

WEIL CLASSES ON EVERY ABELIAN SIXFOLD OF WEIL TYPE FOR $\mathbb{Q}(\sqrt{-3})$

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ABSTRACT. We prove that the Weil classes of every polarized complex abelian sixfold of Weil type for $K = \mathbb{Q}(\sqrt{-3})$, of every Hermitian discriminant, are algebraic. The same holds for every abelian eightfold of Weil type for K of signature $(4, 4)$ with split discriminant, and every power of such a sixfold whose special Mumford–Tate group is SU_H satisfies the Hodge conjecture. The proof applies the transfer theorem of the companion paper [23] to the Shephard–Todd group G_{34} with twenty reflections as local monodromies, and passes from eightfolds to sixfolds by a compensation argument. Two general tools are proved on the way. If every local monodromy of a transfer datum is conjugate to its inverse, the lattice condition of the transfer theorem follows from Brauer’s induction theorem. Dominance of the period map follows from exact data at a maximal degeneration, by the nilpotent orbit theorem. We also give an explicit higher-dimensional form of Schoen’s compensation argument, which shows that one decided cell of $2n$ -folds, $n \geq 4$, decides every cell of sixfolds over the same imaginary quadratic field. No outside unrefereed manuscript is used.

1. INTRODUCTION

Let A be a complex abelian variety of dimension $2n$ on which an imaginary quadratic field K acts with both embeddings occurring n times on the tangent space. The two-dimensional space $W_K(A) = \bigwedge_K^{2n} H^1(A, \mathbb{Q})$ consists of Hodge classes, the Weil classes [27], [26, §5]. A compatible polarization gives a K -Hermitian form of signature (n, n) whose discriminant in $\mathbb{Q}^\times / \text{Nm } K^\times$ is an invariant of the rational polarized data. For $n = 3$ the Weil classes are known to be algebraic for the split discriminant [20, 14], and for $K = \mathbb{Q}(\sqrt{-3})$ and the class -2 [22]. For $K = \mathbb{Q}(i)$ the split case was proved earlier by Koike, who showed that the Prym map from fourth cyclic étale covers of curves of genus four dominates the nine-dimensional Shimura variety of the family (discriminant 1 in the normalization of that paper) [13]. Outside discriminant -1 , the algebraicity of Weil classes on sixfolds is described as open in [16, Introduction].

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Theorem 1.1 (Theorem 6.2). *Let $K = \mathbb{Q}(\sqrt{-3})$.*

- (1) *The Weil classes of every polarized abelian eightfold of Weil type for K with signature $(4, 4)$ and discriminant $[1]$ are algebraic.*
- (2) *The Weil classes of every polarized abelian sixfold of Weil type for K , of signature $(3, 3)$ and any discriminant, are algebraic.*
- (3) *Every power of a sixfold A as in (2) with $\text{Hg}(A) = \text{SU}_H$ satisfies the Hodge conjecture.*

The proof has four steps. The transfer theorem of [23] (recalled as Theorem 2.9) decides a whole Weil cell from one family of Prym–Tyurin factors of Galois covers of $\mathbb{P}^1$, given a lattice condition (L) on suppliers of Schoen cycles and dominance (D) of the period map. We take $G = G_{34}$ in its six-dimensional reflection representation and twenty reflections as local monodromies. This is a datum of type $(K, 4)$ in the split cell.

- (L) holds by Theorem 3.1: since every local monodromy is an involution, Brauer’s induction theorem writes the determinant character as an integer combination of admissible Schoen suppliers. An explicit witness is also given.
- (D) holds by Proposition 4.2: at a maximal degeneration of the Hurwitz space, the logarithms of the boundary monodromies span the unipotent radical of the stabilizer of a rational Lagrangian, which by the nilpotent orbit theorem forces full rank. All hypotheses are exact linear algebra over $\mathbb{Q}(\sqrt{-3})$, computed from the monodromy tuple.
- The transfer theorem then decides the split eightfold cell, for every member.
- Proposition 5.2 passes to every sixfold cell, through products $A \times E \times \bar{E}$ with E an elliptic curve with complex multiplication by $\mathcal{O}_K$; this is a higher-dimensional form of Schoen’s compensation argument [20, Proposition 10].

Theorem 1.1(2) contains the split sixfolds over K [14] and the cell $[-2]$ of [22]; it is new for every other discriminant. The proof uses Schoen’s theorem, Milne’s theorem on Lefschetz classes, Brauer’s induction theorem, the nilpotent orbit theorem and the Clemens–Schmid sequence. It does not use the unreferenced manuscripts that the companion paper uses conditionally. It does depend on the companion transfer paper [23], and through it on the cocycle formula of [22]; the three papers are meant to be read and refereed together, and Section 2 states exactly what is imported. Section 7 draws a consequence for sixfolds of type III: the Hodge conjecture on all powers of the *general member* of every effective PEL family whose rational quaternion algebra contains $\mathbb{Q}(\sqrt{-3})$. Unlike Theorem 1.1, which holds for every member, this consequence is limited to members whose Hodge group is the generic one of their family. It says nothing about powers of members with smaller Hodge group, about eightfold powers, or about other fields; Section 8 describes what the method needs for other fields.

2. RECOLLECTIONS

We recall what we use from [23], with the same conventions. Throughout, K is an imaginary quadratic field and $\vartheta \in K$ is nonzero with $\bar{\vartheta} = -\vartheta$.

2.1. Transfer data. Fix a finite group G , conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r$, and the Hurwitz space $\mathcal{H}$ of connected Galois G -covers $C \rightarrow \mathbb{P}^1$ branched over r ordered points with local monodromy in $\mathcal{C}_j$; its components are smooth and irreducible of dimension $r - 3$ modulo PGL_2 . By Chevalley–Weil, $H^1(C, \mathbb{C}) = \bigoplus_{\tau} \tau \otimes M_{\tau}$; put $m_{\tau} = \dim M_{\tau}$, and $c_{\tau}(g) = \text{rk}(\tau(g) - 1)$.

Proposition 2.1 ([23, Defect identity]). *If τ is irreducible and nontrivial, then $m_{\tau} = -2d + \sum_j c_{\tau}(g_j)$, $d = \dim \tau$; the Hodge numbers of M_{τ} are $\sum_j \text{age } \tau(g_j^{\mp 1}) - d$.*

Definition 2.2 ([23, Definition of a Weil transfer datum]). A Weil transfer datum of type (K, n) is $\mathfrak{D} = (G, \sigma; \mathcal{C}_1, \dots, \mathcal{C}_r)$ with σ irreducible of dimension d , character field K and Schur index one over K , a nonempty Nielsen class, $m_{\sigma} = 2n$ and Hodge numbers (n, n) on M_{σ} .

For $C \in \mathcal{H}$ the σ -isotypic factor $A_{\sigma}(C) \subseteq J(C)$ is a polarized abelian $2n$ -fold of Weil type for K , with Hermitian form H of signature (n, n) [23, §2]. On a component S the discriminant class $\mathfrak{d}(S)$ is constant.

