

WEIL CLASSES ON ABELIAN SIXFOLDS AND EIGHTFOLDS OF WEIL TYPE FOR $\mathbb{Q}(\sqrt{-3})$

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ABSTRACT. Let $K = \mathbb{Q}(\sqrt{-3})$. We prove that the Weil classes of every polarized complex abelian eightfold of Weil type for K with split Hermitian discriminant are algebraic, and deduce the same for every abelian sixfold of Weil type for K , of every discriminant. For sixfolds whose Hodge group is the full special unitary group, the Hodge conjecture follows on all powers. The eightfolds are Prym–Tyurin varieties of Galois covers of the projective line with monodromy group the Shephard–Todd group G_{34} , branched at twenty points with reflections as local monodromy. Algebraic cycles come from Schoen’s construction on cyclic subcovers and reach the Prym–Tyurin factor through Milne’s theorem on Lefschetz classes; a character identity that makes this possible follows from Brauer’s induction theorem. The family dominates the sixteen-dimensional moduli space of split eightfolds; we prove this from exact data at a maximally degenerate point of the Hurwitz space, by the nilpotent orbit theorem. The sixfolds are reached by multiplying with a product of CM elliptic curves of suitable polarization, as in Schoen’s product argument.

1. INTRODUCTION

1.1. Weil classes. Let A be a complex abelian variety of dimension $2n$ with an action $\iota: K \rightarrow \text{End}(A) \otimes \mathbb{Q}$ of an imaginary quadratic field K , such that the two embeddings of K in $\mathbb{C}$ each occur with multiplicity n on the tangent space. Then the K -line

$$W_K(A) = \bigwedge_K^{2n} H^1(A, \mathbb{Q}) \subset H^{2n}(A, \mathbb{Q})$$

is a two-dimensional space of Hodge classes of type (n, n) , the *Weil classes* of A [34]. For a general such A they are not generated by divisor classes, so they are a natural test case for the Hodge conjecture [29, §5]. A polarization compatible with ι determines a K -Hermitian form H on $H_1(A, \mathbb{Q})$ of signature (n, n) , and its discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ is an invariant of the polarized rational Hodge structure. The polarized abelian $2n$ -folds of Weil type for K with a fixed discriminant form one family, parametrized by

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a quotient of the Hermitian symmetric domain of $SU(n, n)$, of dimension n^2 ; we call its members the abelian $2n$ -folds of Weil type for K with that discriminant. The discriminant $(-1)^n$ is called *split*: it is the discriminant of the Hermitian forms that have an n -dimensional totally isotropic subspace.

1.2. What was known. For fourfolds, the Weil classes are algebraic for every imaginary quadratic field and every discriminant: Markman proved the split case [17], and deduced the general case from split sixfolds [18]. For sixfolds, the split case is known: Schoen for $\mathbb{Q}(\sqrt{-3})$ [24], Koike for $\mathbb{Q}(i)$ [15], and Markman for every K [18, Theorem 1.5.1]. Beyond the split class, the only sixfold case we know of is the class -2 for $\mathbb{Q}(\sqrt{-3})$ [27]; Mostaedi describes the non-split sixfold cases as open [21, Introduction]. Voisin's Bourbaki exposé [32] surveys this work.

In dimension eight and above, no full moduli space of abelian varieties of Weil type was previously known to have algebraic Weil classes. Markman's secant-sheaf method is carried out in dimension at most six, and he leaves dimension eight dependent on an open question [19, §§1, 12]. Van Geemen and Verra [30] prove that the Weil classes are algebraic on the general member of a six-dimensional family of eightfolds of quaternion type, which lies inside a Weil family for $\mathbb{Q}(i)$ but does not fill it.

If split eightfolds were known for every K , the product argument below would give every Weil sixfold for every K . That is not known, and this is why the construction with G_{34} is needed.

1.3. Results.

Theorem A. *Let $K = \mathbb{Q}(\sqrt{-3})$. The Weil classes of every polarized abelian eightfold of Weil type for K with split discriminant are algebraic.*

Theorem B. *Let $K = \mathbb{Q}(\sqrt{-3})$. The Weil classes of every polarized abelian sixfold of Weil type for K , of any discriminant, are algebraic.*

Theorem C. *Let A be an abelian sixfold as in Theorem B whose Hodge group is SU_H . Then every Hodge class on every power A^m is algebraic.*

Theorem C holds for every sixfold of Weil type for K outside a countable union of proper closed subvarieties of its moduli space. Section 12 gives a consequence for abelian sixfolds of type III.

1.4. The construction in one page. The sixfolds are reached through eightfolds and a product construction:

$$\begin{array}{ccccc} G_{34}\text{-covers } C \rightarrow \mathbb{P}^1 & \implies & \text{Prym-Tyurin factor} & \implies & \\ \text{branched at 20 points} & & A_\sigma(C) \subset J(C), \dim 8 & & \\ & \implies & \text{every split} & \implies & A \times E \times \bar{E} \\ & & \text{eightfold} & & \text{for every sixfold } A \end{array}$$

The eightfolds. The Shephard-Todd group G_{34} is a complex reflection group of order 39 191 040, acting on K^6 through a representation σ whose

character field is K . Take Galois covers $C \rightarrow \mathbb{P}^1$ with group G_{34} , branched over twenty points, with a reflection as the local monodromy at each. The representation σ occurs in $H^1(C)$ with multiplicity 8, and the multiplicity space is the first cohomology of an abelian eightfold $A_\sigma(C)$, cut out of the Jacobian $J(C)$ by a rank-one idempotent of $\mathbb{Q}[G_{34}]$; the σ -isotypic part of $J(C)$ is isogenous to $A_\sigma(C)$ ⁶. The centre of $\mathbb{Q}[G_{34}]$ acting on σ is K , and it makes $A_\sigma(C)$ an eightfold of Weil type for K (Section 3). As C moves in its 17-dimensional Hurwitz space, $A_\sigma(C)$ moves in the 16-dimensional moduli space of split eightfolds of Weil type for K .

Cycles. For a subgroup $H' \leq G_{34}$ and a character $\chi: H' \rightarrow \mu_m$, the cyclic cover $C/\ker \chi \rightarrow C/H'$ carries algebraic Weil classes on its Prym variety, by Schoen's theorem [23], under a condition on its branch data. These Prym varieties are factors of $J(C)$ cut out by idempotents of $\mathbb{Q}[G_{34}]$. A product of their Weil classes, contracted against a suitable class on a product of factors of $J(C)^R$, gives a nonzero Weil class on $A_\sigma(C)$. The contracting class is algebraic by Milne's theorem on Lefschetz classes [20], provided it is invariant under the centralizer of G_{34} ; that is a condition on characters of a torus, an integral identity in its character group (Section 4). For G_{34} with reflections, the identity follows from Brauer's induction theorem, because every reflection is conjugate to its inverse (Section 5). So the Weil classes of $A_\sigma(C)$ are algebraic for every C .

Dominance and spreading. If the family $A_\sigma(C)$ dominates the moduli space, the set of moduli points with algebraic Weil classes contains an open set; it is a countable union of closed subvarieties, so it is everything (Section 6). Dominance is proved at a maximal degeneration of the twenty branch points. Let $V_{\text{par}} \cong H_1(A_\sigma(C), \mathbb{Q})$ be the eight-dimensional K -space of the family, with its Hermitian form; in Section 7 it is realized as a parabolic cohomology group. The logarithms of the local monodromies of the family there span the full 16-dimensional abelian Lie algebra $\mathfrak{n}_I$ of endomorphisms of V_{par} that map V_{par} into a rational Lagrangian I and kill I , and the nilpotent orbit theorem turns this into full rank of the period map nearby (Section 8). The required data, seventeen nilpotent matrices over K , are computed exactly from the monodromy tuple, which we print in Appendix A. The same data show that the Hermitian form of the family has a rational Lagrangian, so the family consists of split eightfolds.

From eightfolds to sixfolds. If A is a sixfold of Weil type for K of discriminant t , and E is an elliptic curve with complex multiplication by $\mathcal{O}_K$, then $A \times E \times \bar{E}$, with a product polarization of suitable degree on $\bar{E}$, is a split eightfold of Weil type for K . Its Weil classes are algebraic by Theorem A, and contracting with the algebraic Weil classes of $E \times \bar{E}$ gives those of A (Section 10). This is Schoen's product argument [24, Proposition 10], by which he passes from sixfolds to fourfolds, in the dimension-free form of Voisin [32, Lemme 2.9]; our only addition is the explicit choice of the compensating factor.

What is new. The ingredients are classical. Schoen constructed the cycles on cyclic covers [23], and van Geemen and Verra already carried them to a non-abelian Prym factor through a cyclic intermediate quotient [30, Corollary 4.10]; dominance followed by specialization is the method of Schoen and Koike [24, 15]; and Alexeev, Donagi, Farkas, Izadi and Ortega proved dominance of a Prym–Tyurin map onto $\mathcal{A}_6$ at a maximally degenerate point of $\overline{\mathcal{M}}_{0,24}$ [2]. Two things are new. The representation need not be induced from one cyclic subgroup: Weil classes are assembled from several cyclic covers by an integral identity in a character group, which Brauer’s induction theorem supplies whenever the branch classes are real. And dominance of a Weil family is proved at a maximal degeneration, by exact linear algebra over K on the monodromy tuple. The degeneration produces a rational Lagrangian, so this criterion only reaches the split discriminant (Proposition 8.5); that is why the method gives split eightfolds and then needs the product argument.

Inputs. The proof uses Schoen’s theorem on cyclic covers [23], Milne’s theorem on Lefschetz classes [20], Brauer’s induction theorem, the several-variable nilpotent orbit theorem [22, 5], the description of monodromy on parabolic cohomology by Dettweiler and Wewers [7], and a computation over K whose output, a Gram matrix and seventeen 4×4 matrices, is printed in Appendix A together with the basis it refers to. Conditions (H1)–(H4) can be read off the printed matrices, and the matrices can be recomputed from the printed tuple and basis by exact linear algebra in dimension 8 (Subsection 8.1).

1.5. Organization. Section 2 recalls abelian varieties of Weil type and their moduli. Section 3 constructs the Prym–Tyurin factor of a Galois cover. Section 4 produces algebraic Weil classes on it, and Section 5 verifies the needed character identity for data with real local monodromy. Section 6 spreads algebraicity from a dominant family to the whole moduli space. Section 7 identifies the Hermitian form of the family with a parabolic cup product. Section 8 proves the dominance criterion at a maximal degeneration. Section 9 applies everything to G_{34} and proves Theorem A, Section 10 proves Theorem B, and Section 11 proves Theorem C. Section 12 treats abelian sixfolds of type III. Appendix A gives the data of the computation.

2. ABELIAN VARIETIES OF WEIL TYPE

Throughout, K is an imaginary quadratic field and $\vartheta \in K$ is a nonzero element with $\vartheta = -\bar{\vartheta}$.

Definition 2.1. A *polarized abelian $2n$ -fold of Weil type for K* is a triple (A, ι, E) : a complex abelian variety A of dimension $2n$, an embedding $\iota: K \rightarrow \text{End}(A) \otimes \mathbb{Q}$ such that both embeddings of K occur with multiplicity n on $T_0 A$, and a polarization E (an alternating form on $H_1(A, \mathbb{Q})$) with $E(\iota(a)x, y) = E(x, \iota(\bar{a})y)$.

Following van Geemen [29, Lemma 5.2], the form $H(x, y) = E(x, \iota(\vartheta)y) + \vartheta E(x, y)$ is K -Hermitian of signature (n, n) on the $2n$ -dimensional K -space $H_1(A, \mathbb{Q})$. Its discriminant

$$\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$$

does not depend on ϑ , and it is unchanged by a K -linear isogeny that multiplies the polarization by a positive rational number, because such a change multiplies $\det H$ by a $2n$ -th power of a rational number, which is a norm. The sign of $\text{disc } H$ is $(-1)^n$. By Landherr's theorem [29, §5.4], a nondegenerate K -Hermitian space is determined up to isometry by its dimension, signature and discriminant; in particular the Hermitian spaces of signature (n, n) with a totally isotropic subspace of dimension n are exactly those of discriminant $(-1)^n$, which we call *split*.

2.1. Moduli. Fix a K -Hermitian space (V, H) of dimension $2n$, signature (n, n) and discriminant $\mathfrak{d}$, and an $\mathcal{O}_K$ -stable lattice $\Lambda \subset V$ on which the alternating form $\text{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$, scaled by a positive rational, is integral. The space $\mathcal{H}_n$ of n -dimensional complex subspaces of $V_\mathbb{R}$ (for the complex structure given by $\iota(\vartheta)/|\vartheta|$) on which H is positive is the Hermitian symmetric domain of $SU(n, n)$, connected of dimension n^2 . A point $V_+ \in \mathcal{H}_n$ determines a complex structure J' on $V_\mathbb{R}$ commuting with K , hence a Hodge structure of weight one on V of Weil type for K , polarized by $\text{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$; and every such polarized Hodge structure arises in this way [29, 5.10 and the proof of 6.11]. Write SU_H for the special unitary group of (V, H) , an algebraic group over $\mathbb{Q}$; it is the Hodge group of a very general member of the family below [29, Theorem 6.11]. For $\ell \geq 3$ the congruence subgroup $\Gamma(\ell)$ of the unitary group of (Λ, H) is torsion free, and $\mathcal{M} = \Gamma(\ell) \backslash \mathcal{H}_n$ is a smooth connected quasi-projective variety with a universal abelian scheme $f: \mathcal{A} \rightarrow \mathcal{M}$ with K -action [29, §§5.5–5.11].

Lemma 2.2 (Isogeny invariance). *If $\psi: A' \rightarrow A$ is a K -linear isogeny, then $W_K(A)$ is spanned by algebraic classes if and only if $W_K(A')$ is. Every polarized abelian $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}$ is K -isogenous to a fibre of $\mathcal{A} \rightarrow \mathcal{M}$.*

Proof. Pullback and pushforward by ψ are algebraic correspondences, and ψ^* maps $W_K(A)$ isomorphically onto $W_K(A')$, since ψ^* is a K -linear isomorphism on $H^1(-, \mathbb{Q})$. For the second statement, Landherr's theorem gives a K -isometry from $(H_1(A, \mathbb{Q}), H)$ to (V, H) ; the image of $H_1(A, \mathbb{Z})$ is commensurable with Λ , so a multiple of it lies in Λ , and multiplication by that integer is a K -linear isogeny from A to the fibre of $\mathcal{A}$ at the point given by the complex structure of A . $\square$

3. PRYM–TYURIN FACTORS OF GALOIS COVERS

3.1. Hurwitz spaces. Fix a finite group G and conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r$ of G , $r \geq 4$. The *Nielsen class* is the set of tuples $(g_1, \dots, g_r)$ with $g_j \in \mathcal{C}_j$,

$g_1 \cdots g_r = 1$, and $\langle g_1, \dots, g_r \rangle = G$. A connected Galois cover $C \rightarrow \mathbb{P}^1$ with group G , branched over r ordered points $p_1, \dots, p_r$, determines such a tuple up to simultaneous conjugation, once standard generators of $\pi_1(\mathbb{P}^1 - \{p_j\})$ are chosen: g_j is the monodromy along a loop around p_j . The Hurwitz space Hur of such covers is finite and étale over the configuration space of r distinct ordered points; its connected components are smooth of dimension $r - 3$ modulo PGL_2 , and correspond to orbits of the pure braid group on the Nielsen class modulo conjugation [10].

