

Convolution Smoothing in the Turán Hierarchy: The Blind Zone Extends from Order 43 to Order 61, and Empirically to Order 68

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Statement on the use of generative artificial intelligence. Generative AI assistance was used for literature retrieval, notation cross-checking, $\text{\LaTeX}$ typesetting, and for the independent recomputation of every numerical constant recorded in Sections 4–6. The mathematical content—the multiplicity analysis of the β -alphabet, the convolution identity for the shift-1 row, the Chebyshev derivative identity $W_r = \frac{1}{2}U'_{r+1}(\cos \theta)$, the sign criterion of Lemma 3.4, the proof of Theorem 3.9, and the design of the numerical experiments—was produced by the author. Every statement that is not proved is labelled as such; see Section 7 for a complete ledger of what is proved and what is numerical.

Abstract

Let $\Xi(t) = \xi(\frac{1}{2}+it)$ be the Riemann ξ -function on the critical line, $\Xi(t) = \sum_{k \geq 0} (-1)^k c_k t^{2k}$ its even Taylor expansion, and $D(n, r) = \det[c_{n+i-j}]_{i,j=0}^{r-1}$ the associated Toeplitz minors. Writing $G(z) = \sum_{k \geq 0} c_k z^k$ and $\beta_j = -1/z_j$ for the zeros of G , the dual Jacobi–Trudi identity gives $D(n, r) = c_0^r s_{(r^n)}(\beta)$, the Schur function of the rectangular partition (r^n) in the alphabet β .

Positivity of $D(n, r)$ for all $n \geq 0$ and all $r \geq 1$ is equivalent to the Riemann hypothesis (RH) by the Schoenberg–Edrei characterisation of Pólya frequency sequences; for each fixed r , the inequalities $D(n, r) \geq 0$ are a necessary condition, and for $r = 2$ they hold unconditionally [1]. A previous paper of the author [3] proved that every level $r \leq 43$ is *blind*: the minors $D(n, r)$ stay positive even in the presence of non-real zeros. The mechanism there was the two-variable Chebyshev criterion $U_k(\cos \theta) > 0 \iff (k+1)\theta < \pi$, together with the uniform bound $\theta \leq \theta_{\max} = 2 \arctan(1/(2\gamma_1)) = 0.07071826 \dots$

The present paper shows that the bound 43 is not the truth but an artefact of ignoring two structural features of the true alphabet. First, the evenness $\Xi(-t) = \Xi(t)$ forces every zero of G to occur with *multiplicity two* in β , so the relevant constituent is not U_r but the self-convolution

$$W_r = \sum_{i=0}^r U_i(\cos \theta) U_{r-i}(\cos \theta), \quad \text{with generating function} \quad \sum_{r \geq 0} W_r z^r = \frac{1}{(1 - 2z \cos \theta + z^2)^2}.$$

We prove the identity $W_r = \frac{1}{2}U'_{r+1}(\cos \theta)$ and the pole-free sign criterion of Lemma 3.4, and we show that $W_r > 0$ for all $r \leq 61$ while $W_{62} < 0$: the convolution of the Chebyshev sequence against itself *smooths away* the sign change that occurs at U_{44} , pushing the threshold from 43 to 61. Secondly, the zeros on the critical line contribute a positive real sub-alphabet whose complete symmetric functions $h_i(X) > 0$ convolve with the violating part, $h_r(X \sqcup Y) = \sum_{i+j=r} h_i(X)h_j(Y)$, pushing the threshold further.

Main theorem (proved). *For every $r \leq 61$ and every $n = 1$, and regardless of whether RH is true, $D(1, r) > 0$. Consequently the first 61 levels of the Turán hierarchy are blind on the shift-1 row.*

High-precision computation of the full rectangular Schur functions $s_{(r^n)}(\beta)$ on the true alphabet (truncated at N critical-line zeros) shows that the first discriminating order is $r^* = 69$ at shift $n = 1$ and $n = 3$, so that *all* orders $r \leq 68$ are blind; the value is stable under truncation ($N = 45, 60, 80$ all give 69). A scaling law $r^* \approx 1.5 \pi / \theta$ is observed across two and a half decades of the violating offset. At shift $n = 2$ no negative minor was found for any $r \leq 400$, suggesting a qualitatively stronger phenomenon. The number 61 is proved; the number 68 is numerical, and Section 7 separates the two without ambiguity. No claim of a proof of RH is made.

Keywords. Riemann zeta function; ξ -function; Turán inequalities; Toeplitz minors; Schur functions; Chebyshev polynomials; convolution; blind zone.

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1 Introduction

1.1 The Turán hierarchy and its blind zone

Let ζ be the Riemann zeta function and let

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

be the completed zeta function, entire of order one and satisfying $\xi(s) = \xi(1-s)$. On the critical line put

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right).$$

Then Ξ is an even real entire function of t and admits an even Taylor expansion at the origin

$$\Xi(t) = \sum_{k \geq 0} (-1)^k c_k t^{2k}, \quad c_k > 0. \quad (1)$$

Define the *Toeplitz array* $C = (c_{i-j})_{i,j \geq 0}$ (with $c_k = 0$ for $k < 0$) and its finite minors

$$D(n, r) = \det [c_{n+i-j}]_{i,j=0}^{r-1}, \quad n \geq 1, \quad r \geq 1. \quad (2)$$

Here n is the *shift* (or *level displacement*) and r is the *order* (the size of the minor; note that r is the number of columns of the corresponding partition, and the matrix in (2) is $r \times r$).

Remark 1.1 (Toeplitz, not Hankel). *The array (c_{i-j}) is Toeplitz, not Hankel. The distinction is not cosmetic: for $r = 2$ the Hankel minor built from (c_{i+j}) equals $-D(n+1, 2)$, so the two families differ by a sign at the very first non-trivial level. All minors in this paper are Toeplitz minors in the sense of (2).*

The relevance of (2) to RH comes from the theory of Pólya frequency sequences.

Theorem 1.2 (Schoenberg–Edrei; see [2]). *The sequence $(c_k)_{k \geq 0}$ of (1) is a Pólya frequency sequence, i.e. $D(n, r) \geq 0$ for all $n \geq 0$ and all $r \geq 1$, if and only if*

$$\Xi(t) = \Xi(0) \prod_j \left(1 - \frac{t^2}{t_j^2}\right)$$

with all t_j real—that is, if and only if RH holds.

Thus the full family of inequalities $D(n, r) \geq 0$ is *equivalent* to RH. For a *fixed* order r , however, the family $\{D(n, r) \geq 0 : n \geq 0\}$ is only a *necessary* condition. The case $r = 2$ is the classical Turán inequality:

Theorem 1.3 (Csordas–Norfolk–Varga [1]). *$D(n, 2) > 0$ for all $n \geq 0$, unconditionally.*

Theorem 1.3 is a statement of the first importance for the present work, because it shows that level 2 is *not discriminating*: the inequality holds whether or not RH is true, so it carries no information about RH. The question studied in [3] and here is:

How far up the hierarchy does this failure of discrimination persist?

1.2 The previous bound $r \leq 43$

Put $G(z) = \sum_{k \geq 0} c_k z^k$, so that $G(-t^2) = \Xi(t)/\Xi(0)$, and write the zeros of G as z_j . Define

$$\beta_j = -\frac{1}{z_j}. \quad (3)$$

If t_j is a zero of Ξ then $z_j = -t_j^2$ and $\beta_j = 1/t_j^2$. The fundamental algebraic identity used throughout is the dual Jacobi–Trudi formula

$$D(n, r) = c_0^r s_{(r^n)}(\beta), \quad (4)$$

where (r^n) denotes the rectangular partition with n rows each of length r , and s_λ is the Schur function.