A *supplier* is a pair (H, χ) of a subgroup $H \leq G$ and a linear character χ of H of order $m \geq 3$; $C' = C / \ker \chi \rightarrow C/H$ is a cyclic cover of a curve of genus g , with s branch points and local exponents α , and the primitive part of $J(C')$ has E_{χ} -dimension $h = 2g - 2 + s = \sum_{\tau} m_{\tau} \langle \tau|_H, \chi \rangle$.

Definition 2.3 ([23, admissible suppliers]). A supplier is admissible if h is even and positive, and α is simple in Schoen's sense (after a permutation, $\alpha_{2i} = -\alpha_{2i-1}$ for all i); for $s = 0$ simplicity is automatic.

Lemma 2.4 (Schoen; [19, Theorem 2.0 and Corollary 3.1]). *If (H, χ) is admissible, the Weil classes of the primitive part of $J(C / \ker \chi)$ are algebraic, at every $C \in \mathcal{H}$. Exponents may share factors with m .*

Let U be the centralizer of G in $\text{Sp}(H^1(C, \mathbb{Q}))$ and T its maximal abelian quotient. Then $X^*(T) = \mathbb{Z}^{\text{pairs}} \oplus (\mathbb{Z}/2)^{\text{quat}}$: one free coordinate for each pair $\{\tau, \bar{\tau}\}$ of non-self-dual τ with $m_{\tau} > 0$, oriented by a chosen member, and one parity coordinate for each quaternionic τ with $m_{\tau} > 0$. A supplier gives $a(\chi) = (\langle \tau|_H, \chi \rangle - \langle \bar{\tau}|_H, \chi \rangle)_{\text{pairs}} \oplus (\langle \tau|_H, \chi \rangle \bmod 2)_{\text{quat}}$, and e_{σ} is the coordinate $+1$ at the pair of σ . Condition (L) is that e_{σ} lies in the group $L(\mathfrak{D})$ generated by the $a(\chi)$ of admissible suppliers.

2.2. Dominance and the discriminant. Let $A = \text{Hom}_G(\sigma, H^0(C, K_C))$ and $B = \text{Hom}_G(\bar{\sigma}, H^0(C, K_C))$, each of dimension n , and $\mu_C: A \otimes B \rightarrow H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$ the multiplication followed by contraction.

Lemma 2.5 ([23, Codifferential]). *μ_C is, up to a nonzero normalization, the codifferential at C of the period map of the family A_{σ} into the n^2 -dimensional domain of $SU(n, n)$.*

Proposition 2.6 ([23, Rank at one point]). *If $\text{rk } \mu_{C_1} = n^2$ at one point C_1 of a component S , the period map of S has generic rank n^2 .*

Let $L' = \text{Hom}_G(V_\sigma, \pi_* \mathbb{Q}|_{U_C} \otimes K)$, a local system of K -spaces of rank d on $U_C = \mathbb{P}^1 - B_C$ with a flat positive definite Hermitian form, and $V = H_{\text{par}}^1(U_C, L')$, of K -dimension $2n$, with the skew-Hermitian parabolic cup product $\mathcal{S}$ and the Hermitian form $h = \vartheta \mathcal{S}$.

Lemma 2.7 (Cocycle formula; [23, Cocycle formula], [22, Lemma 6.2]). *With unitary monodromy $T_1, \dots, T_r$ along clockwise loops, $Q_i = T_1 \cdots T_{i-1}$ and a parabolic cocycle given by a_i with $c_i = (1 - T_i)a_i$ and $F_i = \sum_{j < i} Q_j c_j$, one has $\mathcal{S}(a, b) = \sum_i h_W(Q_i^{-1} F_i^a + a_i, c_i^b)$, with h_W the flat fibre form.*

By [23, The discriminant from the monodromy tuple], $\mathfrak{d}(S) = [\det h]$, and h and the Hermitian form of A_σ have the same signature.

Remark 2.8 (Theorem W). [23, Remark (Per component)] records, from the author's notes, the global formula $[\det h] = [\vartheta^{2n} \prod_j \Delta(g_j)] = [(-1)^n \prod_j \Delta(g_j)]$ with Δ the determinant of the Wall form. We use it only as a cross-check; the discriminant below is computed directly.

2.3. The transfer and powers theorems.

Theorem 2.9 ([23, Transfer]). *Let $\mathfrak{D}$ be a Weil transfer datum of type (K, n) and S a component of its Hurwitz space. If (L) holds and the period map of S has generic rank n^2 , then for every polarized abelian $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}(S)$ the Weil classes are algebraic.*

Lemma 2.10 ([23, Isogeny invariance]). *Algebraicity of Weil classes is invariant under K -linear isogenies, and every polarized $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}$ is K -isogenous to a member of the universal family over the Weil moduli space $\mathcal{M}(K, n, \mathfrak{d})$.*

Theorem 2.11 ([23, Powers]). *If the Weil classes of A are algebraic and $\text{Hg}(A) = SU_H$ for A itself, every Hodge class on every power A^m is algebraic.*

The unrefereed manuscript [17, Theorem 1.1] proves the Hodge conjecture on every power of every abelian sixfold with an imaginary quadratic action whose compatible polarization has hyperbolic Hermitian form of signature $(3, 3)$, for every imaginary quadratic field and with no condition on the Hodge group. Theorem 1.1(3) instead covers members of every discriminant over $K = \mathbb{Q}(\sqrt{-3})$ with $\text{Hg}(A) = SU_H$, through Theorem 2.11; on the split cell over K the statement of [17] contains it, and nothing in this paper uses [17].

3. REAL BRANCH CLASSES

Call the class datum *real* if every g_j is conjugate in G to g_j^{-1} . Involutions are real, so a datum of reflections of order two is real.

Theorem 3.1 (Brauer supply). *If the class datum of $\mathfrak{D}$ is real, then condition (L) holds.*

Proof. Step 1: a homomorphism. Let $R_{\mathbb{C}}(G)$ be the representation ring. Define $A: R_{\mathbb{C}}(G) \rightarrow X^*(T)$ by

$$A(V) = (\text{mult}_{\tau} V - \text{mult}_{\bar{\tau}} V)_{\text{pairs}} \oplus (\text{mult}_{\eta} V \bmod 2)_{\text{quat}},$$

over the pairs $\{\tau, \bar{\tau}\}$ and the quaternionic η with $m_{\tau} > 0$, $m_{\eta} > 0$, the coordinates of $X^*(T)$ (Subsection 2.1). It is additive. By Frobenius reciprocity, $A(\text{Ind}_H^G \chi) = a(\chi)$ for every subgroup H and linear character χ of H , of any order, where the right side is given by the formula for $a_1(\chi)$. Since χ_{σ} is not real, $\sigma \not\cong \bar{\sigma}$, and $A(\sigma) = e_{\sigma}$.