For an irreducible complex representation τ of G write $H^1(C, \mathbb{C}) = \bigoplus_{\tau} \tau \otimes M_{\tau}$ (Chevalley–Weil), and $m_{\tau} = \dim M_{\tau}$, the multiplicity of τ in $H^1(C, \mathbb{C})$. The space M_{τ} carries a Hodge structure of weight one. For $g \in G$ write $c_{\tau}(g) = \text{rk}(\tau(g) - 1)$, and $\text{age } \tau(g) = \sum_k \alpha_k$ when the eigenvalues of $\tau(g)$ are $e^{2\pi i \alpha_k}$ with $0 \leq \alpha_k < 1$.

Proposition 3.1. *If τ is irreducible and nontrivial of dimension d , then*

$$m_{\tau} = -2d + \sum_{j=1}^r c_{\tau}(g_j), \quad \dim \text{Hom}_G(\tau, H^0(C, K_C)) = \sum_{j=1}^r \text{age } \tau(g_j^{-1}) - d.$$

Proof. Let $U = \mathbb{P}^1 - \{p_j\}$ and $\mathbb{V}$ the local system on U attached to τ . Then m_{τ} is the dimension of the parabolic cohomology $H^1(\mathbb{P}^1, j_* \mathbb{V})$, the image of $H_c^1(U, \mathbb{V}) \rightarrow H^1(U, \mathbb{V})$. Since τ is irreducible and nontrivial, $H^0(U, \mathbb{V}) = 0$, and $H^2(U, \mathbb{V}) = 0$ because U is an affine curve; so $\dim H^1(U, \mathbb{V}) = (r - 2)d = \dim H_c^1(U, \mathbb{V})$. The compactly supported dimension is also $(r - 2)d$, since the Euler characteristic of $\mathbb{V}$ with compact supports is $\chi_c(U)d = (2 - r)d$ and $H_c^0 = H_c^2 = 0$, the latter by duality with $H^0(U, \mathbb{V}^{\vee}) = 0$. The long exact sequence of the pair (the boundary circles of small punctured discs Δ_j^* around the p_j) begins

$$0 = H^0(U, \mathbb{V}) \rightarrow \bigoplus_j H^0(\Delta_j^*, \mathbb{V}) \rightarrow H_c^1(U, \mathbb{V}) \rightarrow H^1(U, \mathbb{V}) \rightarrow \dots,$$

so the image of $H_c^1 \rightarrow H^1$ has dimension $(r - 2)d - \sum_j \dim H^0(\Delta_j^*, \mathbb{V})$; and $\dim H^0(\Delta_j^*, \mathbb{V})$ is the dimension $d - c_{\tau}(g_j)$ of the fixed space of $\tau(g_j)$. This gives the first formula. The second is the Chevalley–Weil formula [9, Theorem 2.10]. $\square$

3.2. The Prym–Tyurin factor. Now let σ be an irreducible representation of G whose character field is K and which is realizable over K , on $V_{\sigma} = K^d$. Here the character field is the field generated by the values of the character; since it is the imaginary quadratic field K , the character takes non-real values, so $\sigma \not\cong \bar{\sigma}$. The simple factor of $\mathbb{Q}[G]$ attached to σ is $\mathbb{Q}[G]\varepsilon \cong \text{End}_K(V_{\sigma}) \cong M_d(K)$, with centre K . Averaging gives a G -invariant positive definite K -Hermitian form h_V on V_{σ} ; the Rosati involution $g \mapsto g^{-1}$ acts on $\mathbb{Q}[G]\varepsilon$ as the h_V -adjoint and on its centre K as complex conjugation.

Definition 3.2. Choose $v \in V_{\sigma}$ with $h_V(v, v) \neq 0$, and let $e \in \mathbb{Q}[G]\varepsilon$ be the h_V -orthogonal projection onto the line Kv ; it is a primitive idempotent

fixed by the Rosati involution. For a cover C , the *Prym–Tyurin factor* is

$$A_\sigma(C) = \text{im} (N_0 e: J(C) \rightarrow J(C)),$$

for an integer $N_0 \geq 1$ with $N_0 e \in \mathbb{Z}[G]$, with the restricted polarization E and the action ι of the centre K .

Different choices of v give K -isogenous factors. The σ -isotypic part of $J(C)$ is isogenous to $A_\sigma(C)^d$, one copy for each line of an orthogonal basis of V_σ . Prym–Tyurin varieties of this kind are classical [3, Ch. 12]; Weil classes on them were first exploited for the quaternionic case by van Geemen and Verra [30].

Lemma 3.3. *$(A_\sigma(C), \iota, E)$ is a polarized abelian variety of dimension m_σ , and both embeddings of K occur on its tangent space, with multiplicities the Hodge numbers (p_σ, q_σ) of M_σ . If $m_\sigma = 2n$ and $(p_\sigma, q_\sigma) = (n, n)$, it is a polarized abelian $2n$ -fold of Weil type for K .*

Proof. The σ -isotypic part of $H_1(J(C), \mathbb{Q})$ is $V_\sigma \otimes_K M$ with $M = \text{Hom}_{K[G]}(V_\sigma, H_1(J(C), \mathbb{Q})_\varepsilon)$, and $H_1(A_\sigma, \mathbb{Q}) = Kv \otimes_K M \cong M$. After extending scalars along the two embeddings of K , M becomes M_σ and $M_{\bar{\sigma}}$. The Rosati involution of E restricts on $\iota(K)$ to that of $\mathbb{Q}[G]$ on the centre, which is complex conjugation. $\square$

Remark 3.4 ($A_\sigma(C)$ is not of CM type). In our application $(p_\sigma, q_\sigma) = (4, 4)$ (Lemma 9.1), and the family $A_\sigma(C)$ dominates the moduli space of split eightfolds (Section 9). Since the Hodge group of a very general point of that moduli space is SU_H [29, Theorem 6.11], so is that of $A_\sigma(C)$ for C outside a countable union of proper analytic subsets of the component; in particular $A_\sigma(C)$ is not of CM type and not isogenous to a power of an elliptic curve. The equality $p_\sigma = q_\sigma$ is forced by the branch data: if every g_j is conjugate to g_j^{-1} , then $\text{age } \sigma(g_j) = \text{age } \sigma(g_j^{-1})$, and Proposition 3.1, applied to σ and to $\bar{\sigma}$, gives $p_\sigma = q_\sigma$. This is consistent with Section 5: for real branch classes the line $\det M_\sigma$ is shown to be algebraic, so it must be of Hodge type.

Lemma 3.5. *On a connected component S of Hur, the discriminant of the Hermitian form of $A_\sigma(C)$ does not depend on $C \in S$.*

Proof. The spaces $H_1(A_\sigma(C), \mathbb{Q})$ with ι and E form a local system over S , since e and ι act by flat endomorphisms and E is flat; parallel transport is a K -isometry, and the discriminant is an isometry invariant. $\square$

4. ALGEBRAIC WEIL CLASSES ON THE PRYM–TYURIN FACTOR

Keep the notation of Section 3, and assume $m_\sigma = 2n$ and $(p_\sigma, q_\sigma) = (n, n)$. The idea of this section is the following. Schoen’s theorem gives algebraic Weil classes on Prym varieties of cyclic covers. For a subgroup $H' \leq G$ and a character χ of H' , the cyclic cover $C/\ker \chi \rightarrow C/H'$ is a quotient of C , so its Prym variety is a factor of $J(C)$. We look for an algebraic correspondence between a product of such Prym varieties and $A_\sigma(C)$ that

carries the product of their Weil classes to a nonzero Weil class of $A_\sigma(C)$. Milne's theorem on Lefschetz classes (classes in the subalgebra generated by divisor classes) implies that a cohomology class on a product of factors of $J(C)$ is algebraic as soon as it is invariant under the centralizer of G , since that centralizer contains the relevant Lefschetz group. Invariance under the centralizer reduces, through its maximal abelian quotient, to an identity between characters, which is condition (L) below.

4.1. Cyclic suppliers and Schoen's theorem.

Definition 4.1. A *supplier* is a pair (H', χ) of a subgroup $H' \leq G$ and a character $\chi: H' \rightarrow \mathbb{C}^\times$ of order $m \geq 3$.

For $C \in \text{Hur}$, the curve $C' = C/\ker \chi$ is a connected cyclic cover of C/H' with group $H'/\ker \chi \cong \mathbb{Z}/m$. Its primitive part $P_\chi \subset J(C)$, the part on which $\mathbb{Z}/m$ acts through primitive characters, is cut out by a rational idempotent of $\mathbb{Q}[H']$, and the cyclotomic field $F_\chi = \mathbb{Q}(\mu_m)$ acts on it. For $t \in (\mathbb{Z}/m)^\times$ let $V_t \subset H^1(P_\chi, \mathbb{C})$ be the eigenspace where H' acts by χ^t ; then

$$V_t = \bigoplus_{\tau} \text{Hom}_{H'}(\chi^t, \tau|_{H'}) \otimes M_\tau,$$

all V_t have the same dimension h , and the Weil classes $W_{F_\chi}(P_\chi) = \bigwedge_{F_\chi}^h H^1(P_\chi, \mathbb{Q})$ satisfy $W_{F_\chi}(P_\chi) \otimes \mathbb{C} = \bigoplus_t \det V_t$.

The branch data of $C' \rightarrow C/H'$ are read off from the tuple. The points of C/H' over p_j correspond to the double cosets $H' \backslash G / \langle g_j \rangle$; over the double coset of x the local monodromy is $x g_j^\ell x^{-1} \in H'$, with ℓ the least positive integer for which this lies in H' . Let q be the genus of C/H' , s the number of branch points of $C' \rightarrow C/H'$, and $\alpha \in (\mathbb{Z}/m - \{0\})^s$ their exponents with respect to a generator; then $h = 2q - 2 + s$, which is the common dimension of the eigenspaces V_t above (Chevalley–Weil for $C' \rightarrow C/H'$, [23, Lemma 1.5]). Following Schoen [23, §1], α is *simple* if, after reordering, $\alpha_{2i} = -\alpha_{2i-1}$ for all i . A supplier is *admissible* if h is even and positive and α is simple (automatic when $s = 0$).

Lemma 4.2. *For a supplier (H', χ) the numbers q, s, h , the exponents α up to $(\mathbb{Z}/m)^\times$ and order, and hence admissibility, depend only on G and the classes $\mathcal{C}_j$.*

Proof. Replacing g_j by a conjugate yg_jy^{-1} replaces the double coset of x by that of xy^{-1} , with the same local monodromy; Riemann–Hurwitz for $C \rightarrow C/H'$ gives q . $\square$

Lemma 4.3 (Schoen). *If (H', χ) is admissible, then $W_{F_\chi}(P_\chi)$ is spanned by classes of algebraic cycles, for every $C \in \text{Hur}$.*

Proof. We quote Schoen [23, §2]. Let C' be a smooth projective irreducible curve over $\mathbb{C}$ with a faithful action $\mathbb{Z}/m \hookrightarrow \text{Aut}(C')$, $m > 1$, whose invariant h is even, so that the quotient map has an even number $2r$ of branch points,

with exponents $1 \leq \alpha_j \leq m-1$. *Theorem 2.0*: if the $(\mathbb{Z}/m)^\times$ -orbit of $(\alpha_1, \dots, \alpha_{2r})$ is simple, then the relevant subspace of $H^h(C'^h)$ is generated by fundamental classes of algebraic cycles. *Corollary 3.1*, for $m > 2$: under the same hypotheses, the corresponding subspace of $H^h(B, \mathbb{Q})$ is generated by classes of algebraic cycles of codimension $h/2$, where $B \subseteq \text{Alb}(C')$ is the image of $\prod_{1 \neq Z' \leq \mathbb{Z}/m} \sum_{t \in Z'} (\text{Id} - t)$, the primitive part. Neither statement asks the exponents to be prime to m : Schoen's §1 treats a branch point with $d = \gcd(\alpha_j, m) > 1$ through the local model $y^{m/d} = w^{\alpha_j/d}$. Simplicity is automatic for $r \leq 1$, and the unramified case $r = 0$ is included, with its own proof [23, §2].

Apply this to $C' = C/\ker \chi \rightarrow C/H'$, which is smooth, projective and irreducible as a quotient of C , with $H'/\ker \chi \cong \mathbb{Z}/m$ acting faithfully. The subspace of Corollary 3.1 is W_{F_χ} of the primitive part, which is isogenous to P_χ compatibly with F_χ . Admissibility does not depend on C by Lemma 4.2. When every exponent is $m/2$ and $4 \mid m$, one step of Schoen's proof is replaced by Lemma 4.4 below. $\square$

For the group G_{34} of Section 9 every local monodromy is an involution, so every branch point of every supplier cover has exponent $m/2$, and $4 \mid m$ occurs: three of the 84 suppliers of the explicit identity of Remark 9.2 have $m = 12$ and $s > 0$ (rows 42–44 of the deposited rebuild log). For such covers one step of Schoen's proof needs a replacement, which we now give.

