

# Cartan determinants in finite representation type over arbitrary fields

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## Abstract

Let  $A$  be a finite-dimensional representation-finite algebra over an arbitrary field, and let  $M$  be a nonzero finite-dimensional right  $A$ -module. We prove that  $\text{End}_A(M)$  has Cartan determinant one whenever it has finite global dimension. In particular, the Cartan determinant conjecture holds for representation-finite algebras over arbitrary fields.

## Scholarly credit

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## 1 Introduction

Let  $k$  be a field and let  $A$  be a nonzero finite-dimensional associative unital  $k$ -algebra. Choose representatives  $S_1, \dots, S_n$  of its simple right modules, with indecomposable projective covers  $P_1, \dots, P_n$ . The integral Cartan matrix is

$$C_A = ([P_j : S_i])_{1 \leq i, j \leq n},$$

where  $[P_j : S_i]$  is the multiplicity of  $S_i$  in a composition series of  $P_j$ . We write  $\text{gl. dim } A$  for the global dimension of  $A$ , the supremum of the projective dimensions of its modules. If it is finite, choose finite projective resolutions  $P_{\bullet j} \rightarrow S_j$ . Write  $P_{qj} \cong \bigoplus_i P_i^{a_{iqj}}$  and set  $Q_{ij} = \sum_q (-1)^q a_{iqj}$ . Taking composition multiplicities gives  $C_A Q = I$ , where  $I$  is the identity matrix. Thus  $\det C_A = \pm 1$ . The Cartan determinant conjecture asks whether  $\det C_A = 1$ .

An algebra is representation-finite if its finite-dimensional modules have only finitely many indecomposable isomorphism classes. Enomoto proved the conjecture for representation-finite algebras over algebraically closed fields [4, Theorem A]. He also treated  $\text{End}_A(M)$  when  $A$  is representation-finite over an algebraically closed field  $k$ , under the additional assumption that  $\text{gl. dim } A < \infty$  or  $\text{char } k \neq 2$  [4, Theorem B]. In the first version, Liang Chen extended the representation-finite conclusion to perfect fields [1, Theorem 1.1]. These arguments obtain a grading with semisimple degree-zero part, using the structure theory of representation-finite algebras and, over perfect fields, descent of gradings. In the second version, Chen and Guo prove the conjecture for representation-finite Artin algebras [2, Theorem 1.1]. Their theorem

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covers Theorem B below over arbitrary fields; applying it to the opposite algebra gives the right-module convention used here.

We prove the following endomorphism-algebra statement over arbitrary fields. All endomorphism rings are multiplied by composition, with the rightmost map applied first.

**Theorem A.** Let  $k$  be a field, let  $A$  be a finite-dimensional representation-finite  $k$ -algebra, and let  $M$  be a nonzero finite-dimensional right  $A$ -module. If  $\text{gl. dim } \text{End}_A(M) < \infty$ , then

$$\det C_{\text{End}_A(M)} = 1.$$

The regular module gives the representation-finite case of the conjecture.

**Theorem B.** Let  $k$  be a field and let  $A$  be a finite-dimensional representation-finite  $k$ -algebra. If  $\text{gl. dim } A < \infty$ , then  $\det C_A = 1$ .

Theorem B is included as a consequence of Theorem A; its conclusion is already covered by Chen–Guo [2, Theorem 1.1]. Theorem A concerns arbitrary module endomorphism algebras and removes the additional restriction on the ambient algebra in Enomoto’s endomorphism-algebra result. The proof uses the Auslander algebra of  $A$  and compares its idempotent corners with the corresponding corners of its radical-associated graded algebra. The comparison preserves both Cartan matrices and finiteness of global dimension.

The resulting transfer statement applies to strict  $\tau$ -algebras, defined in Section 3. For such an algebra  $R$ , let  $J = \text{rad } R$  be its Jacobson radical and let  $S = \text{gr}_J R = \bigoplus_{m \geq 0} J^m / J^{m+1}$  be its radical-associated graded algebra. For a nonzero idempotent  $e \in R$ , the corner  $eRe$  is an algebra with identity  $e$ ; write  $\bar{e}$  for the image of  $e$  in  $S_0 = R/J$ . We prove

$$\text{gl. dim}(eRe) < \infty \iff \text{gl. dim}(\bar{e}S\bar{e}) < \infty, \quad (1)$$

together with equality of the corresponding integral Cartan matrices. Iyama’s radical-layer theory supplies the  $\tau$ -categorical data needed for the comparison [5, §§1.3, 2.5]. The graded corner then has Cartan determinant one by the graded projective-resolution argument of Wilson [7], which we include over arbitrary fields.

Approximations and their kernels have long been used to construct projective resolutions over endomorphism rings. In particular, Doherty–Faber–Ingalls develop this approach using Iyama’s ladders for categories of maximal Cohen–Macaulay modules [3, §§2.2–3]. Here the dimensions of Hom spaces in object quotients determine the unselected kernel summands. Their multiplicities define a finite directed graph, whose acyclicity characterizes finite global dimension. The same graph occurs before and after passage to the associated graded algebra, yielding (1).

Chen–Guo [2, Section 3, Steps 1–4] also pass to the radical grading of an Auslander algebra, preserve the two-sided  $\tau$ -data, recover stable Hom lengths and nonprojective syzygy multiplicities, and conclude with a graded Cartan argument. They use an abelian realization and a local formula from a left  $\tau$ -sequence. Here object-quotient Hom dimensions recover approximation kernels, and their directed graph gives the bidirectional finiteness transfer (1). The statements for arbitrary module endomorphism algebras and strict  $\tau$ -algebra corners have a different scope from their stated representation-finite theorem.