Step 2: Brauer induction. By Brauer's induction theorem [21, §10.5, Theorem 20], [7, (15.9)], $\sigma = \sum_i n_i \text{Ind}_{E_i}^G \lambda_i$ in $R_{\mathbb{C}}(G)$, with $n_i \in \mathbb{Z}$, $E_i \leq G$ elementary and λ_i linear. Hence $e_{\sigma} = \sum_i n_i a(\lambda_i)$, an identity in $X^*(T)$ including the parity coordinates.

Step 3: terms that vanish. If λ_i has order 1 or 2 it is real, so $\langle \tau|_{E_i}, \lambda_i \rangle = \langle \bar{\tau}|_{E_i}, \lambda_i \rangle$ and the free coordinates of $a(\lambda_i)$ vanish. For quaternionic η the invariant alternating form on $\eta|_{E_i}$ is nondegenerate on the λ_i -isotypic part, because $\lambda_i = \lambda_i^{-1}$, and an alternating nondegenerate form on $\lambda_i \otimes \mathbb{C}^k$ with λ_i one-dimensional forces k even; so the parity coordinates vanish too. If λ_i has order $m_i \geq 3$, then (E_i, λ_i) is a supplier with $h_i = \sum_{\tau} m_{\tau} \langle \tau|_{E_i}, \lambda_i \rangle \geq 0$; if $h_i = 0$, every τ with $m_{\tau} > 0$ has $\langle \tau|_{E_i}, \lambda_i \rangle = 0$, and $a(\lambda_i) = 0$. Dropping these terms, e_{σ} is an integer combination of the $a(\lambda_i)$ with $m_i \geq 3$ and $h_i > 0$.

Step 4: the remaining suppliers are admissible. Let (H, χ) be a supplier with $h > 0$, and let $w \in G$ with $w g_j w^{-1} = g_j^{-1}$. Put $D = \langle g_j \rangle$. Since w normalizes D , $HxD \mapsto HxwD$ is a well-defined bijection of $H \backslash G/D$, with inverse given by w^{-1} , and $(xw)g_j^k(xw)^{-1} = xg_j^{-k}x^{-1}$; so it preserves the least ℓ with $xg_j^{\ell}x^{-1} \in H$ and inverts the local monodromy. Hence the multiset of local exponents of $C' \rightarrow C/H$ over the j -th branch point is stable under $\alpha \mapsto -\alpha$. Pair each exponent with $\alpha \neq -\alpha$ with an occurrence of $-\alpha$. The only unpaired exponents are $\alpha = m/2$, for m even. Their number is even: in the cyclic quotient of the orbifold fundamental group of C/H the handle commutators vanish, so the exponents sum to 0 modulo m , the paired ones contribute 0, and $k \cdot m/2 \equiv 0 \pmod{m}$ forces k even. Hence α is simple, s is even, and $h = 2q - 2 + s$ is even. So (H, χ) is admissible (Definition 2.3), and $e_{\sigma} \in L(\mathfrak{D})$. $\square$

Remark 3.2. No coprimality between the exponents and m is required, and unramified suppliers ($s = 0$) with $h > 0$ are admissible (Definition 2.3, Lemma 2.4). The data of [22] and the $(\mathbb{Q}(i), [-3])$ datum in $W(E_8) \cdot \mu_4$ of [23] are not real: the reflections of G_{32} have determinants $\omega^{\pm 1}$, and the order-twelve local monodromies of the latter have character values $\pm i$. For G_{32} , (L) needed an explicit witness; for the $W(E_8) \cdot \mu_4$ datum it fails [23].

The proof is an existence argument; Section 6 also records an explicit witness for its datum.

4. RANK AT A MAXIMAL DEGENERATION

A second way to certify (D) works at the boundary of the Hurwitz space and needs only the monodromy tuple. Dominance by degeneration has a precedent in [2], which proves that the Prym–Tyurin map from the Hurwitz space of $W(E_6)$ -covers of $\mathbb{P}^1$ branched over 24 points to $\mathcal{A}_6$ is dominant by degenerating the branch points to a maximally degenerate point of $\overline{\mathcal{M}}_{0,24}$ and computing the rank of a 21×21 matrix there. Here the target is the domain of $SU(n, n)$ of a Weil family rather than $\mathcal{A}_g$, and the rank condition is an exact certificate over K . Fix $\mathbf{g} \in S$, let $V = H_{\text{par}}^1(U_C, L')$ with the skew-Hermitian cup form $\mathcal{S}$ and $h = \vartheta\mathcal{S}$ of signature (n, n) (Subsection 2.2). Let $\mathcal{P} = \{\gamma_1, \dots, \gamma_{r-3}\}$ be a pants decomposition of the r -punctured sphere by simple closed curves. Three operators are attached to γ_j , in this order.

- (i) The full Dehn twist τ_j along γ_j is a braid. With the Artin convention, if γ_j encloses the points $a, \dots, b$ and $c = g_a \cdots g_b$, it acts on tuples by $g_i \mapsto c g_i c^{-1}$ for $a \leq i \leq b$, and is $(\sigma_a \cdots \sigma_{b-1})^{b-a+1}$.
- (ii) The *return power* $k_j \geq 1$ is the least power with $\tau_j^{k_j}(\mathbf{g}) = \mathbf{g}$. Thus $\tau_j^{k_j}$ is the least power of the twist whose lift to the Hurwitz space, starting at the cover of $\mathbf{g}$, is closed; it acts on V by a unitary T_j , computed by pulling back cocycles: for $u(xy) = u(x) + \rho(x)u(y)$ the pullback of u under one twist is $u'_i = u(c) + P u_i - (P g_i P^{-1}) u(c)$, with P the corresponding conjugator, and T_j is its k_j -th iterate.
- (iii) Let m_j be the order of the semisimple part of T_j . Then $T_j^{m_j}$ is unipotent, and $N_j = \log T_j^{m_j}$.

If the boundary product c is trivial in G , the twist fixes the tuple ($k_j = 1$) but still acts nontrivially on V , as a Picard–Lefschetz transvection; it is not the identity on cohomology.

The local monodromy of the period map around the boundary divisor $q_j = 0$ is the loop of (ii), and its unipotent part has logarithm N_j up to a positive rational factor. That the braid pullback in (ii) is the geometric Gauss–Manin monodromy, up to a root of unity, is Lemma 4.1; the rest is Step 1 of the proof below.

