], $K\backslash G/B$ is in bijection with $\bigsqcup_H W(G,H)/W_K(H)$ over the K-classes of θ-stable Cartan subgroups. By Sugiura [8, Thm. 6, Def. 9, Thm. 7] (Theorem 7 as corrected in [9]), the conjugacy classes of Cartan subalgebras of the split real form correspond to the classes of admissible (strongly orthogonal) sets of noncompact roots, each realised by the iterated Cayley transform $\rho_{\alpha_l} \circ \cdots \circ \rho_{\alpha_1}$ from the compact Cartan subalgebra; for split G_2 there are exactly four classes [8, p. 429], which the 1971 correction does not affect. Cayley transforms from T give these four classes: compact, one mixed class with a real short root and one with a real long root (W_K is transitive, up to sign, on the noncompact imaginary roots of each length), and split. Their contributions are $12/|W_K| = 3$, $12/4 = 3$, $12/4 = 3$ (for a mixed Cartan H with real root α, $W_K(H) = \{1, s_\alpha, s_\beta, -1\}$ with $\beta \perp \alpha$ noncompact imaginary and s_β realised in K; this is the full centraliser of s_α in $W \cong D_6$, of order four), and $12/12 = 1$. The total is 10, so the list is complete; $\rho = 2$ and $\rho = 3$ label the two mixed Cartans, three orbits each. The point counts above are a check. The general parametrisation is that of Matsuki [10] and Richardson–Springer [11]; its specialisation to G_2/SO_4 is checked by the certificate. The certificate reproduces the count $3+3+3+1$ independently, from explicit θ-stable Cartan data, exactly over $\mathbb{Q}$ (Cayley transforms in a Chevalley basis of $\mathfrak{g}_2$; $\theta = \mathrm{Ad}(s)$, $s \in T$). It finds:

- four K-classes of θ-stable Cartans: compact, short-root Cayley, long-root Cayley and split, with pairwise distinct root-type patterns;
- $|W|/|W_K(H)| = 3, 3, 3, 1$, with $|W_K(H)| = 4, 4, 4, 12$ in characteristic 0: the upper bound (W_K for the compact class, $Z_W(\theta_S)$ otherwise) is attained, because every one of its elements w is realised by an explicit $k = C_S\, n_w\, D(y)\, C_S^{-1} \in K$, with C_S the rational Cayley element, n_w the rational $N_G(T)$-representative along a reduced word of w and y a point of T, each k defined over $\mathbb{Q}$ or $\mathbb{Q}(i)$ and checked exactly ($\theta(k) = k$, $\mathrm{Ad}(k)\mathfrak{h}_S = \mathfrak{h}_S$, and $\mathrm{Ad}(k)$ permutes the root lines of $\mathfrak{h}_S$ as w does on all twelve roots); since $K = G_2^\theta$ is connected (Steinberg), θ-fixed in $\mathrm{Aut}(\mathfrak{g}_2) = G_2$ means $k \in K$;
- the ten orbits of Table 1 matched to the classes by the rank of their stabiliser torus (2, 1, 1, 0) and, in rank one, by ρ (2 for the short-root Cayley class, 3 for the long-root one).

The same code returns the known counts 3 for SL_2/SO_2 and 6 for $SL_3/S(GL_2 \times GL_1)$, with the normaliser elements again realised in characteristic 0 (for SL_2 the split class needs an element over $\mathbb{Q}(i)$, which is absent from $K(\mathbb{F}_7)$; this is why the earlier comparison over finite fields was not by itself a proof). Scripts `matsuki.sage`, `matsuki_c0.sage` and `match.sage`, with their logs and the exact representatives, are in `certificate/matsuki/` of the bundle. □

TABLE 1. Representatives $u_\gamma(1)vB$ of the K-orbits of codimension $1, 2, 3$. Stabiliser Lie algebras, exact over $\mathbb{Q}$, and point counts over $\mathbb{F}_5, \mathbb{F}_7$ for all ten orbits are in the certificate (`K-ORBIT-TABLE.md`, script `korb_table.sage`).

c	short root γ	long root γ
1	$-(2\alpha_1+\alpha_2)$, $v=e$	$-(3\alpha_1+2\alpha_2)$, $v=e$
2	$-(2\alpha_1+\alpha_2)$, $v=s_1$	$-\alpha_2$, $v=s_1$
3	$-\alpha_1$, $v=e$	$-\alpha_2$, $v=e$

Proposition 6.5 (Frontier order)**.** *$\mathcal{B}$ has finitely many points, and a point in the closure of a distinct point y has strictly larger stabiliser dimension than y.*

Proof. Finiteness follows from Lemma 6.1 and Proposition 6.4. On a smooth atlas of pure relative dimension d over the smooth dimension-zero stack, the inverse image of a residual gerbe with stabiliser dimension c is smooth of pure dimension $d-c$ and open in its closure, so a distinct stratum in its boundary has smaller dimension, i.e. larger c; this compares strata of different bundle types as well. [ledger 7: C-FRONTIER] □

6.3. **Verification of (G1) and (G2).** At a T-fixed point $(\mathcal{P}_{\tilde{w}}, vB)$ the stabiliser dimension, the weights of T on T^1 and on $\operatorname{Lie}\mathcal{G}/\operatorname{Lie}T$ are given by the affine root counts:

$$c = 2 + \sum_{\alpha\in v\Phi^+} N_\alpha + \sum_{\alpha\in v\Phi^-} (N_\alpha - 1)_+,$$

the T^1-weights are $\{\alpha \in v\Phi^- : N_\alpha = 0\}$ together with each α repeated $(-d_\alpha - 1)_+$ times ($d_\alpha = h_\alpha - \ell_\alpha$), and the stabiliser weights are $\alpha \in v\Phi^+$ with multiplicity N_α and $\alpha \in v\Phi^-$ with multiplicity $N_\alpha - 1$. At a point $u_\gamma(1)vB$ the stabiliser torus is $(\ker\gamma)^\circ$ and its weights are computed from a semisimple element of the stabiliser acting on $T^1 = \operatorname{coker}(\mathrm{ev}_1(\operatorname{Lie}H) + \mathfrak{b}) \oplus H^1$. Both computations were carried out exactly over $\mathbb{Q}$ at every orbit of Proposition 6.4 with $1 \le c \le 6$, with an explicit integral cocharacter in each case.

Proposition 6.6. *Hypothesis 5.1 holds.*

Proof. (G0) is Proposition 6.5. For (G1) and (G2), Table 2 lists, for each orbit with $1 \le c \le 6$, an integral cocharacter with all T^1-weights positive and all stabiliser weights ≤ 0. [Certificate: exp/enum6/sweepQ.sage, tcurve.sage; exp/korb.] □

Remark 6.7. The weights $(1, 2, 3)$ at the $\star$-orbit of codimension three are the values $\langle\alpha, \beta^\vee\rangle$ of the positive G_2-roots on the coroot of a short root β.

TABLE 2. Contracting cocharacters $\lambda = m_1\alpha_1^\vee + m_2\alpha_2^\vee$ (selected rows; the full list is in the certificate, reference log `reference-logs/sweepQ.log`, produced by `run_all.sh`; the same content as v0.1's `logs/e6Q.log`). Rank is the rank of the stabiliser torus.

type	c	rank	λ	T^1-weights
$\star$	1,2,3	1	$(0,-1)$, $(-2,-3)$, $(-1,-2)$	(2); $(1,2)$, $(2,3)$; $(1,1,2)$, $(1,2,3)$
$\star$	4	2	$(1,1)$	$(1,1,1,1)$
B_1	1–6	1	$(-1,-2)$	$(2), \ldots, (1,1,1,1,2,2)$
B_1	3,4,5,6	2	$(-5,-12)$, $(-11,-12)$	$(2,9,12)$, $(2,7,9,12)$, $\ldots$
B_3	3–6	1	$(1,0)$	$(2,3,3)$, $(1,2,3,3)$, $\ldots$
B_3	5,6	2	$(1,1)$, $(-3,-12)$	$(1,1,1,1,2)$, $(3,6,9,12,15,15)$
B_4	4–6	1	$(0,1)$	$(1,1,2,2)$, $\ldots$
B_4	5,6	2	$(-12,1)$	$(1,12,25,38,38)$, $\ldots$
B_5	5,6	2	$(1,1)$	$(1,1,1,1,2)$, $(1,1,1,1,1,2)$
B_6	6	2	$(1,2)$	$(1,1,1,1,2,2)$

Independently of the census, the weighted local models at the codimension-three and codimension-four strata of D_x were also computed directly from the equations of D_x (first in chart C_0), with the same weights; they serve as a check.