Recall Schoen's construction [23, §2, pp. 11–19], for $C' \rightarrow X$ as in Lemma 4.3 with branch points ordered so that $\alpha_{2i} = -\alpha_{2i-1}$, $1 \leq i \leq r$. Let $W_0 = C'^h/G_0$, where $G_0 \subset (\mathbb{Z}/m)^h \rtimes S_h$ is the kernel of the sum map extended by S_h ; the residual group $\mathbb{Z}/m$ acts on W_0 , and $\beta_0: W_0 \rightarrow S^h X$ is a $\mathbb{Z}/m$ -cover branched along the divisors $S_j = b_j + S^{h-1}X$. Schoen chooses a linear subspace $P_0 \subset S^h X$ of dimension $h/2$ (his Lemma 2.3) and, for $i = 1, \dots, r$, blows up $(S^h X)_{i-1}$ along $K_{i-1} = S_{2i-1, i-1} \cap S_{2i, i-1}$, the intersection of the strict transforms of S_{2i-1} and S_{2i} ; the exceptional divisor E_i is a $\mathbb{P}^1$ -bundle over K_{i-1} . Let W_i be the normalization of the pulled-back cover, $\beta_i: W_i \rightarrow (S^h X)_i$, and $F_i = \beta_i^{-1}(E_i)$. After r steps the strict transform of P_0 avoids the branch locus, its inverse image splits into m components, and these give the cycles of Theorem 2.0 (his Lemmas 2.4–2.6). To see that their classes on W_0 are nonzero, his Lemma 2.8 shows that no primitive character of $\mathbb{Z}/m$ occurs in $H^*(W_i, \mathbb{C})/H^*(W_{i-1}, \mathbb{C})$, and this reduces to the statement that no primitive character occurs in $H^*(F_i, \mathbb{C})$ [23, p. 18]. Schoen proves the latter (Sublemma 2.9, p. 19) by choosing a prime $p \mid m$ such that the $\mathbb{Z}/p$ -quotient of $C' \rightarrow X$ is ramified over b_{2i-1} . When every exponent equals $m/2$ and $4 \mid m$, no such prime exists.

Lemma 4.4 (Schoen's Theorem 2.0 for exponents $m/2$). *In the situation of Schoen's Theorem 2.0, for every $i \leq r$ and every primitive character χ of $\mathbb{Z}/m$, the χ -part of $H^*(F_i, \mathbb{C})$ vanishes. Hence Theorem 2.0 and Corollary 3.1 of [23] hold under their stated hypotheses, without the choice of a*

prime in Sublemma 2.9; in particular they hold when every exponent is $m/2$ and $4 \mid m$.

Proof. Since β_i is finite, $H^*(F_i, \mathbb{C})_\chi = H^*(E_i, (\beta_{i*}\mathbb{C})_\chi|_{E_i})$. The stalk of $\beta_{i*}\mathbb{C}$ at a point is the permutation module on the points above it, and its χ -part vanishes at every point whose stabilizer contains nontrivial inertia, because a primitive character is faithful. By Schoen's Lemma 2.5, E_i is not a branch divisor of β_i : its multiplicity in the branch locus is $\alpha_{2i-1} + \alpha_{2i} \equiv 0 \pmod{m}$. So on E_i the sheaf $(\beta_{i*}\mathbb{C})_\chi$ is $j_!\mathcal{L}$, where $\mathcal{L}$ is a rank-one local system on the complement in E_i of the strict transforms $S_{j,i}$, with monodromy $\chi(\alpha_j) \neq 1$ around $S_{j,i}$. The strict transforms $S_{2i-1,i}$ and $S_{2i,i}$ meet E_i in two disjoint sections of $p: E_i \rightarrow K_{i-1}$, and every other $S_{j,i}$ meets E_i in a union of whole fibres. Such a fibre lies in the closed set $S_{j,i}$, so the restriction of $j_!\mathcal{L}$ to it is the zero sheaf. On any other fibre $\mathbb{P}^1$ it is the extension by zero of a rank-one local system on $\mathbb{C}^\times = \mathbb{P}^1 - \{0, \infty\}$ with monodromy $\chi(\alpha_{2i-1}) \neq 1$, whose compactly supported cohomology vanishes: $H_c^0 = 0$, H_c^2 is dual to the invariants of the dual system, which are 0, and the Euler characteristic is 0. Since p is proper and $j_!\mathcal{L}$ is constructible, proper base change identifies the stalk of $R^b p_*(j_!\mathcal{L})$ at $k \in K_{i-1}$ with $H^b(p^{-1}(k), j_!\mathcal{L}|_{p^{-1}(k)})$ [25, Theorem 11.3.6], which we have just shown to vanish. Hence $Rp_*(j_!\mathcal{L}) = 0$ and $H^*(E_i, j_!\mathcal{L}) = H^*(K_{i-1}, Rp_* j_!\mathcal{L}) = 0$. This uses only $\alpha_{2i-1} \not\equiv 0 \pmod{m}$. The rest of Schoen's proof is unchanged. $\square$

4.2. The character condition. Let U be the centralizer of G in the symplectic group of $(H^1(C, \mathbb{Q}), E)$, an algebraic group over $\mathbb{Q}$. We only need its complex points. Recall that an irreducible complex representation τ of a finite group is *orthogonal* or *quaternionic* if it is self-dual and carries a nonzero invariant symmetric, respectively alternating, bilinear form (Frobenius–Schur indicator $+1$ or -1), and that otherwise $\tau \not\cong \bar{\tau}$ [26, §13.2]. In $H^1(C, \mathbb{C}) = \bigoplus_\tau \tau \otimes M_\tau$ the G -invariant alternating cup-product form pairs $\tau \otimes M_\tau$ with $\bar{\tau} \otimes M_{\bar{\tau}}$ and is zero between other summands. By Schur's lemma its restriction is the tensor product of the invariant form on τ , unique up to scalar, with a nondegenerate form on the multiplicity spaces: a duality between M_τ and $M_{\bar{\tau}}$ if $\tau \not\cong \bar{\tau}$; a symmetric form on M_τ if τ is quaternionic; an alternating one if τ is orthogonal. An automorphism commuting with G acts through $\prod_\tau GL(M_\tau)$, and preserving the form means preserving these forms. Hence $U(\mathbb{C})$ is the product of the groups $GL(M_\tau)$ over the pairs $\{\tau, \bar{\tau}\}$ of non-self-dual irreducible representations with $m_\tau > 0$ (acting on M_τ and contragrediently on $M_{\bar{\tau}}$), of $O(M_\tau)$ over the quaternionic τ with $m_\tau > 0$, and of $Sp(M_\tau)$ over the orthogonal τ with $m_\tau > 0$; representations with $m_\tau = 0$ do not occur in $H^1(C)$ and play no role. Let $T = U(\mathbb{C})/[U(\mathbb{C}), U(\mathbb{C})]$ be its maximal abelian quotient. Its character group is

$$X^*(T) = \mathbb{Z}^{\text{pairs}} \oplus (\mathbb{Z}/2)^{\text{quat}},$$

where “pairs” and “quat” run over the pairs and the quaternionic τ with $m_\tau > 0$: one free coordinate for each pair (the determinant on M_τ , for

a chosen member τ of the pair; for the pair of σ we choose σ), and one coordinate of order two (the determinant of $O(M_\tau)$) for each quaternionic τ . Each line $\det V_t$ above is an eigenline of U ; its character is

$$a_t(\chi) = (\langle \tau|_{H'}, \chi^t \rangle - \langle \bar{\tau}|_{H'}, \chi^t \rangle)_{\text{pairs}} \oplus (\langle \tau|_{H'}, \chi^t \rangle \bmod 2)_{\text{quat}},$$

and we put $a(\chi) = a_1(\chi)$. The line $\det M_\sigma \subset H^{2n}(A_\sigma(C), \mathbb{C})$, one of the two lines of $W_K(A_\sigma) \otimes \mathbb{C}$, has character e_σ : coordinate +1 at the pair of σ and 0 elsewhere. Complex conjugation exchanges χ and χ^{-1} , so $a_{-1}(\chi) = -a(\chi)$ on the free coordinates and $a_{-1}(\chi) = a(\chi)$ on the others.

Definition 4.5. Let $L(G; \mathcal{C}_1, \dots, \mathcal{C}_r) \subseteq X^*(T)$ be the subgroup generated by the $a(\chi)$ of the admissible suppliers. *Condition (L)* is that $e_\sigma \in L$.

By Lemma 4.2, condition (L) depends only on G , σ and the classes $\mathcal{C}_j$.

Proposition 4.6. *Let $C \in \text{Hur}$ and let $X = \prod_k X_k$ be a finite product of abelian subvarieties $X_k = \text{im}(N_k e_k) \subseteq J(C)$, with e_k idempotents of $\mathbb{Q}[G]$. Let $\ell \subset H^p(X, \mathbb{C})$ be an eigenline of U whose character lies in L . Then ℓ lies in the complex span of classes of algebraic cycles on X .*

Proof. Spaces, classes and degrees. Write the character of ℓ as $\xi = \sum_i n_i a(\chi_i)$ with admissible (H'_i, χ_i) and $n_i \neq 0$. Put $t_i = \text{sign } n_i$, $P_i = P_{\chi_i}$, $m_i = \text{ord } \chi_i$, $h_i = h(\chi_i)$, so $\dim P_i = \varphi(m_i) h_i / 2$, and let $V_{i,t}$, $t \in (\mathbb{Z}/m_i)^\times$, be the eigenspaces of $H^1(P_i, \mathbb{C})$, each of dimension h_i . Let

$$B = \prod_i P_i^{|n_i|}, \quad d_B = \dim B = \sum_i |n_i| \varphi(m_i) h_i / 2.$$

By Lemma 4.3 choose a nonzero rational algebraic class $y_i \in W_{F_{\chi_i}}(P_i) \subset H^{h_i}(P_i, \mathbb{Q})$, and put $\mathbf{y} = \boxtimes_i y_i^{\boxtimes |n_i|} \in H^{h_B}(B, \mathbb{Q})$, $h_B = \sum_i |n_i| h_i$, an algebraic class. Choose generators $z_{i,t}$ of the lines $\det V_{i,t}$ and put $Z_i = \prod_{t \neq t_i} z_{i,t} \in H^{(\varphi(m_i)-1)h_i}(P_i, \mathbb{C})$. Let x generate $\ell \subset H^p(X, \mathbb{C})$ and

$$\Gamma = \left(\boxtimes_i Z_i^{\boxtimes |n_i|} \right) \boxtimes x \in H^{2d_B - h_B + p}(B \times X, \mathbb{C}).$$

For a class Γ on $B \times X$, the associated correspondence is $\Gamma_*: H^k(B) \rightarrow H^{k+\deg \Gamma - 2d_B}(X)$, $\Gamma_*(\beta) = \text{pr}_{2*}(\text{pr}_1^* \beta \cup \Gamma)$. If Γ is a complex combination of algebraic classes and β is algebraic, $\Gamma_*(\beta)$ is a complex combination of algebraic classes. Here $\Gamma_*(\mathbf{y}) \in H^p(X, \mathbb{C})$.

Step 1: Γ is U -invariant. The product $\prod_t z_{i,t}$ spans the top cohomology of P_i , on which U acts trivially, since U preserves the polarization; so Z_i has character $-a(\chi_i^{t_i})$, which is $-t_i a(\chi_i)$ on the free coordinates and $a(\chi_i)$ on the others. Hence Γ has character $\xi - \sum_i n_i a(\chi_i) = 0$ on the free coordinates, and $\xi - \sum_i |n_i| a(\chi_i) \equiv 0$ on the coordinates of order two.

Step 2: Γ is algebraic. Let R be the number of factors of $B \times X$, namely $\sum_i |n_i|$ copies of the P_i and the factors X_k . Each factor Y' is $\text{im}(N_0 e')$ for an idempotent $e' \in \mathbb{Q}[G]$ (for P_i , the rational idempotent of $\mathbb{Q}[H'_i]$ cutting out P_{χ_i}), with one integer N_0 for all factors such that every $N_0 e' \in \mathbb{Z}[G]$. Let $\iota_{Y'}: Y' \rightarrow J(C)$ be the inclusion and $\pi_{Y'}: J(C) \rightarrow Y'$ the map $N_0 e'$

with its target restricted to its image; then $\pi_{Y'} \circ \iota_{Y'} = [N_0]$, because e' acts as the identity on its image. Taking products,

$$B \times X \xrightarrow{\iota = \prod \iota_{Y'}} J(C)^R \xrightarrow{\pi = \prod \pi_{Y'}} B \times X, \quad \pi \circ \iota = [N_0].$$

The group U commutes with $\mathbb{Q}[G]$, so it preserves each $H^1(Y') = e' H^1(J(C))$, and ι^* and π^* are U -equivariant, with U acting diagonally on $J(C)^R$. So $\pi^* \Gamma \in H^*(J(C)^R, \mathbb{C})$ is U -invariant. Milne defines $S(A)$, for an abelian variety A and the symplectic form e_D of an ample divisor, as the largest algebraic subgroup of $Sp(e_D)$ whose elements commute with $\text{End}^0(A)$ [20, §1]; it acts diagonally on $H^*(A^r)$, and his Theorem 3.2 states: *for any abelian variety A and integer $r \geq 0$, the algebra $H^*(A^r)^{S(A)}$ is generated by divisor classes*; by his Lemma 3.1 this holds with complex coefficients. For $A = J(C)$ with the canonical polarization, $S(J(C))$ commutes with $\mathbb{Q}[G] \subseteq \text{End}^0(J(C))$, so $S(J(C)) \subseteq U$, and $\pi^* \Gamma$ is $S(J(C))$ -invariant. By Theorem 3.2 with $r = R$, $\pi^* \Gamma$ is a complex combination of products of divisor classes on $J(C)^R$. Since $[N_0]^*$ acts on H^k of an abelian variety as multiplication by N_0^k , we get $\iota^* \pi^* \Gamma = [N_0]^* \Gamma = N_0^{\deg \Gamma} \Gamma$, so Γ is a complex combination of algebraic classes.

Step 3: the contraction is nonzero. Write $y_i = \sum_t c_{i,t} v_{i,t}$ with $v_{i,t} \in \det V_{i,t}$ normalized by $\int_{P_i} v_{i,t_i} \cup Z_i = 1$. For $t \neq t_i$ the product $v_{i,t} \cup Z_i$ contains $\det V_{i,t}$ twice and vanishes. By the Künneth formula and the projection formula,

$$\Gamma_*(\mathbf{y}) = \pm \prod_i c_{i,t_i}^{|n_i|} x,$$

the sign coming from reordering graded factors. Each $c_{i,t}$ is nonzero, because $W_{F_{\chi_i}}(P_i)$ is a one-dimensional F_{χ_i} -vector space and a nonzero vector in it has a nonzero component under every embedding of F_{χ_i} . So x lies in the complex span of algebraic classes. Neither the complements Z_i nor Γ 's factors are claimed to be algebraic individually. $\square$

Corollary 4.7. *If condition (L) holds, then $W_K(A_\sigma(C))$ is spanned by classes of algebraic cycles for every $C \in \text{Hur}$.*

Proof. $W_K(A_\sigma) \otimes \mathbb{C} = \det M_\sigma \oplus \det M_{\bar{\sigma}}$, two eigenlines of U with characters $\pm e_\sigma \in L$. Proposition 4.6 puts both in the complex span of algebraic classes. If $\mathcal{Z}$ is the rational span of algebraic classes and $W \otimes \mathbb{C} \subseteq \mathcal{Z} \otimes \mathbb{C}$ for a rational subspace W , then $W \subseteq \mathcal{Z}$. $\square$

5. REAL BRANCH CLASSES AND BRAUER'S INDUCTION THEOREM

Call the classes $\mathcal{C}_j$ *real* if every g_j is conjugate in G to g_j^{-1} ; involutions are real.

