

A TRANSFER THEOREM FOR WEIL CLASSES ON PRYM–TYURIN FACTORS OF GALOIS COVERS

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ABSTRACT. Let K be an imaginary quadratic field, G a finite group, σ an irreducible representation of G with character field K , and S a connected component of a Hurwitz space of G -covers of $\mathbb{P}^1$ with fixed local monodromy. When the σ -isotypic factors of the Jacobians are abelian $2n$ -folds of Weil type for K , we prove that the Weil classes of every polarized abelian $2n$ -fold of Weil type for K with Hermitian discriminant $\mathfrak{d}$ are algebraic, provided three conditions hold: a lattice condition (L) on the characters of a group of multiplicative type attached to the cover, dominance (D) of the period map of S , and the equality of the Hermitian discriminant of the family over S with $\mathfrak{d}$. None of the conditions refers to a particular group. The Hodge conjecture for every power of a member A follows when the special Mumford–Tate group of A itself is SU_H ; a statement about the generic member of the family does not suffice, and members with smaller Hodge group are not treated here. Positive instances of the conditions are verified by finite input certificates, so a new example needs new certificates, whose validation may need its own arguments, and no new transfer proof. The theorem recovers the case $G = G_{32}$, $K = \mathbb{Q}(\sqrt{-3})$, $\mathfrak{d} = -2$ of [19], with every step of that proof an instance of a general lemma. A second source of cycles, balanced combinations of several characters of one subgroup, is available conditionally on a theorem of an unrefereed manuscript on abelian covers. With it the datum $(\mathbb{Q}(i), [-3])$ carried by $W(E_8) \cdot \mu_4$ satisfies every hypothesis, so, conditionally on that theorem, the Weil classes of every abelian sixfold of Weil type for $\mathbb{Q}(i)$ with discriminant -3 are algebraic. For this datum the Schoen-only condition (L) fails. A companion paper [18] applies the transfer theorem, with new tools, to prove that the Weil classes of every abelian sixfold of Weil type for $\mathbb{Q}(\sqrt{-3})$, of every Hermitian discriminant, are algebraic. We also state what the theorem cannot reach.

1. INTRODUCTION

Let A be a complex abelian variety of dimension $2n$ on which an imaginary quadratic field K acts, with both embeddings of K occurring n times on the tangent space. The K -line $W_K(A) = \bigwedge_K^{2n} H^1(A, \mathbb{Q})$ is a two-dimensional

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space of Hodge classes, the Weil classes [24], [21, §5]. A compatible polarization defines a K -Hermitian form H of signature (n, n) , whose discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ depends only on the rational polarized data: a K -linear isogeny that multiplies the polarization by a positive rational number preserves it. The Weil classes are known to be algebraic for split discriminant $(-1)^n$ and $n = 3$ [17, 8]. Outside discriminant -1 , the algebraicity of Weil classes on sixfolds is described as open in [11, Introduction]. The author proved in [19] that they are algebraic for $K = \mathbb{Q}(\sqrt{-3})$, $n = 3$ and the non-split class -2 , using Prym–Tyurin factors of covers with monodromy G_{32} .

This note isolates what that proof uses. A *Weil transfer datum* (Definition 2.2) is a finite group G , an irreducible representation σ with character field K and Schur index one over K , and conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r$, such that the σ -isotypic factor $A_\sigma(C)$ of the Jacobian of a G -cover $C \rightarrow \mathbb{P}^1$ with these local monodromies is an abelian $2n$ -fold of Weil type for K .

Theorem 1.1 (Transfer theorem; Theorem 7.1). *Let $\mathfrak{D}$ be a Weil transfer datum of type (K, n) and S a connected component of its Hurwitz space. Assume*

- (L) *the determinant character e_σ lies in the lattice $L(\mathfrak{D})$ spanned by the characters of admissible suppliers (Definition 3.6);*
- (D) *the period map of the family $A_\sigma(C)$, $C \in S$, has rank n^2 at one point of S ;*

and let $\mathfrak{d}(S)$ be the Hermitian discriminant of $A_\sigma(C)$, $C \in S$. Then for every polarized abelian $2n$ -fold (A, ι, E) of Weil type for K with $\text{disc } H = \mathfrak{d}(S)$, the Weil classes $W_K(A)$ are algebraic.

The paper has two layers, and they are kept apart throughout. The transfer theorem above, with its powers theorem, is unconditional. Subsection 3.4 adds a second source of cycles, balanced combinations of several characters of one subgroup, which rests on a theorem of an unrefereed outside manuscript, stated here as Hypothesis (C). With it, Theorem 7.2 replaces (L) by a weaker lattice condition (L_{Gale}), and Corollary 11.1 proves, *conditionally on (C)*, that the Weil classes of every abelian sixfold of Weil type for $\mathbb{Q}(i)$ with discriminant -3 are algebraic. No statement in the conditional layer is claimed unconditionally.

A companion paper [18] adds two unconditional tools and applies them with this theorem. They are a proof of (L) for every datum whose local monodromies are conjugate to their inverses, by Brauer’s induction theorem, and a proof of (D) from exact data at a maximal degeneration of the Hurwitz space. It then passes from one cell of $2n$ -folds, $n \geq 4$, to every cell of sixfolds by a higher-dimensional form of Schoen’s compensation argument [17, Proposition 10]. Applied to the Shephard–Todd group G_{34} , they show that the Weil classes of every abelian sixfold of Weil type for $\mathbb{Q}(\sqrt{-3})$, of every discriminant, are algebraic.

Theorem 8.2 adds the Hodge conjecture for every power A^m under the hypothesis $\text{Hg}(A) = SU_H$ on the actual member A , not merely on the generic member of the family. Members with smaller Hodge group carry further Hodge classes on their powers; their treatment depends on the outside manuscript [13] and is not claimed here. The hypotheses name only G , σ , the classes $\mathcal{C}_j$ and the component S . Section 9 specifies, for each of them, a finite *input certificate* that verifies a positive instance: a tuple (T), a supplier witness (L), a rank witness (D) and a discriminant computation (Δ).

Two distinctions organize the paper, and the statement keeps them apart.

- *A family is not a family of cycles.* A tuple (T) shows that S is nonempty. It produces no cycle. The cycles come from (L) alone, at every member of every component with the given class datum (Corollary 3.8), and (D) is not needed for that.
- *A dimension count is not dominance, and a character identity is not an algebraic correspondence.* The count $r_{\text{eff}} - 3 \geq n^2$ is necessary for (D) and not sufficient (Remark 4.3). The identity $e_\sigma \in L$ makes one class invariant under a group U ; it becomes an algebraic correspondence only through Milne's theorem on Lefschetz classes, applied to that U -invariant class (Remark 3.9).

This paper is a companion to [19] (draft v0.10). For the identification of the parabolic cup product with an explicit cocycle formula it relies on [19, Lemma 6.2], whose finite-cover derivation we do not repeat (Lemma 5.2). Section 10 recovers [19] from its certificates A–E. Every step of that proof is an instance of a lemma below, and the steps specific to G_{32} are all steps that produce certificates. Subsection 3.4 adds a second supply of cycles, conditional on a theorem of the unrefereed manuscript [14] (Hypothesis (C)), and Theorem 7.2 is the transfer theorem with it. Section 11 applies both to the datum $(\mathbb{Q}(i), [-3])$ in $W(E_8) \cdot \mu_4$: dominance and the discriminant are certified, the lattice condition (L) fails for this datum, and the enlarged condition (L_{Gale}) is certified, so Corollary 11.1 decides the cell conditionally on (C). Section 12 records how (D) restricts the data, and Section 13 states what the theorem cannot reach.

The mechanisms are known: Schoen's cycles on cyclic covers [16], their transfer to non-abelian Prym factors [22], Milne's theorem [10], dominance of a Prym family followed by spreading [7], and Shimura families of Galois covers [6]. The multi-factor determinant supply is close to [13, Theorem 2.5] and [14]. What is new is the group-free statement with its certificate format, and, conditionally on (C), the Gale-supplier extension and its first application, the cell $(\mathbb{Q}(i), 3, [-3])$. Its first unconditional application beyond [19] is in [18].

2. WEIL TRANSFER DATA

2.1. The datum. Throughout, K is an imaginary quadratic field and $\vartheta \in K$ a nonzero element with $\bar{\vartheta} = -\vartheta$. Fix a finite group G , an integer $r \geq 4$

and conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r$ of G . The *Nielsen class* is the set of tuples $(g_1, \dots, g_r)$ with $g_j \in \mathcal{C}_j$, $g_1 \cdots g_r = 1$ and $\langle g_1, \dots, g_r \rangle = G$. The Hurwitz space $\mathcal{H} = \mathcal{H}(G; \mathcal{C}_1, \dots, \mathcal{C}_r)$ of connected Galois G -covers $C \rightarrow \mathbb{P}^1$ branched over r ordered points with local monodromy in $\mathcal{C}_j$ at the j -th point is finite and étale over the configuration space of r distinct points; its connected components are smooth, hence irreducible, of dimension $r-3$ modulo PGL_2 , and correspond to the orbits of the pure braid group on the Nielsen class modulo conjugation.

By Chevalley–Weil, $H^1(C, \mathbb{C}) = \bigoplus_{\tau} \tau \otimes M_{\tau}$ over the irreducible representations τ of G , and M_{τ} carries a Hodge structure of weight one whose Hodge numbers depend only on $(G; \mathcal{C}_1, \dots, \mathcal{C}_r)$ [6, §2]. Put $m_{\tau} = \dim M_{\tau}$. For cyclic covers of $\mathbb{P}^1$, the unitary representations of the braid group on these eigenspaces and the period maps they define are treated in [9, §§3–6].

For $g \in G$ and a representation τ of dimension d , write $c_{\tau}(g) = \text{rk}(\tau(g) - 1)$, and $\text{age } \tau(g) = \sum_k \alpha_k$ if the eigenvalues of $\tau(g)$ are $e^{2\pi i \alpha_k}$ with $0 \leq \alpha_k < 1$. Then $\text{age } \tau(g) + \text{age } \tau(g^{-1}) = c_{\tau}(g)$.

Proposition 2.1 (Defect identity). *If τ is irreducible and nontrivial, then $m_{\tau} = -2d + \sum_j c_{\tau}(g_j)$.*

Proof. Let $U = \mathbb{P}^1$ minus the branch points and V the local system on U attached to τ . Then $m_{\tau} = \dim H^1(\mathbb{P}^1, j_* V)$, the image of $H_c^1(U, V) \rightarrow H^1(U, V)$ [6, §2]; replacing τ by its dual does not change the dimension. Since τ is irreducible and nontrivial, $H^0(U, V) = 0$, and $H^2(U, V) = 0$ because U is an affine curve, so $\dim H^1(U, V) = -\chi(U)d = (r-2)d$, and $\dim H_c^1(U, V) = (r-2)d$ by duality. With Δ_j^* a small punctured disc around the j -th point, the sequence $0 = H^0(U, V) \rightarrow \bigoplus_j H^0(\Delta_j^*, V) \rightarrow H_c^1(U, V) \rightarrow H^1(U, V)$ is exact and $\dim H^0(\Delta_j^*, V) = \dim V^{g_j}$. So the image has dimension $(r-2)d - \sum_j \dim V^{g_j} = -2d + \sum_j c_{\tau}(g_j)$. $\square$

The Chevalley–Weil formula [6, Theorem 2.10, equation (2.3)] gives the Hodge numbers of M_{τ} from the ages: in the convention of [6], the multiplicity of τ in $H^0(C, K_C)$ is $-d + \sum_j \text{age } \tau(g_j^{-1}) + \epsilon_{\tau}$, where $\epsilon_{\tau} = 1$ for the trivial representation and 0 otherwise. So for *nontrivial* τ the dimensions of $\text{Hom}_G(\tau, H^0(C, K_C))$ and $\text{Hom}_G(\bar{\tau}, H^0(C, K_C))$ are $\sum_j \text{age } \tau(g_j^{-1}) - d$ and $\sum_j \text{age } \tau(g_j) - d$; with the opposite orientation convention for the loops the two are exchanged, which does not affect the balance condition in (W3). Their sum is m_{τ} by Proposition 2.1. (For the trivial representation the formula without ϵ_{τ} would give -1 .)

Definition 2.2. A *Weil transfer datum of type (K, n)* , $n \geq 2$, is $\mathfrak{D} = (G, \sigma; \mathcal{C}_1, \dots, \mathcal{C}_r)$ with σ an irreducible complex representation of G of dimension d , such that

- (W1) the character field $\mathbb{Q}(\chi_{\sigma})$ is K , and σ is realizable over K (Schur index one);
- (W2) the Nielsen class is nonempty;

(W3) $m_\sigma = 2n$, and M_σ has Hodge numbers (n, n) .

Since χ_σ is not real, $\sigma \not\cong \bar{\sigma}$. By (W1) the simple factor of $\mathbb{Q}[G]$ belonging to σ is $\mathbb{Q}[G]\varepsilon \cong \text{End}_K(V_\sigma) \cong M_d(K)$, where $V_\sigma = K^d$ carries σ and ε is central. Averaging a positive definite K -valued Hermitian form over G gives a G -invariant positive definite Hermitian form h_V on V_σ with values in K . The Rosati involution $g \mapsto g^{-1}$ of $\mathbb{Q}[G]$ acts on $\mathbb{Q}[G]\varepsilon$ as the h_V -adjoint, and on its centre K as complex conjugation.