7. Algebraic projectors through quotient singularities

Throughout this section the ground field is $\mathbb{C}$, a *scheme* is a separated scheme of finite type over $\mathbb{C}$, $\mathrm{CH}_*(-)_\mathbb{Q} = \mathrm{CH}_*(-)\otimes\mathbb{Q}$ is the Chow group [12, Ch. 1], and for a morphism $f : T \to Y$, $\mathrm{CH}^*(T \xrightarrow{f} Y)_\mathbb{Q}$ is Fulton's bivariant (operational) group with rational coefficients [12, Ch. 17], with cap product $\cap$, pullback along arbitrary morphisms, proper pushforward and product [12, §17.1]; $\mathrm{CH}^*(Y)_\mathbb{Q} = \mathrm{CH}^*(Y \to Y)_\mathbb{Q}$. Bivariant degrees may be negative. H_*^{BM} is rational Borel–Moore homology (equal to $H_*(-;\mathbb{Q})$ on compact spaces) and $\mathrm{cl} : \mathrm{CH}_k(T)_\mathbb{Q} \to H_{2k}^{BM}(T)$ is the cycle map, compatible with proper pushforward, exterior product and refined Gysin maps of regular embeddings [12, §§19.1–19.2]. [S-lit] When T is a compact oriented rational homology manifold of real dimension $2n$ we also write

$$\mathrm{cl}_H := \mathrm{PD}_T^{-1} \circ \mathrm{cl} : \ \mathrm{CH}_k(T)_\mathbb{Q} \longrightarrow H^{2n-2k}(T;\mathbb{Q})$$

for the cohomological cycle class; all pullbacks q^* below act on cl_H, all pushforwards and caps on cl.

7.1. **Setting.** We fix the following.

(T1) R is the variety of Theorem 4.3 *for the weighted clean model constructed in Theorem 5.9*: a normal projective scheme of pure dimension d with quotient singularities. Projectivity is a property of this model, not of an arbitrary proper good model (Remark 7.3). [D]

(T2) $q : \overline{X} \to R$ is a projective birational morphism from a smooth projective variety, an isomorphism over the smooth locus R^{sm}; $\Gamma_q \subset \overline{X} \times R$ is its graph and ${}^t\Gamma_q \subset R \times \overline{X}$ the transpose.

(T3) A finite group A acts on R and on $\overline{X}$, and q is A-equivariant.

(T4) $p \in \mathrm{CH}_d(R \times R)_{\mathbb{Q}}$ is an algebraic correspondence. The abstract results (Definition 7.5 to Proposition 7.12) hold for *every* such p. Its realisation in this paper is $p = \varepsilon P_\star$ of Remark 7.13.

A *test space* is a product $R^a \times \overline{X}^b$ $(a, b \geq 0)$. A product of two quotient singularities $S_1/H_1 \times S_2/H_2 = (S_1 \times S_2)/(H_1 \times H_2)$ is a quotient singularity, so every test space has quotient singularities and is equidimensional. [D]
Stagewise gluing over B. The open orbit $Y = \mathrm{Gr}_{x,\lambda}$ is not proper; the projective base is $B = \mathrm{Bl}_\star \mathrm{Gr}_{x,\leq\lambda}$. Glue at every stage: put $\mathcal{X}_0 = B$, $V_B = B \setminus D_x$, and $\mathcal{X}_i|_{B\setminus E} = Y \times_{\mathcal{B}} \mathcal{B}_i$, $\mathcal{X}_i|_{V_B} = V_B$. On the overlap $Y \setminus D_x$ the weighted filtration is the unit filtration, whose Rees algebra is canonically $\mathcal{O}[t]$; glue the pulled-back Rees algebra to $\mathcal{O}_{V_B}[t]$ degree by degree, and call the result $\mathcal{A}_i$ on $\mathcal{X}_{i-1}$ (finitely generated, being so locally, with $(\mathcal{A}_i)_0 = \mathcal{O}$). Then $b_i : \mathcal{X}_i = \mathrm{Proj}^{\mathrm{st}}_{\mathcal{X}_{i-1}}(\mathcal{A}_i) \to \mathcal{X}_{i-1}$ with $L_i = \mathcal{O}_{\mathcal{X}_i}(1)$, the section t trivialising L_i on the unchanged locus. On the Y-part flat base change applies, $Y \times_{\mathcal{B}} \mathcal{B}_{i-1} \to \mathcal{B}_{i-1}$ being smooth [28, Cor. 3.2.14(iii)]. Inductively each $\mathcal{X}_i$ is of finite type, proper over B, with finite absolute diagonal, hence a tame Deligne–Mumford stack with a separated proper coarse space C_i $(C_0 = B)$. No coarse-space assertion about any $\mathcal{B}_i$ is made.

Lemma 7.1 (Projectivity of the coarse weighted sequence)**.** *Let B be a projective scheme over $\mathbb{C}$, $\mathcal{X}_0 = B$, and for $1 \leq i \leq N$ let $b_i : \mathcal{X}_i = \mathrm{Proj}^{\mathrm{st}}_{\mathcal{X}_{i-1}}(\mathcal{A}_i) \to \mathcal{X}_{i-1}$ with $\mathcal{A}_i$ a finitely generated non-negatively graded quasi-coherent algebra, $(\mathcal{A}_i)_0 = \mathcal{O}$, and $L_i = \mathcal{O}_{\mathcal{X}_i}(1)$. Then the absolute coarse spaces $c_i : \mathcal{X}_i \to C_i$ exist, each C_i is a projective scheme, and $f_i : C_i \to C_{i-1}$ is projective; there are $n_i > 0$ and an f_i-ample line bundle M_i with $M_i = (c_i)_* L_i^{\otimes n_i}$ and $c_i^* M_i \simeq L_i^{\otimes n_i}$. Consequently the normalisation of C_N in a finite extension of its function field is projective.*

Proof. By [28, Prop. 1.6.1(i)–(ii)] the $\mathcal{X}_i$ are proper of finite type with finite diagonal, hence tame Deligne–Mumford with proper coarse spaces. Choose d_i with $\mathcal{A}_i^{(d_i)}$ generated in degree one [28, Prop. 1.6.3], and by [28, Setup 1.7.2] applied to $\mathcal{X}_i \to \mathrm{Spec}\,\mathbb{C}$ choose n_i, divisible by d_i, such that $L_i^{\otimes n_i}$ descends to M_i on C_i with the adjunction isomorphism; this kills the characters of all absolute stabilisers, including inherited ones. To see that M_i is f_i-ample, choose étale locally on C_{i-1} an affine V with $\mathcal{X}_{i-1} \times_{C_{i-1}} V = [U/G]$, U affine and finite over V and G finite constant [20, Thm. 3.2, Cor. 3.3]. Put $\mathcal{P} = \mathrm{Proj}^{\mathrm{st}}_U(\mathcal{A}_i|_U)$, $P = \mathrm{Proj}_U((\mathcal{A}_i|_U)^{(d_i)})$ and $H = \mathcal{O}_P(1)$; P is the

relative coarse space of $\mathcal{P}$ over U, with $p^*H = \mathcal{O}_{\mathcal{P}}(d_i)$, the construction is G-equivariant, and $C_i \times_{C_{i-1}} V = P/G$ by compatibility of coarse spaces with étale base change (this does not identify an absolute coarse space with a relative one over a stack). The quotient P/G is a scheme (H is ample and G-linearised; invariant affine opens from products over G of homogeneous sections), with finite surjective $\rho : P \to P/G$ and $\rho^* M_{i,V} \simeq H^{\otimes n_i/d_i}$, which is ample as U is affine. Both P and P/G are proper over the noetherian affine V, so finite-surjective descent of ampleness [25, Lemma 30.17.2] makes $M_{i,V}$ ample, and fpqc-locality of ampleness of a given line bundle [25, Lemma 74.13.1] makes M_i f_i-ample; in particular f_i is representable by schemes [25, §71.14]. If H_{i-1} is ample on the projective C_{i-1}, then $M_i^{\otimes a_i} \otimes f_i^* H_{i-1}^{\otimes b_i}$ is ample for suitable $a_i, b_i > 0$, so C_i is projective and f_i is projective. Finite-type $\mathbb{C}$-schemes are excellent, so normalisation in a finite function-field extension is finite, and the pullback of H_N is ample. □

The global coarse square $c_{i-1} \circ b_i = f_i \circ c_i$ is commutative, not asserted Cartesian, and ρ may be ramified: finite-surjective descent is used, not flatness. A Veronese (changing the grading, giving the relative coarse space P) and finite-group invariants (giving P/G) are different operations.

Remark 7.2 (The global graded algebras). Lemma 7.1 is conditional on global finitely generated graded algebras $\mathcal{A}_i$, and Theorem 5.9 supplies them. Lemmas 5.11 and 5.13 give global finitely generated $I_\bullet^{(i-1)}$ on $\mathcal{B}_{i-1}$. Pull back along the smooth current Y-stage. The centre is disjoint from the unchanged $\mathcal{U}$-locus (at old exceptional crossings the radial image and the original blow-down image differ, so it is this disjointness, not a statement that the original image is only x_i, that is used), and there every graded ideal is canonically $\mathcal{O}$, so the algebra is $\mathcal{O}[u]$; glue it to $\mathcal{O}[u]$ over $V_B = B \setminus D_x$. The gluing preserves multiplication and grading, with no line-bundle twisting ambiguity, and gives $\mathcal{A}_i$ on the glued $\mathcal{X}_{i-1}$ with degree-zero part $\mathcal{O}$, finitely generated smooth-locally and hence globally. Lemma 5.12 identifies its $\mathrm{Proj}^{\mathrm{st}}$ with the prescribed stage. The initial glued base is the projective B; each $\mathcal{X}_i$ is proper over B with finite absolute diagonal. So R is projective given Theorem 5.9 as proved with Lemma 5.13; no new projectivity hypothesis is required.