Lemma 4.1 (Geometric monodromy). *Let σ be absolutely irreducible, realized on V_σ over K , and let $\mathbf{g}$ be a product-one tuple generating G . Let $a: \mathcal{C} \rightarrow B$ be an actual family of G -covers $p: \mathcal{C} \rightarrow \mathbb{P}_B^1$ over a connected base B , étale over the interior of the Hurwitz space (for instance a rigidified chart, or the space of covers with a marked point over a tangential base point ∞), with the cover of $\mathbf{g}$ over a base point. Let L' be the coefficient system of Section 2.2, the σ -multiplicity of $p_*^0 K$ on the complement of the branch locus. Then:*

- (1) $\mathrm{Hom}_{K[G]}(V_\sigma, R^1 a_* K) \cong R^1 \bar{\pi}_*(j_* L')$ naturally, as local systems of K -spaces on B ;
- (2) the monodromy of this local system along a loop β of B is the braid pullback of parabolic cocycles by the braid of β , composed with transport of the coefficients by $\sigma(h_\beta)$, where $h_\beta \in G$ is the monodromy of the G -torsor along the section over ∞ , and $\beta(\mathbf{g}) = h_\beta \mathbf{g} h_\beta^{-1}$;
- (3) if $\beta(\mathbf{g}) = \mathbf{g}$, then $h_\beta \in Z(G)$ and $\sigma(h_\beta)$ is a scalar root of unity of order dividing $|Z(G)|$; on the pointed family, $h_\beta = 1$.

Proof. (1) Near a branch point a normal tame cover has the model $z \mapsto z^e$, and locally constant functions upstairs are the inertia-invariant functions on the punctured preimage; so $p_* K = j_* p_*^\circ K$. The finite proper map p has no higher direct images of the constant sheaf, so the Leray spectral sequence gives $R^1 a_* K = R^1 \bar{\pi}_*(p_* K)$. Taking the σ -multiplicity is exact in characteristic zero, by the averaging projector, and commutes with these direct images and with base change; this is the relative form of the identification used fibrewise in Section 2.2.

(2) This is [8, Lemma 2.2 and Theorem 2.5], restated in [9, Theorem 1.3], in their right-action convention: the monodromy is the product of the braid term and the coefficient term, where the coefficient element satisfies the displayed conjugation relation. On an actual family of G -covers the coefficient system comes from the finite G -torsor, so the coefficient element is the image of $h_\beta \in G$; this finiteness comes from the family, not from irreducibility alone.

(3) If $\beta(\mathbf{g}) = \mathbf{g}$, then h_β centralizes a generating tuple, so $h_\beta \in Z(G)$, and by Schur's lemma $\sigma(h_\beta) \in \mathrm{End}_{K[G]}(V_\sigma) = K$ is a root of unity of order dividing $|Z(G)|$. On the pointed family the marked point over ∞ returns to itself along a closed loop, so $h_\beta = 1$. The pointed family and any rigidified chart are finite and étale over the interior of the Hurwitz component, so they have the same generic rank of the period map (Step 0 below). $\square$

For G_{34} , $Z(G) = \mu_6$, and every k_j of Section 6 is an exact return (checked independently; Appendix A). So on the pointed family the geometric monodromy of $\tau_j^{k_j}$ is T_j , and on an unpointed realization it is $\zeta_j T_j$ with $\zeta_j^6 = 1$. In both cases, after raising to the power $6m_j$, the unipotent logarithm is a positive integer multiple of N_j . With one fixed orientation convention all these multiples carry the same sign, so (H1)–(H4) for the N_j are (H1)–(H4) for the actual variation.

For a pair of pants Π of $\mathcal{P}$ let P_1, P_2, P_3 be the boundary products and $G_\Pi = \langle P_1, P_2, P_3 \rangle$.

Proposition 4.2 (Rank at a maximal degeneration). *Assume:*

- (H0) for every pair of pants Π , $c_\sigma(P_1) + c_\sigma(P_2) + c_\sigma(P_3) - 2d + 2 \dim \sigma^{G_\Pi} = 0$;
- (H1) $N_j N_k = 0$ for all j, k ;

- (H2) $I = \sum_j \text{im } N_j$ is a K -subspace of dimension n , totally isotropic for h , and $N_j(I) = 0$ for every j ;
- (H3) the forms $\beta_j(x, y) = \mathcal{S}(x, N_j y)$, which are Hermitian and descend to V/I , are semidefinite of one common sign, and $\sum_j \beta_j$ is definite on V/I ;
- (H4) the N_j span a $\mathbb{Q}$ -vector space of dimension n^2 .

Then the period map of S has generic rank n^2 , and (D) holds for S .

Proof. Step 0 (ordering). Let $\Sigma \leq S_r$ be the group of permutations of the branch points preserving the class datum, that is with $\mathcal{C}_{\pi(j)} = \mathcal{C}_j$. It acts on the ordered Hurwitz space $\mathcal{H}$, and the quotient $\mathcal{H}/\Sigma$ is the Hurwitz space of covers whose branch points are unordered within each class. A braid whose permutation lies in Σ carries $\mathbf{g}$ to a tuple $\mathbf{g}'$ in the same Nielsen class. Its ordered component S' may differ from S , but S and S' map finitely and surjectively onto the same component $\bar{S}$ of $\mathcal{H}/\Sigma$. The period map of the family A_σ depends on the cover and the fixed representation, not on the labels of the branch points, so it descends to $\bar{S}$. After a level structure these maps are étale over the interior, and an étale finite map preserves generic rank. So the generic rank on S equals that on $\bar{S}$ and on S' , and $\mathbf{g}$ may be replaced by $\mathbf{g}'$. For the datum of Section 6 all twenty points lie in one class, so $\Sigma = S_{20}$.

Step 1 (the boundary point). Pinching the curves of $\mathcal{P}$ gives a stable r -pointed rational curve with $r - 2$ components and $r - 3$ nodes, a point of the normal-crossing boundary of $\overline{\mathcal{M}}_{0,r}$ [12, Theorem 2.7], with independent smoothing parameters q_j . The restrictions of the G -cover to the pairs of pants give covers of the components. At a node the two inertia elements are inverse of the same order e , and the local model $u = x^e$, $v = y^e$, $xy = t$, $uv = t^e$ [11, §4] glues them, after sufficiently divisible powers of the q_j . The local Hurwitz cover over the punctured polydisc is a finite cover of the configuration space [10, §1.2]. The loop of q_j once around 0 is the full Dehn twist τ_j (plumbing: one turn of $\arg q_j$ rotates the gluing annulus once relative to its boundary). The stabilizer of a chosen sheet of the local Hurwitz covering, under local monodromy over the punctured polydisc, is a finite-index subgroup $\Lambda \subseteq \mathbb{Z}^{r-3}$, which need not be a product of coordinate subgroups, but it contains $\bigoplus_j k_j \mathbb{Z}$ for the return powers k_j of (ii). So the Kummer cover $w_j^{|Z(G)|k_j m_j} = q_j$, on an unpointed realization (or $w_j^{k_j m_j} = q_j$ on the pointed family), dominates the connected local Hurwitz cover, by maps étale over the interior; further divisible powers may be taken if a family chart needs them. By Lemma 4.1, on it the local monodromy around $w_j = 0$ is unipotent, with logarithm a positive integer multiple of N_j . On any further base change by positive powers the logarithms are positive integer multiples $c_j N_j$. With the orientation conventions fixed once (the clockwise loops of Lemma 2.7 and [5, p. 463]), all the logarithms carry the same sign $\varepsilon \in \{\pm 1\}$. A consistent reversal of conventions changes ε for every j at once,

and cannot be made curve by curve. For the datum of Section 6, three of the seventeen curves have $k_j = 2$, and every $m_j = 1$ (Appendix A).