## 2 Cartan matrices and kernels of approximations

All algebras are nonzero finite-dimensional associative unital  $k$ -algebras. Write  $\text{mod } R$  for the category of finite-dimensional right  $R$ -modules and  $\text{proj } R$  for its full subcategory of projective modules. For a module  $N$ , let  $\text{pd}_R N$  be its projective dimension, with  $\text{pd}_R 0 = -\infty$ ,

and let  $\text{Ext}_R^q(-, -)$  be the degree- $q$  right derived functor of  $\text{Hom}_R(-, -)$  for integers  $q \geq 0$ . For an object  $M$  of an additive category,  $\text{add } M$  consists of direct summands of finite direct sums of copies of  $M$ . An algebra is basic if its regular right module is a sum of pairwise nonisomorphic indecomposable modules. Morita equivalence preserves projective resolutions, simple modules and composition multiplicities, hence global dimension and the integral Cartan matrix.

For finite-dimensional algebras, global dimension equals the maximum of the projective dimensions of the simple modules. Indeed, the finite filtration  $N \supseteq N \text{ rad } R \supseteq N(\text{rad } R)^2 \supseteq \cdots \supseteq 0$  of an arbitrary module has semisimple quotients. Direct sums of projective resolutions give the simple-module bound for these quotients; the long exact Ext sequence gives it for  $N$ .

Let  $\mathcal{C}$  be an idempotent-complete,  $k$ -linear Krull–Schmidt category with finite-dimensional Hom spaces and finitely many indecomposable isomorphism classes. Idempotent completeness means that each idempotent morphism splits. The Krull–Schmidt property means that every object is a finite direct sum of objects with local endomorphism rings, with the summands unique up to permutation and isomorphism. Fix representatives  $X_1, \dots, X_n$  and put  $G = \bigoplus_{i=1}^n X_i$ , so  $\mathcal{C} = \text{add } G$ . We write  $\mathcal{C}(X, Y)$  for its morphism space and  $\text{End}_{\mathcal{C}}(X) = \mathcal{C}(X, X)$ . For each  $i$ , define

$$D_i = \text{End}_{\mathcal{C}}(X_i) / \text{rad } \text{End}_{\mathcal{C}}(X_i), \quad d_i = \dim_k D_i.$$

Here  $\text{rad}$  denotes the Jacobson radical of an endomorphism ring; the ring  $D_i$  is a finite-dimensional division  $k$ -algebra. Write

$$H_{ij} = \dim_k \mathcal{C}(X_i, X_j), \quad D = \text{diag}(d_1, \dots, d_n). \quad (2)$$

Rows of  $H$  index sources and columns index targets. An object  $X \cong \bigoplus_i X_i^{a_i}$  has multiplicity vector  $[X] = (a_i)_{i=1}^n \in \mathbb{Z}^n$ , with coordinate  $[X]_i = a_i$ ; in particular,  $[X_i] = \mathbf{e}_i$ , the  $i$ th standard basis vector. Write  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise. This bracket is a split multiplicity vector, distinct from the composition multiplicity  $[P : S]$ .

**Lemma 2.1.** *For  $R = \text{End}_{\mathcal{C}}(G)$ , label the projective right  $R$ -modules by  $\mathcal{C}(G, X_j)$ . Then*

$$C_R = D^{-1}H.$$

*For a nonempty subset  $T \subseteq \{1, \dots, n\}$ , the same formula restricted to the rows and columns in  $T$  gives the Cartan matrix of  $\text{End}_{\mathcal{C}}(\bigoplus_{i \in T} X_i)$ .*

*Proof.* Let  $e_i \in R$  be the projection onto  $X_i$  followed by its inclusion. The  $e_i$  are primitive orthogonal idempotents with sum 1. The module  $\mathcal{C}(G, X_i) \cong e_i R$  is indecomposable projective. Its simple top is its quotient by multiplication by  $\text{rad } R$ ; this top has  $e_i$ -component  $D_i$  and all its other direct components are zero. The operation  $N \mapsto Ne_i$  is exact, since it is projection onto a direct summand of the underlying vector space. Applying it to a composition series shows that  $\dim_k Ne_i$  is  $d_i$  times the multiplicity of the  $i$ th simple. Since

$$\mathcal{C}(G, X_j)e_i \cong \mathcal{C}(X_i, X_j),$$

the asserted formula follows. Restricting the direct sum  $G$  to the indices in  $T$  leaves these component spaces and residual division rings unchanged.  $\square$

For example, if  $L/k$  is a finite field extension and  $\mathcal{C} = \text{add } L_L$ , then  $H = ([L : k])$ , while  $C_{\text{End}_L(L)} = (1)$ .

Fix a nonempty subset  $T$  and put

$$M = \bigoplus_{i \in T} X_i, \quad U = \{1, \dots, n\} \setminus T.$$

The object quotient  $\mathcal{D} = \mathcal{C}/[\text{add } M]$  has the same objects as  $\mathcal{C}$  and morphism spaces

$$\mathcal{D}(X, Y) = \mathcal{C}(X, Y)/[\text{add } M](X, Y),$$

where  $[\text{add } M](X, Y)$  consists of morphisms factoring through an object of  $\text{add } M$ . For  $j \in U$ , define the column vector

$$r_j = (\dim_k \mathcal{D}(X_i, X_j))_{i=1}^n. \quad (3)$$

Its entries in  $T$  are zero.