A zero of ζ off the critical line, say at $s = \sigma + i\gamma$ with $\sigma > \frac{1}{2}$, produces in the t -variable the four points

$$t = \pm\gamma \mp i\delta, \quad \delta = \sigma - \frac{1}{2} \in (0, \frac{1}{2}),$$

the four arising from the symmetries $\Xi(-t) = \Xi(t)$ (evenness) and $\Xi(\bar{t}) = \overline{\Xi(t)}$ (real coefficients). Passing to $\beta = 1/t^2$ one obtains a conjugate pair

$$\beta = \rho e^{\pm i\theta}, \quad \rho = \frac{1}{\gamma^2 + \delta^2}, \quad \theta = 2 \arctan \frac{\delta}{\gamma}, \quad (5)$$

each with multiplicity two. The angle θ is exactly twice the angle subtended at the origin by the violating zero in the t -plane; we call it the *vertical angle*.

Zeros on the critical line have $t = \pm\gamma$ real, hence $\beta = 1/\gamma^2 > 0$, again with multiplicity two.

Since $|\delta| < \frac{1}{2}$ and the lowest-lying zero has $\gamma_1 = 14.134725141734\dots$, the largest possible vertical angle is

$$\theta_{\max} = 2 \arctan \frac{1}{2\gamma_1} = 0.07071826294293218\dots, \quad \frac{\pi}{\theta_{\max}} = 44.42406420707728\dots \quad (6)$$

For an alphabet of cardinality two the Schur function has the closed form

$$s_{(a,b)}(\rho e^{i\theta}, \rho e^{-i\theta}) = \rho^{a+b} \frac{\sin((a+1-b)\theta)}{\sin \theta} = \rho^{a+b} U_{a-b}(\cos \theta), \quad (7)$$

U_k being the Chebyshev polynomial of the second kind. In particular the sign is governed by

$$U_k(\cos \theta) > 0 \iff (k+1)\theta < \pi. \quad (8)$$

The argument of [3] bounds the index of every constituent arising from the coproduct expansion by $k \leq r$, whence $(r+1)\theta_{\max} < \pi$ gives $r \leq 43$.

1.3 What the previous argument missed

The bound 43 is correct but not sharp, for two reasons.

- (i) **Multiplicity.** The alphabet β never contains a violating pair with multiplicity one: evenness forces multiplicity two, as in (5). For the shift-1 row the relevant quantity is therefore not $h_r = \rho^r U_r(\cos \theta)$ but the *self-convolution*

$$W_r = \sum_{i=0}^r U_i(\cos \theta) U_{r-i}(\cos \theta).$$

Section 3 proves $W_r = \frac{1}{2}U'_{r+1}(\cos \theta)$ and exhibits the closed generating function. The first sign change of W_r occurs at $r = 62$, not 44: convolving the Chebyshev sequence with itself *delays* the sign change by 18 orders. This is the phenomenon we call *convolution smoothing*.

- (ii) **The critical-line background.** The alphabet also contains infinitely many positive real variables (the critical-line zeros). Because $h_r(X \sqcup Y) = \sum_{i+j=r} h_i(X)h_j(Y)$ and $h_i(X) > 0$ for positive real X , this second convolution delays the sign change further, to $r = 69$ at shift 1.

Both effects are monotone in the direction of *more* blindness: the true blind zone is larger than the proved one, and much larger than 43.

1.4 Results of this paper

- (a) **A proved improvement.** Theorem 3.9: for every $r \leq 61$, $D(1, r) > 0$ unconditionally. This is a strict improvement of the corresponding statement of [3] ($r \leq 43$) on the shift-1 row, and it is proved, not observed. The proof is short and uses only the convolution identity and the positivity of complete symmetric functions on positive real alphabets.
- (b) **Exact structure of the shift-1 row.** Theorem 3.8 gives the closed form $h_r(X \sqcup Y) = \sum_{i+j=r} h_i(X) \rho^j W_j$ with $W_j = \frac{1}{2}U'_{j+1}(\cos \theta)$; Lemma 3.4 gives a pole-free criterion for its sign.
- (c) **A numerical determination of the true threshold.** Section 4: on the truncated true alphabet the first negative minor occurs at $r^* = 69$ (shift 1 and 3), stable under truncation. Conservatively, orders $r \leq 68$ are blind.
- (d) **A scaling law.** Section 5: $r^* \approx 1.5 \pi / \theta$ over two and a half decades of the offset δ , with the ratio drifting slowly from 1.553 down to 1.477.
- (e) **Two structural anomalies.** Shift $n = 2$ shows no negative minor up to $r = 400$; and increasing the number of violating pairs makes the hierarchy *more* blind, not less.

Remark 1.4 (Logical status). *Item (a) is a theorem. Items (c)–(e) are numerical findings obtained with 60–80 digit arithmetic; they are reproducible from the scripts supplied in Appendix A, but no proof is offered. In particular the number 43 of [3] remains the largest bound valid for all shifts n that has been proved. Section 7 gives the ledger in full.*

1.5 Relation to prior work

The study of positivity properties of the coefficients of Ξ goes back to Pólya [4], who observed that the *Jensen polynomials*

$$J_n^{(d)}(X) = \sum_{j=0}^d \binom{d}{j} c_{n+j} X^j$$

should be hyperbolic for all n and all d if RH holds, and asked what the failure of hyperbolicity would imply. Pólya’s question was resolved by Csordas, Norfolk and Varga [1] in the case $d = 2$: the quadratic Jensen polynomials are always hyperbolic, which is equivalent to Theorem 1.3. The full Pólya program—all d —is equivalent to RH.

The order r of the present paper indexes a different filtration. The Toeplitz minors $D(n, r)$ of (2) are not the discriminants of the Jensen polynomials; they are the minors of the moment matrix itself, and by the Schoenberg–Edrei theorem [2] their simultaneous positivity is the total positivity of the sequence (c_k) , again equivalent to RH. Whereas the Jensen filtration is by the degree d of a polynomial, the Turán filtration here is by the size r of a determinant. Both filtrations have the property that finitely many levels do not suffice, but they fail in different ways, and the author is not aware of a result transferring a bound from one to the other.

Higher-order Turán inequalities have been studied by Dimitrov and Lucas [5], and the Jensen-polynomial side of the program has seen substantial recent progress in the work of Griffin, Ono, Rolin and Zagier [6], who prove hyperbolicity of $J_n^{(d)}$ for $1 \leq d \leq 4$ and all sufficiently large n . Our results are complementary and, in the author’s view, mutually consistent: results of the form “level r holds unconditionally” and results of the form “level r is blind” both assert that level r carries no information about RH, the former because the inequality is always true and the latter because it is true whether or not RH holds. They are the same assertion reached from opposite sides.

Remark 1.5 (Novelty). *We do not claim that the unconditional positivity of any specific level $r \geq 3$ is new; such statements, where known, belong to the authors cited above. What is new here is (i) the multiplicity analysis of Lemma 2.1, which appears to have been overlooked in the minor-based literature; (ii) the convolution identity $W_r = \frac{1}{2}U'_{r+1}$ and the resulting exact shift-1 analysis (Theorem 3.2, Lemma 3.4); and (iii) the numerical determination of the true threshold, including the scaling law of Section 5 and the shift-2 phenomenon of Section 6.*

2 The alphabet β and its multiplicities

2.1 Symmetry and multiplicity

We record the exact multiplicity structure, since it is the source of the improvement.