Theorem 5.1. *If the classes $\mathcal{C}_1, \dots, \mathcal{C}_r$ are real, condition (L) holds.*

Proof. *A homomorphism.* Define $\mathbf{a}: R_{\mathbb{C}}(G) \rightarrow X^*(T)$ on the representation ring by

$$\mathbf{a}(V) = (\text{mult}_\tau V - \text{mult}_{\bar{\tau}} V)_{\text{pairs}} \oplus (\text{mult}_\eta V \bmod 2)_{\text{quat}}.$$

It is additive, $\mathfrak{a}(\sigma) = e_\sigma$, and by Frobenius reciprocity $\mathfrak{a}(\text{Ind}_{H'}^G \chi) = a(\chi)$ for every subgroup H' and linear character χ of H' .

Brauer induction. By Brauer's theorem [26, §10.5, Theorem 20], $\sigma = \sum_i n_i \text{Ind}_{B_i}^G \lambda_i$ in $R_{\mathbb{C}}(G)$ with $n_i \in \mathbb{Z}$, $B_i \leq G$ elementary (a product of a cyclic group and a p -group for some prime p) and λ_i linear. Hence $e_\sigma = \sum_i n_i a(\lambda_i)$.

Terms that vanish. If λ_i has order 1 or 2 it is real, so the free coordinates of $a(\lambda_i)$ vanish; for quaternionic η the coordinate of order two is $\langle \eta|_{B_i}, \lambda_i \rangle \bmod 2$, and it vanishes. Indeed, let $\langle \ , \ \rangle_\eta$ be the nondegenerate G -invariant alternating form on η , and decompose $\eta|_{B_i} = \bigoplus_\rho \eta_\rho$ into isotypic parts over the irreducible representations ρ of B_i . An invariant pairing between η_ρ and $\eta_{\rho'}$ is a B_i -map $\eta_{\rho'} \rightarrow \eta_\rho^\vee$, which vanishes unless $\rho' \cong \rho^\vee$; so the form pairs η_ρ only with $\eta_{\rho^\vee}$. Since $\lambda_i^\vee = \lambda_i^{-1} = \lambda_i$, the space η_{λ_i} is orthogonal to all the other isotypic parts, so the form is nondegenerate on it; an alternating nondegenerate form lives on an even-dimensional space, so $\langle \eta|_{B_i}, \lambda_i \rangle = \dim \eta_{\lambda_i}$ is even. If λ_i has order at least 3 and $h(\lambda_i) = 0$, then every τ with $m_\tau > 0$ has $\langle \tau|_{B_i}, \lambda_i \rangle = 0$ and $a(\lambda_i) = 0$. So e_σ is an integer combination of $a(\lambda_i)$ over suppliers with $h > 0$.

These suppliers are admissible. Let (H', χ) be a supplier with $h > 0$, and $w \in G$ with $w g_j w^{-1} = g_j^{-1}$. Since w normalizes $D = \langle g_j \rangle$, the map $H' x D \mapsto H' x w D$ is a bijection of $H' \backslash G / D$, and $(xw) g_j^k (xw)^{-1} = x g_j^{-k} x^{-1}$; so it preserves the integer ℓ above and inverts the local monodromy. Hence the exponents of $C' \rightarrow C/H'$ over p_j form a multiset stable under $\alpha \mapsto -\alpha$. Pair each exponent $\alpha \neq -\alpha$ with an occurrence of $-\alpha$; the unpaired ones equal $m/2$. Their number k is even, because the exponents sum to 0 modulo m and the pairs contribute 0, so $k \cdot m/2 \equiv 0 \pmod{m}$. Hence α is simple, s is even, and $h = 2q - 2 + s$ is even. $\square$

6. DOMINANCE AND SPREADING

Fix a component S of Hur and let $\mathfrak{d}(S)$ be the discriminant of $A_\sigma(C)$, $C \in S$ (Lemma 3.5). Let $\mathcal{M} = \mathcal{M}(K, n, \mathfrak{d}(S))$ be the moduli space of Section 2.1. Locally on S the factors $A_\sigma(C)$ give a holomorphic period map to $\mathcal{H}_n$, well defined up to $\Gamma(\ell)$; we say that S is *dominant* if this map has rank n^2 at some point, hence on a dense open subset.

Lemma 6.1. *Let $\mathcal{W} = \bigwedge_K^{2n} R^1 f_* \mathbb{Q} \subset R^{2n} f_* \mathbb{Q}$ on $\mathcal{M}$. The set $\mathcal{M}^{\text{alg}}$ of points t at which $\mathcal{W}_t$ is spanned by classes of algebraic cycles is a countable union of closed algebraic subsets of $\mathcal{M}$.*

Proof. This is the standard argument with relative Hilbert schemes [33, §3.3.2, pp. 90–94, and the proofs of Theorems 7.19 and 8.26]; we give the details. Fix a relatively ample line bundle on the smooth projective family $f: \mathcal{A} \rightarrow \mathcal{M}$. For each Hilbert polynomial, the relative Hilbert scheme of n -dimensional closed subschemes of the fibres is projective over $\mathcal{M}$; there are countably many Hilbert polynomials, and each relative Hilbert scheme has

finitely many irreducible components [33, pp. 90–91]. A rational algebraic class in $H^{2n}(\mathcal{A}_t, \mathbb{Q})$ is a rational combination of classes of n -dimensional subschemes of $\mathcal{A}_t$, so it suffices to consider finite tuples of such subschemes.

Fix a finite tuple $\mathcal{H}_1, \dots, \mathcal{H}_k$ of components, let T be an irreducible component of $\mathcal{H}_1 \times_{\mathcal{M}} \dots \times_{\mathcal{M}} \mathcal{H}_k$, let $\tilde{T} \rightarrow T$ be a resolution of singularities, and let $g: \tilde{T} \rightarrow \mathcal{M}$ be the composite. Then $\tilde{T}$ is smooth and connected, and g is proper, so $g(\tilde{T})$ is a closed algebraic subset of $\mathcal{M}$. The universal subschemes pull back to subschemes $\mathcal{Z}_l \subset \mathcal{A} \times_{\mathcal{M}} \tilde{T}$, flat over $\tilde{T}$, and the classes $[\mathcal{Z}_{l,u}] \in H^{2n}(\mathcal{A}_{g(u)}, \mathbb{Q})$, $u \in \tilde{T}$, are global flat sections of the local system $g^* R^{2n} f_* \mathbb{Q}$ on $\tilde{T}$: the fibre classes of a flat family of subschemes of a smooth projective family are locally constant, as used in [33, proof of Theorem 3.33, p. 94]. Let $\mathcal{F}_u$ be the span of $[\mathcal{Z}_{1,u}], \dots, [\mathcal{Z}_{k,u}]$. Parallel transport along a path from u to u' is an isomorphism $(g^* R^{2n} f_* \mathbb{Q})_u \rightarrow (g^* R^{2n} f_* \mathbb{Q})_{u'}$ carrying each of these global sections to itself, so it carries $\mathcal{F}_u$ onto $\mathcal{F}_{u'}$ and $g^* \mathcal{W}_u$ onto $g^* \mathcal{W}_{u'}$; in particular $\dim \mathcal{F}_u$ does not jump. Hence the set of u with $g^* \mathcal{W}_u \subseteq \mathcal{F}_u$ is either all of $\tilde{T}$ or empty. Call T *good* in the first case; then $\mathcal{W}_t$ is algebraic for every $t \in g(\tilde{T})$.

Conversely, if $\mathcal{W}_t$ is spanned by algebraic classes, choose subschemes $Z_1, \dots, Z_k$ of $\mathcal{A}_t$ whose classes span a space containing $\mathcal{W}_t$; the point $(Z_1, \dots, Z_k)$ lies on some component T , which is then good, and $t \in g(\tilde{T})$. So $\mathcal{M}^{\text{alg}}$ is the union of the closed sets $g(\tilde{T})$ over the good T , of which there are countably many. $\square$

Theorem 6.2. *Assume condition (L), and let S be a dominant component of Hur. Then the Weil classes of every polarized abelian $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}(S)$ are algebraic.*

Proof. Choose $C_1 \in S$ where the period map has rank n^2 and a small polydisc $D \ni C_1$ over which the covers form an analytic family. Over the simply connected D the spaces $H_1(A_\sigma(C), \mathbb{Q})$ with ι and H form a trivial flat bundle of Hermitian spaces of discriminant $\mathfrak{d}(S)$, so Landherr's theorem gives a K -isometry $\mu: H_1(A_\sigma(C_1), \mathbb{Q}) \rightarrow (V, H)$, which we transport over D . Sending C to the complex structure of $H_1(A_\sigma(C), \mathbb{R})$, transported by μ , gives the period map $\Phi: D \rightarrow \mathcal{H}_n$, a submersion at C_1 . Shrinking D , $\Phi(D)$ contains a ball $\mathcal{B}$, whose image in $\mathcal{M}$ is open, since $\Gamma(\ell)$ acts properly and freely.

The fibres over the ball. Let $C \in D$ with $z = \Phi(C) \in \mathcal{B}$, and let A_z be the fibre of $\mathcal{A}$ over the image of z : $H_1(A_z, \mathbb{Z}) = \Lambda$, with complex structure z and polarization $\text{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$ scaled as in Subsection 2.1. The lattice $\mu(H_1(A_\sigma(C), \mathbb{Z}))$ is commensurable with Λ ; if $N\mu(H_1(A_\sigma(C), \mathbb{Z})) \subseteq \Lambda$ for an integer $N \geq 1$, then $N\mu$ is a K -linear map of lattices compatible with the complex structures, so it is a K -linear isogeny $A_\sigma(C) \rightarrow A_z$. Since μ is an isometry of Hermitian forms, it pulls the polarization of A_z back to a positive rational multiple of that of $A_\sigma(C)$; neither the Weil plane nor the class of the discriminant changes. By Corollary 4.7 and Lemma 2.2, the Weil classes of A_z are algebraic. So $\mathcal{M}^{\text{alg}}$ contains a nonempty open set.

A nonempty open subset of $\mathcal{M}$ is a Baire space, so by Lemma 6.1 one of the closed sets has nonempty interior, and since $\mathcal{M}$ is irreducible it is all of $\mathcal{M}$. Every polarized $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}(S)$ is K -isogenous to a fibre of $\mathcal{A}$ (Lemma 2.2). $\square$

7. THE HERMITIAN FORM AS A PARABOLIC CUP PRODUCT

To compute with the family $A_\sigma(C)$ we need its first cohomology and Hermitian form in terms of the monodromy tuple alone.

Let $\pi: C \rightarrow \mathbb{P}^1$ be a cover in Hur, $U_C = \mathbb{P}^1 - \{p_j\}$, and $\mathcal{R} = \pi_* \mathbb{Q}|_{U_C}$, a local system with fibre $\mathbb{Q}[G]$. Put $L' = \text{Hom}_G(V_\sigma, \mathcal{R} \otimes K)$, a local system of K -vector spaces of rank d whose monodromy is σ composed with that of π . The fibre pairing of $\mathcal{R}$ (multiplication followed by the sum over the fibre), made conjugate-linear in the first variable, restricts on $V_\sigma \otimes L'$ to $h_V \otimes h_{L'}$ for a flat positive definite Hermitian form $h_{L'}$ on L' , unique up to a positive rational factor. Let

$$V_{\text{par}} = H_{\text{par}}^1(U_C, L'), \quad \mathcal{S}(a, b) = \int_{U_C} h_{L'}(a \cup b), \quad h = \vartheta \mathcal{S},$$

the parabolic cohomology (the image of compactly supported cohomology), a K -space of dimension m_σ , with its skew-Hermitian cup product $\mathcal{S}$ and the Hermitian form h .

Proposition 7.1. *There is a K -linear or K -antilinear isomorphism $H_1(A_\sigma(C), \mathbb{Q}) \cong V_{\text{par}}$ under which the Hermitian form H of $A_\sigma(C)$ is a nonzero rational multiple of h , or of $\bar{h}$. In particular H and h have the same discriminant, and one is split if and only if the other is.*

Proof. Step 1: the skew-Hermitian lift. On $H_1(A, \mathbb{Q})$ the form $T = H/2\vartheta$ is the unique K -sesquilinear skew-Hermitian form with $\text{Tr}_{K/\mathbb{Q}} T = E$.

Step 2: the multiplicity space. Let $N' = H_1(J(C), \mathbb{Q})_\varepsilon$ and $M = \text{Hom}_{K[G]}(V_\sigma, N')$. Evaluation $V_\sigma \otimes_K M \rightarrow N'$ is a G -equivariant isomorphism, because $\text{End}_{K[G]}(V_\sigma) = K$. A G -invariant sesquilinear form on N' pulls back to $h_V \otimes \psi$ for a unique sesquilinear form ψ on M , since the G -invariant sesquilinear forms on V_σ are the multiples of h_V . Applied to T , which is G -invariant by the uniqueness in Step 1, this gives $T = h_V \otimes \psi$, and $T|_{A_\sigma} = h_V(v, v) \psi$ under $m \mapsto v \otimes m$, because e is Rosati-fixed.

Step 3: the cup product. Since π is finite and C is normal, $H^1(C, K) = H_{\text{par}}^1(U_C, \mathcal{R} \otimes K)$, with the cup product given by the fibre pairing. The functor $\text{Hom}_{K[G]}(V_\sigma, -)$ is exact and commutes with these operations, so it identifies the σ -part of the skew-Hermitian lift of the cup product with $h_V \otimes \mathcal{S}$.

Step 4: comparison. Poincaré duality on C and the identification of H_1 with H^1 by E are G -equivariant and carry T to $\pm$ the lift of Step 3. They carry the σ -component either to the σ -component or to the $\bar{\sigma}$ -component, depending on sign conventions that we do not need to fix; accordingly the induced isomorphism $M \cong V_{\text{par}}$ is K -linear or K -antilinear, and $\psi = c\mathcal{S}$ or

$\psi = c\bar{S}$ with $c \in K^\times$. Both forms are skew-Hermitian, so $c = \bar{c}$ is rational. In either case the discriminants agree: scaling a Hermitian form of even dimension by a rational number multiplies its determinant by a norm, and $\det \bar{h} = \det h$. $\square$

The form $\mathcal{S}$ is computable from the tuple. Let W be a local system of K -vector spaces on $\mathbb{P}^1$ minus r points with a flat Hermitian form h_W and unitary monodromy $T_1, \dots, T_r$ along standard loops with $\ell_1 \cdots \ell_r = 1$, generating a finite group. Put $Q_i = T_1 \cdots T_{i-1}$. A parabolic cocycle is given by vectors $a_1, \dots, a_r$ with $\sum_i Q_i(1 - T_i)a_i = 0$; its value on ℓ_i is $c_i = (1 - T_i)a_i$. Put $F_i = \sum_{j < i} Q_j c_j$.