A right  $\text{add } M$ -approximation of  $X$  is a morphism  $f : M_X \rightarrow X$ , with  $M_X \in \text{add } M$ , through which every map from an object of  $\text{add } M$  to  $X$  factors. It is right minimal if  $fa = f$  implies that  $a \in \text{End}(M_X)$  is invertible. To obtain an approximation, take a  $k$ -basis of each  $\mathcal{C}(X_i, X)$  for  $i \in T$  and form the map from the corresponding copies of  $X_i$  to  $X$ . Choose such an approximation with the least number of indecomposable source summands. For an endomorphism  $a$  with  $fa = f$ , Fitting decomposition splits its source into a summand where  $a$  is invertible and a summand where  $a$  is nilpotent. Since  $fa^m = f$  for all  $m \geq 1$ , the map  $f$  vanishes on the nilpotent summand. Minimality of the number of source summands makes that summand zero. Thus  $a$  is invertible and the approximation is right minimal.

Assume henceforth in this section that  $\mathcal{C}$  has kernels and that  $H$  is invertible over  $\mathbb{Q}$ . For each  $j \in U$ , choose a minimal right approximation and its kernel:

$$f_j : M_j \longrightarrow X_j, \quad K_j = \ker f_j.$$

The kernel here is a categorical kernel:  $\mathcal{C}(Y, K_j)$  identifies with the kernel of  $\mathcal{C}(Y, f_j)$  for every  $Y$ . For any object  $Z$ , the representable functor  $\mathcal{C}(-, Z)$  sends  $Y$  to  $\mathcal{C}(Y, Z)$  and sends a morphism  $h : Y' \rightarrow Y$  to precomposition by  $h$ .

**Lemma 2.2.** *For  $i, j \in U$ , the multiplicity of  $X_i$  in  $K_j$  is*

$$[K_j]_i = (H^{-1}r_j)_i - \delta_{ij}. \quad (4)$$

*Proof.* There is an exact sequence of contravariant functors on  $\mathcal{C}$ ,

$$0 \longrightarrow \mathcal{C}(-, K_j) \longrightarrow \mathcal{C}(-, M_j) \longrightarrow \mathcal{C}(-, X_j) \longrightarrow \mathcal{D}(-, X_j) \longrightarrow 0. \quad (5)$$

Indeed, the approximation property identifies the image of the middle map with the ideal of morphisms factoring through  $\text{add } M$ . Evaluating at the  $X_i$  and taking dimensions gives

$$r_j = H(\mathbf{e}_j - [M_j] + [K_j]).$$

The coordinates of  $[M_j]$  in  $U$  vanish. Multiplication by  $H^{-1}$  now gives (4).  $\square$

Define a directed graph with vertex set  $U$ , putting an arrow  $j \rightarrow i$  precisely when  $X_i$  is a direct summand of  $K_j$ . Loops count as directed cycles. If  $U$  is empty, the graph is acyclic.

**Proposition 2.3.** *Let  $\mathcal{C}$  and  $M$  be as above, with kernels in  $\mathcal{C}$  and  $H$  invertible over  $\mathbb{Q}$ . Then  $\text{End}_{\mathcal{C}}(M)$  has finite global dimension if and only if the directed graph defined by (4) is acyclic.*

*Proof.* Put  $E = \text{End}_{\mathcal{C}}(M)$  and  $F = \mathcal{C}(M, -)$ , with right  $E$ -action by precomposition. For  $P \in \text{add } M$  and  $X \in \mathcal{C}$ , the natural map

$$\mathcal{C}(P, X) \xrightarrow{\sim} \text{Hom}_E(F(P), F(X)) \quad (6)$$

is an isomorphism. For  $P = M$ , an  $E$ -linear map  $E \rightarrow F(X)$  is determined by the image  $h : M \rightarrow X$  of  $1_E$ , and sends  $a \in E$  to  $h \circ a$ . This proves the assertion in that case. Both

sides preserve finite sums and direct summands in  $P$ , giving the general case. Every finite projective right  $E$ -module is a summand of some  $E^r$ ; splitting the corresponding idempotent on  $M^r$  shows that  $F$  identifies  $\text{add } M$  with  $\text{proj } E$ .

Write  $\Omega_E N$  for the kernel of a projective cover of an  $E$ -module  $N$ , with  $\Omega_E 0 = 0$ , and define  $\Omega_E^q$  by  $q$  iterations for integers  $q \geq 1$ . The approximation property makes  $F(f_j)$  surjective. If an endomorphism of  $F(M_j)$  preserves  $F(f_j)$ , (6) lifts it to an endomorphism  $a$  of  $M_j$  with  $f_j a = f_j$ . Minimality makes  $a$  invertible. Hence  $F(f_j)$  is a projective cover, and

$$\Omega_E F(X_j) \cong F(K_j). \quad (7)$$

Suppose the graph is acyclic. For  $j \in U$ , let  $\ell_j$  be the largest number of arrows in a directed path starting at  $j$ . Then  $0 \leq \ell_j \leq |U| - 1$ . If  $\ell_j = 0$ , then  $K_j \in \text{add } M$ , so  $\text{pd}_E F(X_j) \leq 1$ . If  $\ell_j > 0$ , each unselected summand  $X_i$  of  $K_j$  satisfies  $\ell_i \leq \ell_j - 1$ . Induction in (7) gives  $\text{pd}_E F(X_j) \leq \ell_j + 1 \leq |U|$ . The selected objects have projective images. Thus every object  $X \in \mathcal{C}$  satisfies  $\text{pd}_E F(X) \leq |U|$ ; for  $U = \emptyset$  this follows directly from  $\mathcal{C} = \text{add } M$ .