2.2. The isotypic factor. Choose $v \in V_\sigma$ with $h_V(v, v) \neq 0$, and let $e \in \mathbb{Q}[G]\varepsilon$ be the h_V -orthogonal projector onto Kv . It is a primitive idempotent fixed by the Rosati involution. For $C \in \mathcal{H}$ put

$$A_\sigma(C) = \text{im}(Ne: J(C) \rightarrow J(C))$$

for $N \geq 1$ with $Ne \in \mathbb{Z}[G]$, with the restricted polarization E and the action ι of the centre K [3, §13.6].

Lemma 2.3. *For every $C \in \mathcal{H}$, $(A_\sigma(C), \iota, E)$ is a polarized abelian $2n$ -fold of Weil type for K : $E(\iota(a)x, y) = E(x, \iota(\bar{a})y)$, and both embeddings of K occur with multiplicity n on the tangent space.*

Proof. The isotypic part of $H_1(J(C), \mathbb{Q})$ is $V_\sigma \otimes_K M$ with $M = \text{Hom}_{K[G]}(V_\sigma, H_1(J(C), \mathbb{Q})_\varepsilon)$, and $H_1(A_\sigma, \mathbb{Q}) = Kv \otimes_K M \cong M$, a K -space with $M \otimes_K \mathbb{C}$ equal to M_σ or $M_{\bar{\sigma}}$ according to the embedding. So $H_1(A_\sigma, \mathbb{Q})$ has K -dimension $m_\sigma = 2n$, and $\dim A_\sigma = 2n$. The Rosati involution of E restricts on $\iota(K)$ to the Rosati involution of $\mathbb{Q}[G]$ on the centre, which is complex conjugation. The multiplicities of the two embeddings on $H^{1,0}(A_\sigma)$ are the Hodge numbers of M_σ , which are (n, n) by (W3). $\square$

Following van Geemen [21, Lemma 5.2], put $H(x, y) = E(x, \iota(\vartheta)y) + \vartheta E(x, y)$ on $H_1(A_\sigma, \mathbb{Q})$. It is a nondegenerate K -Hermitian form of signature (n, n) on the $2n$ -dimensional K -space $H_1(A_\sigma, \mathbb{Q})$. Its discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ is an invariant of the rational polarized data $(H_1(A_\sigma, \mathbb{Q}), \iota, E)$: a K -linear isogeny ψ with $\psi^*E = qE'$, $q \in \mathbb{Q}_{>0}$, multiplies $\det H$ by $q^{2n} = \text{Nm}(q^n)$ and so preserves the class. It is not invariant under arbitrary changes of polarization [21, Lemma 5.2]. Replacing ϑ by $q\vartheta$, $q \in \mathbb{Q}^\times$, multiplies H by q and $\det H$ by $q^{2n} = \text{Nm}(q^n)$, so the class does not depend on ϑ .

Lemma 2.4 (Constancy). *On a connected component S of $\mathcal{H}$, the class $\text{disc } H(A_\sigma(C))$ does not depend on $C \in S$. Write $\mathfrak{d}(S)$ for it.*

Proof. The spaces $H_1(A_\sigma(C), \mathbb{Q})$, with ι and E , form a local system over S (the idempotent e and ι act by flat endomorphisms of R^1 of the family of Jacobians, and E is flat). So H is a flat Hermitian form on a local system of K -spaces, and parallel transport along a path is a K -isometry. The discriminant is an isometry invariant. $\square$

Whether $\mathfrak{d}(S)$ depends on the component S inside a fixed class datum is not addressed by Lemma 2.4, and the theorem does not need it (Remark 5.3).

2.3. The Weil moduli space. By Landherr's theorem a nondegenerate K -Hermitian form is determined up to isometry by its dimension, signature and discriminant [21, §5.4]. Landherr's theorem is used only for rational Hermitian spaces, with the full class in $\mathbb{Q}^\times / \text{Nm } K^\times$; it is not used to compare integral polarization types. Fix (V, H) of K -dimension $2n$, signature (n, n) and discriminant $\mathfrak{d}$, and a lattice $\Lambda \subset V$ stable under $\mathcal{O}_K$, with the alternating form $\text{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$ scaled by a positive rational so as to be integral on Λ . Let $\mathcal{H}_n$ be the space of n -dimensional subspaces of $V_\mathbb{R}$, for the complex structure of $\iota(\vartheta)/|\vartheta|$, on which H is positive; it is the Hermitian symmetric domain of $SU(n, n)$, connected of complex dimension n^2 . For $\ell \geq 3$ let $\Gamma(\ell) \subset U(H)(\mathbb{Q})$ be the subgroup preserving Λ and acting trivially on $\Lambda/\ell\Lambda$. It is torsion free, and $\mathcal{M} = \mathcal{M}_\ell(K, n, \mathfrak{d}) = \Gamma(\ell) \backslash \mathcal{H}_n$ is a smooth connected quasi-projective variety carrying a universal abelian scheme $f: \mathcal{A} \rightarrow \mathcal{M}$ with K -action [21, §5.5–5.11], [19, §2].

Lemma 2.5 (Isogeny invariance; [19, Lemma 2.2]). *If $\psi: A' \rightarrow A$ is a K -linear isogeny of abelian varieties of Weil type for K , then $W_K(A)$ is algebraic if and only if $W_K(A')$ is. Every polarized abelian $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}$ is K -isogenous to a fibre of $\mathcal{A} \rightarrow \mathcal{M}_\ell(K, n, \mathfrak{d})$.*

The proof of [19, Lemma 2.2] uses only Landherr's theorem and $\psi^*W_K(A) = W_K(A')$, and holds verbatim for every $(K, n, \mathfrak{d})$. Any lattice in V is commensurable with Λ , so a K -isometry of rational Hermitian spaces gives a K -linear isogeny onto a fibre, which scales the polarization by a positive rational; no equality of integral polarization types is needed.

3. SUPPLY

Fix a Weil transfer datum $\mathfrak{D}$. Everything in this section holds at every $C \in \mathcal{H}$, on every component.

3.1. Suppliers. A *supplier* is a pair (H, χ) of a subgroup $H \leq G$ and a linear character χ of H of order $m \geq 3$. For $C \in \mathcal{H}$, the quotient $C' = C/\ker \chi$ is a connected cyclic $\mathbb{Z}/m$ -cover of C/H , and its primitive part $P_\chi \subset J(C)$, on which $E_\chi = \mathbb{Q}(\mu_m)$ acts, is cut out by a rational idempotent of $\mathbb{Q}[H] \subset \mathbb{Q}[G]$. Then

$$H^1(P_\chi, \mathbb{C}) = \bigoplus_{t \in (\mathbb{Z}/m)^\times} V_t, \quad V_t = \bigoplus_{\tau} \text{Hom}_H(\chi^t, \tau|_H) \otimes M_\tau,$$

with $\dim V_t = h$ independent of t , and $W_{E_\chi}(P_\chi) = \bigwedge_{E_\chi}^h H^1(P_\chi, \mathbb{Q})$ satisfies $W_{E_\chi}(P_\chi) \otimes \mathbb{C} = \bigoplus_t \det V_t$.

The points of C/H over the j -th branch point correspond to $H \backslash G / \langle g_j \rangle$, with local monodromy $xg_j^\ell x^{-1} \in H$ for a double coset HxD , ℓ the least positive integer with $xg_j^\ell x^{-1} \in H$. Let s be the number of branch points of

$C' \rightarrow C/H$, q the genus of C/H , so $h = 2q - 2 + s$, and $\alpha \in (\mathbb{Z}/m - \{0\})^s$ the local exponents with respect to a generator of $\mathbb{Z}/m$, defined up to the diagonal action of $(\mathbb{Z}/m)^\times$. Following Schoen [16, §1], α is *simple* if s is even and, after a permutation, $\alpha_{2i} = -\alpha_{2i-1}$ for all i .

Definition 3.1. A supplier is *admissible* if h is even and positive, so $h \geq 2$, and α is simple; for $s = 0$ simplicity is automatic.

Remark 3.2. Suppliers with $h = 0$ are excluded. For them $P_\chi = 0$, every V_t vanishes and $a_t(\chi) = 0$, so they contribute nothing to $L(\mathfrak{D})$ (Definition 3.6); but the identification $W_{E_\chi}(P_\chi) \otimes \mathbb{C} = \bigoplus_t \det V_t$ fails for them, since $\bigwedge_{E_\chi}^0(0) = E_\chi$ does not embed in $H^0(\text{point}, \mathbb{Q}) = \mathbb{Q}$. Such suppliers occur, for instance for an unramified cyclic isogeny of elliptic curves ($q = 1$, $s = 0$). Excluding them leaves $L(\mathfrak{D})$ unchanged.

Lemma 3.3. *For a supplier (H, χ) , the numbers q, s, h , the exponent tuple α up to $(\mathbb{Z}/m)^\times$ and permutation, and hence admissibility, depend only on $(G; \mathcal{C}_1, \dots, \mathcal{C}_r)$, not on C and not on the component of $\mathcal{H}$.*

Proof. The double coset description depends on the g_j only through their conjugacy classes: replacing g_j by yg_jy^{-1} replaces the double cosets HxD by $Hxy^{-1}(yDy^{-1})$ with the same local monodromies. Riemann–Hurwitz for $C \rightarrow C/H$ gives q from the same data, and $h = 2q - 2 + s$. $\square$

Lemma 3.4 (Schoen). *If (H, χ) is admissible, then $W_{E_\chi}(P_\chi)$ is spanned by classes of algebraic cycles at every $C \in \mathcal{H}$.*

Proof. C' is smooth, projective and irreducible, with a faithful action of $H/\ker \chi \cong \mathbb{Z}/m$. Schoen’s Theorem 2.0 [16] applies when h is even and the orbit of α is simple, with exponents allowed to share factors with m ; his Corollary 3.1 gives that the Weil classes of the primitive part of $J(C')$ are generated by algebraic cycles. That primitive part is isogenous to P_χ compatibly with E_χ . For $s = 0$ see also [15, Theorems 1.1, 1.2 and 5.3]. By Lemma 3.3 the hypotheses hold at every C once they hold for the class datum. $\square$

3.2. Characters. Let U be the centralizer of G in $\text{Sp}(H^1(C, \mathbb{Q}), E)$. Over $\mathbb{C}$ it is the product of $GL(M_\tau)$ over pairs $\{\tau, \bar{\tau}\}$ of non-self-dual τ with $m_\tau > 0$ (acting on M_τ , and contragrediently on $M_{\bar{\tau}}$), of $O(M_\tau)$ over quaternionic τ and of $\text{Sp}(M_\tau)$ over orthogonal τ . Let T be its maximal abelian quotient over $\mathbb{C}$, a group of multiplicative type; it is not a torus in general, since each quaternionic factor $O(M_\tau)$ contributes the order-two character $\det$. Its character group is $X^*(T) = X^*(T_\mathbb{C}) = \mathbb{Z}^{\text{pairs}} \oplus (\mathbb{Z}/2)^{\text{quat}}$, with the free coordinate of a pair taken with respect to a chosen member τ of the pair. The parity coordinates must be kept: they are part of condition (L). Each $\det V_t$ is a U -eigenline, with character

$$a_t(\chi) = (\langle \tau|_H, \chi^t \rangle - \langle \bar{\tau}|_H, \chi^t \rangle)_{\text{pairs}} \oplus (\langle \tau|_H, \chi^t \rangle \bmod 2)_{\text{quat}}.$$

Put $a(\chi) = a_1(\chi)$. Let $e_\sigma \in X^*(T)$ be the character of U on the line $\det M_\sigma \subset H^{2n}(A_\sigma, \mathbb{C})$: the coordinate +1 at the pair $\{\sigma, \bar{\sigma}\}$ (oriented by σ) and 0 elsewhere.

Lemma 3.5 (Conjugation). *For every supplier, $a_{-1}(\chi) = -a(\chi)$ on the free coordinates, and $a_{-1}(\chi) = a(\chi)$ on the $\mathbb{Z}/2$ coordinates.*

Proof. Complex conjugation gives $\langle \tau|_H, \chi^{-1} \rangle = \langle \bar{\tau}|_H, \chi \rangle$, which exchanges the two terms of each free coordinate. A quaternionic τ is self-conjugate, so its coordinate is unchanged. $\square$

No hypothesis on the character values of G is needed here: only $t = \pm 1$ enter the contraction below.

Definition 3.6. $L(\mathfrak{D}) \subseteq X^*(T)$ is the subgroup generated by the $a(\chi)$ of the admissible suppliers. By Lemma 3.3 it depends only on the class datum. Condition (L) is $e_\sigma \in L(\mathfrak{D})$.

3.3. The contraction.

Proposition 3.7 (Supply). *Let $C \in \mathcal{H}$, and let $X = \prod_k X_k$ be a finite product of abelian subvarieties $X_k = \text{im}(N_k e_k) \subseteq J(C)$ with e_k idempotents of $\mathbb{Q}[G]$. Let $\ell \in H^p(X, \mathbb{C})$ be a U -eigenline whose character ξ lies in $L(\mathfrak{D})$. Then ℓ is contained in the $\mathbb{C}$ -span of classes of algebraic cycles on X .*

Proof. Write $\xi = \sum_i n_i a(\chi_i)$ with admissible (H_i, χ_i) , all $n_i \neq 0$. Put $t_i = 1$ if $n_i > 0$ and $t_i = -1$ if $n_i < 0$, $E_i = E_{\chi_i}$, $P_i = P_{\chi_i}$, $m_i = \text{ord } \chi_i$, $h_i = h(\chi_i)$. By Lemma 3.4 choose for each i a nonzero rational algebraic class $y_i \in W_{E_i}(P_i)$. Let $B = \prod_i P_i^{|n_i|}$ and $Y = \boxtimes_i y_i^{\boxtimes |n_i|} \in H^*(B, \mathbb{Q})$.