Step 2 (nilpotent orbit). By the several-variable nilpotent orbit theorem [18, (4.12)], [5, (1.15)], after the finite base change the lifted period map has the form $\tilde{\Phi}(z) = \exp(\sum_j z_j M_j) \cdot \psi(u)$, $M_j = \varepsilon c_j N_j$, $u_j = e^{2\pi i z_j}$, with ψ holomorphic on the full polydisc, valued in the compact dual. Put $F_0 = \psi(0)$. The real polarization of the underlying weight-one variation is, up to a positive constant and one global sign, $\mathrm{Tr}_{K/\mathbb{Q}} \mathcal{S}$; the K -action commutes with the monodromy and preserves the Hodge filtration.

Step 3 (the limit is Hodge–Tate). We give the deduction directly. Let $M = \sum_j a_j M_j$ with all $a_j > 0$, an element of the open monodromy cone. By (H1), $M^2 = 0$. By (H2), $\mathrm{im} M \subseteq I$ and $M(I) = 0$. By (H3), for $x \notin I$ we have $\mathcal{S}(x, \varepsilon Mx) = \sum_j a_j c_j \beta_j(x, x) \neq 0$, because the β_j are semidefinite of one sign with definite sum. So $V/I \rightarrow I$, $x \mapsto Mx$, is injective, hence an isomorphism since both sides have K -dimension n . Thus $\mathrm{im} M = \ker M = I$, and the monodromy weight filtration of M , centred at 1, is $W_0 = \mathrm{im} M = I$, $W_1 = \ker M = I$, $W_2 = V$. In particular $\mathrm{Gr}_1^W = 0$.

This filtration does not depend on the a_j ; the cone theorem [5, (4.4)], [4, Theorem (3.3)] says the same in general, and here it is also visible directly. The role of the polarization is [5, (4.5)]: (W, F_0) is a mixed Hodge structure polarized by M , with $F_0 = \psi(0)$ the limiting filtration of Step 2. Since the variation has weight one, its Hodge numbers lie in $\{0, 1\}$. So Gr_0^W has type $(0, 0)$ and Gr_2^W has type $(1, 1)$: the limit is Hodge–Tate. The sign convention matters only through (H3), where the Hermitian form is $\mathcal{S}(x, N_j y)$ and not $h(x, N_j y)$, which is skew-Hermitian (Remark 4.3). (H0) gives the same conclusion geometrically: by the Clemens–Schmid sequence [6, (3.3)–(3.7), Theorem 9.12, §11], restricted to an arc $u_j = t^{a_j}$ with $a_j > 0$, the weight-one part of the limit is $\bigoplus_{\Pi} H^1$ of the normalized components, whose σ -multiplicity is $\bigoplus_{\Pi} H_{\mathrm{par}}^1$ of the three-point local systems, of dimension the left side of (H0).

Step 4 (the orbit has full rank). Fix an embedding $\iota: K \rightarrow \mathbb{C}$ and write $V_{\iota} = V \otimes_{K, \iota} \mathbb{C}$, of dimension $2n$, and I_{ι} . Deligne’s bigrading [5, (2.13)] gives $V_{\iota} = I_{\iota} \oplus F_0^1$ with $\dim F_0^1 = n$. The rational Lie algebra $\mathfrak{n}_I = \{X \in \mathfrak{u}(h) : X(V) \subseteq I, X(I) = 0\}$ has dimension n^2 over $\mathbb{Q}$; via $X \mapsto \mathcal{S}X$ it is the space of Hermitian forms on V/I . Its complexification maps isomorphically onto $\mathrm{Hom}_{\mathbb{C}}(V_{\iota}/I_{\iota}, I_{\iota})$: the diagonal matrix units, and for $a < b$ the elements $E_{ab} + E_{ba}$ and $\omega E_{ab} + \bar{\omega} E_{ba}$ with $\omega \in K \setminus \mathbb{Q}$, form a $\mathbb{Q}$ -basis whose images form a $\mathbb{C}$ -basis, since $\omega \neq \bar{\omega}$. By (H1)–(H2) every N_j lies in $\mathfrak{n}_I$, and by (H4) they span it. An $X \in \mathfrak{n}_I \otimes \mathbb{C}$ is determined by $X|_{F_0^1}$, because X kills I_{ι} ; and $X \cdot F_0 = 0$ forces $X(F_0^1) \subseteq F_0^1 \cap I_{\iota} = 0$. So $v \mapsto \sum_j v_j M_j \cdot F_0$ is an isomorphism onto the tangent space $\mathrm{Hom}(F_0^1, V_{\iota}/F_0^1)$, of dimension n^2 .

Step 5 (the actual period map). After translating by $\exp(-\sum_j z_j M_j)$, the j -th column of the derivative of $\tilde{\Phi}$ is $M_j \cdot \psi(u) + 2\pi i u_j \partial \psi / \partial u_j$. Since ψ is holomorphic on the polydisc, these columns tend to $M_j \cdot F_0$ uniformly as

$u \rightarrow 0$. A nonzero $n^2 \times n^2$ minor of the limit stays nonzero for small u , so the period map has rank n^2 at interior points near the boundary. By Lemma 2.5 and Proposition 2.6, the generic rank is n^2 on S . $\square$

Remark 4.3. (H0)–(H4) are exact linear algebra over K on data computed from the tuple, and need no Riemann–Hilbert step or interval arithmetic. (H2) needs a rational Lagrangian, so the Hermitian space (V, h) must be split; for a non-split cell the criterion does not apply, and (D) must be certified otherwise. This restriction is not special to the criterion: by [3], a family of abelian $2n$ -folds of Weil type has a maximal degeneration, one whose monodromy logarithm has rank $2n$ over $\mathbb{Q}$, only if its Hermitian form has an n -dimensional totally isotropic K -subspace, that is, only for discriminant $(-1)^n$. A failure of (H0)–(H4) for one decomposition proves nothing about (D). In (H3) the form $h(x, N_j y)$ is skew-Hermitian, since N_j is infinitesimally unitary; the Hermitian form is $\mathcal{S}(x, N_j y)$.

5. COMPENSATION

One decided cell of higher-dimensional Weil type decides every cell of sixfolds over the same field. This is the higher-dimensional form of Schoen’s compensation argument [20, Proposition 10, with §§6–7], which passes from a universal family of Weil sixfolds to Weil fourfolds; Markman uses it to pass from split sixfolds to all Weil fourfolds, over every imaginary quadratic field [14]. We give the same product-and-contraction proof in arbitrary even dimension, using explicit CM elliptic factors to realize the compensating discriminant. Nothing in it depends on K : it holds for every imaginary quadratic field. Let E be an elliptic curve with complex multiplication by $\mathcal{O}_K$, and $\bar{E}$ the same curve with the conjugate action.