Lemma 2.1. *Let Ξ be as above.*

- (i) *If $\Xi(t_0) = 0$ with t_0 real, then $\beta = 1/t_0^2$ occurs in the alphabet β with multiplicity exactly two (arising from t_0 and $-t_0$), unless $t_0 = 0$ (which does not occur; ξ has no zero at the centre of the critical strip).*
- (ii) *If $\Xi(t_0) = 0$ with $t_0 = \gamma - i\delta$, $\gamma > 0$, $\delta \in (0, \frac{1}{2})$, then the four zeros $\pm\gamma \mp i\delta$ give exactly two values of β , namely $\rho e^{i\theta}$ and $\rho e^{-i\theta}$, each with multiplicity two, where ρ and θ are given by (5).*

Proof. (i) Ξ is even, so $\Xi(-t_0) = \Xi(t_0) = 0$; both give $z = -t_0^2$ and hence the same $\beta = 1/t_0^2$. No other zero gives the same β unless it is $\pm t_0$.

(ii) Evenness gives $-t_0 = -\gamma + i\delta$; the functional equation $\Xi(\bar{t}) = \overline{\Xi(t)}$ (valid because the c_k are real) gives $\bar{t}_0 = \gamma + i\delta$; and evenness applied to that gives $-t_0 = -\gamma - i\delta$. These four are distinct when $\gamma, \delta > 0$. Now

$$(\gamma - i\delta)^2 = \gamma^2 - \delta^2 - 2i\gamma\delta, \quad (-\gamma + i\delta)^2 = (\gamma - i\delta)^2,$$

so $t = \gamma - i\delta$ and $t = -\gamma + i\delta$ give the same $z = -(\gamma - i\delta)^2$ and hence the same $\beta = 1/(\gamma - i\delta)^2$. Likewise $t = \gamma + i\delta$ and $t = -\gamma - i\delta$ give $\beta = 1/(\gamma + i\delta)^2$. Writing $\gamma - i\delta = \sqrt{\gamma^2 + \delta^2} e^{-i \arctan(\delta/\gamma)}$ we get

$$\beta = \frac{1}{\gamma^2 + \delta^2} e^{2i \arctan(\delta/\gamma)} = \rho e^{i\theta},$$

and the conjugate for the other pair. $\square$

Thus a *single* violating zero of ζ off the critical line contributes to β a four-element multiset

$$Y = \{\rho e^{i\theta}, \rho e^{i\theta}, \rho e^{-i\theta}, \rho e^{-i\theta}\}, \quad (9)$$

and *not* the two-element set $\{\rho e^{i\theta}, \rho e^{-i\theta}\}$. This is the single structural point that separates the present paper from [3].

Remark 2.2 (Why the two-element set is nevertheless relevant). *The two-element analysis is not wrong; it is the analysis of the constituents produced by the coproduct expansion. The point is that those constituents are reassembled by a further convolution (over multiplicity, and then over the critical-line background), and each convolution smooths the sign. Ignoring the reassembly and bounding each constituent separately—the strategy of [3]—is safe but lossy.*

2.2 The critical-line background

Let $\gamma_1 < \gamma_2 < \dots$ be the ordinates of the zeros of ζ on the critical line (assuming, for the purpose of describing the alphabet, that they are all there; if some are not, they are dealt with by Lemma 2.1(ii)). Put

$$X = \left\{ \frac{1}{\gamma_1^2}, \frac{1}{\gamma_1^2}, \frac{1}{\gamma_2^2}, \frac{1}{\gamma_2^2}, \dots \right\}, \quad (10)$$

a multiset of positive real numbers. It is infinite; note that $\sum_k 1/\gamma_k^2 < \infty$ because $\gamma_k \sim 2\pi k / \log k$, so all symmetric functions of X that we use converge absolutely.

The full alphabet is $\beta = X \sqcup Y_{\text{all}}$, where Y_{all} is the union of the multisets (9) over all violating zeros. In the numerical work of Section 4 we truncate X at N ordinates and consider one violating pair attached to γ_1 ; Section 6 varies both the attachment point and the number of pairs.

3 Convolution smoothing: the shift-1 row

For $n = 1$ the partition $(r^1) = (r)$ is a single row and the Schur function is the complete homogeneous symmetric function:

$$D(1, r) = c_0^r h_r(\beta). \quad (11)$$

The shift-1 row is therefore governed entirely by the generating function

$$H_\beta(z) = \sum_{r \geq 0} h_r(\beta) z^r = \prod_{\beta_j} \frac{1}{1 - \beta_j z}, \quad (12)$$

and this is what makes it amenable to exact treatment.

3.1 Layer A: the two-variable constituent

Proposition 3.1. *Let $Y_1 = \{\rho e^{i\theta}, \rho e^{-i\theta}\}$. Then*

$$h_r(Y_1) = \rho^r U_r(\cos \theta) = \rho^r \frac{\sin((r+1)\theta)}{\sin \theta},$$

and consequently $h_r(Y_1) < 0$ first occurs at $r = \lceil \pi/\theta \rceil - 1$, which for $\theta = \theta_{\text{max}}$ is $r = 44$.

Proof. By (12), $H_{Y_1}(z) = ((1 - \rho e^{i\theta} z)(1 - \rho e^{-i\theta} z))^{-1} = (1 - 2\rho z \cos \theta + \rho^2 z^2)^{-1}$, which is the classical generating function $\sum_r \rho^r U_r(\cos \theta) z^r$. The sign criterion (8) gives the first negative index as the least r with $(r+1)\theta > \pi$, i.e. $r = \lceil \pi/\theta \rceil - 1$. For $\theta = \theta_{\max}$, $\pi/\theta_{\max} = 44.424\dots$, so $\lceil \pi/\theta_{\max} \rceil - 1 = 44$. $\square$

Proposition 3.1 is precisely the mechanism behind the bound 43 of [3]: if a constituent of index $k \leq r$ can go negative at $k = 44$, then orders $r \geq 44$ are not covered, and the proved blind zone stops at 43.

3.2 Layer B: multiplicity two

Now take Y as in (9), the multiset with each conjugate value appearing twice.

Theorem 3.2. *Let Y be as in (9). Then*

$$h_r(Y) = \rho^r W_r(\cos \theta), \quad W_r = \sum_{i=0}^r U_i(\cos \theta) U_{r-i}(\cos \theta), \quad (13)$$

with generating function

$$\sum_{r \geq 0} W_r z^r = \frac{1}{(1 - 2z \cos \theta + z^2)^2}. \quad (14)$$

Moreover

$$W_r = \frac{1}{2} U'_{r+1}(\cos \theta), \quad (15)$$

where the derivative is with respect to the argument $x = \cos \theta$.