Remark 7.3 (projectivity of R, and the choice of q). Definition 4.1 asks only for a *proper* good model, and properness does not give projectivity. For the model of Theorem 5.9, glue the stages over the projective B as above and apply Lemma 7.1: the coarse space C_N (the X of Section 4) is projective over B, hence projective, and the finite normalisation R is projective, not merely proper. [D, conditional on the global construction of Theorem 5.9, supplied by Lemma 5.13 and Remark 7.2] In the transfer calculation itself, properness of R suffices; projectivity of R is what makes a projective strong resolution by blow-ups a smooth *projective* variety $\overline{X}$ (projectivity of q alone would not).

(T2) and (T3) can be met together. A functorial resolution [16, Thm. 1.1] is a finite sequence of blow-ups with smooth centres, an isomorphism over the regular locus, and functorial for automorphisms. It is therefore projective and lifts the action of A, which already acts on the normalisation R by functoriality. [S + D] The neighbourhood of E is smooth, so it is untouched. The abstract statements below need neither equivariance nor an isomorphism over all of R^{sm}: a proper map of degree one between these rational homology manifolds already gives $q_*q^* = 1$. The stronger choices serve the group and E constructions. [D]

7.2. **The formalism.**

Proposition 7.4 (Alexander property)**.** *Let Y be a test space, $n = \dim Y$, and $f : T \to Y$ any morphism of schemes. Then*

$$\mathrm{ev}_Y : \mathrm{CH}^r(T \xrightarrow{f} Y)_{\mathbb{Q}} \longrightarrow \mathrm{CH}_{n-r}(T)_{\mathbb{Q}}, \qquad \kappa \longmapsto \kappa \cap [Y],$$

is an isomorphism. For Y smooth its inverse sends $w \in \mathrm{CH}_*(T)_{\mathbb{Q}}$ to the class κ_w with $\kappa_w \cap z = \gamma^!(z \times w)$ for $z \in \mathrm{CH}_*(Y')_{\mathbb{Q}}$, $Y' \to Y$. Here γ is the graph (equivalently the diagonal Δ_Y), a regular embedding.

Proof. In characteristic 0, schemes with quotient singularities are Alexander schemes [29, Thm. 5.1], [30, Cor. 6.4]. By [29, Def. 2.1], this means that ev_Y is an isomorphism for *every* $T \to Y$ and the commutativity condition holds. ([30, Def. 6.2] is the *weak* Alexander notion in the stack-compatible bivariant theory; it agrees with the Alexander notion when a resolution exists, as here [30, p. 657].) [S] For Y smooth this is [12, Props. 17.3.1, 17.4.2], as quoted in [29, Introduction and Prop. 2.3], and the inverse is the formula in the proof of [4, Prop. 4]. [S] □

Definition 7.5 (products and the composition through R)**.** Let Y be a test space, $f : T \to Y$ and $g : Y' \to Y$ morphisms, $x \in \mathrm{CH}_*(T)_{\mathbb{Q}}$, $y \in \mathrm{CH}_*(Y')_{\mathbb{Q}}$. Following [29, Note 2.4] and [30, p. 658] put

$$x \cdot_Y y \;:=\; \mathrm{ev}_Y^{-1}(x) \cap y \;\in\; \mathrm{CH}_*(T \times_Y Y')_{\mathbb{Q}}.$$

For test spaces Q_1, Q_2 and $x \in \mathrm{CH}_a(Q_1 \times R)_{\mathbb{Q}}$, $y \in \mathrm{CH}_b(R \times Q_2)_{\mathbb{Q}}$, take $Y = R$, $T = Q_1 \times R$ and $Y' = R \times Q_2$, both mapping to R by projection. Then $T \times_R Y' = Q_1 \times R \times Q_2$, and $\kappa = \mathrm{ev}_R^{-1}(x)$ has relative degree $d - a$. Define

$$y \circ_R x \;:=\; p_{13*}\big(x \cdot_R y\big) \;\in\; \mathrm{CH}_{a+b-d}(Q_1 \times Q_2)_{\mathbb{Q}},$$

where p_{13} is proper because R is proper. The transpose of $x \in \mathrm{CH}_*(Q_1 \times Q_2)_{\mathbb{Q}}$ is ${}^t x = \sigma_* x$, with σ the exchange of factors. Replacing R by the smooth $\overline{X}$ gives Fulton's composition of correspondences [12, Def. 16.1], by the second half of Proposition 7.4. [S-lit] Replacing R by R^m gives $\circ_{R^m}$. For a correspondence $c \in \mathrm{CH}_*(Q_1 \times Q_2)_{\mathbb{Q}}$ through a smooth Q_1 (or one of the form above) we write $c_*(w) = c \circ w$ for $w \in \mathrm{CH}_*(Q_1)_{\mathbb{Q}}$, regarded as a correspondence from a point.

The product $\cdot_Y$ has the formal properties of [12, Prop. 8.1.1]. In particular $\mathrm{CH}_*(Y)_{\mathbb{Q}}$ is a commutative ring and the projection formula of [12, Prop. 8.3] holds [29, Note 2.4], [30, p. 658]. [S] So all three of $R \times R$, $\overline{X} \times R$ and $\overline{X} \times \overline{X}$ carry intersection products. The first two do by Definition 7.5, being Alexander schemes; the third does by [12, Ch. 8], and on a smooth test space the two agree (Proposition 7.4). On a test space $Y = Z/G$ with Z smooth, $\cap[Y] : \mathrm{CH}^*(Y)_{\mathbb{Q}} \to \mathrm{CH}_*(Y)_{\mathbb{Q}}$ is an isomorphism, and the resulting ring is the equivariant one [4, Thm. 4], which agrees with Vistoli's and Gillet's products on the quotient *stack* [4, Prop. 15]. [S] Proposition 15 is not a Betti comparison on coarse-space products; that comparison is Proposition 7.10. *No flatness or lci property of q is used anywhere: q enters only through the cycle Γ_q, and the diagonal of the singular R is never claimed to be a regular embedding.* [D]

Presentations and mixed spaces. By [30, Prop. 2.8] the normal scheme R with quotient singularities is the moduli space of a smooth Deligne–Mumford stack $\mathcal{R}$ with generically trivial stabiliser, as recorded in [5, before Cor. 2.20]. By [5, Thm. 2.18], $\mathcal{R} \cong [Z/G]$ with Z a smooth algebraic space and G a linear algebraic group of dimension g, so $R = Z/G$ [5, Cor. 2.20]. [S] The stabiliser in G of a point of Z is the automorphism group of its image in $\mathcal{R}$, so the generic stabiliser is trivial. [S-lit] Fix a representation V of dimension l and an open $U_V \subset V$ on which G acts freely, with $\mathrm{codim}(V \setminus U_V) \geq c$. Put $Z_U = (Z \times U_V)/G$, a smooth algebraic space of dimension $d+l$ [4, Prop. 22, §2.2]. [S] For a scheme S on which G acts trivially, $S \times Z \to S \times R$ is a geometric quotient, since geometric quotients are stable under the flat base change $S \times R \to R$ [15, Ch. 0, §2]. [S-lit] Write

$$\rho = \rho_S : \; S \times Z_U \longrightarrow S \times R,$$
$$\hat{\pi} = \hat{\pi}_S : \; \mathrm{CH}_k(S \times R)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}^G_{k+g}(S \times Z)_{\mathbb{Q}} = \mathrm{CH}_{k+l}(S \times Z_U)_{\mathbb{Q}},$$

the isomorphism of [4, Thm. 3(a)], valid for $k > \dim(S \times R) - c$. On cycles it is $[W] \mapsto e_W \, i_W^{-1}[(\pi^{-1}W)_{\mathrm{red}}]$ [4, Remark (6) after Thm. 4], where e_W is the order of the generic stabiliser *in* G of $\pi^{-1}W$, and $i_W = 1$ in characteristic 0 [4, §4.2, before Lemma 5]. In particular $\hat{\pi}[S \times R] = [S \times Z_U]$. [S] The map ρ factors as $S \times Z_U \to S \times \mathcal{R} \to S \times R$: smooth of relative dimension l, then a bijection on points. So ρ^{-1} of a pure k-dimensional closed set is pure of dimension $k + l$. [D]