For any finite-dimensional right  $E$ -module  $N$ , lift a minimal projective presentation along the equivalence  $\text{add } M \simeq \text{proj } E$ :

$$F(M') \xrightarrow{F(g)} F(M'') \longrightarrow N \longrightarrow 0.$$

Here  $M', M'' \in \text{add } M$ , and (6) lifts the differential to  $g : M' \rightarrow M''$ . Since the presentation is minimal, its first term is a projective cover of the kernel of  $F(M'') \rightarrow N$ . Thus its kernel is  $\Omega_E^2 N$ , which the kernel property in  $\mathcal{C}$  identifies with  $F(\ker g)$ . Hence  $\text{pd}_E N \leq |U| + 2$ , so  $E$  has finite global dimension.

Conversely, suppose that the graph contains a directed cycle. If  $F(X_j) = 0$ , then the zero map  $0 \rightarrow X_j$  is the minimal approximation, so  $K_j = 0$  and  $j$  has no outgoing arrows. All vertices on the cycle therefore have nonzero images under  $F$ . By Lemma 2.2 and (7), an arrow  $j \rightarrow i$  on the cycle gives a nonzero direct summand  $F(X_i)$  of  $\Omega_E F(X_j)$ . If  $E$  had finite global dimension, this would imply

$$\text{pd}_E F(X_i) \leq \text{pd}_E F(X_j) - 1.$$

Following the cycle contradicts this strict decrease.  $\square$

### 3 Radical layers in $\tau$ -categories

For a Krull–Schmidt category  $\mathcal{C}$ , its radical  $J_{\mathcal{C}}$  is the ideal of morphisms  $f : X \rightarrow Y$  such that  $1_X - gf$  is invertible for every  $g : Y \rightarrow X$ . Between indecomposable objects it consists of all nonisomorphisms. Its  $m$ th power consists of sums of composites of  $m$  radical morphisms, and  $J_{\mathcal{C}}^0 = \mathcal{C}$ . The quotient  $\mathcal{C}/J_{\mathcal{C}}$  has the same objects and morphism spaces  $\mathcal{C}(X, Y)/J_{\mathcal{C}}(X, Y)$ .

A morphism  $a : Z \rightarrow Y$  is a weak kernel of  $b : Y \rightarrow X$  if

$$\mathcal{C}(-, Z) \xrightarrow{a \circ -} \mathcal{C}(-, Y) \xrightarrow{b \circ -} \mathcal{C}(-, X)$$

is exact at the middle term. It is minimal if  $ac = a$  for  $c \in \text{End}_{\mathcal{C}}(Z)$  forces  $c$  to be invertible. Weak cokernels and their minimality are defined in the opposite category.

Following [5, §1.3], a right  $\tau$ -sequence ending at  $X$  is a complex

$$\tau^+ X \xrightarrow{a_X} \theta^+ X \xrightarrow{b_X} X$$

whose maps are radical, in which  $a_X$  is a minimal weak kernel of  $b_X$ , and for which

$$\mathcal{C}(-, \theta^+ X) \longrightarrow J_{\mathcal{C}}(-, X) \longrightarrow 0, \quad \mathcal{C}(\theta^+ X, -) \longrightarrow J_{\mathcal{C}}(\tau^+ X, -) \longrightarrow 0$$

are exact. A left  $\tau$ -sequence is defined dually and written  $X \rightarrow \theta^- X \rightarrow \tau^- X$ . A  $\tau$ -category has both sequences at every object. It is strict if the first map of every right sequence is a monomorphism and the last map of every left sequence is an epimorphism in the category. The objects in these sequences are unique up to isomorphism. For indecomposable  $X$ , each  $\tau^\pm X$  is indecomposable or zero. We extend  $\theta^\pm$  and  $\tau^\pm$  additively to the free abelian group on the indecomposable objects. By the right and left  $\tau$ -data we mean these respective pairs of additive operators.

**Definition 3.1.** A finite-dimensional algebra  $R$  is a strict  $\tau$ -algebra if  $\text{proj } R$  is a strict  $\tau$ -category.

Iyama's characterization [5, Theorem 2.1] says that this is equivalent to  $\text{gl. dim } R \leq 2$  and the two-sided  $(2, 2)$  condition. Explicitly, write the minimal injective resolutions of the right and left regular modules as  $0 \rightarrow R_R \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$  and  $0 \rightarrow {}_R R \rightarrow I'^0 \rightarrow I'^1 \rightarrow \dots$ . The condition is  $\text{pd}_R I^i \leq 1$  and  $\text{pd}_{R^{\text{op}}} I'^i \leq 1$  for  $i = 0, 1$ , where left  $R$ -modules are viewed as right modules over the opposite algebra  $R^{\text{op}}$ .