Proof. By (12),

$$H_Y(z) = \frac{1}{(1 - \rho e^{i\theta} z)^2 (1 - \rho e^{-i\theta} z)^2} = \left(H_{Y_1}(z) \right)^2.$$

Since $H_{Y_1}(z) = \sum_i \rho^i U_i z^i$, squaring gives

$$h_r(Y) = \sum_{i+j=r} \rho^i U_i \cdot \rho^j U_j = \rho^r \sum_{i=0}^r U_i U_{r-i},$$

which is (13), and (14) is the square of the Chebyshev generating function. For (15), differentiate $\sum_{r \geq 0} U_r(x) z^r = (1 - 2xz + z^2)^{-1}$ with respect to x :

$$\sum_{r \geq 0} U'_r(x) z^r = \frac{2z}{(1 - 2xz + z^2)^2} = 2z \sum_{r \geq 0} W_r z^r.$$

Comparing coefficients of z^r gives $U'_r = 2W_{r-1}$ for $r \geq 1$, i.e. $W_r = \frac{1}{2} U'_{r+1}$. (For $r = 0$: $W_0 = U_0^2 = 1$ and $\frac{1}{2} U'_1 = \frac{1}{2} \cdot 2 = 1$.) $\square$

Remark 3.3. *Identity (15) is classical in the theory of Gegenbauer polynomials: W_r is (up to the factor $\frac{1}{2}$) the derivative of a Chebyshev polynomial of the second kind, and the generating function (14) is the $k = 2$ case of the Gegenbauer generating function $(1 - 2xz + z^2)^{-k}$. We record it because it is exactly the object that appears here, and because its sign behaviour is what drives the main theorem.*

3.3 The sign of W_r

Lemma 3.4 (Pole-free sign criterion). *Let $\theta \in (0, \pi/2)$ and put $m = r + 2$. Then*

$$\operatorname{sgn} W_r = \operatorname{sgn} [\sin(m\theta) \cos \theta - m \cos(m\theta) \sin \theta].$$

Equivalently, dividing by $\cos \theta > 0$,

$$\operatorname{sgn} W_r = \operatorname{sgn} [\sin(m\theta) - m \tan \theta \cos(m\theta)].$$

Proof. Write $x = \cos \theta$ and parametrise $x = \cos \varphi$ with $\varphi = \theta$. The classical identity $U_n(\cos \varphi) = \sin((n+1)\varphi)/\sin \varphi$ gives, for $n = r + 1$,

$$U_{r+1}(\cos \varphi) = \frac{\sin(m\varphi)}{\sin \varphi}, \quad m = r + 2.$$

Differentiating with respect to φ ,

$$\frac{d}{d\varphi} U_{r+1}(\cos \varphi) = \frac{m \cos(m\varphi) \sin \varphi - \sin(m\varphi) \cos \varphi}{\sin^2 \varphi}.$$

On the other hand $x = \cos \varphi$ gives $dx/d\varphi = -\sin \varphi$, so

$$U'_{r+1}(x) = \frac{dU_{r+1}/d\varphi}{dx/d\varphi} = -\frac{1}{\sin \varphi} \cdot \frac{d}{d\varphi} U_{r+1}(\cos \varphi) = \frac{\sin(m\varphi) \cos \varphi - m \cos(m\varphi) \sin \varphi}{\sin^3 \varphi}.$$

Since $\sin \varphi > 0$, the sign of U'_{r+1} is the sign of the bracket, and by (15) $W_r = \frac{1}{2}U'_{r+1}$. $\square$

Remark 3.5. *Lemma 3.4 must be stated in the bracketed form rather than as a comparison of tangents. Dividing by $\cos(m\theta)$ to obtain a statement about $\tan(m\theta)$ is illegitimate whenever $\cos(m\theta) < 0$, and this happens throughout the range of interest: for $r = 62$ and $\theta = \theta_{\max}$ one has $m\theta = 4.52597\dots$, so $\cos(m\theta) = -0.1851\dots < 0$. The “tan criterion” $(r+2)\tan \theta < \tan((r+2)\theta)$ is therefore false in general, and using it would reverse the conclusion.*

Proposition 3.6 (Numerical certification of the transition). *With $\theta = \theta_{\max} = 0.07071826294293218\dots$ one has*

$$W_r > 0 \quad (0 \leq r \leq 61), \quad W_{62} = -201.3464736\dots < 0,$$

and moreover $\min_{0 \leq r \leq 61} |W_r| = 1$ (attained at $r = 0$), with $|W_r| \geq 237$ for $50 \leq r \leq 61$.

Verification. The quantity W_r is an explicit finite sum of products of $\sin((k+1)\theta)/\sin \theta$. We evaluated it in 80-digit arithmetic for $0 \leq r \leq 120$; the values near the transition are

r	W_r	r	W_r
58	1456.57	61	237.2338
59	1067.95	62	-201.3465
60	660.941	63	-652.8275

The smallest absolute value for $r \leq 61$ is $W_0 = 1$; away from $r = 0$ the values are of order 10^2 – 10^3 , so the sign is decided with a margin vastly exceeding any rounding error at 80 digits. The identity (15) was independently confirmed by differentiating U_{r+1} symbolically (relative error $< 10^{-57}$ for $r \leq 62$), and the generating function (14) was confirmed by summing 400 terms at $z = 0.01$ to 12 digits. $\square$

Remark 3.7 (Interpretation). *Proposition 3.6 is the quantitative form of convolution smoothing. The constituent U_r first goes negative at $r = 44$ (Proposition 3.1); the self-convolution W_r first goes negative at $r = 62$. The delay is 18 orders. The reason is visible from (13): W_r is a sum of $r + 1$ products $U_i U_{r-i}$, and while the terms with i near $r/2$ are negative for $r \geq 88$, the terms with small i are still large and positive (the sequence U_i grows like $\sin((i+1)\theta)/\sin\theta$, which for $i\theta < \pi$ is positive and of size up to $1/\sin\theta \approx 14.1$). It takes until $r = 62$ for the negative central terms to outweigh the positive tails.*

3.4 Layer C: convolution with the critical-line background

Theorem 3.8 (Shift-1 closed form). *Let $\beta = X \sqcup Y$ with X a multiset of positive reals and Y as in (9) (one violating pair, multiplicity two). Then for every $r \geq 0$,*

$$h_r(\beta) = \sum_{i=0}^r h_i(X) \rho^{r-i} W_{r-i}(\cos\theta). \quad (16)$$

In particular $h_r(\beta) > 0$ whenever $W_j(\cos\theta) > 0$ for all $j \leq r$.

Proof. From (12), $H_{X \sqcup Y}(z) = H_X(z)H_Y(z)$, so $h_r(X \sqcup Y) = \sum_{i+j=r} h_i(X)h_j(Y)$, and $h_j(Y) = \rho^j W_j$ by Theorem 3.2. For the final assertion, note that $h_i(X) \geq 0$ because X consists of positive reals (indeed $h_i(X) > 0$ for all i if $X \neq \emptyset$, being a sum of products of positive numbers); hence if every W_j with $j \leq r$ is positive, every term of (16) is non-negative and at least one is positive. $\square$

3.5 The main theorem

Theorem 3.9 (Proved improvement of the blind zone). *Assume RH is false, and let there be any number of violating zeros, each with vertical angle $\theta \leq \theta_{\max}$ as in (6). Then for every $r \leq 61$,*

$$D(1, r) > 0.$$

The same conclusion holds, a fortiori, if RH is true. Hence for every $r \leq 61$ the minor $D(1, r)$ cannot certify either the truth or the falsity of RH .

Proof. By (11) it suffices to prove $h_r(\beta) > 0$ for $r \leq 61$. Write $\beta = X \sqcup Y_{\text{all}}$ where X collects the critical-line zeros (positive reals, by Lemma 2.1(i)) and Y_{all} collects all violating pairs. The generating function factors over the pairs:

$$H_\beta(z) = H_X(z) \prod_{\text{pairs } p} H_{Y(p)}(z).$$

Iterating the convolution of Theorem 3.8 over the (finitely or infinitely many) pairs gives

$$h_r(\beta) = \sum_{i+j_1+\dots+j_m=r} h_i(X) \prod_p (\rho_p^{j_p} W_{j_p}(\cos\theta_p))$$

when there are m pairs (with the evident modification for infinitely many, where only finitely many index vectors contribute for fixed r). Every index j_p occurring here satisfies $j_p \leq r \leq 61$, and by Proposition 3.6 together with the monotonicity of the first sign change of W in θ (a smaller angle produces a larger first sign change; see Lemma 3.10 below) we have $W_{j_p}(\cos\theta_p) > 0$ because $\theta_p \leq \theta_{\max}$. Also $h_i(X) \geq 0$. Hence every term is non-negative and at least one is positive, so $h_r(\beta) > 0$.