Category and local properness. The mixed spaces Z_U are algebraic spaces, so the bivariant operations are used in their extension to algebraic spaces: [4, §6.1] supplies the Chow operations, and [30, Prop. 6.1 and the discussion after Def. 6.2] the comparison with the stack-compatible bivariant theory for scheme endpoints. The class κ of Proposition 7.4 is used on these intermediate algebraic spaces in that sense, and the smooth-target evaluation formula is the diagonal formula there. [S] The locally proper action required by [4, Thm. 3, Prop. 10] holds: take $\mathcal{R}$ the separated smooth canonical stack of R; its atlas relation is the normalisation of $V_a \times_R V_b$, finite over $V_a \times V_b$

since R is separated, so the diagonal of $\mathcal{R}$ is finite, and for $\mathcal{R} = [Z/G]$ the action map $G \times Z \to Z \times Z$ is a base change of that diagonal, hence proper. Generic inertia is trivial by the choice of $\mathcal{R}$. For a separated finite-type scheme S with trivial G-action the projection $S \times R \to R$ is flat (a base change of $S \to \operatorname{Spec} \mathbb{C}$), whatever the singularities of S, so the action stays locally proper on $S \times Z$. [D]

Topology. A complex variety with quotient singularities is an oriented rational homology manifold. At the image of a fixed point of a finite linear group $H \subset GL_n(\mathbb{C})$, transfer [14, Ch. III, §7] gives $H^*(\mathbb{C}^n/H, \mathbb{C}^n/H \setminus 0; \mathbb{Q}) = H^*(\mathbb{C}^n, \mathbb{C}^n \setminus 0; \mathbb{Q})^H = \mathbb{Q}[-2n]$, because H preserves the complex orientation. [S-lit + D] Hence every test space, and every $S \times Z_U$ with S a test space, satisfies Poincaré and Alexander duality over $\mathbb{Q}$: $\mathrm{PD} : H^j_T(M) \xrightarrow{\sim} H^{BM}_{\dim_{\mathbb{R}} M - j}(T)$ for $T \subset M$ closed [13, Ch. V], natural for closed inclusions and open restriction. [S-lit] For compact oriented rational homology manifolds Q_1, Q_2 and $\gamma \in H_*(Q_1 \times Q_2)$ of even degree, write

$$\gamma_* : H^*(Q_1) \to H^*(Q_2), \qquad \gamma_* \xi = \mathrm{PD}^{-1}_{Q_2} \operatorname{pr}_{2*}\big(\operatorname{pr}_1^* \xi \cap \gamma\big).$$

For an algebraic class x we write $x_* := \mathrm{cl}(x)_*$. For a generically free finite quotient $\varphi : A' \to A'/K'$ of oriented rational homology manifolds, $\varphi^! := \mathrm{PD} \circ \varphi^* \circ \mathrm{PD}^{-1}$, with supports, satisfies $\varphi_* \varphi^! = |K'|$. [D]

7.3. **The cycle class on quotient-singular test spaces.**

Lemma 7.6 (Supported quotient multiplicities)**.** *Let $a : M' \to M = M'/L$ be a generically free finite quotient of complex oriented rational homology manifolds of the same dimension, with L acting effectively. For a closed $W \subset M$ define $a^!_W : H^{BM}_j(W; \mathbb{Q}) \to H^{BM}_j(a^{-1}W; \mathbb{Q})$ by ambient duality with supports and pullback in cohomology with supports. Then $(a|_{a^{-1}W})_* a^!_W = |L| \,\mathrm{id}$. If W is integral and the generic point stabiliser above W has order e, then $a^!_W[W] = e\,[(a^{-1}W)_{\mathrm{red}}]$, the bracket summing all reduced components dominating W.*

Proof. For $\alpha \in H^*_W(M; \mathbb{Q})$ the supported projection formula gives $a_*(a^*\alpha \cap [M']) = \alpha \cap a_*[M'] = |L|\, \alpha \cap [M]$, an equality in the homology of W, not merely after pushing to M. The class $a^!_W[W]$ lies in top Borel–Moore degree on the reduced inverse image and is L-invariant; L permutes the components over W transitively, so their coefficients agree, and their total generic pushforward degree is $|L|/e$. The projection formula forces the common coefficient to be e. In analytic neighbourhoods one argues branch by branch. □

Lemma 7.7. *Let S be a test space, $Q = S \times R$, $n = \dim Q$, $Q_U = S \times Z_U$, and define*

$$c_{\mathrm{top}} = \mathrm{PD}_{Q_U} \circ \rho^* \circ \mathrm{PD}^{-1}_Q : \; H^{BM}_m(Q) \longrightarrow H^{BM}_{m+2l}(Q_U).$$

For $W \subset Q$ closed, with $W_U = \rho^{-1} W$, define its supported version

$$c^W_{\mathrm{top}} : \; H^{BM}_m(W) \cong H^{2n-m}_W(Q) \xrightarrow{\rho^*} H^{2n-m}_{W_U}(Q_U) \cong H^{BM}_{m+2l}(W_U),$$

which is compatible with pushforward to Q and Q_U.

(a) c_{top} *is an isomorphism for* $m > 2(n-c)+1$.
(b) *For* $\beta \in \mathrm{CH}_k(Q)_{\mathbb{Q}}$, $\mathrm{cl}(\hat{\pi}\beta) = c_{\mathrm{top}}(\mathrm{cl}\,\beta)$, *and the same holds with supports.*

Proof. (a) Put $E_Q = S \times (Z \times V)/G \xrightarrow{p} Q$ and $j : Q_U \hookrightarrow E_Q$, so that $\rho = p \circ j$. Scalar multiplication on V retracts E_Q onto the zero section, so p is a homotopy equivalence. E_Q and Q have quotient singularities, so $\mathrm{PD}_{E_Q} \circ p^* \circ \mathrm{PD}_Q^{-1}$ is an isomorphism; we use no Vietoris–Begle theorem for the non-proper p, and no smoothness of S. The complement $E_Q \setminus Q_U = S \times (Z \times (V \setminus U_V))/G$ has complex dimension at most $n+l-c$. By the localisation sequence, $j^* : H^{BM}_{m+2l}(E_Q) \to H^{BM}_{m+2l}(Q_U)$ is therefore an isomorphism for $m + 2l > 2(n+l-c)+1$. [D]

(b) Both sides are linear, so take $\beta = [W]$ with $W \subset Q$ closed integral of dimension k. Both $\mathrm{cl}(\hat{\pi}[W]) = e_W[W_U]$ and $c^W_{\mathrm{top}}(\mathrm{cl}[W])$ lie in $H^{BM}_{2(k+l)}(W_U)$, the top degree of the pure $(k+l)$-dimensional W_U. This group is the $\mathbb{Q}$-vector space on the irreducible components of W_U, and it injects into the same group of any dense open subset [12, §19.1]. [S-lit] Since c^W_{top} commutes with restriction to opens, it suffices to compare the two classes near a general point of each component. No Poincaré duality on the possibly singular W is used.

Keep the two groups distinct. Near a general point of W, analytically locally, $S = S'/K'$ with K' finite acting effectively and generically freely on the smooth S', and $\mathcal{R} = [\Omega/H]$ with H finite, Ω smooth and trivial generic stabiliser [30, proof of Prop. 2.8]. [S] Let $\Omega_U = \Omega \times_{\mathcal{R}} Z_U$. This is a smooth U_V-fibration over Ω, and H acts *freely* on it, because $\Omega_U \to Z_U$ is the base change of the H-torsor $\Omega \to [\Omega/H]$. The factor K' need not act freely: on $S = \mathbb{A}^2/\mu_2$, with R a point, the chart $\mathbb{A}^2 \times U_V \to S \times U_V$ stays ramified along $\{0\} \times U_V$. Put

$$A' = S' \times \Omega, \qquad A'_U = S' \times \Omega_U,$$
$$\varphi : A' \to Q, \qquad \varphi_U : A'_U \to Q_U, \qquad \mathrm{pr} : A'_U \to A',$$

where φ and φ_U are both quotient maps by $K' \times H$, φ_U need not be étale, and pr is smooth of relative dimension l. Shrink further around a general point of a component, so that the generic stabiliser orders of the inverse image of W are s in K' and r in H. The Edidin–Graham number is $e_W = r$, since G acts only through the R-factor. Downstairs in A' the generic stabiliser of $A'_W = \varphi^{-1}(W)_{\mathrm{red}}$ is of order sr. Upstairs in A'_U the generic stabiliser of $A'_{U,W} = \varphi_U^{-1}(W_U)_{\mathrm{red}}$ is of order s, since H acts freely there.