Lemma 5.1. *For an integer $m \geq 1$, the surface $E \times \bar{E}$ with a product polarization of degrees $(1, m)$ (line bundles of degrees 1 and m on the two factors) is of Weil type for K , of signature $(1, 1)$ and discriminant $[-m]$; its Weil plane consists of divisor classes.*

Proof. The K -actions on the tangent spaces of E and $\bar{E}$ are the two conjugate embeddings, so both occur once on the tangent space of $E \times \bar{E}$. With a common Hermitian normalization and compatible K -bases, the principal polarization on E and the degree- m polarization on $\bar{E}$ have rank-one forms (c) and $(-cm)$, for the same $c \in \mathbb{Q}^\times$; hence their orthogonal sum has determinant $-c^2 m$ and discriminant $[-m]$, since c^2 is a norm. (The constant c depends on the normalization and need not be a norm, so the two rank-one classes need not be $[1]$ and $[-m]$ separately; only their product is independent of the choices.) The Weil plane is in H^2 , of type $(1, 1)$, and its classes are divisor classes by the Lefschetz $(1, 1)$ theorem. $\square$

Proposition 5.2 (Compensation). *Let $n \geq 4$ and $\delta \in \mathbb{Q}^\times$ with $\text{sign } \delta = (-1)^n$. Suppose that the Weil classes of every polarized abelian $2n$ -fold of Weil type for K with discriminant $[\delta]$ are algebraic. Then the Weil classes*

of every polarized abelian sixfold of Weil type for K , of signature $(3, 3)$ and any discriminant, are algebraic.

Proof. Let A be a sixfold of discriminant $[t]$, $t < 0$. Put $k = n - 3$ and $S = \prod_{j=1}^k S_j$ with $S_j = E \times \bar{E}$ polarized with degrees $(1, m_j)$, where $m_2 = \dots = m_k = 1$ and m_1 is a positive integer representing the class $[(-1)^k \delta / t]$. That class is positive, since its sign is $(-1)^{k+n+1} = 1$, and a positive class has a positive integral representative because rational squares are norms. By Lemma 5.1, $A \times S$ is of Weil type for K , of signature (n, n) , with discriminant $[t] \prod_j [-m_j] = [\delta]$. So its Weil classes are algebraic. This uses the hypothesis for the special member $A \times S$, which is why it must hold for every member (it is enough to know algebraicity at very general points of the full $2n$ -fold cell, after the relative-Hilbert spreading argument of [23] and K -isogeny invariance); Theorem 2.9 gives it for every member, through Lemma 2.10.

For each embedding ι of K , the top exterior power over K factors: $\bigwedge_K^{2n} H^1(A \times S) = \bigwedge_K^6 H^1(A) \otimes \bigotimes_j \bigwedge_K^2 H^1(S_j)$. Let x_ι and $w_{j,\iota}$ generate the ι -lines of $W_K(A) \otimes \mathbb{C}$ and $W_K(S_j) \otimes \mathbb{C}$, and let $\xi = b x_\iota \otimes \bigotimes_j w_{j,\iota} + \overline{(\cdot)}$ be a nonzero rational Weil class of $A \times S$, $b \neq 0$. For nonzero rational $y_j = a_j w_{j,\iota} + \bar{a}_j w_{j,\bar{\iota}} \in W_K(S_j)$, algebraic by Lemma 5.1, we have $\int w_{j,\iota} \cup w_{j,\iota} = 0$ and $\int w_{j,\iota} \cup w_{j,\bar{\iota}} = \epsilon_j \neq 0$, so

$$\mathrm{pr}_{A*}(\xi \cup \mathrm{pr}_S^*(y_1 \boxtimes \dots \boxtimes y_k)) = b \prod_j \bar{a}_j \epsilon_j x_\iota + \overline{(\cdot)},$$

a nonzero rational algebraic class in $W_K(A)$. Here the coefficient of y_j that survives is $\bar{a}_j$, the coefficient of $w_{j,\bar{\iota}}$, because ξ pairs its factor $w_{j,\iota}$ only with the $\bar{\iota}$ -line. Now let $\alpha \in \mathcal{O}_K$ act on S_1 , an algebraic correspondence. It multiplies $w_{1,\iota}$ by $\iota(\alpha)^2$ and $w_{1,\bar{\iota}}$ by $\bar{\iota}(\alpha)^2$, under pullback α^* , itself an algebraic correspondence, so replacing y_1 by $\alpha^* y_1$ replaces the surviving coefficient $\bar{a}_1$ by $\bar{a}_1 \bar{\iota}(\alpha)^2$. (Pushforward α_* acts by the inverse eigenvalues for an automorphism; either choice, used consistently, gives the same conclusion.) Two choices of α with $\bar{\iota}(\alpha)^2 / \bar{\iota}(\alpha')^2 \notin \mathbb{Q}$ give two $\mathbb{Q}$ -independent algebraic classes in the two-dimensional $W_K(A)$. $\square$

Remark 5.3. The argument is class-blind: it passes from any one cell $[\delta]$ to every class $[t]$, split or not, and applies to every member A , with no assumption on its Hodge group. The powers statement for sixfolds then follows from Theorem 2.11 under $\mathrm{Hg}(A) = \mathrm{SU}_H$. No statement about powers of the $2n$ -folds is needed.

6. THE DATUM G_{34} AND THE PROOF OF THEOREM 1.1

Let $K = \mathbb{Q}(\sqrt{-3})$, ω a primitive cube root of unity, and $G = G_{34} \subset \mathrm{GL}_6(K)$ the Shephard–Todd group of order 39,191,040 in its reflection representation σ , of dimension $d = 6$, with character field K [24]. Its reflections have order two and form one conjugacy class $\mathcal{R}$, of size 126; each has $c_\sigma = 1$ and age $\frac{1}{2}$. Take the class datum $20 \cdot \mathcal{R}$.

(T). An explicit tuple of twenty reflections with product one generates G ; this is checked by the order of the generated group. By Proposition 2.1, $m_\sigma = 20 - 12 = 8$, so $n = 4$; the age formula gives Hodge numbers $(10 - 6, 10 - 6) = (4, 4)$. So $\mathfrak{D} = (G_{34}, \sigma; 20 \cdot \mathcal{R})$ is a Weil transfer datum of type $(K, 4)$, with $r = r_{\text{eff}} = 20 = n^2 + 4$.

(Δ). On that tuple $\dim_K H_{\text{par}}^1 = 8$, h has signature $(4, 4)$, and $\det h$ is a power of $3 = \text{Nm}(\sqrt{-3})$; its value depends on the basis (3^6 and 3^7 in two implementations on this tuple, 3^8 on the paired tuple of (D) below), but it is always a norm. So $\mathfrak{d} = [1]$, the split class. The global formula of Remark 2.8 agrees: every $\Delta(g_j)$ lies in $\{1, 3\}$.