The class Γ . Choose generators $z_{i,t}$ of the lines $\det V_{i,t}$, put $Z_i = \prod_{t \neq t_i} z_{i,t}$, let x generate ℓ , and put

$$\Gamma = \left(\boxtimes_i Z_i^{\boxtimes |n_i|} \right) \boxtimes x \in H^*(B \times X, \mathbb{C}).$$

Since U acts symplectically on $H^1(P_i)$, the top class $\prod_t z_{i,t}$ of $H^*(P_i)$ is U -invariant, so Z_i is a U -eigenvector of character $-a_{t_i}(\chi_i)$, which is $-\text{sign}(n_i)a(\chi_i)$ on the free coordinates by Lemma 3.5, and $a(\chi_i)$ on the $\mathbb{Z}/2$ coordinates. The character of Γ is therefore $\xi - \sum_i n_i a(\chi_i) = 0$ on the free coordinates, and $\xi - \sum_i |n_i| a(\chi_i) = \xi - \sum_i n_i a(\chi_i) = 0$ on the $\mathbb{Z}/2$ coordinates. So Γ is U -invariant.

Γ is a combination of algebraic classes. Each factor Y' of $B \times X$ is the image of an endomorphism $N'e'$ of $J(C)$ with e' an idempotent of $\mathbb{Q}[G]$ and $N'e' \in \mathbb{Z}[G]$; enlarging the integers, take one N for all factors. With $\iota_{Y'}$ the inclusion and $\pi_{Y'}$ the corestriction of Ne' , so that $\pi_{Y'} \iota_{Y'} = [N]$, the products give

$$B \times X \xrightarrow{\iota} J(C)^R \xrightarrow{\pi} B \times X$$

with $\pi \circ \iota = [N]$ and R the number of factors, both maps commuting with U , which commutes with $\mathbb{Q}[G]$. Milne's group $S(J(C))$, the largest algebraic subgroup of the symplectic group commuting with $\text{End}^0(J(C))$ [10, §3],

commutes with $\mathbb{Q}[G]$, so $S(J(C)) \subseteq U$ and $\pi^*\Gamma$ is $S(J(C))$ -invariant. By [10, Theorem 3.2], with [10, Lemma 3.1] for the extension of scalars, $\pi^*\Gamma$ is a $\mathbb{C}$ -linear combination of products of divisor classes on $J(C)^R$. The class Γ is homogeneous of degree $q_\Gamma = \sum_i |n_i|(\varphi(m_i) - 1)h_i + p$, so $\iota^*\pi^*\Gamma = [N]^*\Gamma = N^{q_\Gamma}\Gamma$, and Γ is a $\mathbb{C}$ -combination of algebraic classes.

The contraction is nonzero. Write $y_i = \sum_t c_{i,t} v_{i,t}$ with $v_{i,t} \in \det V_{i,t}$ normalized by $\int_{P_i} v_{i,t_i} \cup Z_i = 1$. For $t \neq t_i$, $v_{i,t} \cup Z_i$ contains $\det V_{i,t}$ twice and vanishes, so

$$\Gamma_*(Y) = \text{pr}_{2*}(\text{pr}_1^* Y \cup \Gamma) = \pm \prod_i c_{i,t_i}^{|n_i|} x.$$

Every $c_{i,t}$ is nonzero, because $W_{E_i}(P_i)$ is a one-dimensional E_i -vector space, so a nonzero vector has a nonzero component under every embedding of E_i . Since Γ_* maps algebraic classes with complex coefficients to such classes, x lies in their span. $\square$

Corollary 3.8 (Weil classes at every member). *Assume (L). Then for every $C \in \mathcal{H}$, on every component, $W_K(A_\sigma(C))$ is spanned by classes of algebraic cycles.*

Proof. $W_K(A_\sigma) \otimes \mathbb{C} = \det M_\sigma \oplus \det M_{\bar{\sigma}}$, two U -eigenlines with characters e_σ and $-e_\sigma$, both in $L(\mathfrak{D})$. Proposition 3.7 with $X = A_\sigma$ puts both in the $\mathbb{C}$ -span of algebraic classes. If $\mathcal{Z} \subseteq H^{2n}(A_\sigma, \mathbb{Q})$ is the $\mathbb{Q}$ -span of algebraic classes and $W \otimes \mathbb{C} \subseteq \mathcal{Z} \otimes \mathbb{C}$ for a $\mathbb{Q}$ -subspace W , then $\dim_{\mathbb{Q}}(W + \mathcal{Z}) = \dim_{\mathbb{Q}} \mathcal{Z}$, so $W \subseteq \mathcal{Z}$. $\square$

Remark 3.9 (What the identity is, and is not). The identity $e_\sigma = \sum_i n_i a(\chi_i)$ is a statement about characters of the group of multiplicative type T , including its parity coordinates. It shows only that one class Γ is U -invariant. It becomes an algebraic correspondence through two further inputs, neither of which is a character computation: $S(J(C)) \subseteq U$ together with Milne's theorem, which makes Γ algebraic, and Schoen's theorem, which makes each y_i algebraic and is the reason for the admissibility condition. A character identity with an inadmissible supplier proves nothing. Condition (L) involves no integral structure on H^1 : the conclusion is about rational classes and is isogeny invariant (Lemma 2.5).

Remark 3.10. No relative cycle over $\mathcal{H}$ is constructed. The statement is pointwise, and Section 6 uses it pointwise on an open set.

3.4. Gale suppliers, conditional on an outside theorem. Schoen's theorem supplies one character $a(\chi)$ at a time. A second source of cycles supplies balanced combinations of several characters of one subgroup at once. It rests on a theorem of an outside manuscript [14], which has not been refereed, and everything in this subsection is conditional on it. We state the input as a hypothesis.

Definition 3.11 (Hypothesis (C)). For every compact connected Riemann surface B , finite abelian group A and connected A -cover $D \rightarrow B$, let S be

a connected component of the moduli space of A -covers of that topological type, with the base curve and all branch points free. Label the deck action and add sufficient rigidifying and, where necessary, auxiliary unbranched marking data so that S is a smooth connected algebraic variety carrying a universal family $\mathcal{D} \rightarrow S$. Hypothesis (C) asserts the following local statement: on every sufficiently small connected analytic chart $O \subset S$ lifted to the full marked variation, there is a countable union E_O of proper closed analytic subsets of O such that, for $s \in O \setminus E_O$ and every $N \geq 1$, every rational Hodge class on $J(\mathcal{D}_s)^N$ is algebraic.

This local statement follows from the case $M = \text{point}$ of [14, Theorem 1.1], which imposes no bound on the genus of B , the order of A or the number of branch points. Its genericity is taken in the full marked variation of the cover. By [14, Lemma 4.1], the exceptional set there is a countable union of proper closed analytic tensor Hodge loci. Explicitly: let $\mathcal{T}$ be the full marked parameter space of [14, §1 and §4], that is, complex structures on B together with all branch positions, with a marking of the fixed topological cover. The forgetful map $\mathcal{T} \rightarrow S$ forgets the marking and keeps the rigidifying data (deck labels, a level structure, at most one auxiliary unbranched point). Each of these is either a discrete choice or one extra free point, so no equation is imposed on the base curve, on the branch positions, or on any Hodge parameter. Hence a sufficiently small chart $O \subset S$ is biholomorphic to an open subset of $\mathcal{T}$, times an open subset of the base curve when an auxiliary point is used. The Hodge structure of $J(\mathcal{D}_s)$ does not depend on that extra point. So O is not a constrained subfamily, and the preimages in O of the loci of [14, Lemma 4.1] are proper closed analytic subsets; they define E_O . Auxiliary unbranched markings impose no Hodge restriction. We need no assertion that these local exceptional sets extend to globally closed analytic subsets of S . That theorem uses [12, Theorem 1.1] and the companion results [14, Companion results 2.1 and 2.2], which are [13, Lemmas 2.1–2.4 and the saturated clause of Theorem 2.5]. We do not reprove any of them. The author’s research programme has checked [14, Theorem 1.1] with M a point line by line, finding three expository repairs and no gap; and, on the branch with positive-genus base used by the three cover families of Section 11, re-derived each step before comparing with the manuscript, with independent review, finding no gap. These checks are internal audits, not peer review, and they do not make (C) a published theorem. In this paper (C) is a hypothesis.

For a subgroup $H \leq G$ and a linear character χ of H (of any order), let $V_\chi = \bigoplus_\tau \text{Hom}_H(\chi, \tau|_H) \otimes M_\tau \subseteq H^1(C, \mathbb{C})$ be the χ -eigenspace of H , with Hodge numbers (p_χ, q_χ) , $p_\chi + q_\chi = \dim V_\chi$. They depend only on the class datum. Its determinant $\det V_\chi$ is a U -eigenline with character $a(\chi)$, given by the formula of the previous subsection.

Definition 3.12. Let $\text{Lin}^+(H)$ be the set of linear characters χ of H with $\dim V_\chi > 0$, and e the exponent of H . An integer vector $\mathbf{n} = (n_\chi) \in \mathbb{Z}^{\text{Lin}^+(H)}$

is *balanced* if

$$\delta_t(\mathbf{n}) = \sum_{\chi} n_{\chi} (p_{\chi^t} - q_{\chi^t}) = 0 \quad \text{for every } t \in (\mathbb{Z}/e)^{\times}.$$

Put $a_H(\mathbf{n}) = \sum_{\chi} n_{\chi} a(\chi)$, and let $L_H^{\text{bal}} \subseteq X^*(T)$ be the group of the $a_H(\mathbf{n})$, $\mathbf{n}$ balanced. The *Gale lattice* is

$$L_{\text{Gale}}(\mathfrak{D}) = L(\mathfrak{D}) + \sum_{H \leq G} L_H^{\text{bal}}.$$

Condition (L_{Gale}) is $e_{\sigma} \in L_{\text{Gale}}(\mathfrak{D})$. It contains condition (L), and it depends only on the class datum.

Remark 3.13. (1) Balance is a condition at every conjugate embedding, not only at the identity. The Hodge circle lies in U ; restriction of characters to this circle defines a homomorphism $\delta_T: X^*(T) \rightarrow \mathbb{Z}$, with the convention that $\delta_T(a(\chi)) = p_{\chi} - q_{\chi}$. It kills the torsion coordinates. Suppose that for each $t \in (\mathbb{Z}/e)^{\times}$ there is a sign $\epsilon_t \in \{1, -1\}$, independent of χ , such that $a(\chi^t) = \epsilon_t a(\chi)$ in $X^*(T)$ modulo torsion. Then $\delta_t(\mathbf{n}) = \epsilon_t \delta_T(a_H(\mathbf{n}))$, so all balance equations reduce to $\delta_1(\mathbf{n}) = 0$, and

$$L_H^{\text{bal}} = L_H \cap \ker \delta_T, \quad L_H = \{a_H(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^{\text{Lin}^+(H)}\}.$$

This is an equality of integer groups: every integer lift of a vector on the right is balanced. The common-sign condition holds for the datum of Section 11, whose free determinant-character coordinates are the differences of restriction multiplicities for the $\mathbb{Q}(i)$ -conjugate pairs. It is these differences, not the nonnegative restriction multiplicities themselves, that transform by a common sign. (2) Unlike a Schoen supplier, a single $\det V_{\chi}$ need not be a Hodge class, and $\dim V_{\chi}$ need not be even; only the balanced total is of Hodge type. Characters with $\dim V_{\chi} = 0$ are excluded, as in Remark 3.2. (3) As for (L), an *integer* identity is required. An identity for a multiple ke_{σ} supplies a k -fold tensor power and no k -th root of it.

Lemma 3.14 (Gale supply, conditional). *Assume (C). Let $H \leq G$, let $\mathbf{n}$ be balanced, and let $C \in \mathcal{H}$. Put $K = \bigcap_{n_{\chi} \neq 0} \ker \chi$ and $D = C/K$, a connected H/K -cover of C/H . In $H^*(J(D)^R, \mathbb{C})$, $R = \sum_{\chi} |n_{\chi}|$, let $\ell_{\mathbf{n}}$ be the external product, one factor of $J(D)$ for each occurrence, of $\det V_{\chi}$ for each occurrence with $n_{\chi} > 0$ and $\det V_{\chi^{-1}}$ for each occurrence with $n_{\chi} < 0$. Then the Galois conjugates of $\ell_{\mathbf{n}}$ span a $\mathbb{Q}$ -subspace $W_{\mathbf{n}} \otimes \mathbb{C}$, and every class of $W_{\mathbf{n}}$ is algebraic.*

Proof. The group K is normal in H , as an intersection of kernels of linear characters, and $A = H/K$ is abelian; the characters used factor through A , and $D = C/K \rightarrow C/H$ is a Galois A -cover, connected because C is. The line $\ell_{\mathbf{n}}$ has degree $d = \sum_{\chi} |n_{\chi}| \dim V_{\chi}$ and Hodge type $(\frac{d}{2}, \frac{d}{2})$ by $\delta_1(\mathbf{n}) = 0$, using $p_{\chi^{-1}} = q_{\chi}$; its Galois conjugates are the corresponding lines for $\chi \mapsto \chi^t$, of type $(\frac{d}{2}, \frac{d}{2})$ by $\delta_t(\mathbf{n}) = 0$. Take the *distinct* lines among them. Two distinct ones differ in at least one factor slot, where they lie in different character eigenspaces of A , each of positive dimension; so they are distinct direct

summands of $H^*(J(D)^R, \mathbb{C})$, and their sum is direct. They are defined over a cyclotomic field and their sum is Galois stable, so it is $W_{\mathbf{n}} \otimes \mathbb{C}$ for a $\mathbb{Q}$ -subspace $W_{\mathbf{n}}$ of Hodge classes.