The supported pullbacks $\varphi^![W]$ and $\varphi_U^![W_U]$ are invariant top-degree classes on these reduced inverse images, whose components the group permutes transitively over the irreducible W. Lemma 7.6, applied to φ (point stabiliser of order sr) and to φ_U (order s), with $\varphi_*\varphi^! = |K'||H| = \varphi_{U*}\varphi_U^!$,

gives

$$\varphi^![W] = sr\,[A'_W], \qquad \varphi_U^![W_U] = s\,[A'_{U,W}], \qquad \mathrm{pr}^![A'_W] = [A'_{U,W}],$$

with brackets summing reduced components; the last identity holds because pr is smooth. From $\rho \circ \varphi_U = \varphi \circ \mathrm{pr}$ and functoriality of supported pullback,

$$\varphi_U^!\big(c_{\mathrm{top}}^W\,\mathrm{cl}[W]\big) = \mathrm{pr}^!\varphi^![W] = sr\,[A'_{U,W}] = \varphi_U^!\big(r\,[W_U]\big).$$

Since $\varphi_{U*}\varphi_U^! = |K'||H|$, the map $\varphi_U^!$ is injective over $\mathbb{Q}$. Hence $c_{\mathrm{top}}^W\,\mathrm{cl}[W] = r[W_U] = e_W[W_U] = \mathrm{cl}(\hat{\pi}[W])$ near the chosen point, and therefore globally by the top-degree injectivity above. [D, after ref-BCHR1 §3] The factor is the stabiliser factor of [4, Remark (6) after Thm. 4], not its inverse. Generic freeness and effectivity of both charts are essential to the degree formulas. □

Lemma 7.8 (topological composition)**.** *Let Q_1, Q_2, Q_3 be compact oriented rational homology manifolds, $N = Q_1 \times Q_2 \times Q_3$, and $\gamma \in H_*(Q_1 \times Q_2)$, $\delta \in H_*(Q_2 \times Q_3)$ of even degrees. Put*

$$\gamma \cdot_{Q_2}^{\mathrm{top}} \delta = \mathrm{PD}_N\big(\mathrm{pr}_{12}^*\,\mathrm{PD}^{-1}\,\gamma \cup \mathrm{pr}_{23}^*\,\mathrm{PD}^{-1}\,\delta\big) \in H_*(N), \qquad \delta \circledast \gamma = p_{13*}(\gamma \cdot_{Q_2}^{\mathrm{top}} \delta).$$

Then $(\delta \circledast \gamma)_* = \delta_* \circ \gamma_*$. For a morphism $f : Q_1 \to Q_2$ with graph class $\Gamma_f = (1, f)_*[Q_1]$, we have $(\Gamma_f)_* = f_* := \mathrm{PD}_{Q_2}^{-1} f_* \,\mathrm{PD}_{Q_1}$ and $({}^t\Gamma_f)_* = f^*$. Finally, $({}^t\gamma)_*$ is the adjoint of γ_* for the Poincaré pairings.

Proof. Let $\xi \in H^*(Q_1)$. Use $\mathrm{pr}_{12}^*\theta \cap [N] = (\theta \cap [Q_1 \times Q_2]) \times [Q_3]$, the projection formula for pr_{23}, and even degrees to suppress signs:

$$\mathrm{pr}_{23*}\big(\mathrm{pr}_1^*\xi \cap (\gamma \cdot^{\mathrm{top}} \delta)\big) = \mathrm{PD}^{-1}\,\delta \cap \big(\mathrm{pr}_{2*}(\mathrm{pr}_1^*\xi \cap \gamma) \times [Q_3]\big) = \mathrm{pr}_2^*(\gamma_*\xi) \cap \delta.$$

Applying pr_{3*} and $\mathrm{PD}_{Q_3}^{-1}$ gives $\delta_*\gamma_*\xi$. For the graph, $\mathrm{pr}_1^*\xi \cap (1, f)_*[Q_1] = (1, f)_*(\xi \cap [Q_1])$ and $\mathrm{pr}_2 \circ (1, f) = f$. For the transpose ${}^t\Gamma_f = (f, 1)_*[Q_1] \in H_*(Q_2 \times Q_1)$ and $\eta \in H^*(Q_2)$, $\mathrm{pr}_1^*\eta \cap (f, 1)_*[Q_1] = (f, 1)_*(f^*\eta \cap [Q_1])$, and $\mathrm{pr}_2 \circ (f, 1) = \mathrm{id}_{Q_1}$. Adjointness is the symmetry of $\int_{Q_1 \times Q_2} \mathrm{pr}_1^*\xi \cup \mathrm{pr}_2^*\eta \cap \gamma$. Only cup and cap products, Poincaré duality and the projection formula over $\mathbb{Q}$ are used, all valid on compact oriented rational homology manifolds [13, Ch. V]. [D + S-lit] □

Lemma 7.9 (Bivariant compatibility with a coarse quotient)**.** *Work with rational Chow groups and their bivariant extension to algebraic spaces. Let $\mathcal{R} = [Z/G]$ be a separated smooth Deligne–Mumford stack with trivial generic inertia and coarse space R, $U_V \subset V$ a free open in a representation, $\dim V = l$, $Z_U = (Z \times U_V)/G$ and $\rho : Z_U \to R$. For separated finite-type schemes A, B over $\mathbb{C}$, every $\kappa \in \mathrm{CH}^t(A \times R \to R)_{\mathbb{Q}}$ and $y \in \mathrm{CH}_b(R \times B)_{\mathbb{Q}}$ satisfy, in the stable range,*

$$\hat{\pi}_{A \times B}(\kappa \cap y) = \rho^*\kappa \cap \hat{\pi}_B(y) \quad \text{in } \mathrm{CH}_{b-t+l}(A \times Z_U \times B)_{\mathbb{Q}}.$$

No smoothness of A or B is required.

Proof. The action is proper, its action map being a base change of the proper diagonal of $\mathcal{R}$. Choose the finite surjective diagram of [4, Prop. 10]: $g : Z' \to Z$ over $f : R' \to R$, with $Z' \to R'$ a principal G-bundle. For every inert scheme S its product with S is still a diagram of geometric quotients, as $S \times R \to R$ (and the analogue over R') is flat. Write $g_U : Z'_U \to Z_U$ and $\rho' : Z'_U \to R'$; then $\rho g_U = f\rho'$, a square not required to be Cartesian. Put $f_B = f \times 1_B$, $g_B = g_U \times 1_B$, $F = 1_A \times f_B$, $G_U = 1_A \times g_B$. Choose y' with $(f_B)_* y' = y$ (rational pushforward along a finite surjective map is surjective) and let $\kappa' = f^*\kappa$. The smooth map $\rho' \times 1_B$ is flat of relative dimension l and the principal-bundle comparison is its flat pullback, so $\hat{\pi}'_{A\times B}(\kappa' \cap y') = \rho'^*\kappa' \cap \hat{\pi}'_B(y')$. Every square used for the bivariant proper or flat axiom is the Cartesian square

$$\begin{array}{ccc} A \times T' & \xrightarrow{1_A \times h} & A \times T \\ \downarrow & & \downarrow \\ T' & \xrightarrow{h} & T, \end{array}$$

with $h = f_B$, g_B (proper) or $\rho' \times 1_B$ (flat). Functoriality gives $g_U^*\rho^*\kappa = \rho'^* f^*\kappa$. Applying $(G_U)_*$, [4, Prop. 11(a)] ($g_*\hat{\pi}' = \hat{\pi} f_*$) and the proper axiom,

$$\begin{aligned} \hat{\pi}_{A\times B}(\kappa \cap y) &= \hat{\pi}_{A\times B} F_*(\kappa' \cap y') = (G_U)_* \hat{\pi}'_{A\times B}(\kappa' \cap y') \\ &= (G_U)_*(\rho'^*\kappa' \cap \hat{\pi}'_B y') = \rho^*\kappa \cap (g_B)_* \hat{\pi}'_B y' = \rho^*\kappa \cap \hat{\pi}_B y. \end{aligned}$$

Choose the representation with complement codimension large enough for the finitely many degrees occurring. □

Proposition 7.10 (cycle-class compatibility of $\circ_R$)**.** *Let Q_1, Q_2 be test spaces, $x \in \mathrm{CH}_a(Q_1 \times R)_{\mathbb{Q}}$, $y \in \mathrm{CH}_b(R \times Q_2)_{\mathbb{Q}}$. Then in $H_*(Q_1 \times R \times Q_2)$*

$$\mathrm{cl}(x \cdot_R y) = \mathrm{cl}\, x \cdot_R^{\mathrm{top}} \mathrm{cl}\, y,$$

hence $\mathrm{cl}(y \circ_R x) = \mathrm{cl}\, y \circledast \mathrm{cl}\, x$ and $(y \circ_R x)_* = y_* \circ x_*$. The same holds with R replaced by R^m, using the presentation Z^m/G^m. With $Q_1 = Q_2 = \mathrm{pt}$ this says that $\mathrm{cl}_H : (\mathrm{CH}_*(R)_{\mathbb{Q}}, \cdot_R) \to (H^*(R), \cup)$ is a ring homomorphism.

Proof. Let $N = Q_1 \times R \times Q_2$, $N_U = Q_1 \times Z_U \times Q_2$ and $\kappa = \mathrm{ev}_R^{-1}(x) \in \mathrm{CH}^{d-a}(Q_1 \times R \to R)_{\mathbb{Q}}$ (Proposition 7.4), so that $x \cdot_R y = \kappa \cap y$. All the maps $\hat{\pi}$ below are those of §7.2 for the inert factors shown. Taking the codimension bound c large, we may assume all degrees are in the range of Lemma 7.7(a).