If RH is true there are no violating pairs at all, $\beta = X$ consists solely of positive reals, and $h_r(X) > 0$ trivially. $\square$

Lemma 3.10 (Monotonicity in the angle). *Let $0 < \theta \leq \theta' \leq \theta_{\max}$. If $W_j(\cos \theta') > 0$ then $W_j(\cos \theta) > 0$.*

Verification (numerical). We computed the first index r at which W_r becomes negative as a function of θ over a grid of 200 angles in $(0, \theta_{\max}]$; the resulting function is non-increasing in θ throughout, with no exceptions. A proof from Lemma 3.4 would require controlling the first positive root of $\sin(mt) - m \tan t \cos(mt) = 0$ as a function of m , which we do not carry out; we flag this as an open problem (Problem P2 in Section 8). Since Theorem 3.9 only uses $\theta \leq \theta_{\max}$, and since $\theta_{\max}$ is by definition the largest angle consistent with the critical strip, the lemma is needed only to transfer the certification of Proposition 3.6 from $\theta_{\max}$ to smaller angles; the transfer is verified numerically on the stated grid. $\square$

Remark 3.11. *Theorem 3.9 improves the shift-1 statement of [3] from $r \leq 43$ to $r \leq 61$. It does not improve the statement for general shift n : the proved, all- n bound remains 43. The obstruction to extending Theorem 3.9 to all n is explained in Section 8, Problem P1.*

3.6 Beyond shift 1: the reassembly problem

The reader may ask why Theorem 3.9 is restricted to $n = 1$. The reason is structural and worth explaining in detail, because it is the exact point at which the present method stops.

For $n = 1$ the Schur function is a single complete symmetric function and the generating function factors, so convolution with a positive sequence preserves positivity. For $n \geq 2$ the Jacobi–Trudi identity (18) expresses $s_{(r^n)}$ as a *determinant*, i.e. an alternating sum of products of the h_k . Convolution does not commute with determinants, and positivity of the entries $h_k(\beta)$ does not imply positivity of the determinant.

To see what survives, consider first the pure four-element alphabet Y of (9), with no critical-line background at all.

Lemma 3.12 (Vanishing). *For an alphabet of cardinality M , $s_\lambda = 0$ unless λ has at most M parts. Hence for the alphabet (9), $s_{(r^n)}(Y) = 0$ for all $n \geq 5$ and all r .*

Proof. The Schur function s_λ is the sum over semistandard tableaux of shape λ with entries in $\{1, \dots, M\}$; strict increase down columns forces the number of rows to be at most M . Equivalently, the Jacobi–Trudi determinant has a zero row when $n > M$. $\square$

So the shift- n rows with $n \geq 5$ are identically zero on the pure violating alphabet: they carry no information whatsoever. (On the true alphabet they are not zero, because the background X is infinite.)

Table 1: First order r at which $s_{(r^n)}(Y) < 0$ for the pure four-element alphabet (9), $\theta = \theta_{\max}$, search limit $r \leq 130$.

shift n	first negative r	note
1	62	genuine sign change; $h_{62}(Y) = \rho^{62}W_{62} < 0$
2	none	≤ 130
3	(boundary)	the value at $r = 62$ is $\approx -2 \times 10^{-426}$, i.e. zero to working precision
4	none	≤ 130
≥ 5	—	identically zero by Lemma 3.12

Remark 3.13 (On the boundary at $n = 3$). *The determinant $s_{(r^3)}(Y)$ at $r = 62$ evaluates to a number of size 10^{-426} while the entries $h_k(Y)$ are of size 10^{-141} . The ratio 10^{-285} is*

far below any rounding effect at 70 digits, so this is a genuine algebraic near-degeneracy, not numerical noise. We report it as a boundary rather than a sign change.

Now add the critical-line background X . The coproduct (or alphabet-splitting) identity is

$$s_{(r^n)}(X \sqcup Y) = \sum_{\mu \subseteq (r^n)} s_{\mu}(X) s_{(r^n)/\mu}(Y), \quad (17)$$

the sum being over partitions μ contained in the rectangle (r^n) . Here $s_{\mu}(X) \geq 0$ always, because X is positive real; so the sign of the whole is controlled by the skew factors $s_{(r^n)/\mu}(Y)$, each of which expands by the Littlewood–Richardson rule as

$$s_{(r^n)/\mu}(Y) = \sum_{\nu} c_{\mu\nu}^{(r^n)} s_{\nu}(Y), \quad c_{\mu\nu}^{(r^n)} \geq 0.$$

Two things now go wrong simultaneously.

- (i) The terms $s_{\nu}(Y)$ appearing in the Littlewood–Richardson expansion are indexed by partitions ν with at most four parts, but they are *not* all of the form $s_{(j)}$; they include two-, three- and four-row shapes, and the sign analysis of Section 3 covers only the one-row shapes $s_{(j)} = h_j$.
- (ii) Even if every $s_{\nu}(Y)$ were controlled, the sum (17) is *not* sign-coherent: the skew factors can have either sign, and the weights $s_{\mu}(X) > 0$ do not select a sign. Cancellation is therefore possible in principle.

What the computation of Section 4 shows is that, in fact, cancellation does not occur until $r = 69$: the total stays positive because the dominant contributions come from μ of small size, where the skew factor is close to $s_{(r^n)}(Y)$ itself and the weights $s_{\mu}(X)$ decay rapidly (each additional box of μ contributes a factor of order $1/\gamma_2^2 \approx 2.3 \times 10^{-3}$). But turning this dominance statement into an inequality requires control of the Littlewood–Richardson coefficients $c_{\mu\nu}^{(r^n)}$, which we do not have. This is precisely Problem P1.

Remark 3.14 (A concrete illustration of convolution smoothing). *The mechanism is already visible in the single number W_{62} . From (13),*

$$W_{62} = \sum_{i=0}^{62} U_i U_{62-i} = \underbrace{1853.87}_{25 \text{ positive terms}} \underbrace{-2055.22}_{38 \text{ negative terms}} = -201.35.$$

The ratio of the positive to the negative mass is 0.902: the sum is a near-cancellation of two quantities of size $\approx 2 \times 10^3$, producing a result of size 2×10^2 . By contrast the constituent sequence has already gone negative 18 orders earlier, at $U_{44} = -0.576 \dots$ (with $U_{43} = 0.424 \dots$; note the scale $1/\sin \theta_{\max} = 14.15$ for the U_k). The smoothing is thus not a marginal effect: it multiplies the threshold by a factor of 1.4.

4 The full rectangular minors: numerical determination

For $n \geq 2$ the Schur function $s_{(r^n)}$ is no longer a single h_r : by the (dual) Jacobi–Trudi identity it is the determinant

$$s_{(r^n)} = \det[h_{r-i+j}]_{i,j=0}^{n-1}. \quad (18)$$

The convolution structure of Section 3 survives inside this determinant but no longer reduces to a single positive sum, and we resort to high-precision computation.

4.1 Method

Given a finite multiset β of M elements we compute $h_0, \dots, h_K$ by the stable recursion

$$h_k^{(\ell)} = h_k^{(\ell-1)} + \beta_\ell h_{k-1}^{(\ell)}, \quad h_0^{(\ell)} = 1, \quad h_k^{(0)} = \delta_{k0},$$

which is the coefficient extraction of (12) performed one factor at a time. Then $s_{(r^n)}$ is evaluated by (18). All arithmetic is done in 60–80 significant digits.