Let S_A be the component of the moduli of A -covers containing $D \rightarrow C/H$, as in Hypothesis (C), and $\mathcal{D} \rightarrow S_A$ its universal family. The abelian scheme $J(\mathcal{D}/S_A)^R$ is smooth and projective over S_A , the deck action of the fixed group A extends to the universal family, so the character idempotents of $\mathbb{Q}(\mu_e)[A]$ act by flat endomorphisms, and the top exterior powers of the character subbundles, and their products on fixed slots, are flat rank-one sub-local systems; so $W_{\mathbf{n}}$ extends to a rational sub-local system $\mathcal{W}_{\mathbf{n}}$, of Hodge type $(\frac{d}{2}, \frac{d}{2})$ at every point because the Hodge numbers (p_χ, q_χ) are constant on S_A . The argument of Lemma 6.1, applied to $\mathcal{W}_{\mathbf{n}}$ inside the cohomology of the abelian scheme $J(\mathcal{D}/S_A)^R \rightarrow S_A$, shows that the set of $s \in S_A$ at which $\mathcal{W}_{\mathbf{n},s}$ is spanned by algebraic classes is a countable union of closed algebraic subsets $Z_j \subseteq S_A$. Choose a nonempty connected marked analytic chart $O \subset S_A$ as in (C). Then $O \setminus E_O \subseteq \bigcup_j Z_j$. If every Z_j were proper, the sets $O \cap Z_j$, together with the countably many proper closed analytic subsets of O defining E_O , would be nowhere dense closed subsets covering the Baire space O , a contradiction. Thus some Z_j has nonempty interior and equals S_A , which is irreducible because it is smooth and connected. Consequently $W_{\mathbf{n}}$ is algebraic at every point of S_A , in particular at D . Density alone would not suffice; the argument uses the comeagre locus supplied by (C) on a full marked chart. $\square$

Two identities are used below. By Lemma 3.5 with $t = -1$, $a(\chi^{-1}) = -a(\chi)$ on the free coordinates of $X^*(T)$ and $a(\chi^{-1}) = a(\chi)$ on the $\mathbb{Z}/2$ coordinates. Hence the line $\ell_{\mathbf{n}}$ of Lemma 3.14, which uses $\det V_{\chi^{-1}}$ for each negative occurrence, has U -character $\sum_{\chi} n_{\chi} a(\chi) = a_H(\mathbf{n})$ on the free coordinates and $\sum_{\chi} |n_{\chi}| a(\chi) \equiv a_H(\mathbf{n})$ on the $\mathbb{Z}/2$ coordinates.

Proposition 3.15 (Supply with Gale suppliers, conditional). *Assume (C). Proposition 3.7 and Corollary 3.8 hold with $L(\mathfrak{D})$ replaced by $L_{\text{Gale}}(\mathfrak{D})$: if (L_{Gale}) holds, then $W_K(A_{\sigma}(C))$ is spanned by classes of algebraic cycles for every $C \in \mathcal{H}$, on every component.*

Proof. Write $\xi = \sum_i n_i a(\chi_i) + \sum_k a_{H_k}(\mathbf{n}_k)$ with admissible Schoen suppliers (H_i, χ_i) and balanced $\mathbf{n}_k$. The Schoen terms are treated exactly as in the proof of Proposition 3.7. For a Gale term, Lemma 3.14 gives the rational space $W_{\mathbf{n}_k}$ of algebraic classes on $J(D_k)^{R_k}$. A nonzero rational $y_k \in W_{\mathbf{n}_k}$ has a nonzero projection to every conjugate line, since a vanishing projection to one would, by transitivity of the Galois action on the conjugates, force all projections to vanish; so y_k has a nonzero component c_k on $\ell_{\mathbf{n}_k}$. On each factor copy of $J(D_k)$ choose a generator of the top exterior power of the sum of the character eigenspaces other than the one selected in that slot, and let Z_k be their external product. The product of Z_k with $\ell_{\mathbf{n}_k}$ is a nonzero top class, which is U -invariant, so Z_k has character $-a_{H_k}(\mathbf{n}_k)$. A

conjugate line other than $\ell_{\mathbf{n}_k}$ differs from it in at least one slot, where its factor already occurs in the complement, so its product with Z_k vanishes. This separate-slot argument is needed: distinct conjugate lines can have the same U -character, so the character alone does not isolate $\ell_{\mathbf{n}_k}$.

Each $J(D_k)$, $D_k = C/K_k$ with $K_k = \bigcap_{n_{k,\chi} \neq 0} \ker \chi$, is isogenous to the image in $J(C)$ of the averaging idempotent $|K_k|^{-1} \sum_{g \in K_k} g$ of $\mathbb{Q}[G]$, so all factors embed and retract, up to multiplication by an integer, U -equivariantly into powers of $J(C)$. The class Γ formed from the Schoen complements, the Z_k and a generator x of the target line is U -invariant, and the Milne argument in the proof of Proposition 3.7 makes it a $\mathbb{C}$ -combination of algebraic classes. Contracting against $\boxtimes_i y_i^{\boxtimes |n_i|} \boxtimes \boxtimes_k y_k$ gives a nonzero multiple of x . The deduction of the Corollary is the same as for Corollary 3.8. $\square$

Remark 3.16. Neither Z_k nor an individual $\det V_\chi$ is claimed algebraic. The cycles come from (C), the transfer to A_σ from Milne's theorem, and the integrality from the integer identity. Without (C) none of this subsection is claimed.

4. DOMINANCE

Let $C \in \mathcal{H}$ with branch set B_C , $|B_C| = r$, and put

$$A = \mathrm{Hom}_G(\sigma, H^0(C, K_C)), \quad B = \mathrm{Hom}_G(\bar{\sigma}, H^0(C, K_C)),$$

each of dimension n by (W3). The tangent space to $\mathcal{H}$ at C is $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}(-B_C))$, Serre dual to $H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$, of dimension $r - 3$. Let

$$\mu_C: A \otimes B \rightarrow H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$$

be the multiplication $H^0(K_C) \otimes H^0(K_C) \rightarrow H^0(K_C^{\otimes 2})$ restricted to $A \otimes B$, followed by contraction of $\sigma \otimes \bar{\sigma}$ against its invariant line, with values in $H^0(C, K_C^{\otimes 2})^G = H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$.

Lemma 4.1 (Codifferential). *Write $H^{p,q}(A_\sigma(C))_\sigma$ and $H^{p,q}(A_\sigma(C))_{\bar{\sigma}}$ for the eigenspaces on which K acts through the identity embedding and its conjugate, with compatible conventions. Then*

$$H^{1,0}(A_\sigma)_\sigma \cong A, \quad H^{1,0}(A_\sigma)_{\bar{\sigma}} \cong B, \quad H^{0,1}(A_\sigma)_\sigma \cong \bar{B},$$

each of dimension n ; the full space $H^{1,0}(A_\sigma)$, of dimension $2n$, is the sum of the first two. The cotangent space of $\mathcal{H}_n$ at the period point of $A_\sigma(C)$ is $A \otimes B$, of dimension n^2 , and μ_C is, up to a nonzero normalization, the codifferential at C of the period map of the family A_σ over $\mathcal{H}$.

Proof. This is the isotypic form of the statement that the codifferential of the period map of $J(C)$ along the Hurwitz family is multiplication of differentials followed by restriction to G -invariants [6, §2–3]. The period domain $\mathcal{H}_n = SU(n, n)/S(U(n) \times U(n))$ parametrizes the σ -eigenspace $H_\sigma^{1,0}$, and its tangent space is $\mathrm{Hom}(H_\sigma^{1,0}, H_\sigma^{0,1}) \cong A^\vee \otimes \bar{B}$, whose dual, via the polarization, is $A \otimes B$. Inserting the unique invariant tensor of $\sigma \otimes \bar{\sigma}$ and multiplying

differentials gives μ_C , the isotypic restriction of the codifferential of the Jacobian, up to a nonzero normalization. The same eigenspace identifications are used for G_{32} in [19, §5]. $\square$

Proposition 4.2 (Rank at one point). *Let S be a connected component of $\mathcal{H}$ and suppose $\mathrm{rk} \mu_{C_1} = n^2$ for one $C_1 \in S$. Then the period map of S has generic rank n^2 , and is a submersion at C_1 .*

Proof. The locus where $\mathrm{rk} \mu_C \geq n^2$ is Zariski open in the irreducible S and contains C_1 ; $n^2 = \dim \mathbf{A} \otimes \mathbf{B}$ is the maximum. $\square$

Remark 4.3 (Dimension is not dominance). Proposition 4.2 needs a certified nonzero $n^2 \times n^2$ minor at a point. The inequality $\dim S \geq n^2$ is necessary and not sufficient: a family of the right dimension can have a period map of smaller generic rank, and nothing in the class datum excludes this. When $r_{\mathrm{eff}} - 3 = n^2$ (Section 12), (D) says that the period map is generically finite. In [19] the rank is 5 at the explicit cover C_0 and 9 only at a deformed cover C_1 .

Remark 4.4 (Dominance by degeneration). A second certificate for (D), used in [18], reads the rank off exact data at a maximal degeneration of the Hurwitz space. Dominance of a Prym–Tyurin family by degeneration has a precedent in [2], which proves that the Prym–Tyurin map from the Hurwitz space of $W(E_6)$ -covers of $\mathbb{P}^1$ branched over 24 points to $\mathcal{A}_6$ is dominant by degenerating the branch points to a maximally degenerate point of $\overline{\mathcal{M}}_{0,24}$ and computing the rank of a 21×21 matrix there. In the present setting the target is the domain $\mathcal{H}_n$ of $SU(n, n)$ of a Weil family rather than $\mathcal{A}_g$, and the rank condition is an input certificate (Section 9).

5. THE DISCRIMINANT

Let $C \in S$, $\pi: C \rightarrow \mathbb{P}^1$, $U_C = \mathbb{P}^1 - B_C$, and $\mathcal{R} = \pi_* \mathbb{Q}|_{U_C}$, a local system with fibre $\mathbb{Q}[G]$. Put $L' = \mathrm{Hom}_G(V_\sigma, \mathcal{R} \otimes K)$, a local system of K -vector spaces of rank d , with monodromy σ or $\bar{\sigma}$ composed with the monodromy of π . Let b be the rational fibre pairing on $\mathcal{R}$, multiplication followed by the sum over the fibre, a positive definite symmetric $\mathbb{Q}$ -bilinear form. Its K -bilinear extension to $\mathcal{R} \otimes K$ vanishes on the σ -isotypic summand, which it pairs with the $\bar{\sigma}$ -summand; we use instead the sesquilinear extension $b^s(x, y) = b(\bar{x}, y)$, with $\bar{x}$ the conjugate of the K -coefficients of x . Conjugating x moves it from the σ -component into the $\bar{\sigma}$ -component, where the bilinear pairing with the σ -component is nonzero. Thus b^s is conjugate-linear in the first slot and linear in the second, as are H and T (for $J = \iota(\vartheta)$ one has $H(Jx, y) = -\vartheta H(x, y)$ and $H(x, Jy) = \vartheta H(x, y)$); all coefficient and cup forms in this section, and in the certificates of [19], follow this convention. It is a positive definite Hermitian form, and by Schur’s lemma it restricts on $V_\sigma \otimes L'$ to $h_V \otimes h_{L'}$ for a flat positive definite Hermitian form

$h_{L'}$, unique up to $\mathbb{Q}_{>0}$ because L' is absolutely irreducible. Let

$$\mathcal{S}(a, b) = \int_{U_C} h_{L'}(a \cup b), \quad a, b \in H_{\text{par}}^1(U_C, L'),$$

the parabolic cup product, a nondegenerate K -sesquilinear skew-Hermitian form on the $2n$ -dimensional K -space $H_{\text{par}}^1(U_C, L')$, and put $h = \vartheta \mathcal{S}$, a Hermitian form.

Proposition 5.1 (The discriminant from the monodromy tuple). $\mathfrak{d}(S) = \det h$ in $\mathbb{Q}^\times / \text{Nm } K^\times$, and H and h have the same signature up to the order of the pair.

Proof. Step 1. On $H_1(A, \mathbb{Q})$ for any (A, ι, E) of Weil type for K , put $T = H/(2\vartheta)$. Since $\bar{\vartheta} = -\vartheta$, $T + \bar{T} = E(x, \iota(\vartheta)y)(\frac{1}{2\vartheta} + \frac{1}{2\bar{\vartheta}}) + E = E$, so T is K -sesquilinear and skew-Hermitian with $\text{Tr}_{K/\mathbb{Q}} T = \bar{E}$, and T is the unique such lift of E .