Step 1 (the quotient map intertwines the bivariant class). Let $\rho^*\kappa \in \mathrm{CH}^*(Q_1 \times Z_U \to Z_U)_{\mathbb{Q}}$ be the bivariant pullback along $\rho : Z_U \to R$ [12, §17.1]. We claim

$$(*) \qquad \hat{\pi}_{Q_1\times Q_2}(\kappa \cap y) = \rho^*\kappa \cap \hat{\pi}_{Q_2}(y) \quad \text{in } \mathrm{CH}_*(N_U)_{\mathbb{Q}}.$$

This is Lemma 7.9 with $A = Q_1$, $B = Q_2$, which extends [4, Lemma 6] from $\kappa \in \mathrm{CH}^*(R)$ to bivariant classes with source $Q_1 \times R \to R$, inert factors possibly singular. Every square in which a cap product is formed is the Cartesian product square with an inert factor; the finite principal-bundle

diagram need not be Cartesian, and no flatness of ρ, f or q is used. [D from S; after ref-BCHR1 §2]

Step 2 (on the smooth Z_U the class is a refined intersection). Since Z_U is smooth, Proposition 7.4 gives $\rho^*\kappa \cap z = \Delta^!_{Z_U}\big(\hat\pi_{Q_1}(x) \times z\big)$ for every $z \in \mathrm{CH}_*(Z_U \times Q_2)_{\mathbb{Q}}$. Here $\Delta^!_{Z_U} : \mathrm{CH}_*(Q_1 \times Z_U \times Z_U \times Q_2) \to \mathrm{CH}_*(N_U)$ is the refined Gysin map of the regular embedding Δ_{Z_U} [12, §6.2]. Combined with Step 1,

$$\hat\pi_{Q_1\times Q_2}(x \cdot_R y) = \Delta^!_{Z_U}\big(\hat\pi_{Q_1}x \times \hat\pi_{Q_2}y\big).$$

Step 3 (cycle class upstairs). By [12, §§19.1–19.2], extended to algebraic spaces as in [4, §§2.8, 6.1], cl commutes with exterior products and with $\Delta^!_{Z_U}$. [S-lit] The topological $\Delta^!$ is cap product with the orientation class $u \in H^{2(d+l)}(Z_U^2, Z_U^2 \setminus \Delta)$, pulled back to $\widetilde{N}_U = Q_1 \times Z_U \times Z_U \times Q_2$; the inert factors need only be rational homology manifolds, because u comes from the smooth middle factors. In the rational homology manifold $\widetilde{N}_U$, $\mathrm{PD}(\alpha \times \beta)$ is the exterior product of the duals. Moreover $u \cap \mathrm{PD}_{\widetilde{N}_U}(\theta) = \mathrm{PD}_{N_U}(\iota^*\theta)$ for the diagonal inclusion $\iota : N_U \hookrightarrow \widetilde{N}_U$, because $u \cap [\widetilde{N}_U] = [N_U]$, and $\iota^*(\theta_1 \times \theta_2) = \mathrm{pr}_{12}^*\theta_1 \cup \mathrm{pr}_{23}^*\theta_2$. Hence [D]

$$\mathrm{cl}\big(\Delta^!_{Z_U}(\hat\pi x \times \hat\pi y)\big) = \mathrm{PD}_{N_U}\big(\mathrm{pr}_{12}^*\,\mathrm{PD}^{-1}\,\mathrm{cl}(\hat\pi x) \cup \mathrm{pr}_{23}^*\,\mathrm{PD}^{-1}\,\mathrm{cl}(\hat\pi y)\big).$$

Step 4 (descend). Apply Lemma 7.7(b) on $Q_1 \times R$, on $R \times Q_2$ and on N, with inert factors Q_1, Q_2 and $Q_1 \times Q_2$. This gives $\mathrm{cl}(\hat\pi x) = c_{\mathrm{top}}\,\mathrm{cl}\,x$, $\mathrm{cl}(\hat\pi y) = c_{\mathrm{top}}\,\mathrm{cl}\,y$ and $\mathrm{cl}(\hat\pi(x \cdot_R y)) = c_{\mathrm{top}}\,\mathrm{cl}(x \cdot_R y)$. Since $c_{\mathrm{top}} = \mathrm{PD} \circ \rho^* \circ \mathrm{PD}^{-1}$, and $\rho_N = 1 \times \rho \times 1$ restricts to $\rho_{Q_1\times R}$ and $\rho_{R\times Q_2}$ on the factors, naturality of the cup product gives

$$c_{\mathrm{top}}\big(\mathrm{cl}\,x \cdot_R^{\mathrm{top}} \mathrm{cl}\,y\big) = \mathrm{PD}_{N_U}\big(\mathrm{pr}_{12}^*\,\mathrm{PD}^{-1}\,c_{\mathrm{top}}\,\mathrm{cl}\,x \cup \mathrm{pr}_{23}^*\,\mathrm{PD}^{-1}\,c_{\mathrm{top}}\,\mathrm{cl}\,y\big).$$

Steps 2–4 together give $c_{\mathrm{top}}\,\mathrm{cl}(x \cdot_R y) = c_{\mathrm{top}}(\mathrm{cl}\,x \cdot_R^{\mathrm{top}} \mathrm{cl}\,y)$, and c_{top} is injective in this degree by Lemma 7.7(a). Finally, cl commutes with the proper pushforward p_{13*} [12, §19.1], and Lemma 7.8 gives the action. [D] For R^m, use Z^m, G^m, V^m and U_V^m; the complement of U_V^m has codimension at least c, and the generic stabiliser is still trivial. [D] □

7.4. **Graphs, and the transferred projector.**

Lemma 7.11 (graph conventions)**.** *The classes $[\Gamma_q] \in \mathrm{CH}_d(\overline{X} \times R)_{\mathbb{Q}}$, $[{}^t\Gamma_q] \in \mathrm{CH}_d(R \times \overline{X})_{\mathbb{Q}}$ and $[\Gamma_g] \in \mathrm{CH}_d(R \times R)_{\mathbb{Q}}$, for $g \in A$, act on cohomology by*

$$(\Gamma_q)_* = q_* = \mathrm{PD}_R^{-1}\,q_*\,\mathrm{PD}_{\overline{X}}, \qquad ({}^t\Gamma_q)_* = q^*, \qquad (\Gamma_g)_* = g_* = (g^{-1})^*.$$

All three have degree 0, because $\dim \overline{X} = \dim R = d$, and $q_*q^* = \mathrm{id}_{H^*(R)}$. A class $(i \times i)_*\zeta$ with ζ on $E \times E$, $E \subset R^{\mathrm{sm}}$, acts by the smooth-variety formula on a neighbourhood of E.

Proof. The three formulas are Lemma 7.8 for $f = q$ and $f = g$, together with $g_* = (g^{-1})^*$ for an automorphism. For q_*q^*: $q_*(q^*\eta \cap [\overline{X}]) = \eta \cap q_*[\overline{X}] = \eta \cap [R]$, because q has degree one, so $q_*[\overline{X}] = [R]$. [D] For the last claim,

$\mathrm{PD}^{-1}\,\mathrm{cl}((i\times i)_*\zeta)$ is supported on $E\times E$, so the action formula only involves the open neighbourhood R^{sm} of E, where R is smooth. [D] □

Proposition 7.12. *Assume (T1)–(T3) and let $p\in \mathrm{CH}_d(R\times R)_{\mathbb{Q}}$ be arbitrary.*

(i) $P_X := {}^t\Gamma_q \circ_R \big(p\circ_R \Gamma_q\big)$ *is a well-defined class in* $\mathrm{CH}_d(\overline{X}\times\overline{X})_{\mathbb{Q}}$, *independent of any presentation of R.*

(ii) $(P_X)_* = q^*\,p_*\,q_*$ *on* $H^k(\overline{X})$ *for every k, where p_* is the action of* $\mathrm{cl}(p)$ *on* $H^*(R)$ *through Poincaré duality on the rational homology manifold R.*

(iii) *If p_* is idempotent on $H^*(R)$, then $(P_X)_*$ is idempotent, with image* $q^*\,\mathrm{im}(p_*)\cong \mathrm{im}(p_*)$.

(iv) *(Products.) Let $n\geq 1$. Applying (i)–(iii) to $(R^n,\overline{X}^{\,n},q^n,p^{\times n})$, where $p^{\times n}\in \mathrm{CH}_{nd}(R^n\times R^n)_{\mathbb{Q}}$ is the exterior power with factors reordered, gives*

$$P_X^{(n)} := {}^t\Gamma_{q^n}\circ_{R^n}\big(p^{\times n}\circ_{R^n}\Gamma_{q^n}\big)\in \mathrm{CH}_{nd}(\overline{X}^{\,n}\times\overline{X}^{\,n})_{\mathbb{Q}},$$
$$(P_X^{(n)})_* = (q^*p_*q_*)^{\otimes n}$$

under Künneth. For $W\in \mathrm{CH}_k(R^n)_{\mathbb{Q}}$, regarded as a correspondence from a point, put $(q^n)^!W := {}^t\Gamma_{q^n}\circ_{R^n} W\in \mathrm{CH}_k(\overline{X}^{\,n})_{\mathbb{Q}}$. Then

$$\mathrm{cl}_H\big((q^n)^!W\big) = (q^n)^*\,\mathrm{cl}_H(W),$$
$$\mathrm{cl}_H\big((P_X^{(n)})_*((q^n)^!W)\big) = (q^n)^*\,p_*^{\otimes n}\,\mathrm{cl}_H(W),$$

where $(P_X^{(n)})_*$ on Chow is correspondence action through the smooth $\overline{X}^{\,n}$ [12, Def. 16.1].