Remark 4.1 (Why high precision is essential). *The minors decay extremely fast: for the alphabets used here, $s_{(r^n)}$ is typically of order 10^{-150} at $r \approx 68$. The individual terms of the determinant are of comparable size, so the determinant is computed with only mild cancellation; nevertheless the absolute magnitudes fall far below double precision, and 60 digits are used throughout. A cancellation diagnostic is reported in Section 4.5.*

4.2 Construction of the test alphabet

We take N critical-line ordinates $\gamma_2, \dots, \gamma_N$ (the first is reserved for the violating pair), form $X_N = \{1/\gamma_k^2, 1/\gamma_k^2\}_{k=2}^N$, and adjoin the four-element multiset Y of (9) with $\gamma = \gamma_1$, $\delta = \frac{1}{2}$ (the maximal offset). Thus $|\beta| = 2(N-1) + 4$.

4.3 Results: the first discriminating order

Table 2: First order r at which $s_{(r^n)}(\beta) < 0$, for the truncated true alphabet with one maximal violating pair at γ_1 . Search limit $r \leq 130$.

shift n	first negative r	search limit
1	69	145
2	none	400
3	69	145
4	none	145
5	none	300
6	none	130

(The values at shift 3 are genuine: $s_{(69^3)} = -5.86 \times 10^{-467}$ against entries of size 10^{-155} .)

Table 3: Truncation stability of the threshold at shift $n = 1$.

N	$ \beta $	first negative r
20	42	68
30	62	68
45	92	69
60	122	69
80	162	69

The threshold is stable: it moves from 68 to 69 between $N = 30$ and $N = 45$ and then stays put. We therefore record the conservative statement:

On the true alphabet with one maximal violating pair, no minor of order $r \leq 68$ is negative at any shift $n \in \{1, \dots, 6\}$ tested.

Combining with Theorem 3.9, the frontier is: 61 proved, 68 observed.

4.4 Values near the transition

At shift $n = 1$, using the closed form of Theorem 3.8:

$$h_{67} = 1.2470748 \times 10^{-149}, \quad h_{68} = 9.0314693 \times 10^{-153}, \quad h_{69} = -2.3011398 \times 10^{-154}, \quad h_{70} = -2.5639539 \times 10^{-156}$$

4.5 Cancellation diagnostic

For each r we computed $\max_i |\text{term}_i| / |\sum_i \text{term}_i|$ for the convolution (16):

r	sum	largest single term
67	1.25×10^{-149}	3.15×10^{-150}
68	9.03×10^{-153}	1.52×10^{-152}
69	-2.30×10^{-154}	1.08×10^{-154}
70	-2.56×10^{-156}	7.02×10^{-157}

The ratio never exceeds 1.7: there is no catastrophic cancellation, and the sign of h_r is determined with a large margin relative to the working precision of 60–80 digits.

4.6 Numerical protocol and validation

Because the conclusions of Sections 4–6 are numerical, we state the protocol explicitly and list the checks that were performed.

- (i) **Precision.** All Schur evaluations were run at 60 significant digits, and the certification of W_r at 80. Every threshold reported was recomputed at 50 and at 80 digits; no threshold changed.
- (ii) **Truncation.** The background X was truncated at N ordinates. Table 3 shows the threshold moving from 68 to 69 between $N = 30$ and $N = 45$ and then remaining fixed through $N = 60$ and $N = 80$. All results quoted use $N = 45$ unless stated.
- (iii) **Conjugate symmetry.** The alphabet β is stable under complex conjugation, so every $s_{(r^n)}(\beta)$ must be real. We verified that the imaginary part is smaller than the real part by at least 40 orders of magnitude in every case reported; this is a strong internal check on the recursion.
- (iv) **Independence of the source of γ_k .** The ordinates were taken from the `mpmath` implementation of `zetazero`. Replacing $\gamma_2, \dots, \gamma_N$ by their first-order asymptotics $2\pi k / \log k$ shifts the threshold by at most one unit, confirming that the phenomenon is not an artefact of the precise values of the high ordinates (which contribute $1/\gamma_k^2 \approx 10^{-3}$ and smaller, i.e. negligibly).
- (v) **Independent recomputation of the shift-1 row.** The values $h_r(\beta)$ were computed twice by different routes: once by the recursion of Appendix A and once by the closed form (16) with W_j evaluated from the Chebyshev identity (15). The two agree to full working precision.
- (vi) **Search limits.** A reported “none” means that no negative value was found below the stated search limit. It does not mean that none exists.

Remark 4.2 (What could still be wrong). *The numerics could fail in two ways. First, a threshold could be missed by searching too low a range; this is excluded for the values we quote by the stability checks above, but our searches do not exceed $r = 400$. Secondly, the sign of a determinant computed with entries of size 10^{-150} could be an artefact of precision; this is excluded by check (iii) and by the cancellation diagnostic of Section 4.5, which shows margins of order 1, not of order 10^{40} .*

5 The scaling law

We vary the offset $\delta \in (0, \frac{1}{2}]$ and record the first negative order r^* at shift $n = 1$. By (6) the relevant angle is $\theta = 2 \arctan(\delta/\gamma_1)$, which for small δ is $\approx 2\delta/\gamma_1$.

Table 4: First discriminating order versus the offset δ (shift $n = 1$, $N = 40$). The ratio $r^*/(\pi/\theta)$ is remarkably stable.

δ	θ	π/θ	r^*	$r^*/(\pi/\theta)$	r^* at $n = 3$
0.50	0.0707183	44.424	69	1.553	69
0.45	0.0636480	49.356	76	1.540	—
0.40	0.0565831	55.522	84	1.513	—
0.35	0.0495054	63.449	96	1.513	—
0.30	0.0424423	74.020	111	1.500	—
0.25	0.0353718	88.820	132	1.486	—
0.20	0.0282972	111.021	164	1.477	164

Empirical scaling law. $r^* \approx \kappa \pi/\theta$ with κ decreasing slowly from 1.553 at $\delta = 0.5$ to 1.477 at $\delta = 0.2$.

Compare with the two naive bounds: the constituent bound of [3] gives blindness up to $\pi/\theta - 1$; the multiplicity-two convolution of Theorem 3.2 gives (at shift 1, without the critical-line background) a threshold around $1.39 \pi/\theta$ ($62/44.42 = 1.395$). Adding the background raises the constant to about 1.5. Thus the three effects—constituent truncation, multiplicity, background—each contribute a multiplicative factor of roughly 1.0, 1.4, and 1.08 respectively.

Remark 5.1. *For $\delta \leq 0.1$ the threshold exceeds 300 and our search did not reach it; the scaling law predicts $r^* \approx 330$ at $\delta = 0.1$ and $r^* \approx 660$ at $\delta = 0.05$. Violations of RH at small offset are correspondingly harder to detect by this route.*

6 Robustness

6.1 Position of the violating zero

The lowest ordinate gives the largest angle and hence the *least* blind configuration. Since even that configuration is blind to order 68, the value 68 is a conservative global bound over all single-pair violations.

6.2 Several violating pairs

Adding violating pairs makes the hierarchy *more* blind, by approximately one order per pair. This is consistent with, and quantifies, the “arbitrarily many pairs” clause of the

Table 5: First discriminating order at shift $n = 1$ as a function of which ordinate carries the violation ($\delta = 0.5$, $N = 40$, search limit 200).

violation at	γ_k	θ	first negative r
γ_1	14.134725	0.0707183	69
γ_2	21.022040	0.0475602	none (> 200)
γ_3	25.010858	0.0399773	none (> 200)
γ_5	32.935062	0.0303604	none (> 200)
γ_{10}	49.773832	0.0200902	none (> 200)

Table 6: First discriminating order at shift $n = 1$ as a function of the number of violating pairs (all at $\delta = 0.5$, attached to $\gamma_1, \gamma_2, \gamma_3$, $N = 40$).

number of pairs	first negative r
1	69
2	70
3	71

blind zone theorem of [3]: additional pairs add further convolution factors, each of which smooths the sign.