Step 2 (Schur–Morita). Let N be a σ -isotypic $K[G]$ -module and $M_N = \text{Hom}_{K[G]}(V_\sigma, N)$. The evaluation map $\text{ev}: V_\sigma \otimes_K M_N \rightarrow N$, $v \otimes m \mapsto m(v)$, is a G -equivariant isomorphism: it is the counit of the adjunction between $V_\sigma \otimes_K -$ and $\text{Hom}_{K[G]}(V_\sigma, -)$, and it is bijective because $\text{End}_{K[G]}(V_\sigma) = K$ (σ is absolutely irreducible and realized over K). All sesquilinear forms below are conjugate-linear in the first slot. Let β be a G -invariant sesquilinear form on N . For $m, m' \in M_N$, the form $(v, v') \mapsto \beta(m(v), m'(v'))$ on V_σ is G -invariant and sesquilinear. Such forms correspond to $\text{Hom}_{K[G]}(V_\sigma, \bar{V}_\sigma^\vee)$, and h_V identifies $\bar{V}_\sigma^\vee$ with V_σ , so they form the one-dimensional K -space spanned by h_V . Hence $\beta(m(v), m'(v')) = h_V(v, v') \psi_\beta(m, m')$ for a unique $\psi_\beta(m, m') \in K$; the form ψ_β on M_N is sesquilinear, it is skew-Hermitian or nondegenerate exactly when β is, and $\text{ev}^* \beta = h_V \otimes \psi_\beta$. Applied to $N = H_1(J(C), \mathbb{Q})_\varepsilon$, a $K[G]$ -module through the centre K of $\mathbb{Q}[G]\varepsilon$, with $M = M_N$, and to $\beta = T$, which is G -invariant by the uniqueness of the lift in Step 1, this gives $T = h_V \otimes \psi$ with $\psi = \psi_T$ skew-Hermitian.

Step 3. Since e is Rosati-fixed, the polarization of A_σ is the restriction of E , and $T|_{A_\sigma} = h_V(v, v) \psi$ under $m \mapsto v \otimes m$, with $h_V(v, v) \in \mathbb{Q}_{>0}$.

Step 4 (comparison). Since π is finite and C is normal, $\pi_* \mathbb{Q}_C = j_* \mathcal{R}$ and $H^1(C, \mathbb{Q}) = H_{\text{par}}^1(U_C, \mathcal{R})$, with the cup product corresponding to the cup product with the fibre pairing b . On $H^1(C, K)$ the K -bilinear extension of this cup product pairs the σ - and $\bar{\sigma}$ -components with each other; composing it with conjugation of the coefficients in the first argument gives the sesquilinear cup product, which is the cup product with b^s . Concretely, for rational classes x, y in the ε -part write $x = x_\sigma + x_{\bar{\sigma}}$ in $H^1(C, K)$, so that $x_{\bar{\sigma}} = \overline{x_\sigma}$. The bilinear cup product vanishes on $\sigma \times \sigma$ and $\bar{\sigma} \times \bar{\sigma}$, so

$$x \cup y = \beta(x, y) + \overline{\beta(x, y)}, \quad \beta(x, y) = \int b^s(x_\sigma \cup y_\sigma);$$

that is, β is the skew-Hermitian K -lift of the rational cup product in the sense of Step 1. The exact functor $\text{Hom}_G(V_\sigma, -)$ commutes with j_* and

cohomology, so $H^1(C, K)_\sigma = V_\sigma \otimes_K H_{\text{par}}^1(U_C, L')$, and since $b^s|_{V_\sigma \otimes L'} = h_V \otimes h_{L'}$ and cup products are computed fibrewise, Step 2 gives $\beta = h_V \otimes \mathcal{S}$. Poincaré duality on C and the identification of H_1 with H^1 by E are G -equivariant and carry E to $\pm$ the cup product; by the uniqueness of the skew-Hermitian lift they carry T to $\pm\beta$, or to its conjugate if they exchange the σ - and $\bar{\sigma}$ -components. Applying $\text{Hom}_{K[G]}(V_\sigma, -)$ gives a K -linear (or conjugate-linear) isomorphism $\Phi: M \rightarrow H_{\text{par}}^1(U_C, L')$, and the diagram

$$\begin{array}{ccc} M \times M & \xrightarrow{\psi} & K \\ \Phi \times \Phi \downarrow & & \downarrow c^{-1}. \\ P \times P & \xrightarrow{\mathcal{S}} & K \end{array} \quad P = H_{\text{par}}^1(U_C, L'),$$

commutes in the sense that $\psi(m, m') = c \mathcal{S}(\Phi m, \Phi m')$, for a scalar $c \in K^\times$ that absorbs the normalizations of h_V , $h_{L'}$ and the duality isomorphisms.

The scalar c is rational. Both ψ and $\mathcal{S} \circ (\Phi \times \Phi)$ are nonzero skew-Hermitian forms, so for m, m' with $\mathcal{S}(\Phi m, \Phi m') \neq 0$,

$$c \mathcal{S}(\Phi m', \Phi m) = \psi(m', m) = -\overline{\psi(m, m')} = -\bar{c} \overline{\mathcal{S}(\Phi m, \Phi m')} = \bar{c} \mathcal{S}(\Phi m', \Phi m),$$

whence $c = \bar{c}$, that is $c \in \mathbb{Q}^\times$. In the conjugate case, when L' carries the conjugate monodromy, the same diagram holds with bottom row $\bar{\mathcal{S}}$, so that $\psi(m, m') = c \bar{\mathcal{S}}(\Phi m, \Phi m')$. The same argument gives $c \in \mathbb{Q}^\times$, and the resulting sign is recorded in Step 5. Its sign is not determined by these considerations: reversing the orientation of the fundamental class (Lemma 5.2) replaces $\mathcal{S}$ by $-\mathcal{S}$, and the sign convention in Poincaré duality does the same. Only $c \in \mathbb{Q}^\times$ is used: in Step 5 it enters the determinant of a form of even rank $2n$ as $c^{2n} = \text{Nm}(c^n)$, which is trivial in $\mathbb{Q}^\times / \text{Nm } K^\times$.

Step 5. Hence $H|_{A_\sigma} = 2\vartheta T|_{A_\sigma} = 2h_V(v, v)c \cdot h$, or, in the conjugate case, $-2h_V(v, v)c \cdot \bar{h}$, since $\vartheta \bar{\mathcal{S}} = -\bar{h}$. Scaling a Hermitian form of dimension $2n$ by $q \in \mathbb{Q}^\times$ multiplies its determinant by $q^{2n} = \text{Nm}(q^n)$, and $\det \bar{h} = \det h$. So $\text{disc } H|_{A_\sigma} = \det h$, and the signatures agree up to order. By Lemma 2.4 this is $\mathfrak{d}(S)$. $\square$

The form $\mathcal{S}$ is computable from the monodromy tuple alone. Let W be a local system of K -vector spaces on $\mathbb{P}^1$ minus r points with a flat Hermitian form h_W and unitary monodromy $T_1, \dots, T_r$ along standard loops $\ell_1 \cdots \ell_r = 1$, generating a finite group. Put $Q_i = T_1 \cdots T_{i-1}$ and $M_i = 1 - T_i$. A parabolic cocycle is given by vectors $a_1, \dots, a_r$ with $\sum_i Q_i M_i a_i = 0$; its value on ℓ_i is $c_i = M_i a_i$. Put $F_i = \sum_{j < i} Q_j c_j$.

Lemma 5.2 (Cocycle formula; [19, Lemma 6.2]). *For parabolic cocycles a, b , the cup product of their classes with the fibre pairing h_W , on the fundamental class oriented so that the loops run clockwise, is*

$$\mathcal{S}(a, b) = \sum_{i=1}^r h_W(Q_i^{-1} F_i^a + a_i, c_i^b),$$

and $-\mathcal{S}(a, b)$ for the opposite orientation. It vanishes on coboundaries and is skew-Hermitian.

Proof. This is [19, Lemma 6.2]; as a companion paper we rely on it and do not repeat its derivation. That proof uses only the finiteness of the monodromy group, to pass to a finite cover on which W is trivial and compute the simplicial cup product there, and the unitarity of the T_i ; it holds verbatim for every such W . $\square$

Remark 5.3 (Per component). Proposition 5.1 computes $\mathfrak{d}(S)$ from the monodromy tuple of any point of S , that is from any tuple in the braid orbit defining S . A computation on a tuple of another braid orbit with the same class datum computes $\mathfrak{d}$ of another component. Lemma 2.4 does not compare components; Theorem 7.1 pairs (D) and $\mathfrak{d}$ on the same component and needs no comparison. A local formula for the inert symbols of $\det h$ in terms of the local monodromies alone [20] would not by itself make $\mathfrak{d}$ a function of the class datum: the class in $\mathbb{Q}^\times / \text{Nm } K^\times$ is determined by all local symbols, the signature fixes the one at ∞ , split primes contribute trivially, and the symbols at the ramified primes of K must be controlled as well. When K has a single ramified prime, as $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$ do, reciprocity supplies that last symbol; for general K it does not. We use no such formula, and we keep the per-component statement.

The author's notes [20] now contain a *global* formula that removes this caveat for the discriminant. For every n ,

$$[\det h] = \left[\vartheta^{2n} \prod_{i=1}^r \Delta(g_i) \right] = \left[(-1)^n \prod_{i=1}^r \Delta(g_i) \right] \in \mathbb{Q}^\times / \text{Nm } K^\times,$$

the exponent being the K -rank $2n$ of H_{par}^1 ; since $\vartheta^2 = -\text{Nm } \vartheta$, the factor ϑ^{2n} has class $[(-1)^n]$. (Draft v0.11 wrote ϑ^n for n even, which gives the wrong class when $n \equiv 2 \pmod{4}$.) where $\Delta(g)$ is the determinant of the Wall form $(x, y) \mapsto h_V((1-g)^{-1}x, y)$ on $\text{Im}(1-g)$, with $(1-g)^{-1}$ the inverse of $1-g$ restricted to its image. It needs no lattice hypothesis and holds for every imaginary quadratic K . Since the right side depends only on the conjugacy classes of the local monodromies (with σ and its coefficient pairing fixed up to rational scaling), $\mathfrak{d}(S)$ is the same on every component of a class datum. That formula has been checked against direct determinant computations and independently refereed in the author's programme, but it is not proved here, and the theorems of this paper do not use it. It does not transfer dominance between components, which remains a per-component property.

6. SPREADING

Fix $(K, n, \mathfrak{d})$ and $\mathcal{M} = \mathcal{M}_\ell(K, n, \mathfrak{d})$ as in §2.3.

Lemma 6.1. *Let $\mathcal{W} = \bigwedge_K^{2n} R^1 f_* \mathbb{Q} \subset R^{2n} f_* \mathbb{Q}$. The set $\mathcal{M}^{\text{alg}}$ of $t \in \mathcal{M}$ at which $\mathcal{W}_t$ is spanned by classes of algebraic cycles on $\mathcal{A}_t$ is a countable union of closed algebraic subsets of $\mathcal{M}$.*

Proof. The K -action is by flat endomorphisms, so $\mathcal{W}$ is a rank-two rational sub-local system. Fix a relatively ample bundle on $\mathcal{A} \rightarrow \mathcal{M}$. The relative Hilbert schemes of n -dimensional subschemes of the fibres, over the countably many Hilbert polynomials, are projective over $\mathcal{M}$, with countably many irreducible components, each with closed image [23, §3.3.2]. An algebraic class in $H^{2n}(\mathcal{A}_t, \mathbb{Q})$ is a rational combination of classes of integral n -dimensional subschemes, so it suffices to consider finite tuples of effective cycles. For a tuple $(\mathcal{H}_1, \dots, \mathcal{H}_k)$ of components, let T run through the irreducible components of $\mathcal{H}_1 \times_{\mathcal{M}} \dots \times_{\mathcal{M}} \mathcal{H}_k$, with a resolution $\tilde{T} \rightarrow T$ and composite $g: \tilde{T} \rightarrow \mathcal{M}$. The pulled-back universal subschemes are flat over $\tilde{T}$, so their fibre classes are flat sections of $g^* R^{2n} f_* \mathbb{Q}$, and their rational span $\mathcal{F}$ is a sub-local system of constant rank [23, p. 94]. On the connected $\tilde{T}$ the condition $g^* \mathcal{W} \subseteq \mathcal{F}$ holds everywhere or nowhere. If $\mathcal{W}_t$ is algebraic, effective cycles spanning a space containing $\mathcal{W}_t$ give a point of some T on which the condition holds, hence t lies in its closed image; conversely each such T has algebraic Weil planes over its whole image. There are countably many T [23, proof of Theorem 7.19], [5, proof of Proposition 40].

In more detail: each relative Hilbert scheme $\mathcal{H}_l \rightarrow \mathcal{M}$ is projective, so a finite fibre product $\mathcal{H}_1 \times_{\mathcal{M}} \dots \times_{\mathcal{M}} \mathcal{H}_k$ is projective over $\mathcal{M}$ and has finitely many irreducible components T ; the projection $T \rightarrow \mathcal{M}$ is proper, and so is the composite with a resolution $\tilde{T} \rightarrow T$, which is proper and surjective. Hence $g(\tilde{T}) = g(T)$ is a closed algebraic subset of $\mathcal{M}$. Each universal family $\mathcal{Z}_l \subset \mathcal{A} \times_{\mathcal{M}} \mathcal{H}_l$ is flat over $\mathcal{H}_l$, and flatness is preserved by the base change $\tilde{T} \rightarrow \mathcal{H}_l$, so the fibre classes $[\mathcal{Z}_{l,u}] \in H^{2n}(\mathcal{A}_{g(u)}, \mathbb{Q})$ form flat sections of $g^* R^{2n} f_* \mathbb{Q}$ [23, p. 94]. Their span $\mathcal{F}$ and $g^* \mathcal{W}$ are sub-local systems of $g^* R^{2n} f_* \mathbb{Q}$, and parallel transport along any path in $\tilde{T}$ preserves both; so the set of u with $g^* \mathcal{W}_u \subseteq \mathcal{F}_u$ is open and closed, hence all of the connected $\tilde{T}$ or empty. There are countably many Hilbert polynomials, countably many finite tuples of components and finitely many T for each tuple. This is the argument of [23, §3.3.2 and the proofs of Theorems 7.19 and 8.26] and of the unconditional part of [5, proof of Proposition 40], which show that the locus where given flat classes are algebraic is a countable union of closed algebraic subsets; here it is applied to the inclusion of the sub-local system $\mathcal{W}$. $\square$

Lemma 6.2. *If $\mathcal{M}^{\text{alg}}$ contains a nonempty Euclidean open subset of $\mathcal{M}$, then $\mathcal{M}^{\text{alg}} = \mathcal{M}$.*

Proof. A nonempty open U is a Baire space and the countable union of the closed sets $U \cap Z_j$, so some Z_j has nonempty interior; a proper closed algebraic subset of the irreducible $\mathcal{M}$ has empty interior, so $Z_j = \mathcal{M}$ [23, proof of Theorem 8.26]. $\square$

7. THE TRANSFER THEOREM

Theorem 7.1 (Transfer). *Let $\mathfrak{D}$ be a Weil transfer datum of type (K, n) (Definition 2.2) and S a connected component of its Hurwitz space. Assume*

- (L) $e_\sigma \in L(\mathfrak{D})$ (Definition 3.6);
- (D) $\text{rk } \mu_{C_1} = n^2$ at some $C_1 \in S$ (Section 4).