Lemma 7.2. *For parabolic cocycles a, b , the cup product of their classes with the fibre pairing h_W , on the fundamental class oriented so that the loops run clockwise, is*

$$\mathcal{S}(a, b) = \sum_{i=1}^r h_W(Q_i^{-1}F_i^a + a_i, c_i^b).$$

It does not depend on the choice of the vectors a_i , which are determined by the cocycle only modulo $\ker(1 - T_i)$; it vanishes on coboundaries and is skew-Hermitian.

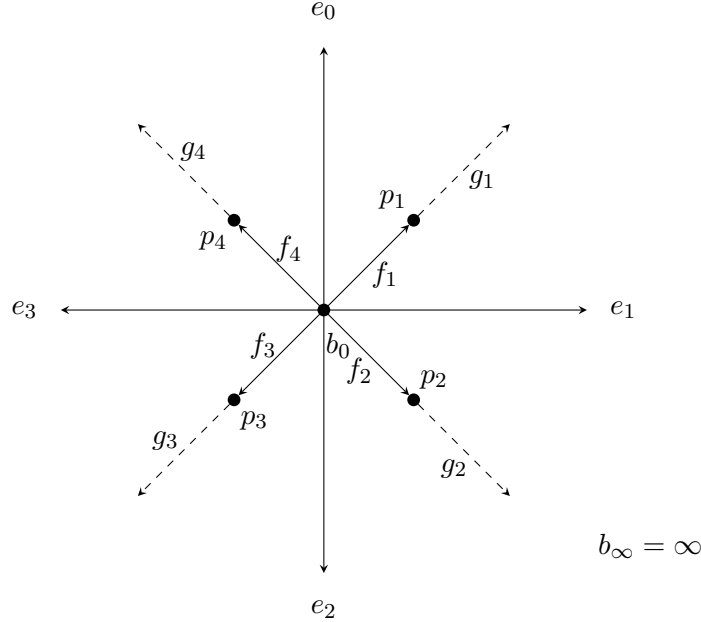


FIGURE 1. The Δ -complex of Lemma 7.2 for $r = 4$, drawn in the plane $\mathbb{P}^1 - \{\infty\}$. The edges e_i and g_i end at $b_\infty = \infty$. The sector between e_{i-1} and e_i contains p_i and is cut by the path $f_i \cup g_i$ into the two triangles at p_i . The loop ℓ_i , out along e_{i-1} and back along e_i , runs clockwise around p_i .

Proof. We use the following Δ -complex structure on $\mathbb{P}^1$ (Figure 1). Its vertices are a base point b_0 , a second point b_∞ , and the punctures $p_1, \dots, p_r$. Its edges are $e_0, \dots, e_{r-1}$ from b_0 to b_∞ , cutting the sphere into r sectors, with the sector between e_{i-1} and e_i containing p_i (indices of e modulo r , so $e_r = e_0$); and, for each i , an edge f_i from b_0 to p_i and an edge g_i from p_i to b_∞ inside that sector. Its faces are the $2r$ triangles with vertices (b_0, p_i, b_∞) in this order: for each i one triangle with edges f_i, g_i, e_{i-1} and one with edges f_i, g_i, e_i . (The Euler characteristic is $(r+2) - 3r + 2r = 2$.) The loop ℓ_i goes out along e_{i-1} and back along e_i , and we orient $\mathbb{P}^1$ so that it runs clockwise around p_i ; then $\ell_1 \cdots \ell_r$ is out along e_0 and back along $e_r = e_0$, which is trivial.

The monodromy group is finite, so there is a finite Galois cover branched over the punctures on which W becomes trivial; lift the complex to it. There the cup product of 1-cochains followed by h_W , evaluated on the fundamental class, is the simplicial cup product $(f \cup f')[v_0, v_1, v_2] = h_W(f[v_0, v_1], f'[v_1, v_2])$ [12, §3.2]. Equivariant cochains contribute the same term on every lift of a triangle, so dividing by the degree gives a sum over the triangles of $\mathbb{P}^1$. Choose the lifts of the two triangles at p_i so that they share their edge g_i ; their edges f_i are then two lifts that differ by the local monodromy at p_i , and the two triangles enter the fundamental class with opposite signs. The terms $h_W(f[b_0, p_i], f'[p_i, b_\infty])$ of the two triangles combine, by the cocycle condition on each triangle, into

$$\mathcal{S}_{\text{simp}}(f, f') = \sum_i h_W(v_{e_i}^f - v_{e_{i-1}}^f, v_{g_i}^{f'}),$$

with v_ϵ^f the value of f on the chosen lift of ϵ . The cocycle conditions on the two triangles at the i -th puncture give $a_i = Q_i^{-1}v_{f_i}$ and $v_{e_{i-1}} = v_{e_0} - F_i$. Substituting and telescoping gives $\mathcal{S}_{\text{simp}}(f, f') = -\sum_i [h_W(c_i, a'_i) + h_W(Q_i c_i, F'_i)]$, the remaining term vanishing by the cocycle relation. By unitarity this equals $-\overline{\mathcal{S}(b, a)}$, and the identity $h_W(c_i, c'_i) = h_W(a_i, c'_i) + h_W(c_i, a'_i)$ together with telescoping $h_W(F_{i+1}, F'_{i+1}) - h_W(F_i, F'_i)$ gives $\mathcal{S}(a, b) + \overline{\mathcal{S}(b, a)} = 0$. A coboundary pairs to zero by the same telescoping. Finally, replacing a_i by $a_i + k$ with $T_i k = k$ changes neither c_i nor any F_j , and changes $\mathcal{S}(a, b)$ by $h_W(k, c'_i) = h_W(k, a'_i) - h_W(T_i^{-1}k, a'_i) = 0$, by unitarity; the second variable enters only through the c'_i . $\square$

8. DOMINANCE AT A MAXIMAL DEGENERATION

We give a criterion for dominance that only needs the monodromy tuple. The idea is to let the branch points collide in pairs until the cover degenerates as far as possible, to a point of the boundary of the Hurwitz space lying over a maximally degenerate stable rational curve. The local monodromies of the family around the boundary divisors there are unipotent after a base change, and their logarithms can be computed exactly from the tuple. If they span

the full unipotent radical of the stabilizer of a rational Lagrangian, the nilpotent orbit theorem shows that the period map has full rank near the boundary. The same method was used by Alexeev, Donagi, Farkas, Izadi and Ortega for Prym–Tyurin maps to $\mathcal{A}_6$ [2].

8.1. Twists and their logarithms. Fix a tuple $\mathbf{g}$ in the component S and V_{par} , $\mathcal{S}$, h as in Section 7. Let $\gamma_1, \dots, \gamma_{r-3}$ be disjoint simple closed curves cutting the r -punctured sphere into pairs of pants. For each j :

- (i) The full Dehn twist τ_j along γ_j is a braid. If γ_j encloses the consecutive points $a, \dots, b$ and $c = g_a \cdots g_b$, it acts on tuples by $g_i \mapsto cg_i c^{-1}$ for $a \leq i \leq b$, and equals $(\sigma_a \cdots \sigma_{b-1})^{b-a+1}$ in the Artin generators.
- (ii) The *return power* k_j is the least $k \geq 1$ with $\tau_j^k(\mathbf{g}) = \mathbf{g}$. The braid $\tau_j^{k_j}$ acts on V_{par} by a unitary operator T_j , computed by pulling back cocycles, as follows. Let $\rho: \pi_1(U_C, b_0) \rightarrow GL(V_\sigma)$ be σ composed with the monodromy of the cover, so $\rho(\ell_i) = \sigma(g_i)$. A cocycle is a map u with $u(xy) = u(x) + \rho(x)u(y)$; it is determined by $u_i = u(\ell_i)$, and it is parabolic when $u_i \in \text{im}(1 - \sigma(g_i))$ (so $u_i = c_i$ in the notation of Lemma 7.2). Put $\gamma_c = \ell_a \cdots \ell_b$, so $\rho(\gamma_c) = \sigma(c)$, and

$$u(\gamma_c) = \sum_{i=a}^b \sigma(g_a \cdots g_{i-1}) u_i.$$

The twist maps ℓ_i to $\gamma_c \ell_i \gamma_c^{-1}$ for $a \leq i \leq b$ and fixes the other ℓ_i , so the pulled-back cocycle $u' = u \circ \tau_j$ has

$$u'_i = u(\gamma_c) + \sigma(c) u_i - \sigma(cg_i c^{-1}) u(\gamma_c) \quad (a \leq i \leq b), \quad u'_i = u_i \text{ otherwise,}$$

by the cocycle identity and $u(\gamma^{-1}) = -\rho(\gamma)^{-1}u(\gamma)$; it is a cocycle for the tuple $\tau_j(\mathbf{g})$. The operator T_j is the k_j -th iterate of this map, which preserves cocycles and coboundaries for $\mathbf{g}$.

- (iii) Let m_j be the order of the semisimple part of T_j ; then $T_j^{m_j}$ is unipotent, and we put $N_j = \log T_j^{m_j}$.

A twist with trivial boundary product c fixes the tuple ($k_j = 1$) but still acts on V_{par} , in general nontrivially: in the example of Section 9 three such twists have logarithms of K -rank 4.

Let B be a connected complex manifold, $\phi: B \rightarrow \text{Hur}$ a holomorphic map, and $\mathcal{C} \rightarrow B \times \mathbb{P}^1$ the pulled-back family of G -covers, with $a: \mathcal{C} \rightarrow B$ and $\bar{\pi}: B \times \mathbb{P}^1 \rightarrow B$ the projections. Let $\mathcal{U} \subset B \times \mathbb{P}^1$ be the complement of the branch divisor, $j: \mathcal{U} \rightarrow B \times \mathbb{P}^1$ its inclusion, and L' the local system on $\mathcal{U}$ defined fibrewise as in Section 7. Fix $b \in B$ whose cover has monodromy tuple $\mathbf{g}$ with respect to a base point $b_0 \in \mathbb{P}^1$ and the loops of Lemma 7.2, and assume that b_0 is not a branch point over any point of B . Fix a point y_0 of the cover over (b, b_0) . For a loop β in B at b , transporting y_0 along $\beta \times \{b_0\}$ in the G -torsor over $\mathcal{U}$ ends at $h_\beta y_0$ for a unique $h_\beta \in G$; and β acts on the loops of $\mathbb{P}^1 - \{p_j\}$ by a braid, also written β .

Lemma 8.1. *The local system $\text{Hom}_{K[G]}(V_\sigma, R^1 a_* K)$ on B is $R^1 \bar{\pi}_*(j_* L')$, and its monodromy along β is the pullback of parabolic cocycles by the braid*

β , as in (ii), composed with $\sigma(h_\beta)$; here $\beta(\mathbf{g}) = h_\beta \mathbf{g} h_\beta^{-1}$. If $\beta(\mathbf{g}) = \mathbf{g}$, then h_β is central and $\sigma(h_\beta)$ is a root of unity in K ; on the family of covers with a marked point over b_0 , $h_\beta = 1$.

Proof. Near a branch point a tame cover is $z \mapsto z^e$, so $p_*K = j_*p_*^\circ K$, and the finite map p has no higher direct images; the Leray spectral sequence gives $R^1a_*K = R^1\pi_*(p_*K)$, and taking the σ -multiplicity is exact. The description of the monodromy is [7, Lemma 2.2 and Theorem 2.5], [8, Theorem 1.3]; on a family of G -covers the coefficient term comes from the G -torsor, so it is $\sigma(h_\beta)$. If $\beta(\mathbf{g}) = \mathbf{g}$, then h_β centralizes a generating tuple, so it is central, and $\sigma(h_\beta) \in \text{End}_{K[G]}(V_\sigma) = K$ by Schur's lemma. $\square$

Lemma 8.2. *Pinching the curves γ_j gives a point of $\overline{\mathcal{M}}_{0,r}$ with $r-3$ nodes and smoothing parameters $q_1, \dots, q_{r-3}$. After the base change $w_j^{|Z(G)|k_jm_j} = q_j$, the family A_σ has unipotent local monodromy around $w_j = 0$, with logarithm $\varepsilon_j N_j$ for positive integers c_j and a sign $\varepsilon \in \{\pm 1\}$ that does not depend on j .*

Proof. The point lies on the normal-crossing boundary of $\overline{\mathcal{M}}_{0,r}$, with independent smoothing parameters [14, Theorem 2.7]. Over it the G -cover degenerates to an admissible cover: at a node the two inertia elements are inverse of the same order e , glued by the model $u = x^e$, $v = y^e$, $xy = t$, $uv = t^e$ [11, §4], after sufficiently divisible powers of the q_j . The loop of q_j around 0 is the full twist τ_j or its inverse. The local Hurwitz cover over the punctured polydisc is finite [10, §1.2], and the stabilizer of a sheet contains $\bigoplus_j k_j \mathbb{Z}$; so the Kummer cover in the statement dominates it, étale over the interior. By Lemma 8.1 the monodromy around $w_j = 0$ is a power of $\zeta_j T_j$ with $\zeta_j^{|Z(G)|} = 1$, raised to a multiple of $|Z(G)|m_j$, hence unipotent with logarithm a positive integer multiple of $\varepsilon_j N_j$, where $\varepsilon_j = \pm 1$ according as the loop of q_j is τ_j or τ_j^{-1} .

It remains to see that ε_j does not depend on j . In plumbing coordinates near the boundary point, the points enclosed by γ_j lie, in a coordinate z centred inside γ_j , at $z = q_j w$ with w in a fixed compact set (for nested curves, at products of the q_k of the curves containing them), and the points outside γ_j do not depend on q_j . As q_j runs once counterclockwise around 0, with the other parameters fixed, the cluster inside γ_j rotates rigidly once counterclockwise about its centre, for the complex orientation of $\mathbb{P}^1$, and nothing outside γ_j moves. This motion is the same kind of full twist for every j , so the loop of q_j is τ_j^ε for one sign ε independent of j , fixed by the orientation of Lemma 7.2 and the convention in (i). $\square$

8.2. The criterion. Consider the following conditions on the logarithms N_j of Lemma 8.2:

- (H1) $N_j N_k = 0$ for all j, k ;
- (H2) $I = \sum_j \text{im } N_j$ is a K -subspace of dimension n , totally isotropic for h , with $N_j(I) = 0$ for all j ;

- (H3) the Hermitian forms $\beta_j(x, y) = \mathcal{S}(x, N_j y)$, which descend to V_{par}/I , are semidefinite of one common sign, and $\sum_j \beta_j$ is definite on V_{par}/I ;
 (H4) the N_j span a $\mathbb{Q}$ -vector space of dimension n^2 .