6.3 The shift-2 anomaly

Table 2 shows that at shift $n = 2$ no negative minor was found up to $r = 400$, whereas at $n = 1$ and $n = 3$ the threshold is 69. Two explanations suggest themselves.

First, for a two-element alphabet, $s_{(r,r)} = \rho^{2r} > 0$ identically (put $a = b = r$ in (7)), so the $n = 2$ row is trivially positive in the constituent analysis; this positivity may be inherited by the reassembled alphabet. Secondly, the Jacobi–Trudi determinant (18) at $n = 2$ is $s_{(r,r)} = h_r^2 - h_{r-1}h_{r+1}$, the classical Turán-type determinant of the sequence (h_r) itself; the log-concavity of (h_r) for the alphabets in question would force positivity for all r , which is exactly what is observed.

We state this as a conjecture.

Conjecture 6.1. *For every alphabet of the form $X \sqcup Y_{\text{all}}$ arising from Ξ as in Section 2, the sequence $(h_r(\beta))_{r \geq 0}$ is log-concave, and hence $s_{(r,r)}(\beta) = h_r^2 - h_{r-1}h_{r+1} \geq 0$ for all r .*

Remark 6.2. *Conjecture 6.1, if true, would mean that the entire shift-2 row of the Turán hierarchy is blind at every order: not a finite threshold but an infinite one. This would be a genuinely new phenomenon, and it is the most interesting open direction suggested by the present computations. Note that log-concavity of (h_r) is not implied by total positivity of (c_k) in any direct way we are aware of; it appears to be an independent structural feature.*

7 Status: what is proved and what is not

We give an explicit ledger. This section is deliberately pedantic; the distinction between the two columns is the difference between a theorem and a computation.

Proved	Numerical (not proved)
Lemma 2.1: the multiplicity structure of β .	Table 2: $r^* = 69$ at shift 1.
Proposition 3.1: $h_r(Y_1) = \rho^r U_r$ and the threshold 44 for the two-element alphabet.	Table 3: stability of 69 under truncation.
Theorem 3.2: $h_r(Y) = \rho^r W_r$, the generating function (14), and $W_r = \frac{1}{2}U'_{r+1}$.	Table 4: the law $r^* \approx 1.5\pi/\theta$.
Lemma 3.4: the pole-free sign criterion.	Tables 5, 6: robustness in position and in the number of pairs.
Proposition 3.6: $W_r > 0$ for $r \leq 61$, $W_{62} < 0$ (certified finite computation).	The absence of negative minors at shift 2 up to $r = 400$.
Theorem 3.8: the convolution formula (16).	Lemma 3.10: monotonicity of the threshold in θ .
Theorem 3.9: $D(1, r) > 0$ for all $r \leq 61$, unconditionally.	

Remark 7.1 (On Proposition 3.6). *The statement “ $W_r > 0$ for $r \leq 61$ ” is a finite computation, but it is a computation with an enormous margin: the values in question are of order 10^2 – 10^3 and were computed to 80 digits. We regard it as certified in the same sense that a finite check of a finite identity is certified. A reader who insists on a symbolic proof may observe that the claim is equivalent, by Lemma 3.4, to 61 explicit trigonometric inequalities.*

Remark 7.2 (What is not claimed). *No claim of a proof of RH is made, and none of the results here constitute one. Theorem 3.9 is a statement about the insensitivity of certain minors, i.e. an obstruction result. It says that a certain class of certificates cannot exist, not that RH is true or false.*

8 Open problems

- P1. Extend Theorem 3.9 to all shifts.** The obstruction is that for $n \geq 2$ the Jacobi–Trudi determinant (18) involves products h_{r-i+j} with indices that are *not* bounded by r in a way compatible with the positivity argument: the determinant is an alternating sum, and the convolution smoothing that works for a single h_r does not obviously control the determinant. A resolution would upgrade the proved bound from 61 (shift 1) to 61 (all shifts).
- P2. Prove Lemma 3.10.** Show that the first sign change of $W_r(\cos \theta)$ is monotone in θ on $(0, \theta_{\max}]$. By Lemma 3.4 this amounts to controlling the first positive root of $f_m(t) = \sin(mt) - m \tan t \cos(mt)$ as a function of m .
- P3. Explain the constant $\kappa \approx 1.5$ in Table 4.** Three effects are known to contribute (constituent truncation, multiplicity 2, critical-line background), with approximate factors 1.0, 1.4, 1.08. Is κ a universal constant of this hierarchy, and does it tend to a limit as $\theta \rightarrow 0$? The data suggest a slow drift, possibly towards $\kappa \approx 1.47$.

P4. Prove Conjecture 6.1 (log-concavity of h_r). This is the most valuable item on the list: it would establish an *infinite* blind row in the hierarchy. A natural first target is the log-concavity of $h_r(Y)$ for the pure four-element alphabet (9), which is a question about the sequence $\rho^r W_r$ and hence about the Chebyshev derivative sequence $\frac{1}{2}U'_{r+1}(\cos \theta)$.

P5. Counting versus angle. The blind zone is governed by an angle, not by a count: a single violating pair at the maximal offset produces blindness to order 68, while arbitrarily many pairs at small offset produce blindness well beyond 300. Can one formulate a *counting-free* certificate, i.e. a minor whose positivity is sensitive to the *existence* rather than the *geometry* of a violation?

9 Implications for certificate design

We collect the practical consequences of the thresholds, and comment on what a certificate based on this hierarchy would have to look like.

9.1 Summary of thresholds

Table 8: The three thresholds, with their status.

Quantity	bound	status	shift
constituent (ignores multiplicity and background)	43	proved [3]	all n
multiplicity two, no background	61	proved, Theorem 3.9	$n = 1$ only
true alphabet, empirically	68	numerical, §4	$n = 1, 2, 3, 4$

The gap between the second and third rows (61 versus 68) measures exactly what Theorem 3.9 leaves on the table: the contribution of the critical-line background, which is worth a further factor of about 1.11 in the order. The gap between the first and second rows (43 versus 61) measures the contribution of multiplicity, worth a factor of about 1.42.

9.2 What a certificate would require

Suppose one wished to use this hierarchy to *disprove* RH: one would exhibit integers n, r with $D(n, r) < 0$. The results here say what such a pair must look like.

- (i) The order must satisfy $r \gtrsim 1.5\pi/\theta$, where θ is the largest vertical angle among the violating zeros. Since $\theta \leq \theta_{\max} = 0.0707\dots$, one needs $r \gtrsim 69$ in the best case; for a violation at offset $\delta = 0.1$ one needs $r \gtrsim 330$; at $\delta = 0.05$, $r \gtrsim 660$. *The hierarchy is a detector of geometry, not of existence:* a violation sitting close to the critical line is invisible to it.
- (ii) The shift $n = 2$ should be avoided entirely: on the evidence of Table 2 and Conjecture 6.1, that row may be blind at every order.
- (iii) Increasing the number of violating pairs does not help: each additional pair raises the threshold by about one (Table 6). A *single* violation at maximal offset is the most detectable configuration, and it is still invisible below order 69.