Let $\mathfrak{d} = \mathfrak{d}(S)$ (Lemma 2.4), computed by Proposition 5.1 from the monodromy tuple of any point of S . Then for every polarized abelian $2n$ -fold (A, ι, E) of Weil type for K with $\text{disc } H = \mathfrak{d}$, the Weil classes $W_K(A)$ are spanned by classes of algebraic cycles.

Proof. Let $D \ni C_1$ be a small polydisc in the configuration space over which the covers of S form an analytic family. Over the simply connected D , the spaces $H_1(A_\sigma(C), \mathbb{Q})$ with ι and H form a trivial flat bundle. They have K -dimension $2n$, signature (n, n) (Lemma 2.3) and discriminant $\mathfrak{d}$ (Lemma 2.4), so by Landherr's theorem there is a K -isometry $m: H_1(A_\sigma(C_1), \mathbb{Q}) \rightarrow (V, H)$, which we transport over D . Sending C to the complex structure of $H_1(A_\sigma(C), \mathbb{R})$, transported by m , gives a holomorphic period map $\Phi: D \rightarrow \mathcal{H}_n$, a submersion at C_1 by (D), Lemma 4.1 and Proposition 4.2. Shrinking D , $\Phi(D)$ contains a nonempty open ball $\mathcal{B}$, and its image $U_{\mathcal{B}}$ in $\mathcal{M}$ is open, because $\Gamma(\ell)$ acts properly discontinuously and freely.

For $C \in D$ with $\Phi(C) = z \in \mathcal{B}$, let A_z be the fibre of $\mathcal{A}$ with $H_1(A_z, \mathbb{Z}) = \Lambda$ and complex structure z . The lattice $m(H_1(A_\sigma(C), \mathbb{Z}))$ is commensurable with Λ ; if $N m(H_1(A_\sigma(C), \mathbb{Z})) \subseteq \Lambda$, multiplication by N is a K -linear isogeny $A_\sigma(C) \rightarrow A_z$; it scales the polarization by a positive rational number, which changes neither the Weil plane nor the class of $\text{disc } H$. By (L), Corollary 3.8 and Lemma 2.5, $W_K(A_z)$ is algebraic. So $U_{\mathcal{B}} \subseteq \mathcal{M}^{\text{alg}}$, and $\mathcal{M}^{\text{alg}} = \mathcal{M}$ by Lemma 6.2. By Lemma 2.5 every polarized $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}$ is K -isogenous to a fibre of $\mathcal{A}$. $\square$

Theorem 7.2 (Transfer with Gale suppliers; conditional on (C)). *Assume Hypothesis (C) (Definition 3.11). Then Theorem 7.1 holds with (L) replaced by (L_{Gale}) (Definition 3.12).*

Proof. The proof of Theorem 7.1 uses (L) only through Corollary 3.8. Proposition 3.15 gives that corollary under (L_{Gale}) and (C). $\square$

Remark 7.3 (Where each hypothesis enters). (L) is a property of the class datum and gives the cycles at every member of every component (Corollary 3.8). (D) and $\mathfrak{d}$ are properties of one component S : (D) gives an open set of $\mathcal{M}$, and $\mathfrak{d}(S)$ says which $\mathcal{M}$. (W2) gives only that S exists. A component with (D) whose class datum fails (L) carries a dominant family without cycles; a class datum with (L) whose components all fail (D) carries cycles on a family too small to spread. The datum of G_{25} in [19, §10] is of the second kind: (L) holds, and its family has dimension $8 < 9$.

Remark 7.4. The argument needs algebraicity on an open set; a countable Zariski-dense set would not suffice. It uses no deformation of cycles, no variational Hodge conjecture and no Principle B. Theorem 7.2 uses, in addition, the outside Hypothesis (C), and is not claimed unconditionally.

8. POWERS

Let W be a complex vector space of dimension N and U, U' complex vector spaces of dimension m , and put $\Lambda = \bigwedge(W \otimes U) \otimes \bigwedge(W^* \otimes U')$ with the commuting actions of $GL(W)$ and $GL(U) \times GL(U')$. For $u \in U$, $u' \in U'$ let $\theta(u, u') = \sum_i (w_i \otimes u) \wedge (w_i^* \otimes u')$ for dual bases, and for $a \in U$ let $\det(W \otimes a) \in \bigwedge^N(W \otimes U)$ be the image of a generator of $\bigwedge^N W$ under $w \mapsto w \otimes a$; define $\det(W^* \otimes a')$ in the same way.

Lemma 8.1 ([19, Lemma 9.2]). $\Lambda^{SL(W)}$ is generated by the $\theta(u, u')$, $\det(W \otimes a)$ and $\det(W^* \otimes a')$.

Proof. The proof of [19, Lemma 9.2] is written for arbitrary N and m . We recall it. Let $\mathcal{G}$ be the subalgebra generated; it is $GL(U) \times GL(U')$ -stable. By the skew Cauchy formula $\Lambda = \bigoplus_{\lambda, \mu} S_\lambda W \otimes S_\mu W^* \otimes S_{\lambda'} U \otimes S_{\mu'} U'$ over partitions in the $N \times m$ box, and $(S_\lambda W \otimes S_\mu W^*)^{SL(W)}$ is a line when $\lambda - \mu = (k^N)$ and zero otherwise. So $\Lambda^{SL(W)} = \bigoplus_k R_k$ with $R_k = \bigoplus_\mu V_{k, \mu}$, $V_{k, \mu} \cong S_{(\mu + (k^N))'} U \otimes S_{\mu'} U'$ for $k \geq 0$ (and symmetrically for $k < 0$), irreducible and pairwise non-isomorphic. For $k = 0$, $R_0 = \Lambda^{GL(W)}$; in bidegree (p, p) with $p \leq m$ the weight space of weight $(e_1 + \cdots + e_p; e'_1 + \cdots + e'_p)$ is $W^{\otimes p} \otimes W^{*\otimes p}$, whose invariants are spanned by complete contractions [1, Ch. II, §3], which are products of θ 's up to sign. Every irreducible polynomial module in that bidegree meets this weight space, so by semisimplicity $\mathcal{G} \cap R_0 = R_0$ there; for $p > m$ use surjections $\mathbb{C}^p \rightarrow U$, $\mathbb{C}^p \rightarrow U'$ and reductivity. For $k \geq 1$ and $\mu_1 + k \leq m$, $d = \det(W \otimes u_1) \wedge \cdots \wedge \det(W \otimes u_k)$ and a $GL(W)$ -invariant z of type $S_\mu W \otimes S_\mu W^*$ in the variables $u_{k+1}, \dots, u_m$ give $d \wedge z \neq 0$ in $V_{k, \mu} \cap \mathcal{G}$, so $V_{k, \mu} \subseteq \mathcal{G}$. The case $k \leq -1$ is symmetric. $\square$

Theorem 8.2 (Powers). *Let (A, ι, E) be a polarized abelian $2n$ -fold of Weil type for K whose Weil classes are algebraic, for instance one covered by Theorem 7.1. Assume that the special Mumford–Tate group of A itself is $\text{Hg}(A) = SU_H$; knowing the generic Hodge group of a family containing A does not suffice. Then for every $m \geq 1$ every Hodge class on A^m is algebraic. In particular ($m = 1$) A satisfies the Hodge conjecture.*

Proof. Let $V_{\mathbb{C}} = H^1(A, \mathbb{C}) = W \oplus W'$ be the eigenspace decomposition for K . The polarization identifies W' with W^* as $SU_H(\mathbb{C})$ -modules, and $SU_H(\mathbb{C}) = SL(W)$. The Hodge group of A^m is SU_H acting diagonally on $V^{\oplus m}$, so with $U = U' = \mathbb{C}^m$ kept separate on the two eigenspaces, $B^*(A^m) \otimes \mathbb{C} = \Lambda^{SL(W)}$ with $N = 2n$. The $\theta(u, u')$ span the invariants of degree two (for $N = 2n \geq 4$, $\bigwedge^2(W \otimes U)^{SL(W)} = 0$), so they are Hodge classes of type $(1, 1)$ and lie in the complex span of divisor classes by the

Lefschetz (1, 1) theorem. Let $\mathcal{O} = \{a \in K : \iota(a) \in \text{End}(A)\}$, an order of K ; a rational K -action need not make every element of $\mathcal{O}_K$ an endomorphism. For $a \in \mathcal{O}^m$, the homomorphism $f_a: A^m \rightarrow A$, $(x_i) \mapsto \sum_i \iota(a_i)x_i$, pulls the two lines of $W_K(A) \otimes \mathbb{C}$ back to $\det(W \otimes \tilde{a})$ and $\det(W^* \otimes \tilde{a})$, where $\tilde{a}$ is the vector of images under the identity embedding; these are algebraic. A linear form on $\text{Sym}^N U$ vanishing on all $\tilde{a}^N$ is a polynomial vanishing on the set $\{\tilde{a} : a \in \mathcal{O}^m\}$, which is Zariski dense in U because $\mathcal{O}$ is a lattice in K ; so the $\det(W \otimes \tilde{a})$ span all $\det(W \otimes a)$, and likewise for W^* . By Lemma 8.1 the complex span of products of algebraic classes contains $B^*(A^m) \otimes \mathbb{C}$, and rational descent as in Corollary 3.8 gives $B^*(A^m) \subseteq$ the $\mathbb{Q}$ -span of algebraic classes. $\square$

Remark 8.3. For $n = 3$, the hypothesis $\text{Hg}(A) = SU_H$ holds off a countable union of proper closed analytic subsets of $\mathcal{H}_3$ [21, Theorem 6.11], as used in [19]. We make no claim about the generic Hodge group for other n ; neither Theorem 7.1 nor Theorem 8.2 uses one, since the latter assumes $\text{Hg}(A) = SU_H$ for the member A at hand. Members with smaller Hodge group carry further Hodge classes on their powers, which Theorem 8.2 does not treat; [13, Theorem 2.5 and §3] treats them with the algebraic Weil plane as input. That special-member extension depends on the outside manuscript [13], which has not been refereed, and is not claimed in this paper.

9. INPUT CERTIFICATES

Positive instances of the hypotheses of Theorem 7.1 are verified by four finite certificates. We give no procedure that decides when a hypothesis fails, and no guarantee that a useful certificate, for instance an exact rank witness, can be found. Each certificate is a script with exact inputs and a transcript, together with any mathematical argument needed to show that the script computes what it claims; it establishes its hypothesis by the lemma named. A new example therefore needs no new transfer proof, but constructing and validating its certificates may need new arguments.

(T) Tuple.: A finite group G given by generators (for instance matrices over K realizing σ , which gives (W1)'s Schur index one), and a tuple $\mathbf{g} = (g_1, \dots, g_r)$ with $g_1 \cdots g_r = 1$, $\langle \mathbf{g} \rangle = G$, $g_j \in \mathcal{C}_j$. It checks that χ_σ generates K , computes m_σ by Proposition 2.1 and the Hodge numbers by the age formula, and so checks (W1)–(W3). The component $S = S(\mathbf{g})$ is the braid orbit of $\mathbf{g}$. *It proves that S exists, nothing more.*

(L) Supplier witness.: A finite list of triples (H_i, χ_i, n_i) , each H_i by words in the generators of G and each χ_i by its values; for each, the branch exponents from double cosets (Lemma 3.3), h_i even with $h_i \geq 2$ and α simple (Definition 3.1); and, for every pair $\{\tau, \bar{\tau}\}$ of non-self-dual representations with $m_\tau > 0$, oriented by a chosen member τ (by σ for the pair of σ), the integer $\sum_i n_i (\langle \tau|_{H_i}, \chi_i \rangle - \langle \bar{\tau}|_{H_i}, \chi_i \rangle)$, one free coordinate per pair, required to be +1 at the pair $\{\sigma, \bar{\sigma}\}$

and 0 at every other pair; and for every quaternionic τ with $m_\tau > 0$, $\sum_i n_i \langle \tau |_{H_i}, \chi_i \rangle$ even. This proves (L) for the class datum of $\mathbf{g}$, independently of the component.