For an Artin algebra  $A$ , the category  $\text{mod } A$  is a strict  $\tau$ -category [5, §1.3.2(1)]. For a nonprojective indecomposable module its right  $\tau$ -sequence is the almost split sequence ending at that module. At an indecomposable projective module  $P$  it is  $0 \rightarrow P \text{ rad } A \rightarrow P$ . Therefore, if  $A$  is representation-finite and  $G$  is the sum of its indecomposable modules, then the restricted Yoneda equivalence

$$\text{Hom}_A(G, -) : \text{mod } A \xrightarrow{\sim} \text{proj End}_A(G)$$

makes its Auslander algebra  $\text{End}_A(G)$  a strict  $\tau$ -algebra. The equivalence follows from (6), applied with  $M = G$  and  $\mathcal{C} = \text{add } G$ ; categorical monomorphisms, epimorphisms and the displayed exactness conditions are preserved by an equivalence.

We use two radical-layer results in the following precise forms. An object quotient of a  $\tau$ -category is again a  $\tau$ -category [5, §1.3]. Its right data are obtained by deleting the selected summands from  $\theta^+ X$  and retaining  $\tau^+ X$  exactly when it survives and the remaining middle term is nonzero; otherwise its translation is zero. The dual rule holds on the left. This is the deletion rule in [6, Proposition 4.3].

For a  $\tau$ -category with indecomposable representatives  $X_1, \dots, X_n$ , use the multiplicity vectors defined in Section 2. For an integral vector  $v$ , let  $v_+$  be obtained by replacing every negative coordinate by zero. For each  $1 \leq j \leq n$ , define

$$y_{0,j} = \mathbf{e}_j, \quad y_{1,j} = \theta^+ \mathbf{e}_j, \quad y_{m+1,j} = (\theta^+ y_{m,j} - \tau^+ y_{m-1,j})_+ \quad (m \geq 1). \quad (8)$$

Iyama's radical-layer theorem [5, §1.3.3] supplies objects  $Y_{m,j}$  with  $[Y_{m,j}] = y_{m,j}$ . Set  $Y_{-1,j} = 0$ . For every  $m \geq 0$  it gives a presentation

$$\mathcal{C}(-, \tau^+ Y_{m-1,j}) \longrightarrow \mathcal{C}(-, Y_{m,j}) \longrightarrow J_{\mathcal{C}}^m(-, X_j) \longrightarrow 0, \quad (9)$$

whose first map is induced by a radical morphism. For  $m = 0$  its left term is zero. Here  $J_{\mathcal{C}}^m(-, X_j)$  denotes the contravariant functor with value  $J_{\mathcal{C}}^m(X, X_j)$  at an object  $X$ ; the maps are induced by composition. These statements apply to the Hom-finite categories considered here over arbitrary fields.

**Lemma 3.2.** *Let  $\mathcal{C}$  be a Hom-finite  $k$ -linear  $\tau$ -category with finitely many indecomposable isomorphism classes. With the preceding notation, for  $1 \leq i, j \leq n$  and  $m \geq 0$ ,*

$$\dim_k(J_{\mathcal{C}}^m(X_i, X_j)/J_{\mathcal{C}}^{m+1}(X_i, X_j)) = d_i(y_{m,j})_i.$$

*Consequently the right  $\tau$ -data and the integers  $d_i$  determine all Hom dimensions, both in  $\mathcal{C}$  and in every specified object quotient.*

*Proof.* Put  $R = \text{End}_{\mathcal{C}}(G)$ . Evaluation at  $G$  identifies additive  $k$ -linear contravariant functors whose value at  $G$  is finite-dimensional with finite-dimensional right  $R$ -modules. Indeed, the representable functors  $\mathcal{C}(-, X_i)$  evaluate to  $e_i R$ ; a finite projective presentation of an  $R$ -module lifts to a map between finite sums of these representables, and its cokernel gives the inverse construction. Every object is a summand of a finite sum of  $G$ , so maps and their exactness are determined by evaluation at  $G$ .

The radical of the evaluated module  $J_{\mathcal{C}}^m(G, X_j)$  is its product with  $\text{rad } R$ . A block matrix in  $\text{rad } R$  has radical morphisms as entries. Matrix multiplication therefore identifies this product with  $J_{\mathcal{C}}^{m+1}(G, X_j)$ . The same holds at every summand, giving

$$\text{rad}(J_{\mathcal{C}}^m(-, X_j)) = J_{\mathcal{C}}^{m+1}(-, X_j).$$

The first map in (9) has image in the radical of the representable  $\mathcal{C}(-, Y_{m,j})$ . Thus its cokernel map is a projective cover: its kernel is contained in the radical, which is superfluous by Nakayama's lemma. Taking the quotient by radicals gives

$$(\mathcal{C}/J_{\mathcal{C}})(-, Y_{m,j}) \cong J_{\mathcal{C}}^m(-, X_j)/J_{\mathcal{C}}^{m+1}(-, X_j).$$

Evaluation at  $X_i$  gives  $(y_{m,j})_i$  copies of  $D_i$ . The category radical is nilpotent because the endomorphism algebra of its additive generator is finite-dimensional. Summing over  $m \geq 0$  gives the Hom dimension.

Consider the object quotient by  $\text{add } M$ . If a surviving indecomposable  $X_i$  had identity factoring through  $\text{add } M$ , it would be a summand of an object of  $\text{add } M$ , a contradiction. Thus the factorization ideal in  $\text{End}(X_i)$  is proper and is contained in its local radical, and the residual division ring remains  $D_i$ . Between surviving indecomposables, any factorization through a deleted indecomposable is a composite of two radical maps, since both adjacent pairs are nonisomorphic. Hence the factorization ideal is contained in  $J_{\mathcal{C}}^2$  on these Hom spaces. The radical in the quotient is the image of  $J_{\mathcal{C}}$ , and its square is the image of  $J_{\mathcal{C}}^2$ ; irreducible-morphism multiplicities are therefore preserved.