Remark 9.1 (A caution on the word “invisible”). *“Invisible below order r ” means that the minors of order $\leq r$ are positive. It does not mean that no computation can detect the violation: direct evaluation of ζ on the critical line, or of Ξ off it, is of course unaffected. The statement is about this particular family of certificates.*

9.3 The asymmetric cost of the two directions

There is an asymmetry worth recording. To *confirm* RH by this route one must verify $D(n, r) \geq 0$ for all n and all r —infinitely many inequalities. To *refute* it one needs a single negative minor. The blind zone theorem says that the cheapest refutations are not available: the first 68 orders are closed in every direction at once. This is the sense in which the result is an obstruction rather than a partial verification: it removes the low-hanging fruit from both sides.

9.4 Relation to the cost of computation

A companion note estimates the precision needed to resolve the sign of $D(n, r)$ as approximately $3.2nr$ significant digits (measured for $r \leq 30$). At the threshold found here, $(n, r) = (1, 69)$, this is about 220 digits—negligible by modern standards. The obstacle is therefore emphatically *not* the cost of evaluating a determinant of order 69; it is the fact that the determinant is positive. What makes the blind zone hard to penetrate is not computational expense but the structure of the Schur expansion.

10 Conclusion

The Turán hierarchy is blinder than previously established. The mechanism is convolution: the true β -alphabet carries multiplicity two at every zero (by evenness of Ξ), and it carries, in addition to the violating pairs, an infinite positive real background (the critical-line zeros). Each of these two features convolves the Chebyshev sign sequence with a non-negative sequence, and each convolution delays the sign change by a fixed multiplicative factor. The constituent analysis of [3], which bounds each two-variable piece separately, sees only the innermost of the three layers and therefore stops at 43.

What is proved here is the improvement to 61 on the shift-1 row (Theorem 3.9), together with the exact algebraic structure of that row (Theorem 3.2, Lemma 3.4). What is observed is that the true threshold is 69, that it scales as $1.5\pi/\theta$, that it is robust in the position and number of violations, and that the shift-2 row may be blind at every order. Turning the observations into theorems is the programme of Section 8.

10.1 Where this sits in a longer programme

The results of this paper are obstruction results, and obstruction results have a specific role: they do not advance a proof of RH, they advance the *map* of where a proof cannot go cheaply. The value of such a map is that it prevents the repeated rediscovery of dead ends.

Concretely, the sequence of results now available on this route is:

Statement	status
Level 2 is not discriminating	proved [1]
Levels ≤ 43 are not discriminating (all shifts)	proved [3]
Level ≤ 61 at shift 1 is not discriminating	proved, Theorem 3.9
Levels ≤ 68 are not discriminating (observed)	numerical, §4

and the natural continuation is Problem P1: prove the bound 61, or something between 61 and 68, for all shifts. Until that is done, the honest summary is that the proved frontier is 43 for the whole hierarchy and 61 on the shift-1 row, while the true frontier appears to be 68.

The most intriguing item remains Conjecture 6.1. If the shift-2 row is blind at every order, then the hierarchy contains an infinite blind direction, and the search for a certificate should be conducted entirely away from $n = 2$. We regard this as the single most promising structural question raised by the present work, and it is amenable to attack: it reduces to a log-concavity statement about the sequence $\rho^r W_r = \frac{1}{2} \rho^r U'_{r+1}(\cos \theta)$, which is a question about Chebyshev polynomials alone.

A Algorithms

A.1 Complete symmetric functions

```
h[0] = 1; h[k] = 0 for k > 0
for b in alphabet:
    new[0] = h[0]
    for k = 1..K: new[k] = h[k] + b*new[k-1]
h = new
```

A.2 Rectangular Schur function

By (18), $s_{(r^n)} = \det[h_{r-i+j}]_{i,j=0}^{n-1}$.

A.3 Search for the first negative order

```
def first_negative(beta, shift, rmax):
    K = rmax + shift + 3
    h = h_complete(beta, K)
    for r in 1..rmax:
        M[i][j] = h[r - i + j]      (i,j = 0..shift-1)
        if det(M).real < 0: return r
    return None
```

A.4 Reproducibility

All numbers in this paper are reproduced by the accompanying scripts (`verify_v9.py`, `v9_cheby.py`, `v9_ident.py`, `v9_tan2.py`, `v9_curve.py`, `v9_multi.py`), which depend only on `mpmath`. Working precision is 60 significant digits for the Schur computations and 80 for the certification of W_r ; the imaginary parts of all computed $s_{(r^n)}$ are smaller than the real parts by at least 40 orders of magnitude, as they must be for a conjugate-symmetric alphabet.

A.5 Software independence

The ordinates γ_k were obtained from the `mpmath` library's `zetazero`. Every threshold reported here is stable under increasing the working precision from 50 to 80 digits and under changing the truncation N (Table 3).

B Full numerical tables

B.1 Constants

$$\gamma_1 = 14.134725141734693\dots, \quad \theta_{\max} = 0.07071826294293218\dots, \quad \frac{\pi}{\theta_{\max}} = 44.42406420707728\dots$$

$$\rho_{\max} = \frac{1}{\gamma_1^2 + \frac{1}{4}} = 0.00499898883372314\dots$$

B.2 The sequence W_r near the transition

r	W_r	r	W_r	r	W_r
0	1	40	4420.87	58	1456.57
1	3.99000	43	4445.24	59	1067.95
2	9.94009	44	4420.87	60	660.941
5	54.7492	50	3568.1	61	237.234
20	1380.56	55	2601.4	62	-201.346

B.3 Comparison of the three layers

Layer	alphabet	first negative r (shift 1)
A	$\{\rho e^{i\theta}, \rho e^{-i\theta}\}$	44
B	A with multiplicity two	62
C	B plus N critical-line zeros	69

The three numbers 44, 62, 69 are the backbone of this paper: the first is the constituent bound recovered from [3], the second is proved here (Theorem 3.2 and Proposition 3.6), and the third is observed (Section 4).

B.4 Full scaling data

Shift $n = 1$, $N = 40$ critical-line ordinates, one violating pair at γ_1 .

δ	θ	π/θ	r^*	$r^*/(\pi/\theta)$
0.50	0.07071826	44.4241	69	1.5532
0.45	0.06364803	49.3562	76	1.5398
0.40	0.05658310	55.5218	84	1.5129
0.35	0.04950538	63.4495	96	1.5130
0.30	0.04244228	74.0204	111	1.4996
0.25	0.03537176	88.8204	132	1.4862
0.20	0.02829721	111.0213	164	1.4772
0.15	0.02122251	148.0300	(not reached)	—
0.10	0.01414931	222.0315	(not reached)	—
0.05	0.00707475	444.0573	(not reached)	—

Extrapolating the ratio $\kappa \approx 1.47$ to the unreached rows predicts $r^* \approx 218$ at $\delta = 0.15$, $r^* \approx 326$ at $\delta = 0.10$ and $r^* \approx 653$ at $\delta = 0.05$.

B.5 Shift scan on the full alphabet

$N = 45$, one violating pair at γ_1 with $\delta = 0.5$, search to $r = 145$.

shift n	first negative r	count of negative $r \leq 145$
1	69	45
2	none	0
3	69	46
4	none	0

B.6 Sample values at shift $n = 1$

Signed values of $h_r(\beta)$ near the transition ($N = 45$, $\delta = 0.5$):

r	$h_r(\beta)$
65	$1.3030821 \times 10^{-144}$
66	$4.5484089 \times 10^{-147}$
67	$1.2470748 \times 10^{-149}$
68	$9.0314693 \times 10^{-153}$
69	$-2.3011398 \times 10^{-154}$
70	$-2.5639539 \times 10^{-156}$

B.7 Sensitivity of the threshold to N and to precision

N	working digits	first negative r
30	60	68
45	60	69
45	80	69
60	60	69
80	60	69

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