By the several-variable nilpotent orbit theorem [22, (4.12)], [5, (1.15)], after the base change of Lemma 8.2 the lifted period map near the boundary point has the form $\tilde{\Phi}(z) = \exp(\sum_j z_j M_j) \cdot \psi(u)$ with $M_j = \varepsilon c_j N_j$, $u_j = e^{2\pi i z_j}$, and ψ holomorphic on the polydisc with values in the compact dual. Put $F_0 = \psi(0)$, the limit Hodge filtration. Fix an embedding $K \subset \mathbb{C}$ and write $V_{\text{par}, \mathbb{C}}$, $I_{\mathbb{C}}$ for the scalar extensions along it.

Lemma 8.3 (The limit is Hodge–Tate). *Assume (H1)–(H3). For every $M = \sum_j a_j M_j$ with all $a_j > 0$, $\text{im } M = \ker M = I$, the weight filtration of M is $W_0 = W_1 = I \subset W_2 = V_{\text{par}}$, the limit mixed Hodge structure is of Hodge–Tate type, and $V_{\text{par}, \mathbb{C}} = I_{\mathbb{C}} \oplus F_0^1$ with $\dim_{\mathbb{C}} F_0^1 = n$.*

Proof. By (H1) $M^2 = 0$, and by (H2) $\text{im } M \subseteq I$ and $M(I) = 0$. By (H3), for $x \notin I$ we have $\mathcal{S}(x, \varepsilon M x) = \sum_j a_j c_j \beta_j(x, x) \neq 0$, so $M: V_{\text{par}}/I \rightarrow I$ is injective, hence bijective since both sides have K -dimension n . Thus $\text{im } M = \ker M = I$, and the monodromy weight filtration of M , centred at 1, is $W_0 = W_1 = I$, $W_2 = V_{\text{par}}$. By [5, (4.5)], (W, F_0) is a mixed Hodge structure polarized by M . Since the variation has weight one, $\text{Gr}_0^W = I$ has type $(0, 0)$ and $\text{Gr}_2^W = V_{\text{par}}/I$ type $(1, 1)$. A Hodge structure of type $(0, 0)$ has $F^1 = 0$, so $F_0^1 \cap I_{\mathbb{C}} = 0$. In the chosen eigenspace the weight-one variation has Hodge numbers (n, n) , so $\dim F_0^1 = n = \dim I_{\mathbb{C}}$ and $V_{\text{par}, \mathbb{C}} = I_{\mathbb{C}} \oplus F_0^1$. $\square$

Lemma 8.4 (The logarithms span the tangent space). *Assume (H1)–(H4). Then $v \mapsto \sum_j v_j M_j \cdot F_0$ maps $\mathbb{C}^{\{j\}}$ onto the tangent space $\text{Hom}(F_0^1, V_{\text{par}, \mathbb{C}}/F_0^1)$ of $\mathcal{H}_n$ at F_0 , of dimension n^2 .*

Proof. The rational Lie algebra $\mathfrak{n}_I = \{X \in \mathfrak{u}(h) : X(V_{\text{par}}) \subseteq I, X(I) = 0\}$, a $\mathbb{Q}$ -vector space ($\mathfrak{u}(h)$ is the Lie algebra of the unitary group of h , an algebraic group over $\mathbb{Q}$), has dimension n^2 over $\mathbb{Q}$: via $X \mapsto \mathcal{S}(\cdot, X \cdot)$ it is the $\mathbb{Q}$ -space of K -Hermitian forms on the n -dimensional K -space V_{par}/I , which has $\mathbb{Q}$ -dimension $n + 2\binom{n}{2} = n^2$ (n rational diagonal entries and $\binom{n}{2}$ entries in K above the diagonal). Its complexification maps isomorphically onto $\text{Hom}(V_{\text{par}, \mathbb{C}}/I_{\mathbb{C}}, I_{\mathbb{C}})$. By (H1)–(H2) every N_j lies in $\mathfrak{n}_I$, and by (H4) they span it. By Lemma 8.3, $V_{\text{par}, \mathbb{C}}/I_{\mathbb{C}} \cong F_0^1$ and $I_{\mathbb{C}} \cong V_{\text{par}, \mathbb{C}}/F_0^1$, so an element X of $\mathfrak{n}_I \otimes \mathbb{C}$ is determined by $X|_{F_0^1}$, and $X \cdot F_0$, the class of $X|_{F_0^1}$ in $\text{Hom}(F_0^1, V_{\text{par}, \mathbb{C}}/F_0^1)$, vanishes only if $X(F_0^1) \subseteq F_0^1 \cap I_{\mathbb{C}} = 0$. So $X \mapsto X \cdot F_0$ is injective on the n^2 -dimensional space $\mathfrak{n}_I \otimes \mathbb{C}$, hence onto the n^2 -dimensional tangent space. $\square$

Proposition 8.5. *Assume (H1)–(H4). Then S is dominant, and the discriminant $\mathfrak{d}(S)$ is split.*

Proof. *The period map near the boundary.* After translating by $\exp(-\sum_j z_j M_j)$, the j -th column of the derivative of $\tilde{\Phi}$ is $M_j \cdot \psi(u) +$

$2\pi i u_j \partial\psi/\partial u_j$, which tends to $M_j \cdot F_0$ as $u \rightarrow 0$. By Lemma 8.4 these limits span the n^2 -dimensional tangent space, and so we may choose n^2 of the $r - 3$ indices j for which the limits $M_j \cdot F_0$ form a basis; the corresponding $n^2 \times n^2$ minor of the limit is nonzero and stays nonzero for small u , so the period map has rank n^2 at interior points near the boundary, and S is dominant.

The discriminant. By (H2), (V_{par}, h) has a totally isotropic subspace of dimension n , so h is split, and so is the Hermitian form of $A_\sigma(C)$ by Proposition 7.1. $\square$

Remark 8.6. By (H2) and Proposition 8.5, the criterion applies only to split discriminants; for context, Brosnan [4] shows more generally that a family of abelian $2n$ -folds of Weil type with a maximal degeneration, one whose monodromy logarithm has rank $2n$ over $\mathbb{Q}$, has split Hermitian form. In the example of Section 9 the parabolic cohomology of every pair of pants vanishes, as it must: by the Clemens–Schmid sequence [6] the weight-one part of the limit is the sum, over the components of the degenerate curve, of their parabolic cohomology with coefficients in L' , and the limit is of Hodge–Tate type (Lemma 8.3).

9. THE GROUP G_{34} AND THE PROOF OF THEOREM A

Let $K = \mathbb{Q}(\sqrt{-3})$ and ω a primitive cube root of unity. The Shephard–Todd group $G_{34} \subset GL_6(K)$ [28] has order 39 191 040 and centre μ_6 . Let σ be its reflection representation on K^6 , of dimension $d = 6$ and character field K ; it preserves the standard Hermitian form $\langle x, y \rangle = \sum_k \bar{x}_k y_k$. Its reflections have order two and form one conjugacy class.

Let $\mathbf{g} = (s_1, s_1, s_2, s_2, \dots, s_{10}, s_{10})$ be the tuple of twenty reflections listed in Appendix A, with s_i the reflection in the root v_i . Its product is 1 because every s_i is an involution, and the s_i generate G_{34} (Appendix A). So $\mathbf{g}$ lies in the Nielsen class of twenty reflections; let S be its component of the Hurwitz space, of dimension 17.

Why this tuple and these curves. With r reflections the count of Lemma 9.1 gives $m_\sigma = r - 12 = 2n$, so eightfolds ($n = 4$) need exactly $r = 20$, and then the Hurwitz space has dimension $r - 3 = 17 \geq n^2 = 16$, which leaves room for dominance. One step further fails: $r = 22$ gives $n = 5$ with $19 < 25$ moduli. We need eightfolds rather than sixfolds because the boundary criterion only reaches split discriminants, and Section 10 turns split eightfolds into all sixfolds. Doubling each reflection makes the product of the tuple trivial with no computation; that the ten distinct reflections still generate G_{34} is checked in Appendix A. Doubling also makes many boundary curves easy: a curve enclosing a pair s_i, s_i , or a block of complete pairs, has trivial boundary product, so its twist fixes the tuple ($k_j = 1$) and its logarithm comes from a single pullback. The seventeen curves of Appendix A are nested or disjoint, so they cut the twenty-punctured sphere into pairs of pants, a maximal degeneration ($r - 3 = 17$ curves).

They were found by a search for a decomposition satisfying the conditions of Proposition 8.5; other components of the Nielsen class and other pants decompositions were not examined.

Lemma 9.1. *The representation σ occurs in $H^1(C)$ with multiplicity $m_\sigma = 8$, with Hodge numbers $(4, 4)$. So $A_\sigma(C)$ is an abelian eightfold of Weil type for K .*

Proof. A reflection has $c_\sigma = 1$, and its eigenvalues are -1 once and 1 five times, so $\text{age}(\sigma(g_j^{\pm 1})) = \frac{1}{2}$. Proposition 3.1 gives $m_\sigma = -12 + 20 = 8$ and $20 \cdot \frac{1}{2} - 6 = 4$ for each Hodge number; conclude with Lemma 3.3. $\square$

Proof of Theorem A. The reflections of G_{34} are involutions, so the branch classes are real, and condition (L) holds by Theorem 5.1. Take the 17 curves of Appendix A, each enclosing consecutive points of the twenty. For them the operators N_j of Section 8, computed exactly over K from $\mathbf{g}$ by Lemma 7.2 and the pullback formula of Section 8, satisfy (H1)–(H4): $N_j N_k = 0$; the images span a four-dimensional K -subspace I , totally isotropic for h and killed by every N_j ; every β_j is negative semidefinite and $\sum_j \beta_j$ is negative definite on V_{par}/I ; and the N_j span a $\mathbb{Q}$ -space of dimension $16 = n^2$. Appendix A lists the data. By Proposition 8.5, S is dominant and $\mathfrak{d}(S)$ is split. Theorem 6.2 gives the Weil classes of every polarized abelian eightfold of Weil type for K with split discriminant. $\square$

Remark 9.2. Theorem 5.1 does not produce the coefficients, but an explicit identity is also available. Here the 120 non-real irreducible representations τ of G_{34} with $m_\tau > 0$ form 60 pairs, and no quaternionic τ has $m_\tau > 0$, so $X^*(T) \cong \mathbb{Z}^{60}$; and $e_\sigma = \sum_i n_i a(\chi_i)$ with 84 admissible suppliers, on cyclic subgroups and on subgroups $\langle x \rangle \times \text{Syl}_p(C_G(x))$, $p = 2, 3$, with $\sum_i |n_i| = 231$. The supplementary files contain the identity and a script that rebuilds every row from its subgroup and character, with its q , s , h and admissibility. Three of the suppliers have $m = 12$ and $s > 0$, so the identity uses Lemma 4.4. Cyclic subgroups alone generate a sublattice of full rank that does not contain e_σ : the smallest k with ke_σ in it is $31\,104 = 2^7 \cdot 3^5$, and explicit linear functionals modulo 2 (on six coordinates) and modulo 3 (on four) vanish on the cyclic sublattice and take the value 1 on e_σ .

Example 9.3 (One supplier of the identity). The term of the identity in Remark 9.2 with the smallest h is the following. Let $H' = \langle -I_6 \rangle \times P$, where P is a Sylow 3-subgroup of G_{34} , so $|H'| = 2 \cdot 3^7 = 4\,374$; H' is elementary in Brauer's sense. Let χ be a linear character of H' of order $m = 3$, trivial on $-I_6$ (the character of row 54 of the deposited rebuild log). The genus of C follows from Riemann–Hurwitz for $C \rightarrow \mathbb{P}^1$: each of the twenty branch points has $|G_{34}|/2$ preimages, so $2g(C) - 2 = |G_{34}|(-2 + 20 \cdot \frac{1}{2}) = 8|G_{34}|$. A point of C fixed by a nontrivial element of G_{34} lies over a branch point, and its stabilizer is generated by a reflection. The only involution in H' is $-I_6$, since $|P|$ is odd, and $-I_6$ is not a reflection; so H' acts freely on C ,

and both $C \rightarrow C/H'$ and $C/\ker \chi \rightarrow C/H'$ are unramified: $s = 0$. Hence $2g(C) - 2 = |H'|(2q - 2)$, that is $8 \cdot 39\,191\,040 = 4\,374(2q - 2)$, so $q = 35\,841$ and $h = 2q - 2 = 71\,680$, even and positive; with $s = 0$ the supplier is admissible (Lemma 4.3). It occurs in the identity with coefficient 3, and it is itself a Brauer term, since H' is elementary and χ linear.

10. FROM EIGHTFOLDS TO SIXFOLDS: PROOF OF THEOREM B

The argument is Schoen's [24, Proposition 10], in the form of Voisin [32, Lemme 2.9]: if a product $A \times Y$ has algebraic Weil classes and Y has algebraic Weil classes, so does A . We need only a factor Y that changes the discriminant to the split one. It works for every imaginary quadratic K . Let E be an elliptic curve with complex multiplication by $\mathcal{O}_K$, and $\bar{E}$ the same curve with the conjugate action of K .