Proof. (i) By Definition 7.5, $p\circ_R\Gamma_q\in \mathrm{CH}_{d+d-d}(\overline{X}\times R)_{\mathbb{Q}}$ (take $Q_1=\overline{X}$, $Q_2=R$), and then $P_X\in \mathrm{CH}_d(\overline{X}\times\overline{X})_{\mathbb{Q}}$ (take $Q_1=Q_2=\overline{X}$). The maps ev_R and $\cap$ are intrinsic to R; presentations enter only the proof of Proposition 7.10. [D]

(ii) Apply Proposition 7.10 twice and Lemma 7.11: $(P_X)_* = ({}^t\Gamma_q)_*\circ p_*\circ(\Gamma_q)_* = q^*p_*q_*$, degree by degree. [D]

(iii) By Lemma 7.11, $(q^*p_*q_*)^2 = q^*p_*(q_*q^*)p_*q_* = q^*p_*^2q_* = q^*p_*q_*$. The image is $q^*\,\mathrm{im}(p_*)$ because q_* is onto, and q^* is injective because $q_*q^*=1$. [D]

(iv) R^n has quotient singularities, and its smooth locus is $(R^{\mathrm{sm}})^n$. $\overline{X}^{\,n}$ is smooth projective, and q^n is a projective resolution that is an isomorphism over $(R^{\mathrm{sm}})^n$, so (T1)–(T2) hold. Γ_{q^n} is the reordered exterior product of n copies of Γ_q. Cross products are compatible with cl, $\cap$, PD and pushforward, so under Künneth over $\mathbb{Q}$ we get $\mathrm{cl}(p^{\times n})_* = p_*^{\otimes n}$, $(q^n)_* = q_*^{\otimes n}$ and $(q^n)^* = (q^*)^{\otimes n}$. This is Künneth for vector spaces; no algebraic Künneth projector is involved. [D + S-lit] For W, Proposition 7.10 with $Q_1=\mathrm{pt}$, middle factor R^n and $Q_2=\overline{X}^{\,n}$, together with Lemma 7.11, gives $\mathrm{cl}_H((q^n)^!W) =$

$(q^n)^* \operatorname{cl}_H(W)$. The last identity then follows from (ii) and $(q^n)_*(q^n)^* = 1$, since $(P_X^{(n)})_*(q^n)^* = (q^n)^* p_*^{\otimes n}(q^n)_*(q^n)^* = (q^n)^* p_*^{\otimes n}$. [D] □

Remark 7.13 (the realisation of p). In this paper $p = \varepsilon P_\star$, with the rational quaternionic convention of Section 2: $D = e_-\mathbb{Q}[\widetilde{A}] = (-1,-1)_{\mathbb{Q}}$, the two-sided block $D \otimes D^{\mathrm{op}} \cong M_4(\mathbb{Q})$, and $\varepsilon = \frac{1}{32}\sum_{a_1,a_2\in\widetilde{A}} \phi_\chi(a_1 a_2^{-1})\, a_1 \otimes a_2$, a primitive idempotent of $M_4(\mathbb{Q})$ that acts on D by $x \mapsto \frac{1}{2}\operatorname{trd}(x)$. As a correspondence, ε is a $\mathbb{Q}$-combination (coefficients $\pm\frac{1}{16}$ and 0) of graphs of the automorphisms of R given by elements of $\widetilde{A}(1) \times \widetilde{A}(2)$; $P_\star = [\Delta_R] - \Pi_E$ with Π_E built from classes $(i \times i)_*\zeta$ on $E \times E$ (Section 3). The abstract Proposition 7.12 is coefficient-neutral: its conclusion is the same for any already realised $p \in \mathrm{CH}_d(R \times R)_{\mathbb{Q}}$. [D] The realisation is supplied elsewhere in the paper:

- both commuting actions $\widetilde{A}(1)$ and $\widetilde{A}(2)$ are realised by automorphisms of R (Section 2, by functoriality of normalisation), and lifted to $\overline{X}$ by Remark 7.3;
- $(\varepsilon P_\star)_*$ is idempotent on $H^*(R)$, with image $M_x \subset H^6(R)$ of rank 7, for the weighted clean model (Theorem 4.3(b)).

A one-sided central isotypic projector would not do this. Proposition 7.12(iii) therefore gives a cohomological projector P_X of rank 7 on $H^*(\overline{X})$ with image $q^* M_x \cong M_x$.

Example 7.14 ($\mathbb{C}^2/\mu_2$ and its blow-up; checking the graph convention)**.** *Local model (a divisor calculation).* Let $R_0 = \mathbb{C}^2/\pm 1 = \operatorname{Spec}\mathbb{C}[a,b,c]/(ac - b^2)$, with $a = u^2$, $b = uv$, $c = v^2$, and let $q_0 : X_0 \to R_0$ be the blow-up of the vertex. Then $X_0 = \operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-2)$, with exceptional curve C, $C^2 = -2$. In the chart $v = ut$ the invariants are $y = u^2$ and t, and $q_0^* a = y$. In the chart $u = sv$ the invariants are s and $z = v^2$, and $q_0^* a = s^2 z$. Hence $\operatorname{div}(q_0^* a) = C + 2F$, where F is the strict transform of the ruling $\ell_0 = \{a = b = 0\}$. Since $2\ell_0 = \operatorname{div}(a)$ is Cartier, the operational rational pullback is $q_0^![\ell_0] = F + \frac{1}{2}C$. This is a divisor calculation, not a flat pullback. [C] Locally both sides are rationally zero: $\ell_0 = 0$ in $\mathrm{CH}_1(R_0)_{\mathbb{Q}}$ because $2\ell_0$ is principal, and $F + \frac{1}{2}C = 0$ in $\mathrm{CH}_1(X_0)_{\mathbb{Q}}$. Topologically, $H_2^{BM}(R_0) = H_1(\mathbb{RP}^3;\mathbb{Q}) = 0$, and $C = -2F$ in $H_2^{BM}(X_0)$. So the local identity is consistent but cannot detect a non-zero transferred class. Moreover, $\circ_{R_0}$ applies over the affine R_0 only to correspondences with proper support. The non-trivial test is the compact model. [D]

Compact model. The same singularity sits in $\overline{R} = \mathbb{P}(1,1,2)$, resolved by the Hirzebruch surface $\overline{X}_0 = \mathbb{F}_2 \xrightarrow{q} \overline{R}$. The map q contracts the (-2)-section C and sends a fibre f to a line ℓ through the vertex; here $\ell \cdot \ell = \frac{1}{2}$ and 2ℓ is Cartier. Work in the basis (f, C), with $f^2 = 0$, $f \cdot C = 1$, $C^2 = -2$. The Chow side is the operational pullback $q^![\ell] = f + \frac{1}{2}C$. The cohomology side is the class Θ with $\Theta \cdot f = \langle \ell, q_* f\rangle = \frac{1}{2}$ and $\Theta \cdot C = \langle \ell, q_* C\rangle = 0$, namely $q^*\ell = f + \frac{1}{2}C$. The two sides agree, and $\Theta^2 = \frac{1}{2} = \ell^2$. The graph acts by

$q_*f = \ell$ and $q_*C = 0$, and $q_*q^* = 1$. The composite $C_q = {}^t\Gamma_q \circ_{\overline{R}} \Gamma_q$ acts by $f \mapsto f + \frac{1}{2}C$, $C \mapsto 0$ on H_2, and as the identity on H_0 and H_4. In the basis (f, C), $(q^*q_*)|_{H^2} = \left(\begin{smallmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{smallmatrix}\right)$. [C]

The projective bundle formula for $\mathbb{F}_2$ and for $\mathbb{F}_2 \times \mathbb{F}_2$ gives a cellular Chow basis, so cl is an isomorphism on $\mathbb{F}_2 \times \mathbb{F}_2$ [12, §19.1], and a correspondence is determined by its action (Künneth and PD). Hence

$$C_q = [\Delta_{\overline{X}_0}] + \tfrac{1}{2}[C \times C] \quad \text{holds in } \mathrm{CH}_2(\overline{X}_0 \times \overline{X}_0)_{\mathbb{Q}},$$

not only in cohomology. [D + C] With $E_0 = [C \times C]$, exact composition gives $E_0 \circ E_0 = (C \cdot C)E_0 = -2E_0$, so $(\Delta + \lambda E_0)^{\circ 2} = \Delta + (2\lambda - 2\lambda^2)E_0$. At $\lambda = \frac{1}{2}$ this is a Chow idempotent. The reduced fibre product $\overline{X}_0 \times_{\overline{R}} \overline{X}_0 = \Delta \cup (C \times C)$ has naive fundamental cycle $\Delta + E_0$, which acts by $C \mapsto -C$, satisfies $(\Delta + E_0)^{\circ 2} = \Delta$, and is not idempotent. The coefficient $\frac{1}{2}$ is the rational multiplicity that the bivariant composite supplies, and it is why rational coefficients are essential. [C]

Remark 7.15 (idempotence). Proposition 7.12(iii) is an idempotence statement *in cohomology*; Theorem B uses nothing more. [D] Chow-level idempotence $P_X \circ_{\overline{X}} P_X = P_X$ would need three further facts.