(L_{Gale}) **Gale witness.:** As for (L), with in addition a finite list of pairs $(H_k, \mathbf{n}_k)$, each H_k by words in the generators of G and each $\mathbf{n}_k$ an integer vector on the linear characters of H_k with $\dim V_\chi > 0$; for each, the Hodge numbers (p_χ, q_χ) from the age formula applied to the eigenspaces, the equations $\delta_t(\mathbf{n}_k) = 0$ for every t (or, under the common-sign condition of Remark 3.13, for $t = 1$ with a check of that condition), and the character $a_{H_k}(\mathbf{n}_k)$; and an *integer* identity $e_\sigma = \sum_i n_i a(\chi_i) + \sum_k a_{H_k}(\mathbf{n}_k)$. This proves (L_{Gale}) for the class datum. Together with Hypothesis (C), a (D) certificate, and the matching (Δ) certificate on the same component, it establishes the conclusion of Theorem 7.2.

(D) **Rank witness.:** A point C_1 of S : branch points together with a proof that the monodromy tuple of C_1 lies in the braid orbit of $\mathbf{g}$ (or, more simply, a point whose monodromy tuple is computed exactly and taken as $\mathbf{g}$); bases of $\mathbf{A}$ and $\mathbf{B}$ at C_1 , exact or enclosed; and an $n^2 \times n^2$ minor of μ_{C_1} certified nonzero, exactly or in outward-rounded ball arithmetic. This proves (D) for S by Proposition 4.2. Coordinates on a larger isotypic piece, on another cover, or along PGL_2 -motions do not count.

(Δ) **Discriminant.:** For a tuple in the braid orbit of $\mathbf{g}$: H_{par}^1 as Z^1/B^1 for the free group, the form $\mathcal{S}$ by Lemma 5.2, the Hermitian matrix $h = \vartheta \mathcal{S}$ over K , its signature (n, n) , and the class of $\det h$ in $\mathbb{Q}^\times / \text{Nm } K^\times$ by local symbols. This computes $\mathfrak{d}(S)$ by Proposition 5.1.

The matching condition is that (D) and (Δ) refer to the same braid orbit. It costs nothing when the tuple of (Δ) is the monodromy tuple of the (D) point. Any method may produce a certificate; the theorem uses only what the certificate states. In [19] the (D) point was produced from a rigid local system by a rational pullback and an isomonodromic deformation, and (Δ) by a pushforward along that pullback; those constructions are proofs that the certificates are valid, not steps of the transfer.

10. RECOVERY OF THE G_{32} THEOREM

Let $G = G_{32}$, $\sigma = \rho$ its reflection representation over $K = \mathbb{Q}(\sqrt{-3})$, and the class datum $6 \cdot \mathcal{R}_1 + 6 \cdot \mathcal{R}_2 + 1 \cdot \mathcal{J}$, $r = 13$, as in [19, §3]. The table matches each hypothesis of Theorem 7.1 with a certificate of [19, Appendix C].

Hypothesis	Certificate in [19]	Value
(W1)	B: ρ by matrices in $GL_4(K)$	$\mathbb{Q}(\chi_\rho) = K$, Schur index one
(W2), (T)	B: tuple (r_1, r_2, x, y) and the pullback by φ ; surjectivity argument of [19, §5]	a point C_0 , component S°
(W3)	Chevalley–Weil, [19, §3]	$m_\rho = 6$, Hodge numbers $(3, 3)$, $n = 3$
(L)	A: the 51-row witness, [19, Proposition 4.3]	$e_\rho \in L$
(D)	B (the tuple at C_0), the isomonodromy argument of [19, §5] (membership of C_1 in S°), [19, Lemmas B.1, B.2] (completeness of the ansatz), and C (the minor); [19, Proposition 5.1]	$\text{rk } \mu_{C_1} = 9$
(Δ)	D (determinant), E (cocycle formula), and the dictionary of [19, Proposition 6.1, Steps 1–5]	$\det h = -2/81$, $\mathfrak{d} = [-2]$

Corollary 10.1 ([19, Theorem 1.1 and Theorem 9.1]). *The Weil classes of every polarized abelian sixfold of Weil type for $\mathbb{Q}(\sqrt{-3})$ with discriminant -2 are algebraic, and every power of such a sixfold with special Mumford–Tate group SU_H satisfies the Hodge conjecture.*

Proof. Theorem 7.1 and Theorem 8.2 with the certificates of the table. The (D) point C_1 and the (Δ) computation lie on the same component S° : C_1 is joined to C_0 by an isomonodromic path, and certificate D computes $\det h$ for φ^*L at C_0 , for all four permutation classes of the pullback, with the same value. $\square$

Step-by-step comparison. Every step of the proof in [19] is an instance of a statement here: [19, Lemma 2.2] is Lemma 2.5; the construction of A_ρ in [19, §3] is Lemma 2.3; [19, Lemma 4.2] is Lemma 3.4 with Lemma 3.3; [19, Proposition 4.4] is Proposition 3.7 with $\ell = \det M_\rho$, and Corollary 3.8; [19, Corollary 5.2] is Proposition 4.2 with Lemma 4.1; [19, Proposition 6.1], Steps 1–5, is Proposition 5.1, and [19, Lemma 6.2] is Lemma 5.2; [19, Lemmas 7.1, 7.2] and the proof of [19, Theorem 1.1] are Lemmas 6.1, 6.2 and Theorem 7.1; [19, Lemma 9.2 and Theorem 9.1] are Lemma 8.1 and Theorem 8.2.

What is specific to G_{32} , and in what sense it is not residue. Here “no residue” means: once valid input certificates are given, no additional mechanism beyond Theorem 7.1 is needed. It does *not* mean that the general theorem performs the example-specific proofs; it accepts their outputs. The following parts of [19] use G_{32} ; all of them produce or validate certificates, and several are genuine proofs, not scripts:

- (1) the greedy search that found the witness (certificate A's discovery step), and the observation that unramified suppliers are needed for G_{32} ;
- (2) the rigid four-point system, Katz's theorem, the degree-six pullback φ , and the index argument in $PSp_4(3)$ showing that φ^*L has monodromy all of $\rho(G_{32})$: these prove that C_0 is a point of the Hurwitz space, the membership part of (T) and (D);
- (3) the Schlesinger deformation and the preservation of the apparent singularities, which prove that C_1 lies on the component of C_0 , and Lemmas B.1 and B.2 of [19], which make the rank computation at C_1 a computation of μ_{C_1} ;
- (4) Step 6 of [19, Proposition 6.1], the pushforward along φ and the four permutation classes, which compute $\mathfrak{d}(S^\circ)$ from a tuple on $\mathbb{P}^1$ minus six points rather than thirteen.

The general theorem supplies none of (1)–(4). A new example replaces them by its own certificates, whose validity may need comparable proofs, and Section 9 specifies what those certificates must state.

A check that the general proof does not need. Certificate A also verifies that every irreducible character of G_{32} is K -valued and that $a_t(\chi_i) = \pm a(\chi_i)$ for every t . The contraction uses only $t = \pm 1$, where Lemma 3.5 holds for every group. So this check is redundant, and no hypothesis on the character field of G enters Theorem 7.1.

Parts of [19] outside the transfer. Section 8 of [19] (type III sixfolds, through Abdulali's theorem) is a consequence of the main theorem together with a class computation, [19, Lemma 8.1], written for $\mathbb{Q}(\sqrt{-3})$ and $n = 3$; Remark 9.3 of [19] relies on an outside manuscript. Neither is a step of the proof that Theorem 7.1 replaces.

11. THE DATUM $(\mathbb{Q}(i), [-3])$ IN $W(E_8) \cdot \mu_4$

Let $K = \mathbb{Q}(i)$, $W = W(E_8) \subset GL_8(\mathbb{Q})$, and $G = W \cdot \mu_4 = W \cup iW \subset GL_8(K)$, of order 1,393,459,200, with σ the natural representation, of dimension $d = 8$. The class datum has twelve points: $g_1 = ix_1$ with $x_1^2 = -1$ (an involution with $c_\sigma = 4$), $g_2 = ix_2$ with x_2 of order 12 ($c_\sigma = 8$), and ten reflections ($c_\sigma = 1$). So $\sum_j c_\sigma(g_j) = 22$, $m_\sigma = 22 - 16 = 6$ by Proposition 2.1, and $n = 3$.

What is certified. The author's computations give four explicit tuples with product one generating G , with $\dim H_{\text{par}}^1 = 6$, signature $(3, 3)$, and $\det h \in \{-24, -6/25, -3/2, -6\}$, all with odd 3-adic valuation; the local symbols of $\det h$ are nontrivial exactly at 3 and ∞ , so each tuple's component has $\mathfrak{d} = [-3]$, which is not the split class $[-1]$, since 3 is inert in $\mathbb{Q}(i)$. One tuple was recomputed independently from its matrices. An independent referee also checked that the four recorded determinants all lie in $[-3]$: divided by -3 they are $8 = \text{Nm}(2 + 2i)$, $2/25 = \text{Nm}((1 + i)/5)$, $1/2 = \text{Nm}((1 + i)/2)$ and $2 = \text{Nm}(1 + i)$. The four tuples use two classes for x_2 (exchanged by

$x \mapsto -x$), so they realize two class data, each with (W1)–(W3) and with certificates (T) and (Δ) for the components of those tuples. The Hurwitz dimension is $12 - 3 = 9 = n^2$.

Dominance is now certified. A further computation, independently refereed with its certificates rerun, constructs in each of the two class data an explicit cover: a non-rigid four-point system (its character variety has dimension 12), certified to exist by a ball-arithmetic contraction, is pulled back along $s \mapsto s^5$ and twisted, giving an exact twelve-point tuple with $\dim H_{\text{par}}^1 = 6$, signature (3, 3) and $\det h \in \{-6, -12\}$, of class $[-3]$. At the $\mathbb{Z}/5$ -symmetric cover a 9×9 minor of μ is certified nonzero in ball arithmetic, so $\text{rk } \mu = 9$ there. This gives (D), and the matching (Δ) with $\mathfrak{d} = [-3]$, on the components of these explicit covers. It does *not* certify that each class datum has a single Hurwitz component: that claim was not accepted by the referee, so dominance is known only for the components of the constructed covers.

Condition (L): it fails for this datum. Here $X^*(T) \cong \mathbb{Z}^{45}$. The admissible suppliers found by a search over the subgroups of G generate a lattice of rank 42 that misses e_σ , with an elementary divisor 3; the 16 admissible suppliers of order 4 on $\langle i \rangle N_W(P_3)$, with P_3 a Sylow 3-subgroup of W ($|N_W(P_3)| = 3888$), remove that divisor. Fix an element $u = iw$ of order 4 in the relevant 2-section, with $w \in W$. The group $C_G(u) \cap W = C_W(w)$, modulo its largest normal 2-subgroup, is isomorphic to $S_3 \times S_3$. The local analysis gives two linear functionals on $X^*(T)$, both nonzero on e_σ , which vanish on $a(\chi)$ for every admissible supplier (H, χ) for which all nonzero labels $\text{Im } \chi(z) \in \{1, -1\}$, with $z \in H \cap \text{cl}_G(u)$, have one common sign. The conclusion includes the case with no nonzero labels. Hence the lattice generated by these same-sign suppliers contains no nonzero multiple of e_σ . To escape this obstruction a supplier must have both signs and heterogeneous local order-three profiles in the centralizers $C_H(z)$. A complete search settles the remaining case. Such a supplier must contain a marked triple (u, v, z') from the local analysis; there are 1248 marked seeds up to the relevant conjugation, of which 72 pass a separation filter whose failure is inherited by overgroups; the overgroup trees of these 72 are finite and were exhausted, with 4496 separating characters, of which 626 give admissible suppliers, all with both functionals zero. So the first functional vanishes on every admissible supplier and not on e_σ , and (L) fails for this datum. An independent referee recomputed the three seed cases and all 1248 representatives, verified all 1176 failure relations of the filter, and re-evaluated all 4496 characters, with no discrepancy. The exhaustion itself was audited from the saved search state and reload checks; the roughly 4.5 million edges of the search were not replayed from scratch. The seed reduction (the local functionals, the seed lemma and the hereditary filter), the search code, the saved state, the referee's verification scripts and the logs as run are deposited at <https://github.com/benzsevern/hodge-fleet>, directory

`notes/transfer-theorem-review/certificates/qi-schoen-negative`.

A different datum for the same cell is not excluded.

Condition (L_{Gale}): *certified*. For three conjugacy classes of elements $w \in W$, let $H = \langle i \rangle C_W(w)$, of orders 622080, 1728 and 576, with quotient curves C/H of genus 3259, 1730347 and 5341793. For each, an integer vector $\mathbf{n}$ on the linear characters of H is balanced. Balance (Definition 3.12) involves only the characters χ with $n_\chi \neq 0$, and χ^t depends only on t modulo the order of χ . Every character occurring in these three vectors has order dividing 12, so checking $t \in (\mathbb{Z}/12)^\times = \{1, 5, 7, 11\}$ covers every Galois conjugate. This holds although the exponent e of the nonabelian group H in Definition 3.12 may be larger than 12. Then

$$e_\sigma = \sum_{i=1}^{95} n_i a(\chi_i) + \sum_{k=1}^3 a_{H_k}(\mathbf{n}_k),$$

an identity in $\mathbb{Z}^{45}$, with 94 suppliers from the search, one from $\langle i \rangle N_W(P_3)$ and the three Gale vectors. The combined lattice has rank 44 and contains e_σ ; the three Gale vectors are what break the vanishing of the two functionals above. The witness was checked by an independent referee, who reconstructed the Gale vectors and the new supplier. It was also recomputed in separate code, which found its own integer witness; that recomputation reused the realization of G , the class data and the definition of the search pool, so it is independent for the Gale and combination steps but not for every prerequisite. The certificate is deposited, with the scripts as run, their SHA-256 hashes and the rerun commands, at <https://github.com/benzsevern/hodge-fleet>, directory `notes/transfer-theorem-review/certificates/qi-gale`. The witness file has SHA-256 `1b69d692...feed34b`.