Lemma 10.1. *For an integer $m \geq 1$, the surface $E \times \bar{E}$ with the product polarization of degrees $(1, m)$ is of Weil type for K , of signature $(1, 1)$ and discriminant $[-m]$, and its Weil classes are divisor classes.*

Proof. The two factors carry the two embeddings of K on their tangent spaces: $\iota(\vartheta)$ acts on $T_0 E$ by ϑ and on $T_0 \bar{E}$ by $\bar{\vartheta} = -\vartheta$. A polarization E_0 of a curve satisfies $E_0(x, Jx) > 0$ for its complex structure J , so the values $H(x, x) = E_0(x, \iota(\vartheta)x)$ have opposite signs on the two factors. Identifying $H_1(\bar{E}, \mathbb{Q})$ with $H_1(E, \mathbb{Q})$, the polarization of degree m is m times that of degree 1, so the two rank-one Hermitian forms are (c) and $(-cm)$ with the same $c \in \mathbb{Q}^\times$, and the sum has determinant $-c^2 m$, of class $[-m]$. The Weil classes lie in H^2 and have type $(1, 1)$, so they are divisor classes by the Lefschetz $(1, 1)$ theorem. $\square$

Proposition 10.2. *Let $n \geq 4$, and suppose the Weil classes of every polarized abelian $2n$ -fold of Weil type for K with split discriminant are algebraic. Then the Weil classes of every polarized abelian sixfold of Weil type for K are algebraic.*

Proof. The product. Let A be a sixfold of discriminant $[t]$, $t < 0$. Put $k = n - 3$ and $Y = \prod_{j=1}^k Y_j$ with $Y_j = E \times \bar{E}$ polarized with degrees $(1, m_j)$, where $m_2 = \dots = m_k = 1$ and m_1 is a positive integer in the class $[(-1)^{k+n}/t]$; that class is positive, and a positive class has a positive integral representative because squares are norms. By Lemma 10.1, $A \times Y$ is of Weil type for K , of signature (n, n) and discriminant $[t] \prod_j [-m_j] = [t] [-m_1] [-1]^{k-1} = [(-1)^n]$, the split class; so its Weil classes are algebraic. In the case used for Theorem B, $n = 4$ and $k = 1$: $Y = E \times \bar{E}$ with degrees $(1, m)$, $m \in [-1/t]$, and $[t] [-m] = [1]$.

Notation. Fix an embedding ι of K in $\mathbb{C}$ and write $\bar{\iota}$ for the other. For a K -module M of Weil type, $W_K(M) \otimes \mathbb{C}$ is the sum of two lines, on which $a \in K$ (acting through ι on H^1) acts on the top exterior power by $\iota(a)^{\text{top}}$ and $\bar{\iota}(a)^{\text{top}}$; we call them the ι -line and the $\bar{\iota}$ -line, and complex conjugation

exchanges them. Let x span the ι -line of $W_K(A) \otimes \mathbb{C}$, so $\bar{x}$ spans the $\bar{\iota}$ -line, and let $w_j, \bar{w}_j$ span those of $W_K(Y_j) \otimes \mathbb{C} \subset H^2(Y_j, \mathbb{C})$. Since $\bigwedge_K^{2n} H^1(A \times Y) = \bigwedge_K^6 H^1(A) \otimes \bigotimes_j \bigwedge_K^2 H^1(Y_j)$, a nonzero rational Weil class of $A \times Y$ has the form $\xi = b x \otimes \bigotimes_j w_j + \bar{b} \bar{x} \otimes \bigotimes_j \bar{w}_j$ with $b \in \mathbb{C}^\times$.

The pairings on Y_j . An element $a \in \mathcal{O}_K$ acts on Y_j by an endomorphism of degree $\text{Nm}(a)^2$, on the ι -line of H^2 by $\iota(a)^2$ and on $H^4(Y_j)$ by $\text{Nm}(a)^2 = \iota(a)^2 \bar{\iota}(a)^2$. So $w_j \cup w_j$, on which a acts by $\iota(a)^4$, vanishes (take a with $\iota(a)^2 \notin \mathbb{R}$), and by Poincaré duality $\epsilon_j = \int_{Y_j} w_j \cup \bar{w}_j \neq 0$.

The contraction. A nonzero rational class $y_j = a_j w_j + \bar{a}_j \bar{w}_j \in W_K(Y_j)$ is algebraic (Lemma 10.1). Then

$$z = \text{pr}_{A*} (\xi \cup \text{pr}_Y^* (y_1 \boxtimes \cdots \boxtimes y_k)) = \pm \left(b \prod_j \bar{a}_j \epsilon_j \right) x + \pm \overline{\left(b \prod_j \bar{a}_j \epsilon_j \right)} \bar{x},$$

with one common sign, a nonzero rational algebraic class in $W_K(A)$.

Two independent classes. For $\alpha \in \mathcal{O}_K$, the class $\alpha^* y_1$ is algebraic, and it multiplies $\bar{a}_1$ by $\bar{\iota}(\alpha)^2$; so the class z_α obtained from $\alpha^* y_1$ has coordinates $(c \bar{\iota}(\alpha)^2, \overline{c \bar{\iota}(\alpha)^2})$ in the basis $x, \bar{x}$, with $c \neq 0$ independent of α . The determinant of the coordinates of z_1 and z_α is

$$\overline{c \bar{\iota}(\alpha)^2} - \bar{c} c \bar{\iota}(\alpha)^2 = |c|^2 (\iota(\alpha)^2 - \bar{\iota}(\alpha)^2),$$

which is nonzero exactly when $\alpha^2 \notin \mathbb{Q}$. Taking $\alpha \in \mathcal{O}_K$ neither rational nor purely imaginary (for instance $\alpha = 1 + \omega$ when $K = \mathbb{Q}(\sqrt{-3})$, with $\alpha^2 = \omega$), z_1 and z_α are two independent algebraic classes in the two-dimensional $W_K(A)$. $\square$

Proof of Theorem B. Proposition 10.2 with $n = 4$ and Theorem A. $\square$

11. POWERS: PROOF OF THEOREM C

Let W be a complex vector space of dimension N , and U, U' complex vector spaces of dimension m . Put $\Lambda = \bigwedge(W \otimes U) \otimes \bigwedge(W^* \otimes U')$, with the commuting actions of $GL(W)$ and $GL(U) \times GL(U')$. For $u \in U$, $u' \in U'$ let $\theta(u, u') = \sum_i (w_i \otimes u) \wedge (w_i^* \otimes u')$ for dual bases, and for $a \in U$ let $\det(W \otimes a) \in \bigwedge^N(W \otimes U)$ be the image of a generator of $\bigwedge^N W$ under $w \mapsto w \otimes a$; define $\det(W^* \otimes a')$ similarly.

Lemma 11.1. $\Lambda^{SL(W)}$ is generated by the $\theta(u, u')$, the $\det(W \otimes a)$ and the $\det(W^* \otimes a')$.

Proof. We deduce it from the first fundamental theorem of invariant theory for $SL(W)$, in the form of Kraft and Procesi [16, §8.4, Theorem (FFT for SL_n), p. 90]: *the ring of invariants $K[V^p \oplus V^{*q}]^{SL(V)}$ is generated by the scalar products $\langle j \mid i \rangle$ and the determinants $[i_1, \dots, i_n]$ and $[j_1, \dots, j_n]^*$, for a vector space V of dimension n over a field K of characteristic zero (see also [35, Ch. II]).* We use it in multilinear form. An $SL(W)$ -invariant element of $W^{\otimes p} \otimes W^{*\otimes q}$ is an invariant multilinear function on $W^{*p} \times W^q$, that is, an invariant polynomial on $W^{*p} \oplus W^q$ that is homogeneous of degree one

in each of the $p + q$ vector variables. Apply the theorem with $V = W^*$. A polynomial in the scalar products and determinants that is homogeneous of degree one in each variable is a linear combination of products of scalar products and determinants in which every variable occurs exactly once. So the invariants in $W^{\otimes p} \otimes W^{*\otimes q}$ are spanned by tensor products of three kinds of elementary invariants, each occupying some of the tensor factors: the contraction $\sum_i w_i \otimes w_i^*$ in one factor W and one factor W^* ; the determinant tensor $\sum_{\pi \in S_N} \text{sgn}(\pi) w_{\pi(1)} \otimes \cdots \otimes w_{\pi(N)}$ in N factors W ; and the dual determinant tensor in N factors W^* . (This is the skew (GL, GL) duality of [13] in another form.)

The homogeneous piece $\bigwedge^p(W \otimes U) \otimes \bigwedge^q(W^* \otimes U')$ of Λ is the image of $(W \otimes U)^{\otimes p} \otimes (W^* \otimes U')^{\otimes q} = W^{\otimes p} \otimes W^{*\otimes q} \otimes U^{\otimes p} \otimes U'^{\otimes q}$ under antisymmetrization, which is $SL(W)$ -equivariant and surjective. Since $SL(W)$ is reductive, the invariants of the image are the image of the invariants, so they are spanned by the images of $I \otimes u_1 \otimes \cdots \otimes u_p \otimes u'_1 \otimes \cdots \otimes u'_q$ with I a product of elementary invariants. Antisymmetrization turns tensor products into wedge products, up to sign. The image of a contraction in the factors $(w \otimes u_k)$ and $(w^* \otimes u'_l)$ is $\theta(u_k, u'_l)$. The image of a determinant tensor in the factors $u_{k_1}, \dots, u_{k_N}$ is $\sum_{\pi} \text{sgn}(\pi) (w_{\pi(1)} \otimes u_{k_1}) \wedge \cdots \wedge (w_{\pi(N)} \otimes u_{k_N})$, which is the full polarization of the homogeneous polynomial map $a \mapsto N! \det(W \otimes a)$ of degree N , evaluated at $(u_{k_1}, \dots, u_{k_N})$; by the polarization identity it is a linear combination of values $\det(W \otimes a)$. The same holds for the dual determinant tensors. So every invariant is a combination of products of the stated generators. $\square$

Theorem 11.2. *Let (A, ι, E) be a polarized abelian $2n$ -fold of Weil type for K , $n \geq 2$, whose Weil classes are algebraic and whose Hodge group is SU_H . Then every Hodge class on every power A^m is algebraic.*

Proof. Let $H^1(A, \mathbb{C}) = W \oplus W'$ be the decomposition into the two eigenspaces of K . The polarization identifies W' with W^* as representations of $SU_H(\mathbb{C}) = SL(W)$. The Hodge group of A^m is SU_H acting diagonally, so the Hodge classes on A^m , tensored with $\mathbb{C}$, are $\Lambda^{SL(W)}$ with $N = 2n$ and $U = U' = \mathbb{C}^m$. The $\theta(u, u')$ are Hodge classes of degree two, hence combinations of divisor classes. Let $\mathcal{O} = \{a \in K : \iota(a) \in \text{End}(A)\}$. For $a \in \mathcal{O}^m$ the homomorphism $A^m \rightarrow A$, $(x_i) \mapsto \sum_i \iota(a_i)x_i$, pulls the Weil classes of A back to $\det(W \otimes \tilde{a})$ and $\det(W^* \otimes \tilde{a})$, with $\tilde{a}$ the image of a under the given embedding; these are algebraic. Since $\mathcal{O}$ is a lattice in K , the set of $\tilde{a}$ is Zariski dense in U , so these classes span all $\det(W \otimes a)$, and likewise for W^* . By Lemma 11.1, products of algebraic classes span all Hodge classes over $\mathbb{C}$, and rational descent as in Corollary 4.7 finishes the proof. $\square$

Proof of Theorem C. Theorem B and Theorem 11.2 with $n = 3$. For sixfolds the hypothesis $\text{Hg}(A) = SU_H$ holds outside a countable union of proper closed subvarieties of the moduli space [29, Theorem 6.11]. $\square$

12. ABELIAN SIXFOLDS OF TYPE III

Theorem B covers every member of every moduli space, including special members, and this gives a result for abelian sixfolds of type III.

Corollary 12.1. *Let D be a totally definite quaternion algebra over $\mathbb{Q}$ containing $\mathbb{Q}(\sqrt{-3})$, that is, ramified at ∞ and at an odd number of primes, each equal to 3 or congruent to 2 modulo 3. Let $\mathcal{A}$ be a family of abelian sixfolds of PEL type for D , that is, the universal family over a connected Shimura variety parametrizing polarized abelian sixfolds with an action of a fixed order of D , whose general member has endomorphism algebra exactly D [1, §2.5]. For every member A of $\mathcal{A}$ whose Hodge group is the generic Hodge group of the family, every Hodge class on every power of A is algebraic.*

Proof. $K = \mathbb{Q}(\sqrt{-3})$ embeds in D as a maximal subfield [31, Proposition 14.6.7]. Restricting the action of D to K makes A an abelian sixfold of Weil type for K with a compatible polarization [1, §4]: an element of D anticommuting with $\sqrt{-3}$ exchanges the two eigenspaces of K on $H^{1,0}(A)$. By Theorem B, $W_K(A)$ is spanned by algebraic classes.

Abdulali's Theorem 4.1 [1, p. 673] states: *Let A be a general member of an effective PEL-family of type III. The Hodge ring of A is generated by divisors, together with the Weil cycles associated to a maximal subfield of $\text{End}_{\mathbb{Q}}(A)$. The algebraicity of the Weil cycles on A implies the usual Hodge conjecture for all powers of A .* In his §4, “general” is used for a fibre whose Hodge group is the group G_1 defined there, the generic Hodge group of the family: every fibre has Hodge group contained in G_1 , with equality for the general fibre. His proof computes the Hodge ring of A^r as the ring of invariants of G_1 and uses nothing else about A , so it applies verbatim to every member whose Hodge group is G_1 . For such a member, $\text{End}^0(A)$ is the commutant of G_1 , which is D , so K is a maximal subfield of $\text{End}^0(A)$ and the Weil cycles of the theorem may be taken to be those for K , which are algebraic. $\square$

Example 12.2. The point of Corollary 12.1 is that type III sixfolds reach non-split discriminants, so it is not a consequence of the split case. Write $D = (a, b)_{\mathbb{Q}}$ with basis $1, u, v, uv$, $u^2 = a$, $v^2 = b$, and identify $K = \mathbb{Q}(u)$. A polarization of a sixfold of type III is the trace of a skew-Hermitian form $T(x, y) = \bar{x}^T \text{diag}(t_1, t_2, t_3) y$ on D^3 with pure quaternions t_k [1, §2.5]. As a right K -space, $D = K \oplus vK$, and the K -component of T is the skew-Hermitian form of Section 7 whose trace is the polarization. For one coordinate with $t = \alpha u + \beta v + \gamma uv$, its Gram matrix in the basis $1, v$ is

$$\begin{pmatrix} \alpha u & b(\beta + \gamma u) \\ -b(\beta - \gamma u) & \alpha bu \end{pmatrix},$$

of determinant $b(\alpha\alpha^2 + b\beta^2 - ab\gamma^2) = -b \text{ nrd}(t)$. Dividing by u to make it Hermitian multiplies the determinant by $1/a$. So the discriminant of A is the class of $\prod_k (-b \text{ nrd}(t_k)/a)$ in $\mathbb{Q}^{\times} / \text{Nm } K^{\times}$. For $a = -3$ this is $[b^3 \prod_k \text{ nrd}(t_k)]$,

since $3 = \text{Nm}(\sqrt{-3})$. Take $D = (-3, -1)_{\mathbb{Q}}$, so $\text{nrd}(\alpha u + \beta v + \gamma uv) = 3\alpha^2 + \beta^2 + 3\gamma^2$. The forms $\text{diag}(v, u, u + 2uv)$ and $\text{diag}(u + 4v + uv, u, u + uv)$ have $\prod_k \text{nrd}(t_k) = 1 \cdot 3 \cdot 15 = 45$ and $22 \cdot 3 \cdot 6 = 396$, so discriminants $[-45] = [-5]$ and $[-396] = [-11]$. The split class is $[(-1)^3] = [-1]$, and $[-5] = [-1]$ would make 5 a norm from K , which fails in $\mathbb{Q}_5$: since 5 is inert in K , the completion $K \otimes \mathbb{Q}_5$ is the unramified quadratic extension of $\mathbb{Q}_5$, whose norms have even 5-adic valuation, and $v_5(5) = 1$ is odd. The same holds for 11. So both discriminants are non-split.