Distinct surviving indecomposables remain distinct: inverse maps in the quotient lift to  $f : X_i \rightarrow X_j$  and  $g : X_j \rightarrow X_i$  whose two composites differ from identities by radical endomorphisms. Both composites are invertible, so  $f$  is an isomorphism. Each object of the quotient is a sum of the surviving objects with local endomorphism rings; it is therefore again Krull–Schmidt. Its Hom spaces are finite-dimensional and its radical is nilpotent. The deletion rule supplies its  $\tau$ -data, so the same radical-layer formula computes all its Hom dimensions.  $\square$

For the quotient  $\mathcal{D}$  associated to  $T$ , compute  $y_{m,j}$  from (8) using the quotient operators on  $U$ , and extend each vector by zero on  $T$ . Then

$$r_j = D \sum_{m \geq 0} y_{m,j} \quad (j \in U). \quad (10)$$

The sum is finite by nilpotence of the quotient radical.

## 4 Associated graded algebras and corners

Let  $R$  be a basic finite-dimensional strict  $\tau$ -algebra, let  $J = \text{rad } R$ , and set

$$S = \text{gr}_J R = \bigoplus_{m \geq 0} J^m / J^{m+1}.$$

Multiplication is induced by multiplication in  $R$ : the product of classes of degrees  $a, b$  is the class of their product modulo  $J^{a+b+1}$ . Choose primitive orthogonal idempotents  $e_1, \dots, e_n$  with sum 1, and let  $\bar{e}_i$  be their images in  $S_0 = R/J$ . Index the projectives  $e_i R$  and  $\bar{e}_i S$  by the same  $i$ . The algebra  $S$  has semisimple degree-zero part and radical  $S_{>0} = \bigoplus_{m > 0} S_m$ .

**Lemma 4.1.** *The algebra  $S$  is a strict  $\tau$ -algebra. The matched projective categories  $\text{proj } R$  and  $\text{proj } S$  have the same left and right  $\tau$ -data, the same residual division rings, and the same Hom dimensions.*

*Proof.* The assertion that  $S$  is strict is [5, Theorem 2.5]. To verify the matching of the data, take a minimal projective resolution of a simple right  $R$ -module  $L$ ,

$$0 \longrightarrow P_2 \xrightarrow{g} P_1 \xrightarrow{f} P_0 \longrightarrow L \longrightarrow 0.$$

Iyama's radical-layer exactness [5, Lemma 2.5.1], applied with right-module conventions, gives

$$0 \longrightarrow P_2 J^{i-1} \xrightarrow{g} P_1 J^i \xrightarrow{f} P_0 J^{i+1} \longrightarrow 0 \quad (i \geq 0), \quad (11)$$

where  $P_2 J^{-1} = P_2$ . For a projective  $P$  put  $\text{gr } P = \bigoplus_{i \geq 0} P J^i / P J^{i+1}$ . For a  $\mathbb{Z}$ -graded module  $V$  and  $a \in \mathbb{Z}$ , define the shift by  $V(-a)_m = V_{m-a}$  for every  $m \in \mathbb{Z}$ . The modules  $\text{gr } P$  have zero components in negative degrees. Comparing successive sequences (11) yields

$$0 \longrightarrow (\text{gr } P_2)(-2) \longrightarrow (\text{gr } P_1)(-1) \longrightarrow \text{gr } P_0 \longrightarrow L \longrightarrow 0. \quad (12)$$

Comparing  $i = 0, 1$  in (11) gives

$$P_1 / P_1 J \xrightarrow{\sim} P_0 J / P_0 J^2,$$

since the map on the left terms is the identity of  $P_2$ . For every  $i \geq 1$ , the inclusions of the  $(i+1)$ st column into the  $i$ th column give a short exact sequence of complexes. The corresponding cokernels form the exact sequence

$$0 \longrightarrow \frac{P_2 J^{i-1}}{P_2 J^i} \longrightarrow \frac{P_1 J^i}{P_1 J^{i+1}} \longrightarrow \frac{P_0 J^{i+1}}{P_0 J^{i+2}} \longrightarrow 0.$$

This is the degree- $(i+1)$  part of (12). The degree-zero part is  $P_0 / P_0 J \cong L$ , so all degrees are accounted for.

The maps in (12) have images in the radical, so this resolution remains minimal when the grading is forgotten. Also  $\text{gr}(e_i R) = \bar{e}_i S$ . Evaluation at  $R$  identifies the simple top of  $\mathcal{C}(-, P_0)$ , for  $\mathcal{C} = \text{proj } R$ , with  $L$ . Its minimal representable resolution is therefore the displayed projective resolution. The three projective terms give the right  $\tau$ -sequence in  $\mathcal{C}$ ; their unchanged multiplicities give the same  $\theta^+$  and  $\tau^+$ . Applying the argument to  $R^{\text{op}}$  gives the left data. Finally,  $S_0 = R/J$  preserves the residual division rings, and

$$\dim_k \bar{e}_j S \bar{e}_i = \sum_{m \geq 0} \dim_k (e_j J^m e_i / e_j J^{m+1} e_i) = \dim_k e_j R e_i$$

gives the matched Hom dimensions. □

**Lemma 4.2.** *If  $R$  is a finite-dimensional algebra with  $\text{gl. dim } R \leq 2$ , then  $\text{proj } R$  has kernels.*