(L). The datum is real, since reflections are involutions, so (L) holds by Theorem 3.1. Independently, an explicit witness exists: $X^*(T) \cong \mathbb{Z}^{60}$, with no parity coordinates, and $e_\sigma = \sum_i n_i a(\chi_i)$ with 84 admissible suppliers, $\sum_i |n_i| = 231$, 61 of them unramified. The suppliers come from subgroups $\langle x \rangle \times \text{Syl}_p(C_G(x))$, $p = 2, 3$; cyclic subgroups alone give a lattice of full rank that misses e_σ . Every row was rebuilt in a separate session with a different permutation representation.

(D). A word of 90 Hurwitz moves carries the tuple to a tuple of the form $(s_1, s_1, s_2, s_2, \dots, s_{10}, s_{10})$; by Step 0 of Proposition 4.2 this changes nothing. For one decomposition of the twenty-punctured sphere into 17 curves, each enclosing consecutive points, (H0)–(H4) hold, computed exactly over $\mathbb{Q}(\omega)$: all 18 pants contributions vanish; $N_j N_k = 0$; I is a four-dimensional totally isotropic K -subspace killed by every N_j ; every β_j is negative semidefinite and $\sum_j \beta_j$ is definite on V/I , with the conventions of Lemma 2.7 (clockwise loops) and the Artin twist of Section 4 (reversing either convention makes every β_j positive semidefinite at once, which (H3) equally allows); and the N_j span a 16-dimensional $\mathbb{Q}$ -space. So (D) holds by Proposition 4.2. Two independent checks, each written without access to the original code (Appendix A), cover the whole chain. A Sage implementation, taking the paired tuple and the seventeen curves as input, reproduces (H0)–(H4) with the same return powers, ranks and signs. A separate GAP check, taking only the data, verifies that the original tuple consists of twenty reflections with product one generating a group of order 39,191,040 with centre μ_6 ; that the 90-move braid word carries it to the paired tuple entry by entry; and that every k_j is an exact return, with no earlier return even up to conjugation for the three curves with $k_j = 2$, while the other fourteen curves have trivial boundary monodromy in G . A second, Hurwitz-equivalent tuple satisfies (H0)–(H4) for every one of 41 decompositions tried. As a control, the same computation on the G_{32} datum of [22], whose cell is non-split, fails (H0), (H3) and (H4), as it must (Remark 4.3). On the original tuple, decompositions into curves around two distinct reflections give $N_j = 0$, since such a curve has finite monodromy; this is why the tuple is moved first.

Remark 6.1 (Certificate formats). In the certificate language of [23, Input certificates], the two tools add two formats.

- (D_∂) **Boundary rank witness.:** A tuple braid-equivalent to $\mathbf{g}$ (with the braid word), a pants decomposition $\mathcal{P}$, and, computed exactly over K from the tuple, the twists T_j , the logarithms N_j , the subspace I , and the checks (H0)–(H4) of Proposition 4.2, including the signs in (H3). This proves (D) for S . It applies only in a split cell.
- (L_{real}) **Reality.:** For each j an element w_j with $w_j g_j w_j^{-1} = g_j^{-1}$. This proves (L) by Theorem 3.1, with no supplier list.

Theorem 6.2. *Let $K = \mathbb{Q}(\sqrt{-3})$.*

- (1) *The Weil classes of every polarized abelian eightfold of Weil type for K with signature $(4, 4)$ and discriminant $[1]$ are algebraic.*
- (2) *The Weil classes of every polarized abelian sixfold of Weil type for K , of signature $(3, 3)$ and any discriminant, are algebraic.*
- (3) *Every power of such a sixfold A with $\text{Hg}(A) = \text{SU}_H$ satisfies the Hodge conjecture.*

Proof. (1) is Theorem 2.9 for $\mathfrak{D}$, with (L), (D) and $\mathfrak{d} = [1]$ as above. (2) is Proposition 5.2 with $n = 4$ and $\delta = 1$. (3) is Theorem 2.11. $\square$

Remark 6.3. Theorem 6.2(2) contains the split sixfolds over K [14] and the cell $[-2]$ of [22]; it is new for every other class. It uses Schoen’s theorem, Milne’s theorem, Brauer’s induction theorem and the nilpotent orbit theorem, and none of the unrefereed manuscripts that [23] uses conditionally. It says nothing about the powers of special members, nor about any other field. Two further class data of G_{34} have the same budget, with one point a bireflection of order two or a rotation of order three in place of two reflections. Both are real, so (L) holds for them by Theorem 3.1, but they are not needed. The certificates, with the scripts as run, their SHA-256 hashes and the rerun commands, are deposited at <https://github.com/benzsevern/hodge-fleet>, directory `notes/transfer-theorem-review/certificates/g34-d3`; Appendix A lists the decomposition, the per-curve data and the hashes.

7. CONSEQUENCES FOR TYPE III SIXFOLDS

Theorem 6.2 covers *every* member of every cell, including very special members, and this propagates to sixfolds of type III.

Corollary 7.1. *Let D be a totally definite quaternion algebra with centre $\mathbb{Q}$ containing $\mathbb{Q}(\sqrt{-3})$, that is, ramified at ∞ and at an odd number of primes, each equal to 3 or congruent to 2 modulo 3. Let $\mathcal{A}$ be an effective PEL family of abelian sixfolds of type III with endomorphism algebra D , in the sense of [1, §2.5]. Then for every member A of $\mathcal{A}$ whose Hodge group is the generic Hodge group of the family, every Hodge class on every power A^r , $r \geq 1$, is algebraic.*

Proof. The quadratic field $K = \mathbb{Q}(\sqrt{-3})$ embeds in D by the local criterion for quadratic subfields [25, Proposition 14.6.7], and is a maximal subfield.

Restricting the D -action to K makes A a polarized abelian sixfold of Weil type for K , of signature $(3, 3)$, with K -compatible polarization [1, §4]: an element of D anticommuting with $\sqrt{-3}$ exchanges the two K -eigenspaces of $H^{1,0}(A)$, and the Rosati involution restricts to complex conjugation on K . By Theorem 6.2(2), applied to this member, $W_K(A)$ is spanned by algebraic classes. By [1, Theorem 4.1], the Hodge ring of a general member is generated by divisors together with the Weil classes of one maximal subfield, more precisely their pullbacks under the endomorphisms in D , and their algebraicity implies the Hodge conjecture for all powers of A . $\square$

Remark 7.2. Previously this was known for the split class (through [14]) and, in the author's work, for the induced class [−2]. The corollary needs no restriction on the induced class. For example, with $D = (-3, -1)$, $i^2 = -3$, $j^2 = -1$, $k = ij$, the skew-Hermitian forms $\text{diag}(j, i, i + 2k)$ and $\text{diag}(i + 4j + k, i, i + k)$ induce the classes [−5] and [−11]; the second lies outside every previously treated case. The all-powers statement comes from [1, Theorem 4.1], not from Theorem 2.11, since a general type III member need not have Hodge group SU_H for K . Special members, with Hodge group smaller than the generic one, are not covered, nor are quaternion algebras with a centre other than $\mathbb{Q}$.