APPENDIX A. THE DATA FOR G_{34}

The tuple. Write elements of $K = \mathbb{Q}(\omega)$ as $a + b\omega$. The reflections $s_i(x) = x - 2 \frac{\langle v_i, x \rangle}{\langle v_i, v_i \rangle} v_i$ in the roots

$$\begin{aligned} v_1 &= (1, -1 - \omega, 1, \omega, -1 - \omega, \omega), & v_6 &= (1, -1 - \omega, \omega, \omega, \omega, \omega), \\ v_2 &= (1, 0, 0, 1 + \omega, 0, 0), & v_7 &= (0, 0, 1, 0, 0, -1), \\ v_3 &= (0, 1, 1 + \omega, 0, 0, 0), & v_8 &= (1, \omega, \omega, -1 - \omega, 1, -1 - \omega), \\ v_4 &= (1, \omega, -1 - \omega, \omega, -1 - \omega, 1), & v_9 &= (1, -\omega, 0, 0, 0, 0), \\ v_5 &= (1, -1 - \omega, -1 - \omega, -1 - \omega, -1 - \omega, \omega), & v_{10} &= (0, 1, 0, 0, 0, -1), \end{aligned}$$

with $\langle \cdot, \cdot \rangle$ the standard Hermitian form, generate a group of order 39 191 040 with centre of order 6. In GAP, with `w:=E(3)`, the ten matrices $1 - 2v_i\bar{v}_i^T / \langle v_i, v_i \rangle$ give these numbers through `Size(Group(gens))` and `Size(Centre(Group(gens)))`. By the classification of Shephard and Todd [28], the only irreducible complex reflection group in dimension 6 of this order is G_{34} : the imprimitive groups $G(m, p, 6)$, $m \geq 2$, have order $720 m^6 / p$, which never equals 39 191 040; the symmetric group $S_7 = G(1, 1, 7)$, acting on its six-dimensional reflection representation, has order 5040; and the only other primitive group in dimension 6, the Weyl group G_{35} of E_6 , has order 51 840; the group acts irreducibly on K^6 , because its reduction modulo the prime $\mathfrak{p} = (7, \omega - 2)$ of $\mathcal{O}_K$ is absolutely irreducible over $\mathcal{O}_K/\mathfrak{p} = \mathbb{F}_7$ (checked by `MTX.IsAbsolutelyIrreducible` on the reduced matrices). Here is why that suffices. Every $\langle v_i, v_i \rangle$ is 2 or 6, so the matrices $1 - 2v_i\bar{v}_i^T / \langle v_i, v_i \rangle$ have entries in $\mathcal{O}_K[\frac{1}{3}]$, which is contained in the local ring $\mathcal{O}_{\mathfrak{p}}$; so $\Lambda = \mathcal{O}_{\mathfrak{p}}^6$ is a stable lattice and the reduction is $\Lambda/\mathfrak{p}\Lambda$. If $0 \neq W \subsetneq K^6$ were a stable subspace, then $W \cap \Lambda$ would be a saturated stable sublattice, and its image in $\Lambda/\mathfrak{p}\Lambda$ would be a stable subspace of dimension $\dim_K W$, neither 0 nor everything. The tuple is $\mathbf{g} = (s_1, s_1, s_2, s_2, \dots, s_{10}, s_{10})$.

The curves. The seventeen curves γ_j each enclose the consecutive points $a, \dots, b$ of the twenty. The table gives, for each, the return power k_j , the order m_j of the semisimple part of T_j , the K -rank of N_j , and the signature of β_j on V_{par}/I as (positive, negative, zero).

j	$[a, b]$	k_j	m_j	$\text{rk } N_j$	signature of β_j
1	[1, 8]	1	1	4	(0, 4, 0)
2	[1, 18]	1	1	1	(0, 1, 3)
3	[2, 8]	2	1	3	(0, 3, 1)
4	[3, 8]	1	1	3	(0, 3, 1)
5	[4, 8]	2	1	2	(0, 2, 2)
6	[5, 6]	1	1	1	(0, 1, 3)
7	[5, 8]	1	1	2	(0, 2, 2)
8	[7, 8]	1	1	1	(0, 1, 3)
9	[9, 10]	1	1	1	(0, 1, 3)
10	[9, 12]	1	1	2	(0, 2, 2)
11	[9, 16]	1	1	4	(0, 4, 0)
12	[9, 17]	2	1	4	(0, 4, 0)
13	[9, 18]	1	1	4	(0, 4, 0)
14	[11, 12]	1	1	1	(0, 1, 3)
15	[13, 14]	1	1	1	(0, 1, 3)
16	[13, 16]	1	1	2	(0, 2, 2)
17	[15, 16]	1	1	1	(0, 1, 3)

Fourteen of the curves have trivial boundary product in G_{34} ; for the three with $k_j = 2$ the tuple first returns after two twists. The sum $\sum_j \beta_j$ has signature (0, 4, 0). The N_j satisfy $N_j N_k = 0$, their images span a four-dimensional totally isotropic K -subspace I killed by every N_j , and they span a $\mathbb{Q}$ -space of dimension 16. Every pair of pants has vanishing parabolic cohomology (Remark 8.6).

The basis. A parabolic cocycle u for $\mathbf{g}$ is determined by its values $u_i = u(\ell_i)$, $i = 1, \dots, 20$, and u_i lies in the image of $1 - s_{[i/2]}$, which is the line $Kv_{[i/2]}$; so $u_i = \lambda_i v_{[i/2]}$ for scalars $\lambda_i \in K$. We use the basis $b_1, \dots, b_8$ of V_{par} given by the following cocycles; all λ_i not listed are 0. For $k = 1, \dots, 4$, b_k has $\lambda_{2k-1} = \lambda_{2k} = 1$, that is, the value v_k on both loops around the two points with monodromy s_k . For $k = 5, \dots, 8$:

i	1	3	5	7	10	12	14	16	18	20
b_5	1				$1 + \omega$	1		$-1 - \omega$		$-1 + \omega$
b_6		1			$1 + \omega$	$\frac{2}{3} - \frac{2}{3}\omega$	-1	$-\frac{2}{3} - \frac{1}{3}\omega$		$1 + 2\omega$
b_7			1		$-\frac{1}{3} - \frac{2}{3}\omega$	$-\frac{2}{3} - \frac{1}{3}\omega$	$1 + \omega$		$1 + \omega$	$-\omega$
b_8				1	$1 + \omega$	1	$-2 - \omega$	$-1 - \omega$		3ω

Each satisfies the cocycle relation $\sum_i Q_i u_i = 0$ of Lemma 7.2; for $b_1, \dots, b_4$ this is $v_k + s_k v_k = 0$. The same basis, with every value u_i written out, is in the supplementary file `g34_data.json`. Then $I = \langle b_1, \dots, b_4 \rangle$.

The matrices. With $\mathcal{S}(x, y) = \bar{x}^T S y$, the Gram matrix of the skew-Hermitian form is

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ -3 & 0 & 0 & 0 & -3-6\omega & -3-3\omega & 4+5\omega & -6-6\omega \\ 0 & -1 & 0 & 0 & -3\omega & -1-2\omega & 1+4\omega & -2-4\omega \\ 0 & 0 & -1 & 0 & 1+5\omega & 3+4\omega & -2-4\omega & 3+6\omega \\ 0 & 0 & 0 & -3 & -6\omega & -2-4\omega & 3+6\omega & -3-6\omega \end{pmatrix}$$

In this basis every N_j has the block form $N_j = \begin{pmatrix} 0 & X_j \\ 0 & 0 \end{pmatrix}$, with the following 4×4 blocks over K , numbered as in the table:

$$\begin{aligned}
X_1 &= \begin{pmatrix} 2 & 1/3-1/3\omega & 1/3+2/3\omega & -\omega \\ 2+\omega & 2 & 0 & 2+\omega \\ -1-2\omega & 0 & 2 & -1+\omega \\ 1+\omega & 1/3-1/3\omega & -2/3-1/3\omega & 2 \end{pmatrix}, & X_2 &= \begin{pmatrix} 2 & -2\omega & -2/3+2/3\omega & 2-2\omega \\ 6+6\omega & 6 & -4-2\omega & 12+6\omega \\ -4-2\omega & -2+2\omega & 2 & -6 \\ 4+2\omega & 2-2\omega & -2 & 6 \end{pmatrix} \\
X_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & -\omega & 3+3\omega \\ 0 & 1+\omega & 3 & 3\omega \\ 0 & -\omega & -1-\omega & 3 \end{pmatrix}, & X_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2+\omega \\ 0 & 0 & 2 & -1+\omega \\ 0 & 1/3-1/3\omega & -2/3-1/3\omega & 2 \end{pmatrix} \\
X_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2+2\omega \\ 0 & 0 & -4/3-2/3\omega & 3 \end{pmatrix}, & X_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
X_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1+\omega \\ 0 & 0 & -2/3-1/3\omega & 2 \end{pmatrix}, & X_8 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \\
X_9 &= \begin{pmatrix} 2 & 2 & -4/3-2/3\omega & 2 \\ 6 & 6 & -4-2\omega & 6 \\ -2+2\omega & -2+2\omega & 2 & -2+2\omega \\ 2 & 2 & -4/3-2/3\omega & 2 \end{pmatrix}, & X_{10} &= \begin{pmatrix} 3 & 2-\omega & -2-\omega & 3 \\ 9+3\omega & 8 & -5-4\omega & 9+3\omega \\ -3+3\omega & -1+4\omega & 3 & -3+3\omega \\ 3 & 2-\omega & -2-\omega & 3 \end{pmatrix} \\
X_{11} &= \begin{pmatrix} 5 & 4/3-7/3\omega & -4/3+1/3\omega & 3-2\omega \\ 11+7\omega & 8 & -5-3\omega & 11+4\omega \\ -5-\omega & -2+3\omega & 4 & -5+2\omega \\ 5+2\omega & 7/3-4/3\omega & -7/3-2/3\omega & 5 \end{pmatrix}, & X_{12} &= \begin{pmatrix} 7 & 2/3-11/3\omega & -2/3+5/3\omega & 3-4\omega \\ 13+11\omega & 9 & -5-2\omega & 13+5\omega \\ -7-5\omega & -3+2\omega & 5 & -7+\omega \\ 7+4\omega & 8/3-5/3\omega & -8/3-1/3\omega & 7 \end{pmatrix} \\
X_{13} &= \begin{pmatrix} 5 & 4/3-7/3\omega & -1/3+4/3\omega & 3-2\omega \\ 11+7\omega & 8 & -3 & 11+4\omega \\ -5-4\omega & -3 & -5-\omega & -5-\omega \\ 5+2\omega & 7/3-4/3\omega & -4/3+1/3\omega & 5 \end{pmatrix}, & X_{14} &= \begin{pmatrix} 2 & 4/3-4/3\omega & -4/3-2/3\omega & 2 \\ 8+4\omega & 8 & -4-4\omega & 8+4\omega \\ -2+2\omega & 4\omega & 2 & -2+2\omega \\ 2 & 4/3-4/3\omega & -4/3-2/3\omega & 2 \end{pmatrix} \\
X_{15} &= \begin{pmatrix} 0 & 2 & -2-2\omega & 4+2\omega \\ 0 & 2\omega & 2 & -2+2\omega \\ 0 & 2/3-2/3\omega & -4/3-2/3\omega & 2 \end{pmatrix}, & X_{16} &= \begin{pmatrix} 2 & -\omega & 1/3+2/3\omega & 1-\omega \\ 3+3\omega & 3 & -2-\omega & 6+3\omega \\ -1-2\omega & -1+\omega & 2 & -3 \\ 2+\omega & 1-\omega & -1 & 3 \end{pmatrix} \\
X_{17} &= \begin{pmatrix} 2 & 2/3-2/3\omega & 0 & 2 \\ 4+2\omega & 2 & 0 & 4+2\omega \\ 0 & 0 & 0 & 0 \\ 2 & 2/3-2/3\omega & 0 & 2 \end{pmatrix}
\end{aligned}$$

So (H1) is the block form itself, and (H2) holds because the upper left 4×4 block of S vanishes. On $V_{\text{par}}/I = \langle b_5, \dots, b_8 \rangle$ the form β_j has matrix DX_j , where $D = \text{diag}(-3, -1, -1, -3)$ is the lower left block of S . So (H3) and (H4) are statements about 4×4 matrices that can be checked by hand: each DX_j is Hermitian and negative semidefinite, of the rank in the table; $\sum_j DX_j$ is negative definite; and the X_j span a $\mathbb{Q}$ -space of dimension 16 in the 32-dimensional $\mathbb{Q}$ -space of 4×4 matrices over K . For example DX_1 has first row $(-6, -1+\omega, -1-2\omega, 3\omega)$. The certificate has two levels. *The printed level:* from S and the X_j alone, (H1)–(H4) are statements about 4×4 matrices over K ; the supplementary script `check_g34.py` (Python 3, exact arithmetic over K , no other dependencies) checks them, and `python3 check_g34.py` ends with the line `ALL (H1)–(H4) HOLD: True` after printing the ranks and signatures of the table. *The derivation level:* from the printed roots, tuple, curves and basis $b_1, \dots, b_8$, the matrices S and X_j are recomputed by the procedure described below; the supplementary files contain two separately written programs that do this, with their outputs and checksums.

How to recompute. $V_{\text{par}} = H_{\text{par}}^1(U_C, L')$ is the space of parabolic cocycles modulo coboundaries for the free group on the twenty loops with monodromy $\sigma(\mathbf{g})$, of K -dimension 8; $\mathcal{S}$ is given by Lemma 7.2; each T_j is the k_j -th iterate of the pullback formula of Section 8, and $N_j = \log T_j$ since every $m_j = 1$. All of this is exact linear algebra over K in dimension 8. The supplementary

files contain two separately written programs that compute the table from the tuple and agree, and the explicit supplier identity of Remark 9.2 with its rebuild script.

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