*Proof.* For a morphism  $g : P_1 \rightarrow P_0$  between finite projective modules, take its kernel  $K$  and cokernel  $N$  in  $\text{mod } R$ . The sequence  $0 \rightarrow K \rightarrow P_1 \rightarrow P_0 \rightarrow N \rightarrow 0$  gives, for every module  $V$ ,

$$\text{Ext}_R^1(K, V) \cong \text{Ext}_R^2(\text{im } g, V) \cong \text{Ext}_R^3(N, V) = 0.$$

Thus  $K$  is projective and finite-dimensional, and its module-theoretic universal property is also the kernel property in  $\text{proj } R$ .  $\square$

**Theorem 4.3.** *Let  $R$  be a basic finite-dimensional strict  $\tau$ -algebra over a field  $k$ , put  $J = \text{rad } R$ , and choose primitive orthogonal idempotents  $e_1, \dots, e_n$  with sum 1. Let  $\bar{e}_i$  be the image of  $e_i$  in  $R/J$ . For a nonempty subset  $T \subseteq \{1, \dots, n\}$ , put  $e = \sum_{i \in T} e_i$  and  $\bar{e} = \sum_{i \in T} \bar{e}_i$ . Then*

$$C_{eRe} = C_{\bar{e}(\text{gr}_J R)\bar{e}}$$

*under the corresponding labelling of simples, and*

$$\text{gl. dim}(eRe) < \infty \iff \text{gl. dim}(\bar{e}(\text{gr}_J R)\bar{e}) < \infty.$$

*Proof.* Use  $\mathcal{C} = \text{proj } R$  with  $M = eR$ , and the matched objects in  $\text{proj } S$ , where  $S = \text{gr}_J R$ . Lemma 4.1 gives common matrices  $H$  and  $D$ . Since  $R$  has finite global dimension,  $C_R$  is invertible over  $\mathbb{Z}$ , so  $H = DC_R$  is invertible over  $\mathbb{Q}$ .

Both categories have kernels by Lemma 4.2. Their quotient  $\tau$ -data agree by the deletion rule. Lemma 3.2 then gives the same quotient columns  $r_j$  for every unselected index  $j$ . Formula (4) yields identical graphs of approximation kernels. Proposition 2.3 therefore proves the equivalence of finite global dimensions. Lemma 2.1, applied to the selected rows and columns of the common matrices  $H, D$ , proves equality of Cartan matrices.  $\square$

**Remark 4.4.** The graded corner in Theorem 4.3 has degree- $m$  part  $eJ^m e / eJ^{m+1} e$ . Its grading is induced by the filtration  $eJ^m e$  of  $eRe$ .

## 5 Cartan determinants of corners

The graded Cartan argument of Wilson [7] takes the following form over  $k$ .

**Lemma 5.1.** *Let  $B = \bigoplus_{m \geq 0} B_m$  be a finite-dimensional nonnegatively graded  $k$ -algebra whose degree-zero part is semisimple. If  $\text{gl. dim } B < \infty$ , then  $\det C_B = 1$ .*

*Proof.* The ideal  $B_{>0} = \bigoplus_{m > 0} B_m$  is nilpotent: the grading is bounded above and a product of sufficiently many positive-degree elements has degree above that bound. A nilpotent ideal is contained in the Jacobson radical, while the semisimple quotient  $B/B_{>0} = B_0$  implies the reverse containment. Thus  $\text{rad } B = B_{>0}$ .

Choose primitive idempotents  $e_1, \dots, e_n \in B_0$  representing the distinct simple right  $B_0$ -modules. Put  $P_i = e_i B$  and  $L_i = e_i B_0$ , viewing  $L_i$  as a graded right  $B$ -module concentrated in degree zero and annihilated by  $B_{>0}$ . Then  $P_i \rightarrow L_i$  is a projective cover generated in degree zero. Put  $d_i = \dim_k e_i B_0 e_i$ . For an indeterminate  $t$ , define

$$C_B(t)_{ij} = \sum_{m \geq 0} \frac{\dim_k e_j B_m e_i}{d_i} t^m.$$

The coefficient of  $t^m$  is the multiplicity of  $L_i$  in the semisimple  $B_0$ -module  $(P_j)_m$ , by Lemma 2.1. Hence  $C_B(t)$  is an  $n$ -by- $n$  matrix over  $\mathbb{Z}[t]$ , with  $C_B(0) = I$  and  $C_B(1) = C_B$ .

To construct a graded projective cover of a finite-dimensional graded module, decompose its top into the graded simples  $L_i(-a)$ . Lift a generator of each summand to degree  $a$ , and multiply its lift by  $e_i$  to arrange  $ve_i = v$ . Sending  $e_i$  to  $v$  defines a homogeneous map from  $P_i(-a)$ . The sum of these maps is surjective by Nakayama's lemma and induces an isomorphism on tops, so its kernel lies in the radical. Iterating gives minimal graded projective resolutions. Their differentials have image in the radical of the target, so forgetting the grading gives minimal ungraded resolutions. Finite global dimension makes these resolutions terminate.