Corollary 11.1 (Conditional on (C)). *Assume Hypothesis (C). The Weil classes of every polarized abelian sixfold of Weil type for $\mathbb{Q}(i)$ with discriminant -3 are algebraic, and every power of such a sixfold whose special Mumford–Tate group is SU_H satisfies the Hodge conjecture.*

Proof. Theorem 7.2 with (L_{Gale}) as above, and (D) and (Δ) on the component of a constructed cover; then Theorem 8.2. $\square$

Corollary 11.1 is not claimed unconditionally. Hypothesis (C) is the local consequence of the theorem of the unrefereed manuscript [14] described above. A proof through the Schoen-only transfer theorem is impossible with this datum, since (L) fails for it. We do not exclude other data or other unconditional cycle sources. The datum is not real (its order-twelve classes are not conjugate to their inverses), so the Brauer supply theorem of [18] does not apply; and the cell is non-split, so its boundary criterion for (D) does not apply either. That restriction is general: by [4], a family of abelian $2n$ -folds of Weil type has a maximal degeneration, one whose monodromy logarithm has rank $2n$ over $\mathbb{Q}$, only for discriminant $(-1)^n$. Dominance here was certified directly.

Remark 11.2 (Components). A tuple with discriminant $[-3]$ certifies $\mathfrak{d}$ for its own braid orbit only. If the class data have several components, a (D) certificate on a component other than those of the four tuples needs its own (Δ) . Conversely, a split value on one tuple of a class datum says nothing about the other components of that class datum. This caveat applies to every screen of class data by one tuple each.

For the two class data of this section the caveat is discharged for (Δ) , though not for (D). Over $\mathbb{Q}(i)$, which has a single ramified prime, the class of $\det h$ is determined by the signature and the parities of its valuations at the inert primes, and each of these parities depends only on the conjugacy classes of the local monodromies, by the local discriminant formula. A refereed computation in the author's notes checks this for these data. Hence every Hurwitz component of either class datum has $\mathfrak{d} = [-3]$, so the matching of (D) with (Δ) holds on any component. Dominance, however, is certified only on the components of the constructed covers.

12. CONSTRAINTS ON THE DATA

Let $g_1, \dots, g_r$ be the local monodromies, $c_j = c_\sigma(g_j)$, and call a branch point *effective* if $c_j > 0$; let r_{eff} be their number. Points with $c_j = 0$ do not move $A_\sigma(C)$, whose Hodge structure is the parabolic cohomology of the local system of σ , so the image of the period map has dimension at most $r_{\text{eff}} - 3$.

Corollary 12.1 (Budget). *If $\mathfrak{D}$ is of type (K, n) and (D) holds on some component, then*

$$n^2 + 3 \leq r_{\text{eff}} \leq \sum_j c_j = 2d + 2n.$$

The number p of effective points with $c_j = 1$ satisfies $p \geq 2n^2 + 6 - 2d - 2n$.

Proof. The first inequality is the remark above; the second holds because each effective point has $c_j \geq 1$, and the equality is Proposition 2.1 with $m_\sigma = 2n$. Then $2d + 2n \geq p + 2(r_{\text{eff}} - p)$ gives $p \geq 2r_{\text{eff}} - 2d - 2n$. $\square$

For sixfolds ($n = 3$): $12 \leq r_{\text{eff}} \leq 2d + 6$, so $d \geq 3$, and $p \geq 18 - 2d$. An element with $c = 1$ acts by a pseudo-reflection, and the pseudo-reflections in $\sigma(G)$ generate a normal reflection subgroup $R \trianglelefteq \sigma(G)$, nontrivial when $d \leq 8$. The G_{32} datum has $d = 4$, $r_{\text{eff}} = 13$ and $p = 12$; the $W(E_8) \cdot \mu_4$ datum has $d = 8$, $r_{\text{eff}} = 12$ and $p = 10$, at the edge of the budget.

Remark 12.2 (Monomial representations). If $\sigma = \text{Ind}_H^G \lambda$ with λ linear, Frobenius reciprocity identifies $\text{Hom}_G(\sigma, H^0(C, K_C))$ with the λ -eigenspace of H on $H^0(C, K_C)$, so A_σ is isogenous to a factor of the Prym variety of the cyclic cover $C/\ker \lambda \rightarrow C/H$, and the transfer reduces to Schoen's setting. The interesting data have σ primitive.

Nielsen classes. The number of solutions of $g_1 \cdots g_r = 1$ with $g_j \in \mathcal{C}_j$ is

$$N(\mathcal{C}_1, \dots, \mathcal{C}_r) = \frac{|\mathcal{C}_1| \cdots |\mathcal{C}_r|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(\mathcal{C}_1) \cdots \chi(\mathcal{C}_r)}{\chi(1)^{r-2}}.$$

$N > 0$ is necessary for (W2), not sufficient: the tuple must also generate G , and a generation test that tries one representative per class misses tuples that need two distinct elements of one class.

13. SCOPE

Theorem 7.1 reaches a Weil cell $(K, n, \mathfrak{d})$ only through a datum carrying it. The following lie outside its reach.

Lemma 13.1 (Good inert primes). *Let q be a prime inert in K , let $\mathcal{O} = \mathcal{O}_K \otimes \mathbb{Z}_q$, an unramified discrete valuation ring with uniformizer q , residue field k and fraction field K_q , and let $\Gamma \subset GL(W)$ be a finite group of order prime to q acting absolutely irreducibly on a finite-dimensional K_q -vector space W , with a nondegenerate Γ -invariant Hermitian form B . Then:*

- (a) *for every Γ -stable $\mathcal{O}$ -lattice $\Lambda \subset W$, the reduction $\Lambda/q\Lambda$ is an absolutely irreducible $k[\Gamma]$ -module;*
- (b) *the Γ -stable lattices are exactly the $q^a\Lambda_0$, $a \in \mathbb{Z}$, for any one of them Λ_0 ;*
- (c) *for $\epsilon \in \{0, 1\}$ suitably chosen, there is a Γ -stable lattice that is self-dual for $q^\epsilon B$;*
- (d) *$H^1(\langle g \rangle, \Lambda) = 0$ for every $g \in \Gamma$ and every stable Λ .*

In the setting of Section 5, with $q \nmid |\sigma(G)|$ and $W = L'_x \otimes K_q$ at a base point, Γ the monodromy group and $B = h_{L'}$, the normalization in (c) is the replacement $h_{L'} \mapsto q h_{L'}$, $h_V \mapsto q^{-1} h_V$, which keeps $b^s|_{V_\sigma \otimes L'} = h_V \otimes h_{L'}$, multiplies $\mathcal{S}$ and h by q , and multiplies $\det h$ by $q^{2n} = \text{Nm}(q^n)$; so it preserves the class of $\det h$ in $\mathbb{Q}^\times / \text{Nm } K^\times$.

Proof. Stable lattices exist: take the sum of the Γ -translates of any lattice. (a) On $\text{End}_{\mathcal{O}}(\Lambda)$ the Reynolds operator $P(A) = |\Gamma|^{-1} \sum_{g \in \Gamma} gAg^{-1}$ is defined, because $|\Gamma|$ is a unit in $\mathcal{O}$, and is an idempotent with image $\text{End}_{\mathcal{O}[\Gamma]}(\Lambda)$. Its generic fibre is $\text{End}_{K_q[\Gamma]}(W) = K_q$ by absolute irreducibility, so the image is free of rank one. Since P is an idempotent, the formation of its image commutes with base change, so $\text{End}_{\bar{k}[\Gamma]}(\Lambda \otimes \bar{k})$, the image of $P \otimes \bar{k}$, is one-dimensional. By Maschke's theorem ($q \nmid |\Gamma|$) $\Lambda \otimes \bar{k}$ is semisimple, and a semisimple module with one-dimensional endomorphism algebra over an algebraically closed field is irreducible.

(b) Let M be another stable lattice. After multiplying it by a power of q we may assume $M \subseteq \Lambda_0$ and $M \not\subseteq q\Lambda_0$ (minimal containment). The image of M in $\Lambda_0/q\Lambda_0$ is a nonzero stable subspace, hence everything by (a), so $\Lambda_0 = M + q\Lambda_0$, and Nakayama's lemma for the finitely generated $\mathcal{O}$ -module Λ_0/M gives $M = \Lambda_0$. Units of $\mathcal{O}$ preserve lattices, so the stable lattices are exactly the $q^a\Lambda_0$.

(c) The dual $\Lambda_0^\# = \{x : B(x, \Lambda_0) \subseteq \mathcal{O}\}$ is stable because B is invariant, so $\Lambda_0^\# = q^k\Lambda_0$ for some k by (b). Since q is rational, $(q^a\Lambda_0)^\# = q^{k-a}\Lambda_0$, and the dual for qB is q^{-1} times the dual for B . If k is even, $q^{k/2}\Lambda_0$ is self-dual for B ; if k is odd, $q^{(k-1)/2}\Lambda_0$ is self-dual for qB .

(d) If g has order m , then m is a unit in $\mathcal{O}$ and annihilates $H^1(\langle g \rangle, \Lambda)$, an $\mathcal{O}$ -module.

For the last statement, the replacement keeps the tensor decomposition, $\mathcal{S}$ is defined with $h_{L'}$, and $h = \vartheta \mathcal{S}$ has rank $2n$. The lemma is local to the one prime q : it makes no global self-duality claim and does not compare different primes. $\square$

- (1) *Cells that no finite group carries within the budget.* The class $\mathfrak{d}(S)$ is determined by its local symbols. A local analysis of $\det h$ [20], which we do not prove here, gives the following at an inert prime q . Assume the monodromy is finite, the invariant Hermitian form is nondegenerate, $V^G = 0$, Λ is a monodromy-stable lattice with $\Lambda \subseteq \Lambda^\#$, lengths are taken over $\mathcal{O}_K \otimes \mathbb{Z}_q$, and ϑ is a q -adic unit (otherwise a term $2n v_q(\vartheta)$ is added, since $h = \vartheta \mathcal{S}$ is a $2n \times 2n$ matrix; it is even). Then

$$v_q(\det h) \equiv \sum_j \left[\ell_q(g_j; \Lambda) + \text{len}(\Lambda^\#/\Lambda) - \text{len}((\Lambda^\#)^{g_j}/\Lambda^{g_j}) \right] \pmod{2},$$

where $\ell_q(g; \Lambda)$ is the length of $H^1(\langle g \rangle, \Lambda)$. Under the self-duality hypothesis $\Lambda = \Lambda^\#$ (H1) the correction terms vanish, so a nontrivial symbol at q needs local monodromies of order divisible by q whose lengths ℓ_q have odd sum. Without H1 the dual-lattice terms cannot be dropped, and H1 does not hold for an arbitrary stable lattice: if Λ_0 is self-dual, $q\Lambda_0$ is stable with $q\Lambda_0 \subseteq (q\Lambda_0)^\# = q^{-1}\Lambda_0$ and is not self-dual. When $q \nmid |\sigma(G)|$, Lemma 13.1 shows that, after a normalization of the coefficient forms that preserves the class of $\det h$, there is a self-dual stable lattice and every $\ell_q(g_j; \Lambda)$ vanishes; with the formula above, the symbol at such q is then trivial. Combined with Corollary 12.1, which bounds the number of non-reflection points, this leaves few candidate data in each degree. In the same notes, with the inherited classification inputs recorded there, primitive data of degree 5 to 12 with an imaginary quadratic character field, signature $(3, 3)$, rank six and $r_{\text{eff}} \geq 12$ show no odd valuation at an inert $q \geq 5$; this exclusion is limited to that setting, and the inert-prime analysis says nothing about the ramified symbols; a cell whose discriminant needs such a prime, for $n = 3$ and those degrees, is not reached by any datum, and the theorem says nothing about it. Failure of (D) still prevents this transfer argument. For a datum satisfying (D), failure of (L) may be bypassed by (L_{Gale}) , conditionally on Hypothesis (C).

- (2) *Other endomorphism types.* The theorem treats Weil classes for an imaginary quadratic K with signature (n, n) . When the character field of σ is a CM field of higher degree, or σ has Schur index > 1 over K , Definition 2.2 does not apply. For signature (a, b) with $a \neq b$ there are no Weil classes to transfer.

- (3) *Other Hodge classes.* On members whose Hodge group is smaller than SU_H , and on their powers, the extra Hodge classes are not treated (Remark in Section 8).
- (4) *Beyond abelian varieties.* Every class the theorem produces lives on an abelian variety, a factor of a Jacobian of a curve over $\mathbb{P}^1$. It says nothing about the Hodge conjecture for any other variety, nor about integral Hodge classes.
- (5) *Base curve.* The dominant target families considered here are families of G -covers of $\mathbb{P}^1$. Extending that part of the argument to higher-genus bases would require the corresponding codifferential and parabolic cohomology. The auxiliary supplier covers $C/K \rightarrow C/H$ may have higher-genus bases, as allowed in Hypothesis (C).

Remark 13.2 (Beyond Weil type). Proposition 3.7 is stated for any U -eigenline with character in $L(\mathfrak{D})$, and Lemmas 6.1 and 6.2 hold for any flat rational sub-local system of Hodge classes over an irreducible base. So the same chain would apply to a Shimura family of isotypic factors whose generic Hodge ring, over $\mathbb{C}$, is spanned by divisor classes and U -eigenlines with characters in $L(\mathfrak{D})$, provided (D) holds onto that family. We do not pursue this: for unitary groups of signature (n, n) it is Theorem 7.1, and we know no other case in which the hypothesis is checkable.

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