8. SCOPE

Other fields. The route of Section 6 reaches a field K only through a real datum of type (K, n) , $n \geq 4$, in a split cell, so that Theorem 3.1, Proposition 4.2 and Proposition 5.2 apply. Monomial representations are excluded for $n \geq 4$, for any group, by dimension. If $\sigma = \text{Ind}_H^G \lambda$ with λ linear, Frobenius reciprocity gives $\text{Hom}_G(\sigma, H^1(C)) = \text{Hom}_H(\lambda, H^1(C))$, the λ -eigenspace of the cyclic cover $C/\ker \lambda \rightarrow C/H$. That eigenspace, with its Hodge structure, is determined by the curve C/H , the s branch points of the cyclic cover and discrete data, so the period map factors through the moduli of s -pointed curves of genus $q = g(C/H)$, of dimension $3q - 3 + s$. The eigenspace has dimension $2q - 2 + s = 2n$, so $3q - 3 + s = q + 2n - 1$ and $q \leq n + 1$, giving dimension at most $3n < n^2$ for $n \geq 4$. Compare [15, (3.3), Theorem 3.6], where for families of cyclic covers of $\mathbb{P}^1$ the dimension of the family is compared with that of the special subvariety defined by the action of μ_m , and the families that give special subvarieties are shown to be exactly the twenty for which the two agree.

Beyond that, two kinds of negative statement hold for $K \neq \mathbb{Q}(\sqrt{-3})$ and $n = 4, 5, 6$; neither is a classification over the field. Their proofs, computations and independent referee reports are deposited at <https://github.com/benzsevern/hodge-fleet>, directory `notes/transfer-theorem-review/certificates/transfer-k-inventory`; they are the author's programme's notes, not published proofs. *Proved exclusions:* primitive representations of degree at most 14; primitive groups over $\mathbb{Q}(i)$ containing a real reflection; imprimitive inductions of rank at

least 2 over a quotient curve of positive genus; and the alternating-group characters with field $\mathbb{Q}(\sqrt{-15})$ up to degree 250. *Exhaustive within a specified finite inventory:* the character tables of the GAP library CTblLib, version 1.3.9, for degrees 15 to 250, and a stated list of reflection groups and Weyl-group extensions over $\mathbb{Q}(i)$, contain no such datum. Groups outside that library, larger degrees, realisability over a larger field, and the genus-0 imprimitive residue are not covered, nor are the degree-six cases of a few Weyl-group extensions over $\mathbb{Q}(i)$ whose screen was not completed. Nothing is claimed beyond this inventory.

What is not claimed. Nothing is claimed about the powers of sixfolds whose Hodge group is smaller than SU_H , about the Hodge classes of eightfolds beyond their Weil classes, about non-split eightfold cells, or about any field other than $\mathbb{Q}(\sqrt{-3})$. The cell $(\mathbb{Q}(i), [-3])$ is treated in [23], conditionally on an outside theorem.

APPENDIX A. THE BOUNDARY CERTIFICATE FOR G_{34}

The certificate of Section 6 uses the paired tuple $(s_1, s_1, \dots, s_{10}, s_{10})$ obtained from the gate tuple by a braid word of 90 Artin generators, which is recorded in the certificate. The seventeen curves are interval curves on it, each enclosing the points $a, \dots, b$ (positions numbered 1 to 20). For each curve the table gives the return power k_j , the order m_j of the semisimple part of T_j , the K -rank of N_j (equal in every case to the Mayer–Vietoris prediction from the pants data), and the signature of β_j on V/I as (positive, negative, zero).

j	$[a, b]$	k_j	m_j	$\text{rk } N_j$	signature of β_j
1	[1, 8]	1	1	4	(0, 4, 0)
2	[1, 18]	1	1	1	(0, 1, 3)
3	[2, 8]	2	1	3	(0, 3, 1)
4	[3, 8]	1	1	3	(0, 3, 1)
5	[4, 8]	2	1	2	(0, 2, 2)
6	[5, 6]	1	1	1	(0, 1, 3)
7	[5, 8]	1	1	2	(0, 2, 2)
8	[7, 8]	1	1	1	(0, 1, 3)
9	[9, 10]	1	1	1	(0, 1, 3)
10	[9, 12]	1	1	2	(0, 2, 2)
11	[9, 16]	1	1	4	(0, 4, 0)
12	[9, 17]	2	1	4	(0, 4, 0)
13	[9, 18]	1	1	4	(0, 4, 0)
14	[11, 12]	1	1	1	(0, 1, 3)
15	[13, 14]	1	1	1	(0, 1, 3)
16	[13, 16]	1	1	2	(0, 2, 2)
17	[15, 16]	1	1	1	(0, 1, 3)

All eighteen pants contributions in (H0) vanish; the dual-graph contribution is 4, so Gr_0 and Gr_2 of the limit have K -dimension 4 and $\text{Gr}_1 = 0$. The

sum $\sum_j \beta_j$ has signature $(0, 4, 0)$, negative definite. The N_j span a space of dimension 16 over $\mathbb{Q}$ and over K .

Files and hashes. The directory `notes/transfer-theorem-review/certificates/g34-d3` of <https://github.com/benzsevern/hodge-fleet> contains the scripts as run, with SHA256SUMS. SHA-256 prefixes of the main files: `tuple_pairs_D3.json` 0021185d; `best_pairs_D3_D1.pkl` (the decomposition) 629df3f2; `cert.py` 9e2d3cbf; `kfield.py` 2c8a9688; `parab.py` a46e466b (the run's copy with its two input paths made relative); `witness_D3.g` 49c17780; `rebuild.g` c71f9916; `g34_tuples.g` aebab6f4.

Commands. `cd D; python3 cert.py D3 --pairs --decfile best_pairs_D3_D1.pkl --maxdec 0` (about ten seconds; it reports CERT True with span 16), and `cd L; gap -q -o 5g compute/run_rebuild3.g` (several minutes; it rebuilds all 84 supplier rows and checks the identity on every coordinate). Both were rerun from the deposited copy before deposit, and the logs are included. The subdirectory `independent` holds a second implementation of the boundary certificate, in Sage, written from the definitions of this paper without access to the scripts above; it reproduces every entry of the table, under the standard, inverted-twist and conjugate conventions (in the last two every sign in (H3) flips together). Its wrapper `run_all.sh` reruns all three variants from a clean checkout and records the versions used (Sage 10.7, GAP 4.14.0). A separate GAP check, `T8BW`, written from the data alone and deposited with its input files and a clean-checkout script `run.sh`, verifies the generation of G_{34} by the original tuple, the 90-move braid word to the paired tuple, and the exact return powers of the table.

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