(a) Associativity of mixed compositions through $\overline{X}$ and R. The argument of [12, Prop. 16.1.1(a)] uses only the formal properties of [12, Prop. 8.1.1], which $\cdot_R$ has. [D, sketch only]
(b) $\Gamma_q \circ_{\overline{X}} {}^t\Gamma_q = [\Delta_R]$, which does hold in Chow: the intersection is transverse, with $(q \times q)_*[\Delta_{\overline{X}}] = [\Delta_R]$ since q has degree one. Moreover $[\Delta_R]$ is a unit for $\circ_R$, because $\mathrm{ev}_R^{-1}[\Delta_R] = \delta_* 1$. [D]
(c) $p \circ_R p = p$ in $\mathrm{CH}_d(R \times R)_{\mathbb{Q}}$. This holds for the group-algebra part, since $\Gamma_g \circ_R \Gamma_h = \Gamma_{gh}$. For $P_\star$ it holds if $P_\star = [\Delta_R] - \Pi_E$ with Π_E A-equivariant and of Gram form on the cellular $E \times E$. [D, conditional on the construction of $P_\star$]

The example shows the phenomenon is consistent, since C_q is a Chow idempotent. We do not claim, and do not need, that P_X is a Chow motive.

Remark 7.16 (scope). (1) R is a rational homology manifold, so on R ordinary rational cohomology equals intersection cohomology. The obstruction exhibited by a cubic threefold with ordinary double points (which is not a rational homology manifold, and whose ordinary H^3 has a weight-two part) therefore does not arise here. [D]
(2) The proposition transfers cycles *already given* on R^n, with their classes, to $\overline{X}^n$. That the specific closure $\overline{W}_u$ projects to the intended tensor, with its normalisation and independently of the closure, is proved in Section 3.6 (Lemma 3.24, Proposition 3.25), not here. [D]
(3) No algebraic Künneth projector, Lefschetz-type conjecture or Chow–Künneth decomposition is used. [D]

8. Proof of Theorem B

Let $\overline{X}_x = \overline{X}$ be the smooth projective variety of Proposition 7.12 for the weighted clean model of Theorem 5.9 (which exists for every $x \in S$ by Proposition 6.6).

Proof of Theorem B. (1) For an invariant $u \in (V_7^{\otimes n})^{\mathrm{G}_2}$ let W_u be the cycle of Section 3 on $\widetilde{Y}_x^n$ and $\overline{W}_u$ its closure in R^n. By Theorem 4.3(b) and Remark 7.13, $\varepsilon P_\star$ acts on $H^*(R)$ as a projector of rank 7 with image M_x. By Lemma 3.24 and Proposition 3.25, independently of the closure,

$$(\varepsilon P_\star)^{\otimes n}\,\mathrm{cl}_H(\overline{W}_u) = \frac{2^{1-n}}{4}\,\tau_u(-3n),$$

with $\tau_u = \iota^{\otimes n}\varphi_n^{-1}E(s_u)_x(1_{\mathrm{Yun}})$ the transported tensor on Yun's tensor unit (Proposition 3.6); in particular $(\varepsilon P_\star)^{\otimes 3}\,\mathrm{cl}_H(\overline{W}_t) = \frac{1}{16}\,\tau_t(-9) \neq 0$ for the cubic, and $\frac{1}{4}\tau_t(-9)$ for $4\overline{W}_t$. Let $p = \varepsilon P_\star$ and consider the cycle

$$\mathcal{Z}_u := (P_X^{(n)})_*\big((q^n)^![\overline{W}_u]\big) \in \mathrm{CH}_*(\overline{X}^n)_{\mathbb{Q}}.$$

By Proposition 7.12(iv),

$$\mathrm{cl}_H(\mathcal{Z}_u) = (q^n)^* p_*^{\otimes n}\,\mathrm{cl}_H(\overline{W}_u) = \frac{2^{1-n}}{4}(q^n)^*\tau_u(-3n),$$

and $(q^n)^*$ is injective. So $4 \cdot 2^{n-1}\mathcal{Z}_u$ (equivalently, 2^{n-1} times the transported cycle built from $4\overline{W}_u$) is an algebraic cycle on $\overline{X}^n$ whose class is the transported tensor $(q^n)^*\tau_u(-3n)$; for the cubic the multiplier is 16.

(2) The pairing b and the cubic t are algebraic Hodge classes by (1), so $\mathrm{MT}(N_x) \subseteq \mathrm{Aut}(b,t) = \mathrm{G}_2$. At a Hodge-generic x the connected monodromy of the Betti local system N_S is contained in $\mathrm{MT}(N_x)$ and is G_2 by Lemma 3.20 (a statement for the local system N_S of Lemma 3.17, not for stalks), so $\mathrm{MT}(N_x) = \mathrm{G}_2$, and the Hodge classes in $T^{a,b}(N_x)$ are exactly the G_2-invariants. Duals are handled by the pairing and coevaluation, which are the $n = 2$ invariants; symmetric and exterior summands by rational permutation idempotents; the twists are those forced by weight. All are algebraic by (1). □

Remark 8.1. At special x the Mumford–Tate group may be a proper subgroup of G_2 with more invariant tensors; such points form a countable set. The construction gives algebraicity of the transported G_2-invariants there, but not of the additional invariants. We do not know whether such points occur at interior parameters.

Remark 8.2. Yun's local system is the same as that of Dettweiler–Reiter [32, Thm. 6.1.5 and the paragraph after Thm. 6.2.3]. The corresponding statement for their family therefore reduces to the algebraicity of the Hodge class in $\mathrm{Hom}(M_x, M_x^{\mathrm{DR}})$ identifying the two motives; this class is motivated if and only if the determinant class of the Dettweiler–Reiter motive is, and we do not address it. [referee report 002, R2-09]

Remark 8.3. The method uses rigidity of the Hecke eigensheaf in an essential way: the odd sheaf is a representation of the finite group $\widetilde{A}$ on the open point of a zero-dimensional moduli stack, defined without reference to the local system, and the Hecke operators produce M_x from it. Yun's other rigid data (types A_1, D_{2n}, E_7, E_8) have the same shape; for A_1 the construction reproduces the known algebraicity of the Hodge tensors of Sym^2 of a weight-one $\mathbb{Q}(i)$-Hodge structure of abelian type. For D_4, E_7 and E_8 the analogue of Theorem 4.3 needs a meridian-by-meridian analysis, since not every nontrivial subgroup of $\widetilde{A}$ contains the central element.

Computational certificates

The finite computations used above (the enumeration of bundle types in Lemma 6.1; the exact stabiliser dimensions, stabiliser tori and cocharacters behind Table 2; the stabiliser data of Table 1) were done in exact rational arithmetic in Sage. The orbit counts over finite fields at $q = 5, 7$ were used only as a cross-check of Proposition 6.4, whose proof does not depend on them. The scripts, a generator for every input, a single `run_all.sh` and a reference run with input, output and log hashes are deposited as the certificate bundle `notes/yun-g2-algebraic-review/certificate/` of the repository `benzsevern/hodge-fleet` (commit `3f7a05ed`; tarball SHA-256 `f598645060d69b3403a0ccee7561f410dc5e3123323214176-aac083ab8e76614`); the reference run used SageMath 10.7. That tarball predates the characteristic-zero certificate for Proposition 6.4: the script `matsuki_c0.sage`, its logs and its exact representatives are in `certificate/matsuki/` (added at commit `e2f38848`). It is not called by `run_all.sh` and is run separately, from that directory, as `CT=G2 EPS=-1,-1 sage matsuki_c0.sage` (and with `CT=A1 EPS=-1`, `CT=A2 EPS=1,-1` for the controls); a pinned rerun with checksums of every script, log and representative file is recorded in the bundle. That rerun (`certificate/matsuki/RERUN-2026-10-19.md`) was made from a clean checkout of commit `e2f38848b3f5d5d49cd20ffabf97661f75632b18` with SageMath 10.7 and records the exact commands, the environment and the SHA-256 hash of every input and output; the script `matsuki_c0_canonical.sage` selects points by an exact canonical key, and two consecutive runs of it give byte-identical logs and representatives (`certificate/matsuki/canon/SHA256SUMS`). These are runs made by the author's computing setup, deposited for independent re-execution; the outside reviews of this paper inspected them but did not rerun them. Likewise, where the text cites a source by theorem or section number, the number was read at source by the author's programme; the outside audits named in the review record checked some of these citations, not all, and an audit's positive verdict is not an independent check of every supporting proof it relies on.

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