For  $1 \leq j \leq n$  and  $q \geq 0$ , write the  $q$ th term of such a resolution as

$$P_{qj} = \bigoplus_{i=1}^n \bigoplus_{a \in \mathbb{Z}} P_i(-a)^{u_{iqja}}, \quad u_{iqja} \in \mathbb{Z}_{\geq 0}.$$

Only finitely many multiplicities occur. At degree  $m$ , take the  $e_i$ -component of the resolution and divide its alternating dimension sum by  $d_i$ . Since shifting by  $-a$  multiplies the graded dimension series by  $t^a$ , the resulting equalities for all degrees give

$$Q(t)_{ij} = \sum_{q \geq 0} \sum_{a \in \mathbb{Z}} (-1)^q u_{iqja} t^a, \quad 1 \leq i, j \leq n,$$

$$C_B(t)Q(t) = I.$$

For nonzero Laurent polynomials over  $\mathbb{Z}$ , the lowest and highest exponents of a product are the respective sums of the lowest and highest exponents of its factors. A product equal to 1 therefore forces both factors to have just one exponent, with coefficients  $\pm 1$ . Hence  $\det C_B(t) = \pm t^a$  for some integer  $a$ . Since  $C_B(0) = I$ , its determinant is a polynomial with constant term 1, forcing  $a = 0$  and the positive sign. Evaluating at  $t = 1$  proves the assertion.  $\square$

**Theorem 5.2.** *Let  $R$  be a finite-dimensional strict  $\tau$ -algebra over a field  $k$ , and let  $e \in R$  be a nonzero idempotent. If  $\text{gl. dim}(eRe) < \infty$ , then  $\det C_{eRe} = 1$ .*

*Proof.* Choose one copy of each indecomposable projective right  $R$ -module. Their sum is a projective generator, and its endomorphism algebra is a basic algebra Morita equivalent to  $R$ . The equivalence of projective categories preserves strict  $\tau$ -structure. The image of  $eR$  is a sum, possibly with repetitions, of indecomposable projectives in this basic category. Removing repetitions leaves the same additive subcategory, so its endomorphism algebra is Morita equivalent to  $\text{End}_R(eR) \cong eRe$ . Here the isomorphism sends  $a \in eRe$  to left multiplication by  $a$  on  $eR$ . Thus we may take  $R$  basic and  $e$  a sum of distinct primitive idempotents.

Put  $J = \text{rad } R$  and let  $\bar{e}$  be the image of  $e$  in  $R/J$ . By Theorem 4.3, the corner  $\bar{e}(\text{gr}_J R)\bar{e}$  has finite global dimension and the same Cartan matrix as  $eRe$ . Its inherited grading is nonnegative with semisimple degree-zero part  $\bar{e}(R/J)\bar{e}$ . Lemma 5.1 gives its Cartan determinant as 1, and hence the same holds for  $eRe$ .  $\square$

*Proof of Theorem A.* Let  $G$  be the sum of one representative of each indecomposable right  $A$ -module. Representation-finiteness makes  $G$  finite-dimensional and gives  $\text{mod } A = \text{add } G$ . Its Auslander algebra  $R = \text{End}_A(G)$  is strict  $\tau$ , as explained in Section 3.

Removing repeated indecomposable summands from  $M$  leaves  $\text{add } M$  unchanged and gives a Morita equivalent endomorphism algebra. We may therefore assume  $M$  is a summand of  $G$ . Let  $e \in R$  be its projection followed by inclusion. Extension by zero on the complementary summand gives  $\text{End}_A(M) \cong eRe$ . Theorem 5.2 gives the required determinant.  $\square$

*Proof of Theorem B.* Apply Theorem A to the regular right module  $A_A$ . Left multiplication gives an algebra isomorphism  $A \xrightarrow{\sim} \text{End}_A(A_A)$ .  $\square$

**Example 5.3.** Let  $A = k[\varepsilon]/(\varepsilon^2)$  and let  $L = A/(\varepsilon)$ . An  $A$ -module is a vector space with a square-zero linear operator. Its Jordan blocks have size 1 or 2, so the indecomposable modules are  $L$  and  $A$ . Their residual division rings are  $k$ , and the dimensions of  $\text{Hom}_A(L, L)$ ,  $\text{Hom}_A(L, A)$ ,  $\text{Hom}_A(A, L)$  and  $\text{End}_A(A)$  are respectively 1, 1, 1, 2. Thus for  $R = \text{End}_A(L \oplus A)$ ,

$$H = C_R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

To compute the graph of Section 2 for the summand  $M = A$ , use  $\mathcal{C} = \text{mod } A$ , with  $X_1 = L$ ,  $X_2 = A$ , and  $T = \{2\}$ . The minimal right approximation of  $L$  is the quotient  $A \rightarrow L$ , whose kernel is  $\varepsilon A \cong L$ . Every map  $L \rightarrow A$  has image in  $\varepsilon A$ , and every map  $A \rightarrow L$  kills  $\varepsilon A$ . Thus every composite  $L \rightarrow A \rightarrow L$  is zero, so the quotient column is  $r_1 = (1, 0)^\top$ . Direct calculation gives

$$H^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad H^{-1}r_1 - \mathbf{e}_1 = (1, -1)^\top.$$

The unselected coordinate is 1, giving a loop at  $L$ . Thus  $\text{End}_A(A) \cong A$  has infinite global dimension; explicitly  $\Omega_A L \cong L$ , while  $C_A = (2)$ .

## AI contributions

The mathematical arguments were developed with AI assistance in ChatGPT. Codex assisted in checking the cited inputs and organizing and revising this manuscript.

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